



Graded integral domains and Nagata rings

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ABSTRACT

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be an integral domain graded by an arbitrary torsionless grading monoid Γ . For any $f \in R$, let $C(f)$ be the ideal of R generated by the homogeneous components of f , and let $N(H) = \{g \in R \mid C(g)_{\nu} = R\}$. In this paper, we study relationships between the ideal-theoretic properties of $R_{N(H)}$ and the homogeneous ideal-theoretic properties of R . For example, we show that R is a graded Krull domain if and only if $R_{N(H)}$ is a Dedekind domain, if and only if $R_{N(H)}$ is a PID; and that if R contains a unit of nonzero degree, then R is a PvMD if and only if $R_{N(H)}$ is a Prüfer domain, if and only if each ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R .

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0. Introduction

0.1. Graded integral domains

Let Γ be a torsionless grading monoid, that is, Γ is a commutative cancellative monoid (written additively), and the quotient group of Γ , $\langle \Gamma \rangle = \{a - b \mid a, b \in \Gamma\}$, is a torsionfree abelian group. It is well known that a cancellative monoid Γ is torsionless if and only if Γ can be given a total order compatible with the monoid operation [20, p. 123]. By a (Γ) -graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$, we mean an integral domain graded by an arbitrary torsionless grading monoid Γ . That is, each nonzero $x \in R_{\alpha}$ has degree α , i.e., $\deg(x) = \alpha$. Thus, each nonzero $f \in R$ can be written as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $x_{\alpha_i} \neq 0$, $\deg(x_{\alpha_i}) = \alpha_i$, and $\alpha_1 < \cdots < \alpha_n$. Also, note that if we let $\text{Supp}(\Gamma) = \{\alpha \in \Gamma \mid R_{\alpha} \neq (0)\}$, then $R = \bigoplus_{\alpha \in \text{Supp}(\Gamma)} R_{\alpha}$ and $\text{Supp}(\Gamma)$ is a submonoid of Γ because R

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is an integral domain; hence we assume that $R_\alpha \neq (0)$ for every $\alpha \in \Gamma$. The simplest example of a Γ -graded integral domain is the monoid domain $D[\Gamma] = \bigoplus_{\alpha \in \Gamma} DX^\alpha$ of Γ over an integral domain D with $\deg(aX^\alpha) = \alpha$ for each $0 \neq a \in D$ and $\alpha \in \Gamma$. In particular, if $\Gamma = \mathbb{Z}_+$, the additive semigroup of nonnegative integers, then the monoid domain $D[\Gamma]$ is just the polynomial ring $D[X]$ over D .

Let H be the saturated multiplicative set of nonzero homogeneous elements of R . Then R_H , called the homogeneous quotient field of R , is a graded integral domain whose nonzero homogeneous elements are units. We know that R_H is a completely integrally closed GCD-domain [2, Proposition 2.1]. For $f \in R_H$, let $C(f)$ denote the fractional ideal of R generated by the homogeneous components of f . For a fractional ideal I of R with $I \subseteq R_H$, let $C(I) = \sum_{f \in I} C(f)$. It is clear that $C(f)$ and $C(I)$ are both homogeneous fractional ideals of R . Let K be the quotient field of an integral domain D , and let $K[\langle \Gamma \rangle]$ be the group ring of $\langle \Gamma \rangle$ over K . For $g = a_0X^{\alpha_0} + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n} \in K[\langle \Gamma \rangle]$, where $a_i \in K$ and $\alpha_j \in \langle \Gamma \rangle$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$, let A_g (resp., E_g) be the fractional ideal of D (resp., Γ) generated by the coefficients (resp., exponents) of g ; so $A_g = (a_0, a_1, \dots, a_n)$ and $E_g = (\alpha_0 + \Gamma) \cup (\alpha_1 + \Gamma) \cup \cdots \cup (\alpha_n + \Gamma)$. Clearly, $C(g) \subseteq A_g D[\langle \Gamma \rangle] \cap K[E_g]$ for any $g \in K[\langle \Gamma \rangle]$. For more on graded integral domains and their divisibility properties, see [2,20].

0.2. Motivations and results

Let $D[X]$ be the polynomial ring over an integral domain D , and let $N_v = \{f \in D[X] \mid (A_f)_v = D\}$. (Definitions related to the v -operation will be reviewed in Section 0.3.) Then the ring $D[X]_{N_v}$, called the (t) -Nagata ring of D , has many interesting ring-theoretic properties. For example, D is a Prüfer v -multiplication domain (PvMD) if and only if $D[X]_{N_v}$ is a Prüfer domain, if and only if each ideal of $D[X]_{N_v}$ is extended from D ; D is a Krull domain if and only if $D[X]_{N_v}$ is a PID; and $Cl(D[X]_{N_v}) = 0$. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ and $N(H) = \{f \in R \mid C(f)_v = R\}$; then $N(H)$ is a saturated multiplicative subset of R by Lemma 1.1(2). It is clear that if we let $R = D[X, X^{-1}]$, then $D[X]_{N_v} = R_{N(H)}$ (see Proposition 3.1). So it is natural to ask what ring-theoretic properties the ring $R_{N(H)}$ satisfies.

In this paper, we study relationships between the ideal-theoretic properties of $R_{N(H)}$ and the homogeneous ideal-theoretic properties of R . More precisely, in Section 1, we study a graded integral domain that satisfies property (#): If $C(I)_t = R$ for a nonzero ideal I of R , then there exists an $f \in I$ such that $C(f)_v = R$. For example, we show that R satisfies property (#) if and only if $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \text{ is a homogeneous maximal } t\text{-ideal of } R\}$. We give some examples of graded integral domains with property (#). We also prove that if R satisfies property (#), then (i) each maximal ideal of $R_{N(H)}$ is a t -ideal, (ii) a nonzero homogeneous ideal I of R is t -invertible if and only if $IR_{N(H)}$ is invertible, (iii) for $0 \neq f \in R$, $C(f)$ is t -invertible if and only if $C(f)R_{N(H)} = fR_{N(H)}$, (iv) if R is a PvMD, then each ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R , and (v) R is a PvMD if and only if $R_{N(H)}$ is a Prüfer domain.

In Section 2, we study graded Krull domains. In particular, we show that R is a graded Krull domain if and only if $R_{N(H)}$ is a PID, if and only if $R_{N(H)}$ is a Dedekind domain. Finally, in Section 3, we study graded integral domains with a unit of nonzero degree, where we generalize some of the results on the ring $D[X]_{N_v}$. Precisely, we show that $Cl(R_{N(H)}) = 0$ and that R is a PvMD if and only if each ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R , if and only if R is integrally closed and $I_w = IR_{N(H)} \cap R$ for all nonzero homogeneous ideal I of R . Also, we generalize the notion of Kronecker function ring, and we then show that this ring is a Bezout domain.

0.3. Definitions related to star operations

To facilitate the reading of this paper, we review some definitions on star operations. Let D be an integral domain with quotient field K . Let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D . A map $*$: $\mathbf{F}(D) \rightarrow \mathbf{F}(D)$, $I \mapsto I^*$, is called a *star operation* on D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$: (i) $(aD)^* = aD$ and $(aI)^* = aI^*$, (ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$, and (iii) $(I^*)^* = I^*$. Given a star operation $*$ on D , we can construct two new star operations $*_f$ and $*_w$ by setting $I^*_{*f} = \bigcup \{J^* \mid J \in \mathbf{f}(D) \text{ and } J \subseteq I\}$ and $I^*_{*w} = \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathbf{f}(D) \text{ with } J^* = D\}$ for all $I \in \mathbf{F}(D)$. Clearly, $(*)_f = *_f$ and $(*)_w = *_w = (*_f)_w$. Also, $I^* = I^{*f}$ for all $I \in \mathbf{f}(D)$ and $I^*_{*w} \subseteq I^{*f} \subseteq I^*$ for all $I \in \mathbf{F}(D)$.

A star operation $*$ on D is said to be of *finite character* if $*_f = *$; hence both $*_f$ and $*_w$ are of finite character. An $I \in \mathbf{F}(D)$ is called a $*$ -ideal if $I^* = I$. A $*$ -ideal $I \in \mathbf{F}(D)$ is said to be of *finite type* if $I = J^*$ for some $J \in \mathbf{f}(D)$. A $*$ -ideal is called a *maximal $*$ -ideal* if it is maximal among proper integral $*$ -ideals. Let $\text{Max}(D)$ be the set of maximal $*$ -ideals of D . It is well known that $*_f\text{-Max}(D) \neq \emptyset$ if D is not a field; each maximal $*_f$ -ideal is a prime ideal; each proper $*_f$ -ideal is contained in a maximal $*_f$ -ideal; and each prime ideal minimal over a $*_f$ -ideal is a $*_f$ -ideal. An $I \in \mathbf{F}(D)$ is said to be *$*$ -invertible* if $(II^{-1})^* = D$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, while D is a *Prüfer $*$ -multiplication domain* (P^*MD) if each nonzero finitely generated ideal of D is $*_f$ -invertible; equivalently, D_P is a valuation domain for all $P \in *_f\text{-Max}(D)$ [15, Theorem 1.1].

Examples of the most well-known star operations include the v -, t -, w - and d -operations. The v -operation is defined by $I_v = (I^{-1})^{-1}$. The t -operation is defined by $t = v_f$, and the w -operation is given by $w = v_w$. The d -operation is just the identity function on $\mathbf{F}(D)$, i.e., $I^d = I$ for all $I \in \mathbf{F}(D)$; so $d_f = d_w = d$. It is well known that $t\text{-Max}(D) = w\text{-Max}(D)$ and $I_w = \bigcap_{P \in t\text{-Max}(D)} ID_P$ for every $I \in \mathbf{F}(D)$. Let $T(D)$ (resp., $\text{Inv}(D)$, $\text{Prin}(D)$) be the group of t -invertible fractional t -ideals (resp., invertible fractional ideals, nonzero principal fractional ideals) of D under the t -multiplication $I * J = (IJ)_t$. It is obvious that $\text{Prin}(D) \subseteq \text{Inv}(D) \subseteq T(D)$. The t -class group of D is the abelian group $\text{Cl}(D) = T(D)/\text{Prin}(D)$ and the *Picard group* (or *ideal class group*) of D is the subgroup $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$ of $\text{Cl}(D)$. It is clear that if each maximal ideal of D is a t -ideal, then $\text{Pic}(D) = \text{Cl}(D)$. For more on the t -class group, see the survey article [6].

1. Preliminary results on the ring $R_{N(H)}$

Let Γ be a torsionless grading monoid, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain graded by Γ , H be the set of nonzero homogeneous elements of R , and $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$. Let D be an integral domain with quotient field K , X be an indeterminate over D , and $D[X]$ be the polynomial ring over D .

The Dedekind–Mertens Lemma says that for $f, g \in D[X]$, there is an integer $n \geq 1$ such that $(A_f)^{n+1}A_g = (A_f)^n A_{fg}$; it plays a very important role when we study the polynomial ring $D[X]$. We begin this section by a graded integral domain analogue of the Dedekind–Mertens Lemma, which was proved by Northcott [19].

Lemma 1.1. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain.

- (1) If $f, g \in R_H$, then $C(f)^{n+1}C(g) = C(f)^nC(fg)$ for some integer $n \geq 1$.
- (2) $N(H)$ is a saturated multiplicative subset of R .

Proof. (1) Since $f, g \in R_H$, there are $a, b \in H$ such that $af, bg \in R$. Hence $a^{n+1}b(C(f)^{n+1}C(g)) = C(af)^{n+1}C(bg) = C(af)^nC(af \cdot bg) = a^{n+1}b(C(f)^nC(fg))$ for some integer $n \geq 1$ (see [19] or the proof of [21, Proposition 3.2] for the second equality). Thus, $C(f)^{n+1}C(g) = C(f)^nC(fg)$.

(2) Let $f, g \in R$. Then $C(f)^{n+1}C(g) = C(f)^nC(fg)$ for some integer $n \geq 1$ by (1) and $C(fg) \subseteq C(f)C(g)$. Thus, $fg \in N(H) \Leftrightarrow C(fg)_v = R \Leftrightarrow C(f)_v = C(g)_v = R \Leftrightarrow f, g \in N(H)$. \square

Lemma 1.2. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain.

- (1) $N(H) = \{0 \neq x \in R \mid (x, s)_v = R \text{ for all } s \in H\}$.
- (2) $R = R_H \cap R_{N(H)}$.
- (3) For $0 \neq f \in R$, $fR_H \cap R = fR$ if and only if $C(f)_v = R$.
- (4) If R is integrally closed, then $fR_H \cap R = fC(f)^{-1}$.
- (5) If Q is a maximal t -ideal of R with $Q \cap H \neq \emptyset$, then Q is homogeneous.
- (6) R is integrally closed if and only if $R_{N(H)}$ is integrally closed.
- (7) If $R_{N(H)}$ is completely integrally closed, then R is completely integrally closed.

Proof. (1) [7, Lemma 1.1].

(2) Let $\frac{f}{x} = \frac{h}{g} \in R_H \cap R_{N(H)}$, where $f, h \in R, x \in H$, and $g \in N(H)$. Then $\frac{f}{x} \cdot g = h$, and so there is an integer $n \geq 1$ such that $C(g)^{n+1}C(\frac{f}{x}) = C(g)^nC(\frac{f}{x} \cdot g) = C(g)^nC(h)$. Hence $\frac{f}{x} \in C(\frac{f}{x})_v = (C(g)^{n+1}C(\frac{f}{x}))_v = (C(g)^nC(h))_v = C(h)_v \subseteq R$. Thus, $R_H \cap R_{N(H)} \subseteq R$. The reverse containment is clear.

(3) (\Rightarrow) Let $u \in C(f)^{-1}$. Then $uC(f) \subseteq R$, and so $u \in R_H$. Hence $uf \in fR_H \cap R = fR$ by assumption. Thus, $u \in R$, which means $C(f)^{-1} = R$ or $C(f)_v = R$. (\Leftarrow) For $x \in H$ and $g \in R$, let $f \cdot \frac{g}{x} = h \in fR_H \cap R$. Then $fg = xh$, and since there is an integer $m \geq 1$ such that $C(f)^{m+1}C(g) = C(f)^mC(fg)$, we have $C(g)_v = C(fg)_v = xC(h)_v$. Thus, $\frac{g}{x} \in C(\frac{g}{x})_v = C(h)_v \subseteq R$. Hence $fR_H \cap R \subseteq fR$. The reverse containment is clear, and therefore $fR_H \cap R = fR$.

(4) Clearly, $fC(f)^{-1} \subseteq fR_H \cap R$. For the reverse containment, let $h = f \cdot \frac{g}{u} \in fR_H \cap R$, where $u \in H$ and $g \in R$. Then $fg = uh$, and so $C(f)^{m+1}C(g) = C(f)^mC(fg) = C(f)^mC(uh) = uC(f)^mC(h)$ for some integer $m \geq 1$. Hence $C(f)^m[C(f)C(\frac{g}{u})] = C(f)^mC(h) \subseteq C(f)^m$. Thus, $C(f)C(\frac{g}{u})$ is integral over R (cf. [18, Theorem 12]), and since R is integrally closed, we have $C(f)C(\frac{g}{u}) \subseteq R$. Hence $\frac{g}{u} \in C(f)^{-1}$ and $h \in fC(f)^{-1}$.

(5) [7, Lemma 1.2].

(6) and (7) These follow directly from (2) because R_H is a completely integrally closed GCD-domain. \square

Proposition 1.3. (Cf. [17, Proposition 2.2].) Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let I be a nonzero homogeneous fractional ideal of R .

- (1) $(IR_{N(H)})^{-1} = I^{-1}R_{N(H)}$.
- (2) $(IR_{N(H)})_v = I_vR_{N(H)}$.
- (3) $(IR_{N(H)})_t = I_tR_{N(H)}$.
- (4) If Q is a homogeneous maximal t -ideal of R , then $QR_{N(H)}$ is a maximal t -ideal of $R_{N(H)}$.

Proof. (1) Clearly, $I^{-1}R_{N(H)} \subseteq (IR_{N(H)})^{-1}$. For the reverse containment, let $u \in (IR_{N(H)})^{-1}$. For any $0 \neq \alpha \in I \cap H$, we have $(IR_{N(H)})^{-1} \subseteq \alpha^{-1}R_{N(H)} \subseteq (R_H)_{N(H)}$; so $u = \frac{f}{g}$ with $f \in R_H$ and $g \in N(H)$. Then $\frac{f}{g}(IR_{N(H)}) \subseteq R_{N(H)}$, and hence $fI \subseteq fIR_{N(H)} \subseteq R_{N(H)}$. Thus $\beta f \in R_{N(H)}$ for any $\beta \in I \cap H$. Now, $\beta fh \in R$ for some $h \in N(H)$. So by Lemma 1.1(1), $R \supseteq C(\beta fh)_v = \beta C(fh)_v = \beta C(f)_v \supseteq \beta C(f)$, where the last equality follows because $C(h)_v = R$. Since I is homogeneous, $C(f) \subseteq I^{-1}$. Hence $f \in I^{-1}$, and thus $u = \frac{f}{g} \in I^{-1}R_{N(H)}$. Therefore $(IR_{N(H)})^{-1} \subseteq I^{-1}R_{N(H)}$.

(2) Since I is homogeneous, I^{-1} is also homogeneous. Thus by (1), we have $(IR_{N(H)})_v = (I^{-1}R_{N(H)})^{-1} = I_vR_{N(H)}$.

(3) Let J be a nonzero finitely generated ideal of R such that $J \subseteq I$. Then, since I is homogeneous, we may assume that J is also homogeneous, and so $J_v \subseteq J_vR_{N(H)} = (JR_{N(H)})_v \subseteq (IR_{N(H)})_t$ by (2). Thus $I_tR_{N(H)} \subseteq (IR_{N(H)})_t$. For the reverse containment, let A be a nonzero finitely generated subideal of $IR_{N(H)}$. Then there is a finitely generated ideal $I_0 \subseteq I$ such that $A = I_0R_{N(H)}$, and by replacing I_0 with $C(I_0)$, we may assume that I_0 is homogeneous. Again by (2), we have $A_v = (I_0R_{N(H)})_v = (I_0)_vR_{N(H)} \subseteq I_tR_{N(H)}$. Thus $(IR_{N(H)})_t \subseteq I_tR_{N(H)}$.

(4) By (3), $(Q_{N(H)})_t = Q_tR_{N(H)} = QR_{N(H)}$; so $QR_{N(H)}$ is a prime t -ideal. Also, since Q is a maximal t -ideal of R , $QR_{N(H)}$ is a maximal t -ideal of $R_{N(H)}$ by [17, Lemma 3.17] that if I is an ideal of R such that $IR_{N(H)}$ is a t -ideal, then $IR_{N(H)} \cap R$ is a t -ideal of R . \square

Let $R = D[X, X^{-1}] = \bigoplus_{n \in \mathbb{Z}} DX^n$; so R is a \mathbb{Z} -graded integral domain. It is easy to show that $R_{N(H)} = D[X]_{N_v}$, where $N_v = \{f \in D[X] \mid (Af)_v = D\}$ (see Proposition 3.1). In particular, by [17, Proposition 2.1],

$$\begin{aligned} \text{Max}(R_{N(H)}) &= \{P[X]_{N_v} \mid P \in t\text{-Max}(D)\} \\ &= \{Q_{N(H)} \mid Q \text{ is a homogeneous maximal } t\text{-ideal of } R\}. \end{aligned}$$

So it is natural to ask when $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$, where Ω is the set of homogeneous maximal t -ideals of R . We next give an answer to this question, which plays a key role in subsequent arguments of this paper.

Proposition 1.4. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let Ω be the set of maximal t -ideals Q of R with $Q \cap H \neq \emptyset$.

- (1) $N(H) = R \setminus \bigcup_{Q \in \Omega} Q$.
- (2) $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ if and only if R has the property that if I is a nonzero ideal of R with $C(I)_t = R$, then $I \cap N(H) \neq \emptyset$.

Proof. (1) Let $x \in R$. Then $x \in N(H) \Leftrightarrow C(x)_v = R \Leftrightarrow C(x) \not\subseteq Q$ for all $Q \in \Omega \Leftrightarrow x \notin Q$ for all $Q \in \Omega \Leftrightarrow x \in R \setminus \bigcup_{Q \in \Omega} Q$ by Lemma 1.2(5).

(2) (\Rightarrow) Let I be a nonzero ideal of R such that $C(I)_t = R$. Then $I \not\subseteq Q$ for all $Q \in \Omega$, and hence $IR_{N(H)} = R_{N(H)}$. Thus $I \cap N(H) \neq \emptyset$.

(\Leftarrow) Let I be a nonzero ideal of R such that $I \subseteq \bigcup_{Q \in \Omega} Q$. If $C(I)_t = R$, then, by assumption, there exists an $f \in I$ with $C(f)_v = R$. But, since $I \subseteq \bigcup_{Q \in \Omega} Q$, we have $f \in Q$ for some $Q \in \Omega$, a contradiction. Thus $C(I)_t \subsetneq R$, and hence $I \subseteq Q$ for some $Q \in \Omega$ by Lemma 1.2(5). Thus $\{Q_{N(H)} \mid Q \in \Omega\}$ is the set of maximal ideals of $R_{N(H)}$ [13, Proposition 4.8]. \square

In this paper, we are mainly interested in graded integral domains $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with the property of Proposition 1.4(2). For convenience, we will say that R **satisfies property (#)** if, for any nonzero ideal I of R , $C(I)_t = R$ implies that there exists an $f \in I$ such that $C(f)_v = R$.

Corollary 1.5. If every element of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is homogeneous, i.e., $\Gamma = \{0\}$, then R satisfies property (#) if and only if each maximal ideal of R is a t -ideal. In this case, $R_{N(H)} = R$.

Proof. It is clear that if every element of R is homogeneous, then $N(H)$ is the set of units of R , and so $R_{N(H)} = R$. Thus the result follows from Proposition 1.4(2). \square

As we noted in the remark before Proposition 1.4, $D[X, X^{-1}]$ satisfies property (#). We next give some examples of graded integral domains with property (#).

Example 1.6. A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ satisfies property (#) if R is one of the following domains.

- (1) R contains a (homogeneous) unit of nonzero degree.
- (2) $R = D[\Gamma]$, where $\Gamma \neq \{0\}$.
- (3) Every homogeneous element of R is a unit. In this case, $R_{N(H)}$ is the quotient field of R .
- (4) R contains a nonzero homogeneous prime element of nonzero degree (for example, $R = D[\{X_\alpha\}]$ for $\{X_\alpha\}$ a nonempty set of indeterminates).

Proof. Let I be a nonzero ideal of R such that $C(I)_t = R$. Then there exist nonzero elements $f_1, \dots, f_k \in I$ such that $C(f_1, \dots, f_k)_v = R$.

(1) Let $x \in R$ be a unit with $\deg(x) > 0$. Then there is a positive integer n_1 such that $C(f_1 + f_2x^{n_1}) = C(f_1) + C(f_2)$. Repeating this process, we have positive integers n_2, \dots, n_{k-1} such that $C(f_1 + f_2x^{n_1} + f_3x^{n_2} + \dots + f_kx^{n_{k-1}}) = C(f_1) + \dots + C(f_k) = C(f_1, \dots, f_k)$. Let $f = f_1 + f_2x^{n_1} + f_3x^{n_2} + \dots + f_kx^{n_{k-1}}$. Then $f \in I$ and $C(f)_v = R$.

(2) By (1), we may assume that $\Gamma \cap -\Gamma = \{0\}$, and hence each nonzero element of Γ is greater than 0.

Case 1. D is infinite. We can choose $\alpha_{i+1} \in \Gamma$ for $i = 1, \dots, k-1$ such that $\deg(f_1) < \alpha_2$ and $\deg(f_iX^{\alpha_i}) < \alpha_{i+1}$ for $i = 2, \dots, k-1$. Let $f = f_1 + f_2X^{\alpha_2} + \dots + f_kX^{\alpha_k}$; then $A_{f_1} + \dots + A_{f_k} = A_f$.

So $D[\Gamma] = C(f_1, \dots, f_k)_v \subseteq (A_f D[\Gamma])_v = (A_f)_v D[\Gamma] \subseteq D[\Gamma]$, and hence $(A_f)_v = D$. Thus $f \notin \bigcup_{P \in t\text{-Max}(D)} PD[\Gamma]$.

Next, since D is infinite, we can choose a $0 \neq a_2 \in D$ such that $E_{f_1+a_2f_2} = E_{f_1} \cup E_{f_2}$. Repeating this process, we can choose $a_3, \dots, a_k \in D$ so that if we let $g = f_1 + a_2f_2 + \dots + a_kf_k$, then $E_g = E_{f_1} \cup \dots \cup E_{f_k}$. Since $D[\Gamma] = C(f_1, \dots, f_k)_v \subseteq D[(E_g)_v] \subseteq D[\Gamma]$, we have $(E_g)_v = \Gamma$.

Finally, choose $\alpha \in \Gamma$ with $\deg(g) < \alpha$, and let $h = g + fX^\alpha$. Then $A_h = A_g + A_f \supseteq A_f$ and $E_h = E_g \cup E_{fX^\alpha} \supseteq E_g$. Hence $(A_h)_v = D$ and $(E_h)_v = \Gamma$; so $h \in I \setminus [(\bigcup_{P \in t\text{-Max}(D)} PD[\Gamma]) \cup (\bigcup_{S \in t\text{-Max}(\Gamma)} D[S])]$. Thus $h \in I$ and $C(h)_v = D[\Gamma]$ (cf. Lemma 1.2(5) and [7, Corollary 1.3]).

Case 2. D is finite. Then D is a field and $\text{char } D = p \neq 0$; so if $h = a_1X^{\alpha_1} + a_2X^{\alpha_2} + \dots + a_sX^{\alpha_s} \in D[\Gamma]$, then $h^{p^m} = a_1^{p^m}X^{p^m\alpha_1} + a_2^{p^m}X^{p^m\alpha_2} + \dots + a_s^{p^m}X^{p^m\alpha_s}$. Hence we can choose an integer $m_2 \geq 1$ so that $E_{f_1+f_2^{p^{m_2}}} = E_{f_1} \cup E_{f_2^{p^{m_2}}}$ because Γ is torsion-free. Repeating this process, we can choose positive integers m_3, \dots, m_k so that if we let $f = f_1 + f_2^{p^{m_2}} + \dots + f_k^{p^{m_k}}$, then $E_f = E_{f_1} \cup E_{f_2^{p^{m_2}}} \cup \dots \cup E_{f_k^{p^{m_k}}}$. Clearly, $f \in I$ and $(E_f)_v = \Gamma$, and thus $C(f)_v = D[\Gamma]$.

(3) Clear.

(4) Let p be a (nonzero) homogeneous prime element of nonzero degree. Then pR is a homogeneous maximal t -ideal of R , and hence at least one of the f_i 's, say f_1 , is not contained in pR ; so $(f_1, p)_t = R$. Since $\deg(p) \neq 0$, there are some positive integers n_1, \dots, n_{k-1} such that $C(f_1 + f_2p^{n_1} + \dots + f_kp^{n_{k-1}}) = C(f_1) + C(f_2p^{n_1}) + \dots + C(f_kp^{n_{k-1}})$. Let $h = f_1 + f_2p^{n_1} + \dots + f_kp^{n_{k-1}}$. Note that if Q is a homogeneous maximal t -ideal of R , then $h \notin Q$, because (i) $f_1 \notin Q$ when $Q = pR$ and (ii) $(f_1, f_2p^{n_1}, \dots, f_kp^{n_{k-1}})R_Q = (f_1, f_2, \dots, f_k)R_Q = R_Q$ when $Q \neq pR$. Thus $h \in I$ and $C(h)_v = R$. \square

In the rest of this section, we study graded integral domains with property (#), with focus on the t -invertibility of homogeneous ideals. Our first result is a simple generalization of the fact that $ID[X]_{N_v} \cap K = I_w$ for all $I \in \mathbf{F}(D)$ [9, Lemma 2.1].

Lemma 1.7. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#). Then $IR_{N(H)} \cap R = I_w$ and $(I_w)R_{N(H)} = IR_{N(H)}$ for all nonzero homogeneous ideals I of R .

Proof. If $IR_{N(H)} \cap R = I_w$, then $(I_w)R_{N(H)} \subseteq IR_{N(H)} \subseteq (I_w)R_{N(H)}$, and thus $(I_w)R_{N(H)} = IR_{N(H)}$. Hence it suffices to show that $IR_{N(H)} \cap R = I_w$.

(\subseteq) Let $f = \frac{g}{h} \in IR_{N(H)} \cap R$, where $g \in I$ and $h \in N(H)$. Then $fh = g \in I$, and since $C(f)C(h)^{m+1} = C(h)^mC(fh)$ for some integer $m \geq 1$ by Lemma 1.1(1), we have $fC(h)^{m+1} \subseteq C(f)C(h)^{m+1} = C(h)^mC(fh) = C(h)^mC(g) \subseteq I$. Also, note that $(C(h)^{m+1})_v = R$ since $C(h)_v = R$. Thus $f \in I_w$.

(\supseteq) Let $f \in I_w$, and let A be a nonzero finitely generated ideal of R such that $A_v = R$ and $fA \subseteq I$. Then $C(A)_t = R$, and since R satisfies property (#), there exists an $h \in A$ with $C(h)_v = R$. Hence $h \in N(H)$ and $fh \in I$. Thus $f = \frac{fh}{h} \in IR_{N(H)} \cap R$. \square

Proposition 1.8. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#).

- (1) Each maximal ideal of $R_{N(H)}$ is a t -ideal.
- (2) If P is a maximal t -ideal of R , then $P \cap H \neq \emptyset$ if and only if $P \cap N(H) = \emptyset$.
- (3) If I is a nonzero homogeneous ideal of R , then I is t -invertible if and only if $IR_{N(H)}$ is invertible.

Proof. (1) This follows directly from Propositions 1.3 and 1.4.

(2) If $P \cap H \neq \emptyset$, then P is homogeneous by Lemma 1.2(5), and thus $P \cap N(H) = \emptyset$. Conversely, if $P \cap H = \emptyset$, then $C(P)_t = R$ because $P \subsetneq C(P)_t$ and P is a maximal t -ideal. Hence there is an $f \in P$ such that $C(f)_v = R$. Thus $f \in P \cap N(H)$.

(3) Assume that I is t -invertible. Then $(II^{-1})_t = R$, and hence $II^{-1} \not\subseteq Q$ for all homogeneous maximal t -ideals Q of R ; so $R_{N(H)} = (II^{-1})R_{N(H)} = (IR_{N(H)})(I^{-1}R_{N(H)}) = (IR_{N(H)})(IR_{N(H)})^{-1} \subseteq R_{N(H)}$ by Propositions 1.3 and 1.4. Thus $(IR_{N(H)})(IR_{N(H)})^{-1} = R_{N(H)}$. Conversely, assume that $IR_{N(H)}$ is invertible. Then $IR_{N(H)}$ is finitely generated, and so there is a finitely generated ideal $J \subseteq I$ of R such that

$IR_{N(H)} = JR_{N(H)}$. Note that I is homogeneous; so by replacing J with $C(J)$, we may assume that J is homogeneous, and hence $I_w = IR_{N(H)} \cap R = JR_{N(H)} \cap R = J_w$ by Lemma 1.7. Also, I_w is t -locally principal by the fact that I is homogeneous, $(I_w)R_{N(H)} = IR_{N(H)}$, and $IR_{N(H)}$ is invertible. Thus I is t -invertible [17, Corollary 2.7]. \square

Corollary 1.9. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#) and $0 \neq f \in R$. Then the following statements are equivalent.

- (1) $C(f)$ is t -invertible.
- (2) $C(f)R_{N(H)}$ is invertible.
- (3) $C(f)R_{N(H)} = fR_{N(H)}$.

Proof. (1) \Leftrightarrow (2) This follows from Proposition 1.8(3) because $C(f)$ is homogeneous.

(2) \Rightarrow (3) Let $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$, where each x_{α_i} is a nonzero homogeneous element and $\deg(x_{\alpha_1}) < \cdots < \deg(x_{\alpha_n})$. By Proposition 1.4, every maximal ideal of $R_{N(H)}$ has the form $QR_{N(H)}$ for a homogeneous maximal t -ideal Q of R ; so it suffices to show that $C(f)R_Q = fR_Q$ [13, Theorem 4.10].

Since $QR_{N(H)}$ is a maximal ideal of $R_{N(H)}$, then $C(f)R_Q = (C(f)R_{N(H)})_Q R_{N(H)} = x_{\alpha_i}R_Q$ for some x_{α_i} [13, Proposition 7.4]. Note that $\frac{x_{\alpha_k}}{x_{\alpha_i}} \in R_Q$ for $k = 1, 2, \dots, n$; so $\frac{f}{x_{\alpha_i}} \in R_Q$. Thus $\frac{f}{x_{\alpha_i}} = \frac{g}{h}$, and hence $fh = gx_{\alpha_i}$ for some $g \in R$ and $h \in R \setminus Q$. By Lemma 1.1(1), there is an integer $m \geq 1$ such that $C(h)^{m+1}C(f) = C(h)^mC(hf) = C(h)^mC(x_{\alpha_i}g) = x_{\alpha_i}C(h)^mC(g)$. Hence $x_{\alpha_i}R_Q = C(f)R_Q = (C(h)^{m+1}C(f))R_Q = (x_{\alpha_i}C(h)^mC(g))R_Q = (x_{\alpha_i}C(g))R_Q$, and so $C(g)R_Q = R_Q$, which means that $\frac{1}{g} \in R_Q$. Hence $\frac{f}{x_{\alpha_i}} = \frac{g}{h}$ is a unit in R_Q . Thus, $C(f)R_Q = fR_Q$.

(3) \Rightarrow (2) Clear. \square

A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is called a *graded Prüfer v -multiplication domain* (graded PvMD) if the monoid of homogeneous v -ideals of finite type forms a group under v -multiplication. Clearly, R is a graded PvMD if and only if every nonzero finitely generated homogeneous ideal of R is t -invertible.

Corollary 1.10. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#). If R is a graded PvMD, then every ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R .

Proof. Let $0 \neq f \in R$. Then $C(f)$ is t -invertible because R is a graded PvMD, and thus $fR_{N(H)} = C(f)R_{N(H)}$ by Corollary 1.9. Hence if A is an ideal of $R_{N(H)}$, then $A = IR_{N(H)}$ for some ideal I of R , and thus $A = (\sum_{f \in I} C(f))R_{N(H)}$. Clearly, $\sum_{f \in I} C(f)$ is a homogeneous ideal of R . \square

A multiplicative subset S of an integral domain D is called a *t -splitting set* if every $0 \neq d \in D$ can be written as $dD = (AB)_t$, where A and B are integral ideals of D such that $A_t \cap sD = sA_t$ (equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$. A t -splitting set S of D is called a *t -lcm t -splitting set* if $sD \cap dD$ is t -invertible for all $s \in S$ and $0 \neq d \in D$. It is known that $D + XD_S[X]$ is a PvMD if and only if D is a PvMD and S is a t -splitting set of D [4, Theorem 2.5]. We know that D is a PvMD if and only if $D[X]_{N_v}$ is a Prüfer domain [17, Theorem 3.7]. The next corollary is an extension of this result to graded integral domains with property (#).

Corollary 1.11. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#). Then the following statements are equivalent.

- (1) H is a t -lcm t -splitting set.
- (2) R is a graded PvMD.
- (3) R is a PvMD.
- (4) $R_{N(H)}$ is a Prüfer domain.
- (5) R_Q is a valuation domain for every homogeneous maximal t -ideal Q of R .

Proof. (1) \Leftrightarrow (2) [7, Theorem 2.6].

(2) \Leftrightarrow (3) [3, Theorem 6.4].

(2) \Rightarrow (4) Let A be a nonzero finitely generated ideal of $R_{N(H)}$. Then by Corollary 1.10, $A = IR_{N(H)}$ for some nonzero finitely generated homogeneous ideal I of R . Since R is a graded PvMD, I is t -invertible, and thus $A = IR_{N(H)}$ is invertible by Proposition 1.8(3).

(4) \Rightarrow (5) Let Q be a homogeneous maximal t -ideal of R . Then $Q_{N(H)} \subsetneq R_{N(H)}$, and thus $R_Q = (R_{N(H)})_{Q_{N(H)}}$ is a valuation domain.

(5) \Leftrightarrow (2) [11, Lemma 2.7]. \square

Theorem 1.12. (Cf. [17, Theorem 3.8].) Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#), let T be an overring of R such that $T \subseteq R_H$, and let $N_T(H) = \{f \in T \mid (C(f)T)_v = T\}$. If R is a PvMD, then $N(H) \subseteq N_T(H)$ if and only if $T = (\bigcap_{Q \in \Phi} R_Q) \cap R_H$, where Φ is a set of homogeneous prime t -ideals of R .

Proof. We first note that $R_{N(H)}$ is a Prüfer domain and each ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R by Corollaries 1.10 and 1.11.

(\Rightarrow) Note that $R_{N(H)} \subseteq T_{N_T(H)}$, so by [13, Theorem 26.1],

$$\begin{aligned} T_{N_T(H)} &= \bigcap_{Q \in \Phi} (R_{N(H)})_{Q_{N(H)}} \\ &= \bigcap_{Q \in \Phi} R_Q, \end{aligned}$$

where Φ is a collection of nonzero homogeneous prime ideals of R . Since $R_{N(H)}$ is a Prüfer domain, every nonzero prime ideal of $R_{N(H)}$ is a t -ideal, and hence each prime ideal in Φ is a t -ideal by Proposition 1.3. Note also that if H' is the set of nonzero homogeneous elements of T , then $R_H = T_{H'}$. Thus by Lemma 1.2(2), $T = T_{N_T(H)} \cap R_H = (\bigcap_{Q \in \Phi} R_Q) \cap R_H$.

(\Leftarrow) Assume that $T = (\bigcap_{Q \in \Phi} R_Q) \cap R_H$, and let $f = x_1 + \cdots + x_n \in N(H)$, where each x_i is homogeneous and $\deg(x_1) < \deg(x_2) < \cdots < \deg(x_n)$. Let $u \in qf(R)$, where $qf(R)$ is the quotient field of R , such that $uC(f) \subseteq T$ (note that $u \in R_H$). Then, for every $Q \in \Phi$, we have $uC(f) \subseteq R_Q$, and so there exists a $g \in R \setminus Q$ such that $ugf \in uC(f) \subseteq R$. Note that $C(ugf)_v = (C(u)C(g)C(f))_v$ [3, Theorem 3.5] because R is integrally closed. Thus $ug \in (C(u)C(g))_v = (C(u)C(g)C(f))_v = C(ugf)_v \subseteq R$. Hence $u \in R_Q$. Thus $(C(f)T)^{-1} \subseteq (\bigcap_{Q \in \Phi} R_Q) \cap R_H = T$, and hence $(C(f)T)_v = T$. Therefore $N(H) \subseteq N_T(H)$. \square

2. Graded Krull domains and the ring $R_{N(H)}$

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, H be the set of nonzero homogeneous elements of R , $N(H) = \{f \in R \mid C(f)_v = R\}$, and Ω be the set of homogeneous maximal t -ideals of R . By Lemma 1.2(5), $Q \in \Omega \Leftrightarrow Q \in t\text{-Max}(R)$ and $Q \cap H \neq \emptyset$. Also, it is clear that each element of R is homogeneous if and only if $\Gamma = \{0\}$; hence if $\Gamma = \{0\}$, then $R_{N(H)} = R$ (cf. the proof of Corollary 1.5).

A graded integral domain is a *graded Krull domain* if it is completely integrally closed with respect to homogeneous elements and satisfies the ascending chain condition on homogeneous integral v -ideals [3, Definition 5.1]. As in the Krull domain case, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Krull domain if and only if every nonzero homogeneous ideal is t -invertible, if and only if every nonzero homogeneous prime (t) -ideal is t -invertible [7, Theorem 2.4]. In this section, we study the ring $R_{N(H)}$ when R is a graded Krull domain. In particular, we show that R is a graded Krull domain if and only if $R_{N(H)}$ is a PID.

Lemma 2.1. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property (#), and $f, g, a \in R$. If $(f, a)_v = R$, then $(f, ag)_{R_{N(H)}} = (f, g)_{R_{N(H)}}$.

Proof. Let $Q \in \Omega$. If $f \in Q$, then $a \notin Q$, and thus $(f, ag)R_Q = (f, g)R_Q$. Next, if $f \notin Q$, then $(f, ag)R_Q = R_Q = (f, g)R_Q$. Hence $(f, ag)R_{N(H)} = (f, g)R_{N(H)}$ by Proposition 1.4 and [13, Theorem 4.10]. \square

Lemma 2.2. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $\Gamma \neq \{0\}$. If the intersection $\bigcap_{Q \in \Omega} R_Q$ is locally finite, then R satisfies property (#).

Proof. Let I be a nonzero ideal of R such that $C(I)_t = R$. Then $I \not\subseteq Q$ for all $Q \in \Omega$. Since $\Gamma \neq \{0\}$, we can choose a $0 \neq x \in H$ such that $\deg(x) \neq 0$. Let $0 \neq f \in I$. By assumption, there are only finitely many prime ideals $Q_1, \dots, Q_n \in \Omega$ that contain xf . Choose another $g \in I \setminus \bigcup_{i=1}^n Q_i$. Clearly, $R = (C(x^m f) + C(g))_t$ for all integers $m \geq 1$. Also, there is an integer $k \geq 1$ such that $C(x^k f + g) = C(x^k f) + C(g)$. Thus if we let $h = x^k f + g$, then $h \in I$ and $C(h)_v = R$. \square

Theorem 2.3. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $\Gamma \neq \{0\}$. Then the following statements are equivalent.

- (1) R is a graded Krull domain.
- (2) $R_{N(H)}$ is a Dedekind domain.
- (3) $R_{N(H)}$ is a PID.

Proof. (1) \Rightarrow (3) First, note that R satisfies property (#) by [2, Proposition 5.6] and Lemma 2.2. Next, let P be a nonzero prime ideal of R such that $P \cap N(H) = \emptyset$. Then there is a prime ideal $Q \in \Omega$ with $P \subseteq Q$ by Proposition 1.4, and since R is a graded Krull domain, Q must be a height-one prime ideal of R [2, Proposition 5.5]. Thus $P = Q$. Hence it suffices to show that $QR_{N(H)}$ is principal for all $Q \in \Omega$.

Let $Q \in \Omega$. Note that R_Q is a rank-one DVR; so there is an $a \in Q$ such that $QR_Q = aR_Q$. Since Q is homogeneous, we may assume that a is homogeneous. Let $S = \{x \in H \mid \deg(x) \neq 0\}$. Then there are two cases that we have to consider.

Case 1. $S \not\subseteq Q$. Choose $x \in S \setminus Q$. Since R is a graded Krull domain, there are only finitely many homogeneous maximal t -ideals Q, Q_1, \dots, Q_k of R that contain ax . Choose $g \in Q \setminus \bigcup_{i=1}^k Q_k$. Clearly, $(x^m, g)_v = R$ for all integers $m \geq 1$, and hence $(ax^m, g)R_{N(H)} = (a, g)R_{N(H)}$ by Lemma 2.1. Since $\deg(x) \neq 0$, there is an integer $n \geq 1$ such that $C(ax^n + g) = C(ax^n) + C(g)$. So if we let $h = ax^n + g$, then $hR_{N(H)} = (a, g)R_{N(H)}$ by Corollary 1.9. Note that Q is the unique homogeneous maximal t -ideal of R containing (a, g) . Also, $QR_Q = aR_Q = (a, g)R_Q$. Thus $hR_{N(H)} = (a, g)R_{N(H)} = \bigcap_{Q' \in \Omega} (a, g)R_{Q'} = \bigcap_{Q' \in \Omega} QR_{Q'} = QR_{N(H)}$ by Proposition 1.4 and [13, Theorem 4.10].

Case 2. $S \subseteq Q$. We can choose a nonzero homogeneous element $b \in Q$ such that $\deg(b) \neq \deg(a)$ and $(a, b)_v = Q$ (cf. the proof of Case 1). Note that $t = w$ on R (cf. [17, Theorem 3.5]) since R is a PvMD; so $Q = (a, b)_w$. Thus if we set $f = a + b$, then $fR_{N(H)} = (a, b)R_{N(H)} = (a, b)_w R_{N(H)} = QR_{N(H)}$ by Corollary 1.9 and Lemma 1.7.

(3) \Rightarrow (2) Clear.

(2) \Rightarrow (1) If $R_{N(H)}$ is a Dedekind domain, then the intersection $\bigcap_{Q \in \Omega} R_Q$ is locally finite, and hence R satisfies property (#) by Lemma 2.2. Let I be a nonzero homogeneous ideal of R . Then $IR_{N(H)}$ is invertible by assumption, and thus I is t -invertible by Proposition 1.8(3). Hence R is a graded Krull domain [7, Theorem 2.4]. \square

Corollary 2.4. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $\Gamma \neq \{0\}$. Then the following statements are equivalent.

- (1) R is a Krull domain.
- (2) R_H is a UFD and $R_{N(H)}$ is a PID.
- (3) R_H and $R_{N(H)}$ are both Krull domains.

Proof. (1) \Rightarrow (2) A Krull domain is a graded Krull domain, and thus $R_{N(H)}$ is a PID by Theorem 2.3. Also, R_H is a UFD since R_H is a GCD-domain and a Krull domain.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) This follows from the fact that $R = R_H \cap R_{N(H)}$ by Lemma 1.2(2). \square

Let S be a saturated multiplicative subset of an integral domain D , and let $N = \{0 \neq x \in D \mid (x, s)_v = D \text{ for all } s \in S\}$. We call S a *splitting set* if $SN = D \setminus \{0\}$, i.e., for every $0 \neq d \in D$, there are $s \in S$ and $t \in N$ so that $d = st$. It is known that if $R = D[\Gamma]$ is a monoid domain, then H is a splitting set of R if and only if R is a GCD-domain [7, Corollary 1.5].

Corollary 2.5. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $\Gamma \neq \{0\}$. Then R is a graded UFD if and only if H is a splitting set and $R_{N(H)}$ is a PID.

Proof. By the proof of Theorem 2.3, R satisfies property (#).

(\Rightarrow) Assume that R is a graded UFD. Then R is a graded GCD-domain, and hence H is a splitting set of R [7, Theorem 1.4]. Also, since a graded UFD is a graded Krull domain, $R_{N(H)}$ is a PID by Theorem 2.3. (\Leftarrow) By Theorem 2.3, R is a graded Krull domain. Moreover, $Cl(R) = 0$ since $Cl(R) = Cl(R_H) \oplus Cl(R_{N(H)}) = 0$ [5, Corollary 3.8]. Thus R is a graded UFD. \square

As we noted in the proof of Corollary 2.5, if R is a graded GCD-domain, then H is a splitting set of R . So we have

Corollary 2.6. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded GCD-domain with $\Gamma \neq \{0\}$. Then R is a graded UFD if and only if $R_{N(H)}$ is a PID.

As in [21], we say that a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded strong Mori domain* (graded SM domain) if R satisfies the ascending chain condition on homogeneous w -ideals, or equivalently, if every homogeneous w -ideal has finite type. It is known that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Krull domain if and only if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is an integrally closed graded SM domain [21, Corollary 3.7].

Theorem 2.7. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $\Gamma \neq \{0\}$ such that every homogeneous prime t -ideal of R has height-one. Then R is a graded SM-domain if and only if $R_{N(H)}$ is a Noetherian domain.

Proof. (\Rightarrow) Assume that R is a graded SM domain. Then the intersection $\bigcap_{Q \in \Omega} R_Q$ is locally finite [21, Theorem 3.5], and hence R satisfies property (#) by Lemma 2.2. Note that each $Q \in \Omega$ has height-one; so $\{Q_{N(H)} \mid Q \in \Omega\}$ is the set of nonzero prime ideals of $R_{N(H)}$ by Proposition 1.4. Hence, by Cohen's theorem, we only have to show that $Q_{N(H)}$ is finitely generated for all $Q \in \Omega$. To show this, let $P \in \Omega$. Then $P = J_w$ for some nonzero finitely generated homogeneous ideal J of R [21, Lemma 3.3]. Hence $P_{N(H)} = (J_w)_{N(H)} = J_{N(H)}$ by Lemma 1.7. Thus $P_{N(H)}$ is finitely generated.

(\Leftarrow) Let I be a (proper) nonzero homogeneous integral w -ideal of R . Then $I_{N(H)}$ is an ideal of $R_{N(H)}$, and so $I_{N(H)} = J_{N(H)}$ for some nonzero finitely generated ideal $J \subseteq I$ of R . Since I is homogeneous, we may assume that J is homogeneous. Hence $I = I_{N(H)} \cap R = J_w$ by Lemma 1.7. Thus I is of finite type, and hence R is a graded SM domain. \square

Let D be an integral domain with quotient field K . A star operation $*$ on D is said to be an *endlich arithmetisch brauchbar* (e.a.b.) star operation if $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*$ for all $A, B, C \in \mathbf{f}(D)$. Clearly, $*$ is an e.a.b. star operation if and only if $*_f$ is an e.a.b. star operation. We know that if D admits an e.a.b. star operation, then D is integrally closed [13, Corollary 32.8]. Conversely, suppose that D is integrally closed, and define

$$I^b = \bigcap \{IV \mid V \text{ is a valuation overring of } D\}$$

for every $I \in \mathbf{F}(D)$. Then the mapping $b : \mathbf{F}(D) \rightarrow \mathbf{F}(D)$, given by $I \mapsto I^b$, is an *e.a.b.* star operation on D [13, Theorem 32.5]. More generally, Chang [10, Lemma 3.1] proved that, for a star operation $*$ on an integrally closed domain D , if we set

$$I^{*c} = \bigcap \{IV \mid V \text{ is a } * \text{-linked valuation overring of } D\},$$

then $*_c$ is an *e.a.b.* star operation of finite character on D . (An overring T of D is said to be **-linked* over D if $I^* = D$ for $I \in \mathbf{f}(D)$ implies $(IT)_v = T$.) It is clear that $d_c = b$.

Let X be an indeterminate over an integrally closed domain D , $*$ an *e.a.b.* star operation on D , and

$$Kr(D, *) = \left\{ \frac{f}{g} \mid f, g \in D[X], g \neq 0, \text{ and } A_f \subseteq (A_g)^* \right\}.$$

It is well known that $Kr(D, *)$, called the *Kronecker function ring of D* (with respect to $*$), is a Bezout domain and $Kr(D, *) \cap K = D$ [13, Theorem 32.7]; D is a Prüfer domain if and only if $Kr(D, b) = D[X]_S$, where $S = \{f \in D[X] \mid A_f = D\}$, [8, Theorem 4]; and D is a PvMD if and only if $Kr(D, v_c) = D[X]_{N_v}$ [10, Corollary 3.8]. For more on $Kr(D, *)$, see [13, Section 32] or Fontana and Loper's interesting survey article [12].

We close this section by introducing a graded integral domain analogue of Kronecker function rings. (For convenience, we use the same notation $Kr(R, *)$.) We first need a simple lemma.

Lemma 2.8. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, $*$ an *e.a.b.* star operation on R , and $0 \neq f, g \in R$. Then $(C(f)C(g))^* = C(fg)^*$.*

Proof. By Lemma 1.1(1), there is a positive integer m such that $C(f)^{m+1}C(g) = C(f)^mC(fg)$. Hence $(C(f)^{m+1}C(g))^* = (C(f)^mC(fg))^*$, and thus $(C(f)C(g))^* = C(fg)^*$ because $*$ is an *e.a.b.* star operation. \square

Theorem 2.9. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, $*$ an *e.a.b.* star operation on R , and*

$$Kr(R, *) = \left\{ \frac{f}{g} \mid f, g \in R, g \neq 0, \text{ and } C(f) \subseteq C(g)^* \right\}.$$

Then

- (1) $Kr(R, *)$ is an integral domain.
- (2) $Kr(R, *) \cap R_H = R$.
- (3) If $f, g \in R$ are nonzero such that $C(f + g)^* = (C(f) + C(g))^*$, then $(f, g)Kr(R, *) = (f + g)Kr(R, *)$. In particular, $fKr(R, *) = C(f)Kr(R, *)$ for all $f \in R$.

Proof. (1) Clearly, $R \subseteq Kr(R, *) \subseteq qf(R)$, where $qf(R)$ is the quotient field of R . Hence it suffices to show that $Kr(R, *)$ is closed under addition and multiplication.

Let $0 \neq f_1, f_2, g_1, g_2 \in R$ be such that $\frac{f_1}{g_1}, \frac{f_2}{g_2} \in Kr(R, *)$; so $C(f_i) \subseteq C(g_i)^*$ for $i = 1, 2$. Then $\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1g_2 + f_2g_1}{g_1g_2}$ and $(\frac{f_1}{g_1})(\frac{f_2}{g_2}) = \frac{f_1f_2}{g_1g_2}$. Note that, by Lemma 2.8, we have $C(f_1g_2 + f_2g_1) \subseteq C(f_1g_2) + C(f_2g_1) \subseteq (C(f_1g_2) + C(f_2g_1))^* = (C(f_1g_2))^* + (C(f_2g_1))^* \subseteq ((C(f_1)C(g_2)))^* + ((C(f_2)C(g_1)))^* \subseteq (C(g_1)C(g_2))^* = C(g_1g_2)^*$ and $C(f_1f_2) \subseteq C(f_1f_2)^* = (C(f_1)C(f_2))^* \subseteq (C(g_1)C(g_2))^* = C(g_1g_2)^*$. Thus $\frac{f_1g_2 + f_2g_1}{g_1g_2}$ and $\frac{f_1f_2}{g_1g_2}$ are in $Kr(R, *)$.

(2) Let $\frac{f}{g} = \frac{h}{x} \in Kr(R, *) \cap R_H = R$, where $0 \neq f, g, h \in R$, $C(f) \subseteq C(g)^*$, and $x \in H$. Then $xf = gh$; so $xC(f)^* = C(gh)^* = (C(g)C(h))^* \supseteq (C(f)C(h))^*$. Hence $xR \supseteq C(h)^* \supseteq C(h)$, and thus $R \supseteq C(\frac{h}{x})$. Hence $\frac{f}{g} = \frac{h}{x} \in R$. The reverse containment is clear.

(3) Since $C(f + g)^* = (C(f) + C(g))^*$, we have $C(f), C(g) \subseteq C(f + g)^*$, and thus $\frac{f}{f+g}, \frac{g}{f+g} \in Kr(R, *)$. So if we set $h = f + g$, then $f, g \in hKr(R, *)$. Hence $(f, g)Kr(R, *) = hKr(R, *)$. \square

Let $*$ be an e.a.b. star operation on $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. It is clear that if $\Gamma = \{0\}$, then $Kr(R, *) = R = R_{N(H)}$. Thus it is reasonable that $\Gamma \neq \{0\}$ is assumed for the study of $Kr(R, *)$.

Corollary 2.10. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Krull domain with $\Gamma \neq \{0\}$. Then $R_{N(H)} = Kr(R, t)$.*

Proof. Note that if R is a PvMD, then t is an e.a.b. star operation on R . Thus $Kr(R, t)$ is well-defined since a graded Krull domain is a PvMD.

Clearly, $R_{N(H)} \subseteq Kr(R, t)$. For the reverse containment, let Q be a homogeneous maximal t -ideal of R . Then $Q_{N(H)} = fR_{N(H)}$ for some $f \in Q$ by Theorem 2.3, and hence $QKr(R, t) = fKr(R, t)$. If $QKr(R, t) = Kr(R, t)$, then $\frac{1}{f} \in Kr(R, t)$, and so $R = C(1)_t \subseteq C(f)_t \subseteq Q_t = Q$, a contradiction. Thus $QKr(R, t) \subsetneq Kr(R, t)$, which implies that there is a prime ideal M of $Kr(R, t)$ such that $M \cap R_{N(H)} = Q_{N(H)}$, and since $R_{N(H)}$ is a PID, $Kr(R, t)_M = (R_{N(H)})_Q R_{N(H)} = R_Q$. Thus $R_{N(H)} = \bigcap_{Q \in \Omega} R_Q \supseteq \bigcap_{M \in \text{Max}(Kr(R, t))} Kr(R, t)_M = Kr(R, t)$. \square

3. Graded integral domains with a unit of nonzero degree

As in Section 2, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ denotes a graded integral domain, H is the set of nonzero homogeneous elements of R , $N(H) = \{f \in R \mid C(f)_v = R\}$, and Ω is the set of homogeneous maximal t -ideals of R .

In this section, we study graded integral domains with a unit of nonzero degree; such domains satisfy property (#) by Example 1.6. Our first result explains why it is valuable to study graded integral domains with a unit of nonzero degree. Also, many of the results of this section are generalizations of results on the ring $D[X]_{N_v}$, where $N_v = \{f \in D[X] \mid (A_f)_v = D\}$ (see [17]).

Proposition 3.1. *Let D be an integral domain, G the quotient group of Γ , $R = D[G]$ the group ring of G over D , and $N_v = \{f \in D[\Gamma] \mid (A_f)_v = D\}$. Then $D[\Gamma]_{N_v} = R_{N(H)}$.*

Proof. We first note that $\text{Max}(D[\Gamma]_{N_v}) = \{P[\Gamma]_{N_v} \mid P \in t\text{-Max}(D)\}$ (cf. [17, Proposition 2.1]) and $(D[\Gamma]_{N_v})_{P[\Gamma]_{N_v}} = D[\Gamma]_{P[\Gamma]} = D[G]_{P[G]}$ for each $P \in t\text{-Max}(D)$. Also, by Example 1.6, R satisfies property (#); so $\text{Max}(R_{N(H)}) = \{P[G]_{N(H)} \mid P \in t\text{-Max}(D)\}$ by [7, Corollary 1.3] and Proposition 1.4. Thus $D[\Gamma]_{N_v} = \bigcap_{P \in t\text{-Max}(D)} D[G]_{P[G]} = \bigcap_{P \in t\text{-Max}(D)} (R_{N(H)})_{P R_{N(H)}} = R_{N(H)}$. \square

Let \mathbb{Z} be the additive group of integers. Clearly, the direct sum $\Gamma \oplus \mathbb{Z}$ of Γ with \mathbb{Z} is a torsionless grading monoid. So if y is an indeterminate over $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, then $R[y, y^{-1}]$ is a graded integral domain graded by $\Gamma \oplus \mathbb{Z}$.

Let $S = \{f \in D[X] \mid A_f = D\}$. Then $D(X) := D[X]_S$ is called the Nagata ring of D . It is known that $\text{Pic}(D(X)) = 0$ [1, Theorem 2]. More generally, if $*$ is a star operation on D , then $\text{Pic}(D[X]_{N_*}) = 0$, where $N_* = \{f \in D[X] \mid (A_f)^* = D\}$ [17, Theorem 2.14]. We next show that $\text{Pic}(R_{N(H)}) = 0$ when R contains a unit of nonzero degree (Theorem 3.3). The proof is similar to that of [1, Theorem 2]. To do this, we first need a simple lemma.

Lemma 3.2. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, y an indeterminate over R , and $T = R[y, y^{-1}]$. If $N_T(H) = \{f \in T \mid C_T(f)_v = T\}$, where $C_T(f)$ is the ideal of T generated by the homogeneous components of f , then $T_{N_T(H)} = (R_{N(H)})(y)$.*

Proof. Note that if M is a maximal t -ideal of T , then either $M \cap R = (0)$ or $M \cap R \neq (0)$ such that $M = (M \cap R)[y, y^{-1}]$ and $M \cap R$ is a maximal t -ideal of R (cf. [16, Proposition 1.1]). Clearly, if $M \cap R = (0)$, then M is not homogeneous because $y, y^{-1} \in T$. Hence if M is homogeneous, then

$M \cap R \in \Omega$ and $M = (M \cap R)[y, y^{-1}]$. Thus $\text{Max}(T_{N_T(H)}) = \{Q T_{N_T(H)} \mid Q \in \Omega\}$ by Proposition 1.4. Next, since $\text{Max}((R_{N(H)})(y)) = \{P(y) \mid P \text{ is a maximal ideal of } R_{N(H)}\}$ [13, Proposition 33.1], we have $\text{Max}((R_{N(H)})(y)) = \{(Q_{N(H)})(y) \mid Q \in \Omega\}$ by Proposition 1.4. Thus

$$\begin{aligned} T_{N_T(H)} &= \bigcap_{Q \in \Omega} (T_{N_T(H)})_Q T_{N_T(H)} \\ &= \bigcap_{Q \in \Omega} (R[y, y^{-1}])_{Q R[y, y^{-1}]} \\ &= \bigcap_{Q \in \Omega} R[y]_Q R[y] \\ &= \bigcap_{Q \in \Omega} R_Q(y) \\ &= \bigcap_{Q \in \Omega} (R_{N(H)}(y))_{Q R_{N(H)}(y)} \\ &= R_{N(H)}(y). \quad \square \end{aligned}$$

Theorem 3.3. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then $\text{Cl}(R_{N(H)}) = \text{Pic}(R_{N(H)}) = 0$.

Proof. By Proposition 1.8(1), $\text{Cl}(R_{N(H)}) = \text{Pic}(R_{N(H)})$; so it suffices to show that every invertible ideal of $R_{N(H)}$ is principal. Let A be an invertible ideal of $R_{N(H)}$, and let $\alpha \in H$ be a unit of nonzero degree. Then $A = (f_1, \dots, f_n)R_{N(H)}$ for some $f_1, \dots, f_n \in R$. Since $\deg(\alpha) \neq 0$, there are positive integers k_1, \dots, k_{n-1} such that if we set $f = f_1 + f_2\alpha^{k_1} + \dots + f_n\alpha^{k_{n-1}}$, then $C(f) = C(f_1) + \dots + C(f_n)$. We claim that $A = fR_{N(H)}$.

Let the notation be as in Lemma 3.2. Let $g = f_1 + f_2y + \dots + f_ny^{n-1}$, and note that $A(y)$ is invertible; so $A(y) = gR_{N(H)}(y)$ [1, Theorem 1]. Hence $fR_{N(H)}(y) \subseteq A(y) = gR_{N(H)}(y) = gT_{N_T(H)}$ by Lemma 3.2, and so $f = g \cdot \frac{h_2}{h_1}$, where $h_1, h_2 \in T$ with $h_1 \in N_T(H)$, i.e., $C_T(h_1)_w = C_T(h_1)_v = T$. Note that if $h = g_1y^{m_1} + \dots + g_ky^{m_k} \in R[y, y^{-1}]$, where $g_i \in R$ and m_i is an integer with $m_1 < \dots < m_k$, then $C_T(h) = (\sum C(g_i))R[y, y^{-1}]$, and thus $C_T(h)_w = (\sum C(g_i))_w R[y, y^{-1}]$ (cf. [14, Proposition 4.3]). Also, $C_T(g) = C_T(f)$ because $y, y^{-1} \in T$. So, since $fh_1 = gh_2$, we have

$$\begin{aligned} C(f)_w R[y, y^{-1}] &= C_T(f)_w = (C_T(f)C_T(h_1))_w \\ &= C_T(fh_1)_w = C_T(gh_2)_w \subseteq (C_T(g)C_T(h_2))_w \\ &= (C_T(f)C_T(h_2))_w = ((C(f)_w R[y, y^{-1}])(I_w R[y, y^{-1}]))_w \\ &= (C(f)I)_w R[y, y^{-1}] \\ &\subseteq C(f)_w R[y, y^{-1}], \end{aligned}$$

where $I = \sum C(g'_i)$ when $h_2 = \sum_{i=1}^s g'_iy^{n_i}$ with $g'_i \in R$ and $n_1 < \dots < n_s$. Hence $C(f)_w \subseteq (C(f)I)_w \subseteq C(f)_w$, and thus $C(f)_w = (C(f)I)_w$.

Let $Q \in \Omega$. Then $C(f)R_Q = C(f)_w R_Q = (C(f)I)_w R_Q = (C(f)I)R_Q = (C(f)R_Q)(IR_Q)$, and since $C(f)R_Q \neq 0$, by the Nakayama Lemma [18, Theorem 78], we have $IR_Q = R_Q$. Note that I is homogeneous. Thus $I_w = R$ and $C_T(h_2)_w = I_w R[y, y^{-1}] = R[y, y^{-1}]$; so $h_2 \in N_T(H)$. Hence $fR_{N(H)}(y) = fT_{N_T(H)} = gT_{N_T(H)} = A(y)$ by Lemma 3.2. Therefore $fR_{N(H)} = fR_{N(H)}(y) \cap R_{N(H)} = A(y) \cap R_{N(H)} = A$ [13, Proposition 33.1]. \square

It is known that if A, B, C are ideals of an integral domain D , then $(A + B)(A + C)(B + C) = (A + B + C)(AB + AC + BC)$. Also, a graded PvMD is a PvMD. Thus $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a PvMD if and only if every nonzero ideal of R generated by two homogeneous elements is t -invertible. We use this result in the proof of the next theorem without comments.

Theorem 3.4. *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with a unit of nonzero degree.*

- (1) R is a PvMD.
- (2) Every ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R .
- (3) Every principal ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R .
- (4) R is integrally closed and $I_t = IR_{N(H)} \cap R$ for every nonzero homogeneous ideal I of R .
- (5) $R_{N(H)}$ is a Prüfer domain.
- (6) $R_{N(H)}$ is a Bezout domain.

Proof. Let $\alpha \in H$ be a unit of nonzero degree.

(1) \Rightarrow (2) Corollary 1.10.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) Let $a, b \in H$. We can choose a positive integer n such that $\deg(a) \neq \deg(\alpha^n b)$; so $C(a + \alpha^n b) = (a, b)R$. By (3), if we set $h = a + \alpha^n b$, then $hR_{N(H)} = IR_{N(H)}$ for some homogeneous ideal I of R . Note that $h \in IR_{N(H)} \cap R = I_w$ by Lemma 1.7, and I_w is homogeneous; hence $C(h) \subseteq I_w$. Thus $hR_{N(H)} = IR_{N(H)} = I_w R_{N(H)} \supseteq C(h)R_{N(H)} \supseteq hR_{N(H)}$, and so $hR_{N(H)} = C(h)R_{N(H)}$. Thus $(a, b) = C(h)$ is t -invertible by Corollary 1.9.

(1) \Rightarrow (4) Clearly, R is integrally closed. Also, we know that if R is a PvMD, then $t = w$ on R (cf. [17, Theorem 3.5]). Thus $I_t = IR_{N(H)} \cap R$ by Lemma 1.7.

(4) \Rightarrow (1) Let $a, b \in H$. As in the proof of (3) \Rightarrow (1) above, we may assume that $\deg(a) \neq \deg(b)$; so $C(a + b) = (a, b)$. Let $f = a + b$ and $g = a - b$. Then $fg = a^2 - b^2$ and $C(fg) = (a^2, b^2)$. Since R is integrally closed, $C(fg)_v = (C(f)C(g))_v$ [3, Corollary 3.6], and thus $ab \in (a^2, b^2)_v \subseteq (a^2, b^2)R_{N(H)}$. Hence $(a, b)^2 R_{N(H)} = (a^2, b^2)R_{N(H)}$, and by [13, Proposition 24.2], $(a, b)R_{N(H)}$ is invertible. Thus (a, b) is t -invertible by Proposition 1.8(3).

(1) \Leftrightarrow (5) Corollary 1.11.

(5) \Leftrightarrow (6) This follows from Theorem 3.3. \square

Theorem 3.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree and $*$ an e.a.b. star operation on R .*

- (1) $Kr(R, *)$ is a Bezout domain.
- (2) $IKr(R, *) \cap R_H = I^*$ for every nonzero finitely generated homogeneous ideal I of R .
- (3) If $*_f = t$, then R is a PvMD if and only if $R_{N(H)} = Kr(R, t)$.

Proof. Let $\alpha \in H$ be a unit of nonzero degree.

(1) It suffices to show that if $f, g \in R$, then $(f, g)Kr(R, *)$ is principal. Note that $C(f + \alpha^m g) = C(f) + C(\alpha^m g) = C(f) + C(g)$ for some integer $m \geq 1$. Thus if we set $h = f + \alpha^m g$, then $(f, g)Kr(R, *) = hKr(R, *)$ by Theorem 2.9(3).

(2) Let $I = (a_1, \dots, a_n)R$, where $a_i \in H$, and let k_i , for $i = 1, \dots, n$, be positive integers such that $\deg(a_i \alpha^{k_i}) \neq \deg(a_j \alpha^{k_j})$ when $i \neq j$. So if we let $h = a_1 \alpha^{k_1} + \dots + a_n \alpha^{k_n}$, then $IKr(R, *) = hKr(R, *)$ by Theorem 2.9(3). Therefore, if $d \in R_H$, then $d \in IKr(R, *)$ if and only if $\frac{d}{h} \in Kr(R, *)$, if and only if $dR \subseteq C(d) \subseteq C(h)^* = I^*$. Thus $IKr(R, *) \cap R_H = I^*$.

(3) (\Leftarrow) This follows from (1) and Theorem 3.4.

(\Rightarrow) Clearly, $R_{N(H)} \subseteq Kr(R, t)$. So if R is a PvMD, then $R_{N(H)}$ is a Bezout domain by Theorem 3.4, and thus $Kr(R, t)$ is a quotient ring of $R_{N(H)}$ [13, Proposition 27.3]. If $Q \in \Omega$, then by (2), $QKr(R, t) \subsetneq Kr(R, t)$, and so there is a maximal ideal M of $Kr(R, t)$ such that $QKr(R, t) \subseteq M$. Thus $M \cap R_{N(H)} = Q R_{N(H)}$ by Proposition 1.4. Hence $R_Q \subseteq Kr(R, t)_M$, and since R_Q is a valuation domain, we have $R_Q = Kr(R, t)_M$. Thus $R_{N(H)} = \bigcap_{Q \in \Omega} R_Q \supseteq \bigcap_{M \in \text{Max}(Kr(R, t))} Kr(R, t)_M = Kr(R, t)$. Hence $R_{N(H)} = Kr(R, t)$. \square

We will say that $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is a *graded Prüfer domain* if every nonzero finitely generated homogeneous ideal of R is invertible. We know that R is a graded PvMD if and only if R is a PvMD, but the next example shows that graded Prüfer domains need not be Prüfer domains.

Example 3.6. Let D be an integral domain, X be an indeterminate over D , and $R = D[X, X^{-1}]$.

- (1) R is a graded Prüfer domain if and only if D is a Prüfer domain,
- (3) (Cf. [13, Exercise 15, p. 287]) R is a Prüfer domain if and only if D is a field.

Proof. (1) Assume that R is a graded Prüfer domain. Let I be a nonzero finitely generated ideal of D . Then $IR = I[X, X^{-1}]$ is a nonzero finitely generated homogeneous ideal of R , and hence $R = (IR)(IR)^{-1} = (I[X, X^{-1}](I[X, X^{-1}])^{-1} = (I[X, X^{-1}](I^{-1}[X, X^{-1}]) = (II^{-1})[X, X^{-1}]$ [17, Proposition 2.2]. Thus $II^{-1} = D$. Conversely, assume that D is a Prüfer domain. Note that each homogeneous element of R is of the form aX^n for some $a \in D$ and integer n . So if A is a nonzero finitely generated homogeneous ideal of R , then $A = (a_1X^{k_1}, \dots, a_nX^{k_n})$ for some $0 \neq a_i \in D$ and integers k_i . Clearly, if we let $J = (a_1, \dots, a_n)D$, then $A = J[X, X^{-1}]$ and $JJ^{-1} = D$. Hence $AA^{-1} = (J[X, X^{-1}](J[X, X^{-1}])^{-1} = (JJ^{-1})[X, X^{-1}] = D[X, X^{-1}]$. Thus, A is invertible.

(2) Assume that D is not a field, and let a be a nonzero nonunit of D . Then $(a, X) \subsetneq R$ and $(a, X)^{-1} = R$; so $(a, X)(a, X)^{-1} = (a, X) \subsetneq R$. Thus (a, X) is not invertible, and hence R is not a Prüfer domain. Conversely, if D is a field, then R is a PID, and thus R is a Prüfer domain. \square

It is well known that if D is integrally closed, then D is a Prüfer domain if and only if $D(X)$ is a Prüfer domain, if and only if $D(X) = \text{Kr}(D, b)$ [13, Theorem 33.4]. We next give a graded Prüfer domain analogue.

Theorem 3.7. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with a unit of nonzero degree, and let $S(H) = \{f \in R \mid C(f) = R\}$. If R is integrally closed, then R is a graded Prüfer domain if and only if $R_{S(H)} = \text{Kr}(R, b)$.

Proof. (\Rightarrow) Clearly, each nonzero homogeneous ideal of R is a t -ideal; so $N(H) = S(H)$ and $R_{S(H)}$ is a Bezout domain by Theorem 3.4. Note that $R_{S(H)} \subseteq \text{Kr}(R, b)$, and hence $\text{Kr}(R, b)$ is a quotient ring of $R_{S(H)}$ [13, Proposition 27.3]. But, if M is a maximal ideal of $R_{S(H)}$, then $M = Q R_{S(H)}$ for some $Q \in \Omega$ by Proposition 1.4 and $Q \text{Kr}(R, b) \cap R_H = Q^b \subsetneq R$ by Theorem 3.5(2), and hence $M \text{Kr}(R, b) \subsetneq \text{Kr}(R, b)$. Thus $\text{Kr}(R, b) = R_{S(H)}$ (see the proof of Theorem 3.5(3)).

(\Leftarrow) Assume that $R_{S(H)} = \text{Kr}(R, b)$, and hence $R_{S(H)}$ is a Prüfer domain. So if I is a nonzero finitely generated homogeneous ideal of R , then $IR_{S(H)}$ is invertible. Note that $(IR_{S(H)})^{-1} = I^{-1}R_{S(H)}$ (cf. Proposition 1.3); so $R_{S(H)} = (IR_{S(H)})(IR_{S(H)})^{-1} = (II^{-1})R_{S(H)}$. Hence $II^{-1} \cap S(H) \neq \emptyset$, and since II^{-1} is homogeneous, we have $II^{-1} = R$. Thus R is a graded Prüfer domain. \square

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