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Right l-groups, geometric Garside groups, and solutions of the quantum Yang–Baxter equation

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Right lattice-ordered groups are introduced and studied as a general framework for Garside groups and related groups without a Garside element. Every right l-group G has a (two-sided) partially ordered subgroup $N(G)$ which generalizes the quasi-centre of an Artin–Tits group. The group $N(G)$ splits into copies of $\mathbb{Z}$ if G is noetherian. The positive cone of a right l-group is described as a structure that is known from algebraic logic: a pair of left and right self-similar hoops. For noetherian right l-groups G, modularity of the lattice structure is characterized in terms of an operation on the set $X(G^-)$ of atoms. It is proved that modular Garside groups are equivalent to finite projective spaces with a non-degenerate labelling. A concept of duality for $X(G^-)$ is introduced and applied in the distributive case. This gives a one-to-one correspondence between noetherian right l-groups with duality and non-degenerate unitary set-theoretic solutions of the quantum Yang–Baxter equation. The description of Garside groups via Garside germs is extended to right l-groups, which yields a one-sided enhancement and a new proof of Dvurečenskij's non-commutative extension of Mundici's correspondence between abelian l-groups and MV-algebras.

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Contents

Introduction

In 1972, a simultaneous breakthrough in understanding the structure of Artin–Tits groups was made by Brieskorn and Saito [10], and independently, by Deligne [26]. The initial spark came from Garside's discovery [36] of the fundamental element Δ_n in Artin's braid group, which led to a solution of the conjugacy problem and the determination of the centre. The fascination about these insights gave birth to the rapidly developing theory of Garside groups (see [25,19,22,23]). According to [22], the philosophy of Garside's theory in its current perception consists in the study of certain groups as groups of fractions of monoids with special utilization of divisibility.

In the present paper, we pick up this view and propose to study Garside groups and similar groups without a Garside element as close relatives of lattice-ordered groups, briefly called *l-groups*, to take profit from their well-developed theory (see, e.g., [1,3,17, 43]). To this end, we introduce *right l-groups*, that is, groups G with a lattice structure such that

$$a \leqslant b \implies ac \leqslant bc \tag{1}$$

holds for all $a, b, c \in G$. In contrast to l-groups, the left-hand version of this law need not be satisfied. Garside groups are then to be viewed as bounded atomic right l-groups with finitely many atoms having a *strong order unit* (Section 6).

With this perspective in mind, we show first that the negative (!) cone of a right l-group can be characterized as a pair of left and right *self-similar hoops* [63] with a common multiplication (Theorem 1). Recall that a *left hoop* [7,63] is a monoid M with a binary operation $\to$ satisfying the equations

$$a \to a = 1$$
$$ab \to c = a \to (b \to c)$$
$$(a \to b)a = (b \to a)b.$$

With the opposite multiplication, M is said to be a *right hoop*. A left hoop M is called *self-similar* [63] if it satisfies $a \to ba = b$ instead of the weaker equation $a \to a = 1$. Due to their origin in algebraic logic [6,5], hoops have a unit element that stands for the truth value 1, the greatest element, which explains our preference of the negative cone. The binary operation "$\to$" stands for the logical implication, while multiplication interprets the logical conjunction.

As in the case of l-groups, *right* l-groups are determined by their positive (or negative) cone, and vice versa. The missing left-hand version of (1) entails that the partial order of the positive cone and that of the negative cone are no longer related by the inversion $x \mapsto x^{-1}$. So there are actually two partial orders, a phenomenon that is well known in the theory of Garside groups. Similar to l-groups, every element of a right l-group G has a unique representation of the form ab^{-1} with orthogonal $a, b \in G$ (Corollary 4 of Theorem 1). However, a and b no longer commute.

Garside groups, and in particular, braid groups, form a class of right l-groups where every element can be factored into a finite product of atoms such that the number of factors is bounded. As a lattice, they satisfy the ascending and descending chain condition. Totally ordered right l-groups, also called *right ordered groups*, form another class of right l-groups. Introduced and first studied by Conrad [16], they play an important rôle in group theory and low-dimensional topology. For example, the mapping class group of a compact surface with finitely many punctures and non-empty boundary is right orderable [59,65]. In particular, braid groups (i.e. mapping class groups of punctured discs), are right orderable, a celebrated result which was first proved by Dehornoy [18]. Using the hyperbolic structure, many right orderings can be constructed by a method of Thurston [65,55]. Right orderability of 3-manifolds is still a mystery that has yet to be revealed. For example, Boyer, Rolfsen and Wiest [9] have shown that the fundamental group of a compact connected P^2-irreducible 3-manifold with positive first Betti number is right orderable. In particular, all knot groups are right orderable [47]. On the other hand, the fundamental group of many L-spaces is not right orderable. Boyer, Gordon, and Watson [8] conjectured that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left orderable. For further results on right orderability, we refer to the excellent account of Navas [54] and the literature cited there.

There is a close relationship between right ordered groups and l-groups. By the Cayley–Holland theorem [17], every l-group can be right ordered, and its partial order is the intersection of right orders. As a consequence, a group is right orderable if and only if it is isomorphic to a subgroup of an l-group. Surprisingly, right l-groups, being closest to both l-groups and right ordered groups, appear to be uncharted in the literature. One purpose of this article is to remedy this shortcoming. For a striking example how proofs can be reduced by using right l-groups, see Proposition 3.

Another purpose and a starting point of this paper has been to clarify the rôle of Garside groups that come from set-theoretic solutions of the quantum Yang–Baxter equation (QYBE). For the theory of such solutions and their occurrence in other parts of mathematics, the reader may consult [39,42,33,50,32,66,40,60,11,56,61,13,12,41]. For our present investigation, we drop the finiteness of the set of atoms to include infinite solutions of the QYBE.

Recently, Chouraqui [15] observed that the structure group of a finite non-degenerate unitary solution of the QYBE is a Garside group. Such Garside groups are of I-type [42], which means that their lattice is distributive and made up from n-cubes, equipped with a labelling so that the group structure can be read off from the labelling of a single cube. We attack the more general problem to characterize modular Garside groups, which leads to an extension and deeper understanding of "I-type". In the modular case, we obtain noetherian right l-groups as a patchwork of isomorphic projective spaces, the rule of attachment being encoded in the labelling of a single space (Sections 7 and 8).

Before giving more details at the end of this introduction, let us consider the special case of distributive right l-groups arising from solutions of the QYBE. The "geometry" of these groups is given by the n-cube of flats: every line has exactly two points, which means that the geometry splits into n distinct points. In other words, there is no geometry – all structure is contained in the labelling. To include infinite solutions of the QYBE, we have to dismiss the Garside element and consider *noetherian* right l-groups G. We show that a nice lattice structure of G depends on the set $\widetilde{X}(G^-) = X(G^-) \sqcup \{1\}$ of elements of length $\leqslant 1$, that is, the set $X(G^-)$ of atoms together with the unit element 1. The labelling is then contained in a binary operation $\rightarrow$ on the negative cone G^-, and the operation $\rightarrow$ is determined by its restriction to the set $X(G^-)$ of atoms. We will show that $\widetilde{X}(G^-)$ is closed with respect to $\rightarrow$ if and only if the lattice of G is lower semimodular (Proposition 5). Under these equivalent assumptions, the equation

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \tag{2}$$

is valid in $\widetilde{X}(G^-)$. Together with the implication

$$x \rightarrow y = x \rightarrow z \implies y = z,$$

Eq. (2) defines a *cycle set* [60], a structure that is known to be equivalent to a left non-degenerate unitary (set-theoretic) solution of the QYBE. We will show that G is distributive if and only if in addition to modularity, the above implication holds for $x, y, z \in X(G^-)$. Thus $\widetilde{X}(G^-)$ is almost a cycle set, with the only exception that the implication does not hold in the special case $x = y \neq z = 1$.

One way to avoid this anomaly is to deal with $X(G^-)$ rather than $\widetilde{X}(G^-)$. For a modular noetherian right l-group G, we define a *duality* to be a bijective map $D\colon X(G^-) \rightarrow X(G^-)$ satisfying

$$D(x \rightarrow y) = (y \rightarrow x) \rightarrow D(y)$$

for atoms $x \neq y$. The existence of a duality D entails its uniqueness and implies that the lattice of G is distributive (Proposition 7). We prove that a duality exists if and only if G

is isomorphic to the structure group of a non-degenerate unitary solution of the QYBE (Theorem 2). If the set of atoms is finite, G is a Garside group.

Apart from its relationship to solutions of the QYBE, Eq. (2) connects l-group theory with algebraic logic [63]. In this context, an element 1 of a set X with a binary operation $\to$ is said to be a *logical unit* if $x \to x = x \to 1 = 1$ and $1 \to x = x$ hold for all $x \in X$. Such an element 1 is unique. (In propositional logic, the unit 1 stands for a "true" proposition.) If X has a logical unit and satisfies Eq. (2) together with $x \to y = y \to x = 1 \Rightarrow x = y$, then X is called an *L-algebra* [63]. Every L-algebra admits a canonical embedding into a self-similar left hoop $S(X)$, the *self-similar closure*, which reveals a close relationship between algebraic logic and l-group theory [63]. On the other hand, the primitive elements as well as the simple elements [25,19] of a Garside group form an L-algebra. If an L-algebra admits a smallest element 0, there is a unique monoid homomorphism $\tau: S(X) \to S(X)$, given by $\tau(a)0 = 0a$ for all $a \in S(X)$. If τ is bijective, every $a \in S(X)$ has a unique *normal decomposition* $a = x_1 \cdots x_n$ with $x_i \in X$ (Proposition 12) similar to the normal decomposition in Garside groups, but without any kind of atomicity or noetherian hypothesis.

Our second main result (Theorem 3) places Garside groups into the context of intervals in l-groups, a theory initiated by Mundici's famous theorem [52] relating abelian l-groups with strong order unit to MV-algebras. This class of algebras was introduced by Chang [14] as the semantics of Łukasiewicz' calculus of infinite-valued logic. Mundici's theorem was extended to non-commutative l-groups by Dvurečenskij [31]. We further extend Dvurečenskij's result to right l-groups G with *strong order unit* (Definition 6). Here the prefix "strong" acquires a new meaning: It is assumed that conjugation by the order unit u maps the positive cone G^+ onto itself, a condition that obviously holds if G is an l-group. For example, every Garside element in the positive cone of a Garside group is a strong order unit. Theorem 3 gives a one-to-one correspondence between the isomorphism classes of right l-groups with strong order unit and pairs of L-algebras with common bounds 0 and 1 satisfying four equations (Definition 5).

Sections 7 and 8 deal with semimodular noetherian right l-groups G. Garside groups among them are those with a finite set of atoms. The Basterfield–Kelly theorem [2] implies that such Garside groups are modular. If $X(G^-)$ denotes the set of atoms, the finite meets of elements in $X(G^-)$ form a (dual) geometric lattice defining the *geometry* of G. The structure of G is then determined by the L-algebra $\widetilde{X}(G) = X(G^-) \sqcup \{1\}$. As the elements of $X(G^-)$ are pairwise incomparable, we call such an L-algebra *discrete*. We show first that discrete L-algebras are equivalent to a class of lower semimodular lattices with a consistent labelling of the edges called a *block labelling* (Proposition 18). If the geometry is finite-dimensional, the labelling gives a map from the points to the hyperplanes. The self-similar closure [63] of such an L-algebra can be viewed as a "negative cone" which, however, need not embed into its group of left fractions. To get an embedding, the block labelling has to be *non-degenerate* which implies that the map from points to hyperplanes is bijective. Matchings between points and hyperplanes of projective geometries have been studied by various authors (see [51,27,45,29,37,49]). At present, the geometric meaning of block labellings is completely unexplored. We show that finite geometric lattices need not admit a block labelling (Example 4) and exhibit a non-degenerate block labelling for the Fano plane (Example 5).

Our third main theorem establishes a one-to-one correspondence between non-degenerate discrete L-algebras and modular geometric right l-groups (see Theorem 4 and its corollary). As a consequence, we get a simple description of modular Garside groups by means of discrete L-algebras (Theorem 5). In particular, this extends and reformulates the correspondence between groups of I-type together with their associated solutions of the QYBE, and cycle sets, which thus turn out to be equivalent to a special class of discrete L-algebras.

The last section gives a brief account on the *quasi-centre* of a right l-group, introduced by Brieskorn and Saito in the case of Artin–Tits groups. For an archimedean right l-group G, the quasi-centre is the "l-group part" of G where the left and right partial order meet together. As in the theory of l-groups, noetherian right l-groups are order-complete, while order-complete right l-groups are archimedean. If G is noetherian, an old theorem of Birkhoff [4] implies that the quasi-centre is abelian, hence a cardinal sum of infinite cyclic groups. For a Garside group G, this decomposition of the quasi-centre parallels Picantin's bi-crossed product decomposition of G (see [57]).

1. Preliminaries on Garside groups

Recall that a monoid M satisfies the *left Ore condition* if for each pair $a, b \in M$, there exist $x, y \in M$ with $xa = yb$. Assume that M is left Ore and *right cancellative*, that is, $ac = bc$ implies $a = b$ for all $a, b, c \in M$. Then we can form the *group of left fractions* $G(M)$ which consists of formal fractions $a^{-1}b$ with $a, b \in M$ (see [38], Chapter I). For $a, b, c \in M$, the equation $a^{-1}b = (ca)^{-1}(cb)$ holds in $G(M)$, and the left Ore condition guarantees that in this way, two arbitrary left fractions can be transformed into left fractions with a common denominator. They are defined to be equal in $G(M)$ if the common denominator can be chosen so that the numerators become equal in M. Therefore, the natural map $M \to G(M)$ is injective if and only if M is *left cancellative*, that is, $ab = ac$ implies $b = c$ for all $a, b, c \in M$. If M is left cancellative and satisfies the *right Ore* condition, then $G(M)$ is a two-sided group of fractions, that is, $G(M) = M^{-1}M = MM^{-1}$.

Let M be a monoid with unit element 1. By $M^\times$ we denote the group of invertible elements, the *unit group* of M. An element $a \in M$ is said to be a *right divisor* of $b \in M$, or equivalently, b is called a *left multiple* of a, if $xa = b$ for some $x \in M$. We write $a \leqslant b$ if a is a right divisor of b. The relation $\leqslant$ is also called the *algebraic* preorder of M. If M is right cancellative, the algebraic preorder is a partial order if and only if the unit group $M^\times$ is trivial. Thus, assume that M is right cancellative with $M^\times = \{1\}$. A meet $a \wedge b$ with respect to the algebraic order in M is then called a *right greatest common divisor*, and a join $a \vee b$ is said to be a *left least common multiple* of a and b (see [19]).

Note that the existence of $a \vee b$ implies that there are $x, y \in M$ with $xa = yb = a \vee b$. In other words, M satisfies the left Ore condition.

Similarly, an element a of a monoid M is a *left divisor* of $b \in M$, written $a \preccurlyeq b$, if $ax = b$ for some $x \in M$. If M is left cancellative with $M^\times = \{1\}$, we write $a \curlywedge b$ and $a \curlyvee b$ for the meet and join with respect to $\preccurlyeq$.

Definition 1. A monoid M is said to be *noetherian* if bounded ascending sequences $a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots$ or $b_0 \preccurlyeq b_1 \preccurlyeq b_2 \preccurlyeq \cdots$ in M become *stationary*, that is, $a_{n+1} \leqslant a_n$, respectively $b_{n+1} \preccurlyeq b_n$, for some $n \in \mathbb{N}$.

An *atom* of a monoid M is an element $x \in M$ with $x \neq 1$ such that $x = ab$ implies that $x = a$ or $x = b$, and M is said to be *atomic* if every $a \in M$ can be represented as a product $a = x_1 \cdots x_n$ of atoms. If the number n of factors in such a representation is bounded, we call M *bounded atomic*. The set of atoms will be denoted by $X(M)$. The following proposition is well known and obvious.

Proposition 1. *Every bounded atomic monoid is noetherian. Conversely, a noetherian left and right cancellative monoid M with $M^\times = \{1\}$ is atomic.*

A monoid M has been called *Gaussian* [19,25] if M is bounded atomic, left and right cancellative, and a lattice with respect to each of the partial orders $\leqslant$ and $\preccurlyeq$. Although the term "Gaussian" has become out of date for some years, we occasionally reuse it by lack of an equivalent and because of its brevity. Since a Gaussian monoid M is cancellative and satisfies the left and right Ore condition, M has a two-sided group of fractions $G(M)$. Accordingly, $G(M)$ has been called a *Gaussian group* [25,19].

Each pair of elements $a, b \in M$ of a Gaussian monoid M determines four elements [19] by the equations

$$a \vee b = (a/b)b = (b/a)a \tag{3}$$

$$a \curlyvee b = a(a\backslash b) = b(b\backslash a). \tag{4}$$

An element $\Delta \in M$ is said to be *Garside* [25,19] if the set of left divisors of Δ is finite, generates M, and coincides with the set of right divisors of Δ. If M is Gaussian having a Garside element Δ, then M is called a *Garside monoid*, and $G(M)$ is said to be a *Garside group* [25,19].

2. Right l-groups

A *right partially ordered group* is a group G with a partial order $\leqslant$ satisfying

$$a \leqslant b \;\Rightarrow\; ac \leqslant bc$$

for all $a, b, c \in G$. Equivalently, G is given by a submonoid P with trivial unit group $P^\times = P \cap P^{-1}$. Namely, $a \leqslant b \iff ba^{-1} \in P$, where $P = G^+ := \{a \in G \mid a \geqslant 1\}$ is the *positive cone* of G. By symmetry, P^{-1} is the *negative cone* $G^- := \{a \in G \mid a \leqslant 1\}$.

Every right partial order $\leqslant$ of a group G corresponds to a *left partial order*

$$a \preccurlyeq b \;:\Longleftrightarrow\; b^{-1} \leqslant a^{-1} \;\Longleftrightarrow\; a^{-1}b \in G^+$$

which can be viewed as a right partial order of G^{op}. We say that a right partially ordered group G is a *right l-group* if G is a lattice. In particular, G is a *lattice-ordered* group [3,17] or simply an *l-group* if G is a right l-group for which the left and right partial orders coincide.

A monoid M with a binary operation $\rightarrow$ is said to be a *left hoop* [63] if

$$a \rightarrow a = 1 \tag{5}$$

$$ab \rightarrow c = a \rightarrow (b \rightarrow c) \tag{6}$$

$$(a \rightarrow b)a = (b \rightarrow a)b \tag{7}$$

hold for $a, b, c \in M$. By [63], Proposition 3, the unit element 1 of a left hoop M is a *logical unit*, that is, every $a \in M$ satisfies

$$a \rightarrow a = a \rightarrow 1 = 1, \qquad 1 \rightarrow a = a. \tag{8}$$

Moreover, M has a natural partial order

$$a \leqslant b \;:\Longleftrightarrow\; a \rightarrow b = 1 \tag{9}$$

which is opposite to the algebraic order:

Proposition 2. *Let M be a left hoop. Every pair of elements $a, b \in M$ satisfies*

$$a \leqslant b \iff \exists x \in M\colon a = xb.$$

Proof. If $a \leqslant b$, then Eq. (7) implies that $a = (b \rightarrow a)b$. Conversely, assume that $a = xb$. Then Eqs. (5) and (6) give $a \rightarrow b = xb \rightarrow b = x \rightarrow (b \rightarrow b) = x \rightarrow 1 = 1$. □

By virtue of (9), Eq. (6) yields the adjointness relation

$$ab \leqslant c \iff a \leqslant b \rightarrow c,$$

which shows that the operation $\rightarrow$ and the multiplication of M determine each other. So the multiplication of a left hoop can be eliminated. For example, by [63], Corollary 2 of Proposition 2, Eq. (7) is equivalent to

$$(a \rightarrow b) \rightarrow (a \rightarrow c) = (b \rightarrow a) \rightarrow (b \rightarrow c). \tag{10}$$

On the other hand, a left hoop can be regarded as a special type of monoid. By [63], Proposition 4, every left hoop M is a $\wedge$-semilattice with meet

$$a \wedge b = (a \to b)a \tag{11}$$

for $a, b \in M$.

Furthermore, by [63], Proposition 5, a left hoop M is right cancellative if and only if it satisfies the equation

$$a \to ba = b. \tag{12}$$

Such a left hoop M is said to be *self-similar* [63]. Note that Eq. (7) implies the left Ore condition. Therefore, every self-similar left hoop M admits a group of left fractions.

Dually, a *right hoop* is a monoid M with a binary operation $\rightsquigarrow$ satisfying

$$a \rightsquigarrow a = 1$$
$$ab \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c)$$
$$a(a \rightsquigarrow b) = b(b \rightsquigarrow a)$$

for $a, b, c \in M$. In other words, M^{op} (opposite multiplication) is a left hoop. Accordingly, a right hoop M is called *self-similar* if M^{op} is self-similar.

Definition 2. We define a *right l-cone* to be a monoid M with binary operations $\to$ and $\rightsquigarrow$ which make M into is a left and right self-similar hoop.

The terminology is justified by the following

Theorem 1. *Let G be a right l-group. The negative cone G^- is a right l-cone with operations*

$$a \to b := ba^{-1} \wedge 1, \qquad a \rightsquigarrow b := (b^{-1}a \vee 1)^{-1}. \tag{13}$$

Conversely, every right l-cone arises in this way.

Proof. Let G be a right l-group. Since 1 is the greatest element of G^-, Eq. (5) follows. By definition, the right multiplications $x \mapsto xa$ are isotone, hence lattice automorphisms. Using (13), this yields Eqs. (6), (7), and (12). Thus G^- is a self-similar left hoop. The group isomorphism $G \xrightarrow{\sim} G^{\mathrm{op}}$ given by $a \mapsto a^{-1}$ maps G^- onto the positive cone G^+. Furthermore, Eqs. (13) yield a binary operation

$$b/a = (a^{-1} \rightsquigarrow b^{-1})^{-1} = ba^{-1} \vee 1$$

on G^+ which can be viewed as a counterpart of $\to$ on G^-. Hence G^+ with this operation and the opposite order $\geqslant$ is a self-similar left hoop. Thus $(G^-;\rightsquigarrow)$ is a self-similar right hoop, which shows that G^- is a right l-cone. Every $g \in G$ can be written as $g = a^{-1}b$ with $a := (g \wedge 1)g^{-1}$ and $b := (g \wedge 1)$ in G^-. So G is the group of left fractions of G^-.

Conversely, let M be a right l-cone, and let $G = G(M)$ be its group of left fractions. Since M is a left and right hoop, it satisfies the left and right Ore condition and is cancellative. Hence G is a two-sided group of fractions, and M can be regarded as a submonoid of G. We make G into a right partially ordered group with negative cone M. This means that G is endowed with the partial order

$$a \leqslant b \iff ab^{-1} \in M.$$

By Proposition 2, this partial order of G induces the partial order (9) on M. Since $G = MM^{-1}$, each pair of elements $a, b \in G$ is of the form $a = ce^{-1}$, $b = de^{-1}$ with $c, d, e \in M$. Hence $(c \wedge d)e^{-1}$ is the meet of a and b in G. Dually, the right hoop M defines the corresponding left partial order $a \preccurlyeq b \Leftrightarrow b^{-1}a \in M$ on G. So the positive cone G^+ becomes a left hoop with partial order $a \leqslant b \Leftrightarrow b^{-1} \preccurlyeq a^{-1} \Leftrightarrow ab^{-1} \in G^- \Leftrightarrow ba^{-1} \in G^+$. Hence G^+ is a $\vee$-semilattice, and the above argument shows that G is a lattice. Thus G is a right l-group with negative cone M. Since $\to$ and $\rightsquigarrow$ are uniquely determined by the multiplication of M which is induced by the multiplication of G, Eqs. (13) follow from the first part of the proof. $\square$

As an immediate consequence, we obtain the first part of

Corollary 1. *Up to isomorphism, there is a one-to-one correspondence between right l-groups and right l-cones. In particular, a Gaussian group is the same as a right l-group with a bounded atomic negative cone.*

Proof. To prove the second part, let M be a Gaussian monoid. With the algebraic ordering, M can be identified with the positive cone of $G := G(M)$. Applying $x \mapsto x^{-1}$ to Eqs. (4) gives Eqs. (7) and (11) with

$$a \to b = (a^{-1} \backslash b^{-1})^{-1}.$$

In particular, $(a \to a)a = a \wedge a = a$, which yields Eq. (5). Furthermore, Eq. (6) follows by [19], Lemma 1.7. Since G^- is right cancellative, this implies that G^- is a self-similar left hoop. By symmetry, G^- is a right l-cone. Whence G is a right l-group. The converse is trivial. $\square$

Remark. By Proposition 2, the partial order (9) of a right l-cone M is opposite to the algebraic order of M. This discrepancy resolves within the group $G = G(M)$ where the partial order on $M = G^-$ and the algebraic order on G^+ are induced by the partial order of the whole group. In other words, a right l-cone M is the *negative* cone in its group of fractions $G(M)$ while a Gaussian monoid M is the *positive* cone in $G(M)$.

For a right l-cone M, Theorem 1 shows that the partial order of M^{op} is given by

$$a \preccurlyeq b \iff a \rightsquigarrow b = 1.$$

Therefore, $M^{\mathrm{op}} = (M^{\mathrm{op}}; \rightsquigarrow, \rightarrow)$ is a right l-cone with corresponding right l-group G^{op}. We call M^{op} the *dual* of M.

One may wonder if the join in M admits an explicit representation like Eq. (11) for the meet. This is indeed the case:

Corollary 2. *Let $M = (M; \rightarrow, \rightsquigarrow)$ be a right l-cone. Then*

$$a \vee b = \big((b \rightarrow a) \rightsquigarrow (a \rightarrow b)\big) \rightsquigarrow b \tag{14}$$

holds for all $a, b \in M$.

Proof. By Eqs. (13) and (11), we have

$$\begin{aligned} \big((b \rightarrow a) \rightsquigarrow (a \rightarrow b)\big) \rightsquigarrow b &= \big((a \rightarrow b)^{-1}(b \rightarrow a) \vee 1\big)^{-1} \rightsquigarrow b \\ &= \Big(b^{-1}\big((a \rightarrow b)^{-1}(b \rightarrow a) \vee 1\big)^{-1} \vee 1\Big)^{-1} \\ &= \Big(\big((a \rightarrow b)^{-1}(b \rightarrow a)b \vee b\big)^{-1} \vee 1\Big)^{-1} \\ &= \Big(\big((a \rightarrow b)^{-1}(a \rightarrow b)a \vee b\big)^{-1} \vee 1\Big)^{-1} \\ &= \big((a \vee b)^{-1} \vee 1\big)^{-1} = a \vee b. \qquad \square \end{aligned}$$

Corollary 3. *Let $M = (M; \rightarrow, \rightsquigarrow)$ be a right l-cone. For all $a, b \in M$,*

$$(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1.$$

Proof. By Theorem 1, we have to verify that $(b^{-1}a \vee 1)^{-1} \vee (a^{-1}b \vee 1)^{-1} = 1$. Multiplying with $(a^{-1}b \vee 1)b^{-1}a = 1 \vee b^{-1}a$ from the right, the equation turns into the valid equation $(b^{-1}a \vee 1)^{-1}(1 \vee b^{-1}a) \vee b^{-1}a = 1 \vee b^{-1}a$. $\square$

For an element g of a right l-group G, we define

$$g^{+} := (g \vee 1)^{-1}, \qquad g^{-} := g(g \vee 1)^{-1}.$$

Note that $g^{+}, g^{-} \in G^{-}$. We call g^{+} the *positive* and g^{-} the *negative part* of g.

Corollary 4. *Let G be a right l-group. Every element $g \in G$ has a unique representation $g = ab^{-1}$ with $a \vee b = 1$, namely, $a = g^{-}$ and $b = g^{+}$.*

Proof. As G is a left group of fractions of G^-, there exist $a, b \in G$ with $g = a^{-1}b$. Since $a(a \rightsquigarrow b) = b(b \rightsquigarrow a)$, this gives $g = (a \rightsquigarrow b)(b \rightsquigarrow a)^{-1}$. By Corollary 3, we have $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$, which yields the required representation. Conversely, assume that $g = ab^{-1}$ with $a \vee b = 1$. Then $b^{-1} = (a \vee b)b^{-1} = ab^{-1} \vee 1$. Hence $b = (g \vee 1)^{-1} = g^+$, and thus $a = gb = g^-$. $\square$

Corollary 4 extends a fundamental property of l-groups to right l-groups. Another basic property of right l-groups is given by

Proposition 3. *Every right l-group G is torsion-free.*

Proof. If $g^n = 1$, then $h := 1 \vee g \vee \cdots \vee g^{n-1}$ satisfies $hg = h$. Whence $g = 1$. $\square$

The proof that braid groups are torsion-free has thus been reduced to a single line.[1]

Example 1. Let Ω be the set of ordinals α with $|\alpha| < \kappa$ for a fixed infinite cardinal κ. With respect to addition, the monoid Ω is a self-similar right hoop with $\alpha \rightsquigarrow \beta := \inf\{\gamma \in \Omega \mid \alpha + \gamma \geqslant \beta\}$. In particular, Ω admits a group of right fractions. Moreover, Eqs. (5)–(7) and (12) hold whenever the operation $\alpha \to \beta := \inf\{\gamma \in \Omega \mid \gamma + \alpha \geqslant \beta\}$ is defined on both sides of the equation. Note that for $\alpha, \beta \in \Omega$, an ordinal γ with $\gamma + \alpha \geqslant \beta$ need not exist. Thus, in general, $\alpha \to \beta$ is not defined everywhere, and Ω is not a right l-cone.

3. Noetherian right l-groups with duality

Much of the vast theory of l-groups has no counterpart for right l-groups. The reason is that the positive and the negative cone are too loosely connected. Nevertheless, there are strong similarities yet to be exploited. The main difference between a right l-group and a classical l-group consists in the splitting of the lattice order into a left and right one. Their intersection is an l-subgroup, the *quasi-centre*, which will be briefly discussed in Section 9.

Let us call a right l-group G *noetherian* if its negative cone G^- is noetherian. Equivalently, this means that every bounded increasing or decreasing sequence becomes stationary. If G is an l-group, it is enough to assume that bounded increasing sequences become stationary.

An old theorem of Birkhoff [4] states that noetherian l-groups are abelian, and that every such group is a cardinal sum $\mathbb{Z}^{(I)}$, where $\mathbb{Z}$ stands for the additive group of integers. Trivially, such groups are Gaussian. In general, a Gaussian group need not even be distributive as a lattice, a phenomenon that is typical for Artin–Tits groups and other

[1] Such a proof was first given by Fadell, Fox and Neuwirth [34,35] by topological arguments. Using Garside calculus, direct proofs became possible [20,58]. Later, Dehornoy eliminated the unnecessary noetherian hypothesis which led to a much shorter proof [21].

Garside groups. However, there is an important class of noetherian groups behaving more regularly and being more closely related to l-groups.

Recall that an element a of a lattice *covers* b if $a > b$ and there are no other elements between a and b. The set of elements b covered by a will be denoted by a^-. A lattice is said to be *upper semimodular* if $a \vee b$ covers b whenever a covers $a \wedge b$. Lattices satisfying the reverse implication are called *lower semimodular*. A *modular* lattice is defined by the implication

$$a \leqslant c \ \Rightarrow\ (a \vee b) \wedge c = a \vee (b \wedge c).$$

Proposition 4. *Let M be a noetherian right l-cone. For $x, y \in X(M)$ with $x \neq y$,*

$$(x \to y) \rightsquigarrow (y \to x) = x = (x \rightsquigarrow y) \to (y \rightsquigarrow x).$$

Proof. Since $(X; \rightsquigarrow)$ is right self-similar, Eq. (14) implies that

$$\begin{aligned}
\big((x \to y) \rightsquigarrow (y \to x)\big)(x \vee y) &= \big((x \to y) \rightsquigarrow (y \to x)\big)\Big(\big((x \to y) \rightsquigarrow (y \to x)\big) \rightsquigarrow x\Big) \\
&= x\big(x \rightsquigarrow \big((x \to y) \rightsquigarrow (y \to x)\big)\big) \\
&= x\big((x \to y)x \rightsquigarrow (y \to x)\big) \\
&= x\big((y \to x)y \rightsquigarrow (y \to x)\big) = x1 = x.
\end{aligned}$$

As $x \neq y$, this gives the first equation. Passage to M^{op} yields the second equation. $\square$

For a monoid M, consider the set

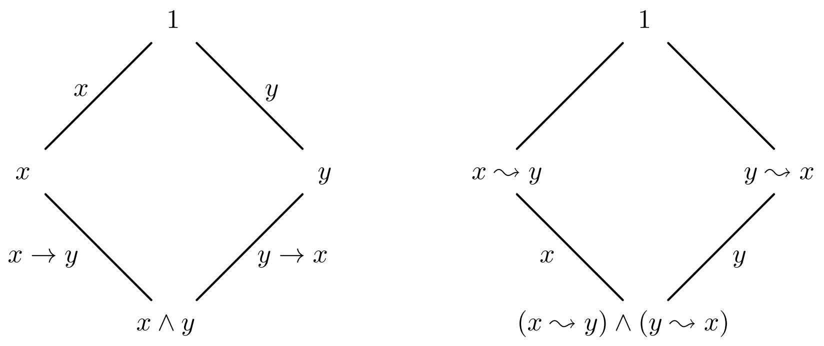
$$\widetilde{X}(M) := X(M) \cup \{1\}$$

of atoms together with the unit element 1.

Proposition 5. *Let M be a noetherian right l-cone with group of fractions $G = G(M)$.*

(a) *G is upper semimodular if and only if $x, y \in \widetilde{X}(M)$ implies that $x \rightsquigarrow y \in \widetilde{X}(M)$.*
(b) *G is lower semimodular if and only if $x, y \in \widetilde{X}(M)$ implies that $x \to y \in \widetilde{X}(M)$.*
(c) *G is modular if and only if $\widetilde{X}(M)$ is closed with respect to $\to$ and $\rightsquigarrow$.*

Proof. Assume first that G is upper or lower semimodular, and let $x, y \in \widetilde{X}(M)$ be given. Since 1 is a logical unit of M as a left or right hoop, we can assume that $x, y \in X(M)$ and $x \neq y$. Then we have the Hasse diagrams



where the label of any edge, multiplied from the left with the upper node yields the lower node of the edge. So the left-hand diagram is equivalent to Eq. (11). As to the right-hand diagram, Corollary 3 of Theorem 1 shows that $(x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$. So it remains to verify that $x(x \rightsquigarrow y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x)$. By Eq. (11), this follows from Proposition 4.

If M is lower semimodular, the first diagram shows that $x \to y$ and $y \to x$ are atoms. Similarly, the second diagram implies that $x \rightsquigarrow y, y \rightsquigarrow x \in X(M)$ if M is upper semimodular. This proves half of (a) and (b). Conversely, assume that $\widetilde{X}(M)$ is closed with respect to the operation $\to$. Then the first diagram, multiplied from the right by any element of G, gives a weak form of lower semimodularity: If a and b are covered by $a \vee b$, they cover $a \wedge b$. Since M is noetherian, every bounded descending chain in G becomes stationary. Hence G is lower semimodular by [28], Lemma 3.3. This establishes (b). Using the right-hand diagram, a similar argument proves (a). Now (c) follows by [28], Lemma 3.4. $\square$

Remark. Note that for an upper or lower semimodular noetherian right l-group G, the length of maximal chains of any interval are equal. So G is a Gaussian group.

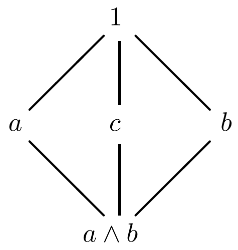
Proposition 6. *Let M be a noetherian right l-cone. The following are equivalent.*

(a) *The group of fractions $G(M)$ is distributive.*
(b) *$G(M)$ is modular and $x \to y = x \to z \;\Rightarrow\; y = z$ holds for $x, y, z \in X(M)$.*
(c) *$G(M)$ is modular and $x \rightsquigarrow y = x \rightsquigarrow z \;\Rightarrow\; y = z$ holds for $x, y, z \in X(M)$.*

Proof. By symmetry, it is enough to verify the equivalence (a) $\Leftrightarrow$ (b).

(a) $\Rightarrow$ (b): Let x, y, z be atoms with $x \to y = x \to z$. Multiplying with x from the right gives $x \wedge y = x \wedge z$. Suppose that $y \neq z$. Then $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = x \wedge y = x \wedge z$, which implies that $x \leqslant y$ and $x \leqslant z$. Hence $x = y = z$, a contradiction.

(b) $\Rightarrow$ (a): Suppose that $G(M)$ is not distributive. Then $G(M)$ contains a diamond sublattice (see [44], Chapter II, Theorem 2). Multiplying with a suitable element from the right, we can assume that the greatest element of the diamond is 1:



Choose an atom $z \in X(M)$ with $c \leqslant z$, and define

$$x := (z \wedge b) \vee a$$
$$y := (z \wedge a) \vee b.$$

Since $G(M)$ is modular, $x, y \in X(M)$. Furthermore, the modularity gives $x \wedge z = (z \wedge b) \vee (a \wedge z) \leqslant y$. Hence $(z \to x)z = z \wedge x = z \wedge y = (z \to y)z$, and thus $z \to x = z \to y$, which yields $x = y$, contrary to $a \vee b = 1$. $\square$

Definition 3. Let G be a noetherian right l-group. We define a *duality* of G to be a bijection $D\colon X(G^-) \to X(G^-)$ which satisfies

$$D(x \to y) = (y \to x) \to D(y) \tag{15}$$

for all $x, y \in X(G^-)$ with $x \neq y$.

Proposition 7. *Every modular noetherian right l-group G with duality D is distributive and satisfies*

$$D^{-1}(x \rightsquigarrow y) = (y \rightsquigarrow x) \rightsquigarrow D^{-1}(y) \tag{16}$$

for $x, y \in X(G^-)$ with $x \neq y$.

Proof. We show first that $x \to y = x \to z$ implies $y = z$ for $x, y, z \in X(G^-)$. Assume first that $x = y$. Then $x \to z = 1$, which yields $z = x = y$. So we can assume that $x \neq y$ and $x \neq z$. Then $D(y \to x) = (x \to y) \to D(x) = (x \to z) \to D(x) = D(z \to x)$. Hence $y \to x = z \to x$, and thus $(y \to x)y = (x \to y)x = (x \to z)x = (z \to x)z = (y \to x)z$, which yields $y = z$. By Proposition 6, it follows that G is distributive.

To verify Eq. (16), suppose that $D^{-1}(y) = y \rightsquigarrow x$. Then $D(y \rightsquigarrow x) = y$, and Corollary 3 of Theorem 1 implies that $x \rightsquigarrow y \neq y \rightsquigarrow x$. Therefore, Proposition 4 yields $D(x) = D\big((x \rightsquigarrow y) \to (y \rightsquigarrow x)\big) = \big((y \rightsquigarrow x) \to (x \rightsquigarrow y)\big) \to D(y \rightsquigarrow x) = y \to y = 1$, which is impossible. Hence $D^{-1}(y) \neq y \rightsquigarrow x$. With $u := D^{-1}(y) \rightsquigarrow (y \rightsquigarrow x)$ and $v := (y \rightsquigarrow x) \rightsquigarrow D^{-1}(y)$, Proposition 4 yields $D^{-1}(y) = u \to v$. Hence $(y \rightsquigarrow x) \to (x \rightsquigarrow y) = y = D(u \to v) = (v \to u) \to D(v) = (y \rightsquigarrow x) \to D(v)$. So Proposition 6 implies that $x \rightsquigarrow y = D(v)$. Thus $D^{-1}(x \rightsquigarrow y) = v = (y \rightsquigarrow x) \rightsquigarrow D^{-1}(y)$, which establishes Eq. (16). $\square$

Next we show that a bijective map which satisfies Eq. (15) must be unique.

Proposition 8. *Let G be a modular noetherian right l-group with duality D. Assume that $x \in X(G)$. Then $D(x)$ is the unique element $y \in X(G^-)$ such that the interval $[yx, 1]$ is a chain.*

Proof. We show first that $[D(x)x, 1]$ is a chain. Suppose that there is an element $y \in X(G^-)$ with $D(x)x = x \wedge y$. Then $D(x) = x \to y$. Hence $D(y \to x) = (x \to y) \to D(x) = 1$, a contradiction. Conversely, let $y \in X(G^-)$ be an atom with $y \neq D(x)$ such that $[yx, 1]$ is a chain. By Proposition 4, we have $D(x) = (D(x) \to y) \rightsquigarrow (y \to D(x))$. Hence $D(x) \to y \neq y \to D(x)$. Therefore, Eq. (16) and Proposition 4 yield $x = D^{-1}D(x) = \big((y \to D(x)) \rightsquigarrow (D(x) \to y)\big) \rightsquigarrow D^{-1}(y \to D(x)) = y \rightsquigarrow D^{-1}(y \to D(x))$. So $yx = y\big(y \rightsquigarrow D^{-1}(y \to D(x))\big) = D^{-1}(y \to D(x))z$ with $z := D^{-1}(y \to D(x)) \rightsquigarrow y \in X(G^-)$. Since $[yx, 1]$ is a chain, we infer that $z = x$, which yields $D^{-1}(y \to D(x)) = y$. Hence $y \to D(x) = D(y)$, and thus $1 = (y \to D(x)) \to D(y) = D(D(x) \to y) \in X(G^-)$, a contradiction. $\square$

4. The quantum Yang–Baxter equation

As an application of Section 3, we now characterize the class of noetherian groups arising from solutions of the quantum Yang–Baxter equation. Let V be a vector space. Every linear map $R: V \otimes V \to V \otimes V$ gives rise to maps $R^{ij}: V^{\otimes n} \to V^{\otimes n}$ where R acts on the ith and jth component of the tensor product $V^{\otimes n}$ with $n \geqslant 2$. For example, $R^{12} = R \otimes 1$, where 1 denotes the identity map on $V^{\otimes(n-2)}$. The quantum Yang–Baxter equation is the equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{17}$$

in $\mathrm{End}(V^{\otimes 3})$. Note that the identity map $1_{V\otimes V}$ is a solution of Eq. (17). To initiate the study of solutions which cannot be obtained as deformations of the trivial solution $R = 1$, Drinfeld [30] introduced *set-theoretic* solutions, that is, solutions R induced by a map $X^2 \to X^2$ for a basis X of the vector space V. Thus $R(x, y) = (x^y, {}^xy)$ is given by two binary operations $(x, y) \mapsto x^y$ and $(x, y) \mapsto {}^xy$ on X.

A set-theoretic solution R of Eq. (17) is said to be *unitary* if $R^{21}R = 1$. It is called *left non-degenerate* if the component map $x \mapsto x^y$ is bijective for all $y \in X$. If the maps $y \mapsto {}^xy$ are bijective, too, the solution R is said to be *non-degenerate* [33]. By [60], Proposition 1, the left non-degenerate unitary solutions $R: X^2 \to X^2$ are equivalent to binary operations $\cdot$ on X with bijective left multiplications $y \mapsto x \cdot y$ such that

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{18}$$

for $x, y, z \in X$. Sets X with such an operation $\cdot$ are called *cycle sets* [60]. Under this correspondence, the map $y \mapsto x \cdot y$ is inverse to $y \mapsto y^x$ for all $x \in X$. The solution R

associated to a cycle set X is non-degenerate if and only if the square map $x \mapsto x \cdot x$ is bijective ([60], Proposition 2). Accordingly, such cycle sets are called *non-degenerate*.

The *structure group* [33] of a solution R of Eq. (17) on a set X is the group G_X generated by X with relations

$$x \circ y = {}^{x}y \circ x^{y} \tag{19}$$

for all $x, y \in X$. In terms of cycle sets, the structure group can be obtained as follows.

A cycle set A with an abelian group structure is said to be *linear* [60] if it satisfies the equations

$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{20}$$

$$(a + b) \cdot c = (a \cdot b) \cdot (a \cdot c). \tag{21}$$

Note that Eq. (21) implies Eq. (18) by the commutativity of addition. Linear cycle sets are equivalent to *braces* [61]. They carry a group structure (the *adjoint group*) $G(A) = (A, \circ)$ given by

$$a \circ b := a^{b} + b, \tag{22}$$

where $a \mapsto a^b$ is inverse to $a \mapsto b \cdot a$.

Every non-degenerate cycle set X admits a unique extension to a brace $\mathbb{Z}^{(X)}$ on the free abelian group generated by X. The operation $\cdot$ on $\mathbb{Z}^{(X)}$ is obtained inductively from equations (20) and (21). The adjoint group $G(\mathbb{Z}^{(X)})$ is isomorphic to the above mentioned structure group G_X (see [60], Section 2). Using the substitution $x \mapsto y \cdot x$, the defining equation (19) turns into $(y \cdot x) \circ y = (x \cdot y) \circ x = x + y$, which connects the adjoint group with the additive group of $\mathbb{Z}^{(X)}$.

Etingof et al. [33] have shown that the structure group G_X is solvable if X is finite. Chouraqui [15] recently observed that G_X is a Garside group. More generally, the following theorem gives a precise characterization of the right l-groups associated to non-degenerate unitary solutions of the quantum Yang–Baxter equation. As our proof does not make use of [15], it includes a new proof and extension of Chouraqui's result.

Theorem 2. *The map $X \mapsto G_X$ defines a one-to-one correspondence between*

(a) *non-degenerate cycle sets X, or equivalently,*
(b) *non-degenerate unitary set-theoretic solutions of the QYBE (17), and*
(c) *modular noetherian right l-groups with duality.*

Proof. The equivalence between (a) and (b) follows by [60], Propositions 1 and 2. Let $R: X^2 \to X^2$ be a non-degenerate unitary solution of Eq. (17), and let X be the corresponding cycle set. Thus $A := \mathbb{Z}^{(X)}$ is a brace, and $G_X = G(A)$ is the structure group of R. To avoid confusion, the inverse of $a \in G_X$ will be denoted by a'. With the positive cone $\mathbb{N}^{(X)}$, the group G_X has a right partial order

$$a \leqslant b \iff b \circ a' \in \mathbb{N}^{(X)}.$$

By Eqs. (20) and (22), and [61], Eqs. (18) and (21), we have $b \circ a' = b^{a'} + a' = (a \cdot b) - (a \cdot a) = a \cdot (b - a)$. Since $c \mapsto a \cdot c$ is a bijection $\mathbb{N}^{(X)} \to \mathbb{N}^{(X)}$, this shows that $a \leqslant b \iff b - a \in \mathbb{N}^{(X)}$. So the right partial order of G_X coincides with the natural partial order of $\mathbb{Z}^{(X)}$. Thus G_X is a noetherian right l-group, with a distributive lattice structure. For $x, y \in X$ with $x \neq y$, Eqs. (11) and (21) give $(x \cdot y) \circ x = x + y = x \wedge y = (x \to y) \circ x$. Hence

$$x \cdot y = x \to y \tag{23}$$

for $x \neq y$ in X. Define a duality $D\colon X \to X$ by

$$D(x) := x \cdot x.$$

Since X is non-degenerate, the map D is bijective. For $x \neq y$ in X, Eq. (23) implies that $D(x \to y) = (x \cdot y) \cdot (x \cdot y) = (y \cdot x) \cdot (y \cdot y) = (y \to x) \to D(y)$.

Conversely, let G be a modular noetherian right l-group with duality D. By Proposition 7, G is distributive. Define an operation $\cdot$ on $X := X(G^-)$ by

$$x \cdot y := \begin{cases} x \to y & \text{for } x \neq y \\ D(x) & \text{for } x = y. \end{cases} \tag{24}$$

Then X is closed under $\cdot$ by Proposition 5, which shows that the operation (24) is well defined. To verify Eq. (18), assume first that $x, y, z \in X$ are distinct. By Proposition 6, this implies that $x \cdot y \neq x \cdot z$ and $y \cdot x \neq y \cdot z$. Hence (18) follows by Eq. (10). Therefore, by symmetry, we can assume that $x \neq y = z$. So we have to deal with the equation $(x \cdot y) \cdot (x \cdot y) = (y \cdot x) \cdot (y \cdot y)$. By Eq. (24), this is just equivalent to Eq. (15).

It remains to verify that the maps $\sigma(x)\colon X \to X$ with $\sigma(x)(y) := x \cdot y$ are bijective for all $x \in X$. Assume that $x \cdot y = x \cdot z$ holds for some $x, y, z \in X$. If $x \neq y$ and $x \neq z$, then $x \to y = x \to z$, which yields $y = z$ by Proposition 6. So we can assume that $x = z$. Suppose that $y \neq z$. Then Eq. (15) gives $D(y \to x) = (x \to y) \to D(x) = (x \cdot y) \to (x \cdot z) = 1$, a contradiction. Thus $\sigma(x)$ is injective. Now let $x, y \in X$ be given. To show that $\sigma(x)$ is surjective, we have to find an atom z with $x \cdot z = y$. If $D(x) = y$, then $x \cdot x = y$. Therefore, assume that $D(x) \neq y$. Then Proposition 4 implies that $(y \to D(x)) \rightsquigarrow (D(x) \to y) = y$ and $(D(x) \to y) \rightsquigarrow (y \to D(x)) = D(x)$. Hence $y \to D(x) \neq D(x) \to y$, and, as in the proof of Proposition 8, Eq. (16) gives $x = y \rightsquigarrow D^{-1}(y \to D(x))$. Consequently, $y \neq D^{-1}(y \to D(x))$, and thus Proposition 4 yields

$$\begin{aligned} y &= \bigl(y \rightsquigarrow D^{-1}(y \to D(x))\bigr) \to \bigl(D^{-1}(y \to D(x)) \rightsquigarrow y\bigr) \\ &= x \to \bigl(D^{-1}(y \to D(x)) \rightsquigarrow y\bigr) = x \cdot \bigl(D^{-1}(y \to D(x)) \rightsquigarrow y\bigr). \qquad \square \end{aligned}$$

5. *L*-algebras and normal decompositions

A set X with a binary operation $\to$ is said to be an *L-algebra* [63] if X has a logical unit 1 (see Eq. (8)) and satisfies Eq. (10) such that the relation $\leqslant$ given by (9) is antisymmetric. Thus every left or right hoop is an L-algebra. By [63], Proposition 2, every L-algebra X is partially ordered by (9), with greatest element 1. A smallest element 0 (if it exists) will be called a *zero element* of X. If X has a zero element, we define

$$\bar{x} := x \to 0$$

for $x \in X$. Note that an L-algebra with 0 satisfies

$$\bar{0} = 1, \qquad \bar{1} = 0.$$

A subset Y of an L-algebra X is said to be an *L-subalgebra* if $1 \in Y$ and Y is closed with respect to $\to$. By [63], Theorem 3, every L-algebra X has a *self-similar closure* $S(X)$, that is, a self-similar left hoop such that X is an L-subalgebra which generates $S(X)$ as a monoid. (The multiplication in $S(X)$ will always be denoted by concatenation.) Up to isomorphism, the self-similar closure is unique. The preorder

$$a \preccurlyeq b \;:\Longleftrightarrow\; \exists\, c \in S(X)\colon a = bc \tag{25}$$

of $S(X)$ induces a preorder on X, and the multiplication in $S(X)$ induces a partial multiplication on X.

Proposition 9. *Let X be an L-algebra with 0. For $x, y \in X$ and $a \in S(X)$, we have $a \to x \in X$ and*

$$0 \preccurlyeq a \implies a \in X \implies 0 \leqslant a \tag{26}$$

$$0 \leqslant xy \iff \bar{y} \leqslant x. \tag{27}$$

Proof. First, we have $1 \to x = x \in X$, and $a \to x \in X$ implies that $ya \to x = y \to (a \to x) \in X$. Hence $a \to x \in X$ for all $a \in S(X)$.

To verify (26), assume that $0 \preccurlyeq a$. Then $0 = ab$ for some $b \in S(X)$. Hence $a = b \to ab = b \to 0 \in X$. Moreover, $a \in X$ implies $0 \to a = 1$, that is, $0 \leqslant a$. By [63], Proposition 5(e), $0 \to xy = \big((y \to 0) \to x\big)(0 \to y) = \bar{y} \to x$, which yields (27). $\square$

In general, the implications (26) are not equivalences. For any $x \in X$, we have $\bar{x}x = (x \to 0)x = x \wedge 0 = 0$ in $S(X)$. Hence $0x = \bar{\bar{x}}\,\bar{x}x = \bar{\bar{x}}\,0$. By iteration, this shows that for any $a \in S(X)$ there is an element $\tau(a) \in S(X)$ with

$$\tau(a)0 = 0a \tag{28}$$

such that $\tau(x) = \bar{\bar{x}}$ for $x \in X$. Since $S(X)$ is self-similar, hence right cancellative, this gives a well-defined monoid homomorphism

$$\tau: S(X) \to S(X). \tag{29}$$

For Garside monoids, the homomorphism τ is well known (see [25], Proposition 2.6).

If τ is bijective, Proposition 9 can be refined.

Proposition 10. *Let X be an L-algebra with 0 such that the map $\tau: S(X) \to S(X)$ is bijective. Then $S(X)$ is cancellative, and $X = \{a \in S(X) \mid 0 \preccurlyeq a\} = \{a \in S(X) \mid 0 \leqslant a\}$. For any $x \in X$, there is a unique $\widetilde{x} \in X$ with $\bar{x}x = x\widetilde{x} = 0$.*

Proof. To verify that $S(X)$ is left cancellative, it is enough to show that $xa = xb$ with $x \in X$ and $a, b \in S(X)$ implies that $a = b$. Note first that $\bar{x}x = (x \to 0)x = x \wedge 0 = 0$. Multiplying $xa = xb$ from the left with $\bar{x}$ then gives $0a = 0b$. Hence $\tau(a)0 = 0a = 0b = \tau(b)0$, and thus $\tau(a) = \tau(b)$. Since τ is injective, this gives $a = b$.

For any $x \in X$, we define

$$\widetilde{x} := \tau^{-1}(\bar{x}).$$

Since $\bar{\bar{x}}\,\bar{x} = 0$, we have $0 = \tau^{-1}(0) = \tau^{-1}(\bar{\bar{x}})\tau^{-1}(\bar{x}) = x\widetilde{x}$. On the other hand, $0 = \bar{x}x$ gives $0 = \tau^{-1}(0) = \tau^{-1}(\bar{x})\tau^{-1}(x) \preccurlyeq \tau^{-1}(\bar{x}) = \widetilde{x}$. Thus Proposition 9 yields $\widetilde{x} \in X$.

Finally, assume that $a \in S(X)$ satisfies $0 \leqslant a$. Then $0 = xa$ for some $x \in S(X)$. Thus $0 \preccurlyeq x$, and Proposition 9 implies that $x \in X$. Hence $x\widetilde{x} = 0 = xa$, and thus $0 \preccurlyeq \widetilde{x} = a$. By (26), this completes the proof. $\square$

Our next result shows that bijectivity of τ can be expressed in terms of X.

Proposition 11. *Let X be an L-algebra with 0. The map $\tau: S(X) \to S(X)$ is bijective if and only if X is $\wedge$-closed and $\tau|_X: X \to X$ is an order isomorphism.*

Proof. Assume that $\tau: S(X) \to S(X)$ is bijective. For $x, y \in X$, Proposition 9 implies that $0 \leqslant x \wedge y$. Thus Proposition 10 shows that $x \wedge y \in X$. Furthermore, Proposition 10 implies that $x\widetilde{x} = 0$ for some $\widetilde{x} \in X$. Hence $x0 = x\widetilde{x}\widetilde{\widetilde{x}} = 0\widetilde{\widetilde{x}}$, which shows that $\tau|_X$ is surjective. Since τ is a monoid homomorphism, τ is monotonic. Assume that $x, y \in X$ satisfy $\tau(x) \leqslant \tau(y)$. Then $\bar{\bar{x}} = a\bar{\bar{y}}$ for some $a \in S(X)$. Hence $0 = \bar{\bar{x}}\,\bar{x} = a\bar{\bar{y}}\,\bar{x} \leqslant \bar{\bar{y}}\,\bar{x}$, and Proposition 10 implies that $0 \preccurlyeq \bar{\bar{y}}\,\bar{x}$. So there is an element $b \in S(X)$ with $0 = \bar{\bar{y}}\,\bar{x}\,b$. Thus $\bar{\bar{y}}\,\bar{x}x = \bar{\bar{y}}\,0 = 0y = \bar{\bar{y}}\,\bar{x}by$. Since $S(X)$ is left cancellative by Proposition 10, we infer that $x = by \leqslant y$. Whence $\tau|_X$ is an order isomorphism.

Conversely, let X be $\wedge$-closed and $\tau|_X$ be an order isomorphism. For $x, y \in X$, this implies that $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$. Hence $\tau(x \to y)\tau(x) = \tau(x \wedge y) = \tau(x) \wedge \tau(y) = (\tau(x) \to \tau(y))\tau(x)$, which shows that

$$\tau(a \to b) = \tau(a) \to \tau(b) \tag{30}$$

holds for all $a, b \in X$. Assume that Eq. (30) has been verified for $a \in X$ and a fixed $b \in S(X)$. For $x, y \in X$, we use [63], Proposition 5(e) to obtain

$$\begin{aligned}\tau(x \to by) &= \tau((y \to x) \to b)\tau(x \to y) = (\tau(y \to x) \to \tau(b))\tau(x \to y) \\ &= \big((\tau(y) \to \tau(x)) \to \tau(b)\big)(\tau(x) \to \tau(y)) = \tau(x) \to \tau(b)\tau(y) \\ &= \tau(x) \to \tau(by).\end{aligned}$$

This proves Eq. (30) for $a \in X$ and $b \in S(X)$. Assume now that (30) has been verified for a fixed $a \in S(X)$ and all $b \in S(X)$. For $x, y \in X$, this yields

$$\begin{aligned}\tau(xa \to b) &= \tau(x \to (a \to b)) = \tau(x) \to \tau(a \to b) = \tau(x) \to (\tau(a) \to \tau(b)) \\ &= \tau(x)\tau(a) \to \tau(b) = \tau(xa) \to \tau(b).\end{aligned}$$

Thus Eq. (30) holds for all $a, b \in S(X)$. Since $\tau|X$ is surjective, $\tau\colon S(X) \to S(X)$ is surjective, too.

Finally, assume that $\tau(a) = \tau(b)$ for some $a, b \in S(X)$. Then $\tau(a \to b) = \tau(a) \to \tau(b) = 1$. Suppose that $a \to b \neq 1$. Then $a \to b \leqslant x < 1$ for some $x \in X$. Hence $\tau(x) = 1$, and thus $x = 1$, a contradiction. Consequently, $a \leqslant b$, and by symmetry, $b \leqslant a$. So τ is bijective. $\square$

Definition 4. Let X be an L-algebra with 0. For an element $a \in S(X) \smallsetminus \{1\}$ we define a *normal decomposition* to be a representation $a = x_1 \cdots x_n$ with $x_i \in X$ such that $x_i \to \overline{x_{i+1}} = \overline{x_i}$ for all $i \in \{1, \dots, n-1\}$. The empty word ($n = 0$) will be regarded as a normal decomposition of $a = 1$.

The following proposition is well known for Garside groups. It will be applied in the next section.

Proposition 12. *Let X be an L-algebra with 0. Assume that the map (29) is bijective. Then every element $a \in S(X)$ has a unique normal decomposition $a = x_1 \cdots x_n$. If $a = xb \neq 1$ with $x \in X$ and $b \in S(X)$, then $x_1 \preccurlyeq x$ and $b \leqslant x_2 \cdots x_n$.*

Proof. By Proposition 10, any $x \in X$ satisfies $\bar{x}x = 0 = \bar{x}\,\widetilde{\bar{x}}$. Since $S(X)$ is left cancellative, this yields

$$\widetilde{\bar{x}} = x.$$

Let $a = x_1 \cdots x_n$ be a normal decomposition with $n \geqslant 2$. With $c := x_1 \cdots x_{n-2}$, this implies that $a \to 0 = cx_{n-1} \to \overline{x_n} = c \to (x_{n-1} \to \overline{x_n}) = c \to \overline{x_{n-1}} = cx_{n-1} \to 0$. Thus, by induction, we get $a \to 0 = x_1 \to 0 = \overline{x_1}$. Hence $x_1 = \widetilde{\overline{x_1}}$ is uniquely determined by a. Since $S(X)$ is left cancellative, the same argument can be applied to $x_2 \cdots x_n$. By induction, this proves that normal decompositions are unique.

Assume that $a = xb$ with $x \in X$ and $b \in S(X)$. Then Proposition 9 implies that $z := x \wedge (b \to 0) \geqslant 0$. Therefore, Proposition 10 shows that $z \in X$. So we obtain $(x \to (b \to 0))x_1 = (a \to 0)x_1 = 0 = z\widetilde{z} = (x \to (b \to 0))x\widetilde{z}$, which yields $x_1 = x\widetilde{z} \preccurlyeq x$. Hence $xb = x_1 \cdots x_n = x\widetilde{z}x_2 \cdots x_n$, and thus $b = \widetilde{z}x_2 \cdots x_n \leqslant x_2 \cdots x_n$.

Finally, let $a \in S(X)$ be given. If $a \neq 1$, we set $x_1 := \widetilde{a \to 0}$. Then

$$(a \to 0)a = (0 \to a)0 = 0\tau^{-1}(0 \to a) = (a \to 0)x_1\tau^{-1}(0 \to a).$$

Hence $a = x_1 b$ with $b := \tau^{-1}(0 \to a)$. If $b \neq 1$, the same argument yields $b = x_2 c$ with $x_2 = \widetilde{b \to 0}$. Furthermore, $x_1 \to \overline{x_2} = x_1 \to (b \to 0) = x_1 b \to 0 = a \to 0 = \overline{x_1}$. By induction, this gives a normal decomposition of a. $\square$

Under the assumptions of Proposition 12, we introduce the *length* $l(a)$ of an element $a \in S(X)$ to be the unique $n \in \mathbb{N}$ such that a has a normal decomposition $a = x_1 \cdots x_n$. Thus $l(1) = 0$ and $l(x) = 1$ for all $x \in X$ with $x \neq 1$. The following corollaries are classical in the context of Garside groups (see, e.g., [19], Section 3).

Corollary 1. *Let X be an L-algebra with 0 such that the map $\tau \colon S(X) \to S(X)$ is bijective. Then $a \leqslant b$ implies $l(a) \geqslant l(b)$ for any $a, b \in S(X)$.*

Proof. Without loss of generality, we can assume that $a = xb$ for some $x \in X$. Furthermore, we exclude the trivial case $b = 1$. Let $a = x_1 \cdots x_n$ and $b = y_1 \cdots y_m$ be normal decompositions. Then $m, n > 0$. Since $x_1 \preccurlyeq x$, there is an element $y \in S(X)$ with $x_1 = xy$. Hence $y \in X$ by Proposition 10. Since $S(X)$ is left cancellative, $yx_2 \cdots x_n = y_1 \cdots y_m$. Now the same argument gives $y_1 = yz$ for some $z \in X$. Hence $zy_2 \cdots y_m = x_2 \cdots x_n$. By induction, this gives $n - 1 \geqslant m - 1$. Thus $n \geqslant m$. $\square$

For an L-algebra X with 0 and $n \in \mathbb{N}$, let $X^n \subset S(X)$ denote the set of all products $x_1 \cdots x_n$ with at most n factors $x_i \in X$. Thus $X^0 = \{1\}$ and $X^1 = X \cup \{1\}$.

Corollary 2. *Let X be an L-algebra with 0 such that the map $\tau \colon S(X) \to S(X)$ is bijective. An element $a \in S(X)$ belongs to X^n if and only if $l(a) \leqslant n$.*

Proof. Assume that $a \in X^n \setminus X$. Then $a = xb$ with $x \in X$ and $b \in X^{n-1}$. We have to show that $l(a) \leqslant n$. By induction, we can assume that $l(b) \leqslant n - 1$. Let $a = x_1 \cdots x_m$ be the normal decomposition of a. Then $x_1 = xy$ for some $y \in X$. Hence $b = yx_2 \cdots x_m \leqslant x_2 \cdots x_m$, and Corollary 1 implies that $m - 1 \leqslant l(b) \leqslant n - 1$. Thus $l(a) = m \leqslant n$. $\square$

Remark. Note that the normal decomposition of Proposition 12 holds without any atomicity or noetherian hypothesis. For example, the closed unit interval $I := [0, 1]$ in $\mathbb{R}$ with $a \to b := \min\{b - a + 1, 1\}$ is an L-algebra which satisfies the assumptions of Proposition 12. The group of fractions of $S(I)$ is then the additive group of $\mathbb{R}$.

6. Right l-groups with strong order unit

In this section, we extend Dvurečenskij's non-commutative version [31] of Mundici's famous equivalence [52,53] between MV-algebras and abelian l-groups with strong order unit to arbitrary right l-groups. In this context, Garside groups can be regarded as Gaussian groups with finitely many atoms and a strong order unit.

Definition 5. We define a *right brick* to be a set X with binary operations $\to$ and $\rightsquigarrow$ such that $(X;\to)$ and $(X;\rightsquigarrow)$ are L-algebras with the same logical unit 1 and a simultaneous zero element 0 such that the equations $\bar{\tilde{x}} = x$ and

$$x \to \overline{x \rightsquigarrow y} = y \to \overline{y \rightsquigarrow x}, \qquad x \rightsquigarrow \widetilde{x \to y} = y \rightsquigarrow \widetilde{y \to x} \tag{31}$$

$$\widetilde{\bar{x} \to \bar{y}} = \overline{\tilde{x} \to \tilde{y}} \tag{32}$$

are satisfied for $x, y \in X$, where $\bar{x} := x \to 0$ and $\tilde{x} := x \rightsquigarrow 0$.

Our terminology appeals to Bosbach's "bricks" which can be represented as intervals in l-groups (see [64]). Note that Eq. (32) is not self-symmetric with respect to $\to$ and $\rightsquigarrow$. The example below gives a standard construction of right bricks.

Definition 6. Let G be a right l-group. We call an element $u \in G$ *normal* if $uG^+u^{-1} = G^+$. We say that $u \in G^+$ is a *strong order unit* if u is normal such that for any $a \in G^+$, there is an $n \in \mathbb{N}$ with $a \leqslant u^n$.

Thus $u \in G$ is normal if and only if $a \leqslant b \iff ua \leqslant ub$ holds for all $a, b \in G$. For l-groups, the concept of strong order unit coincides with the usual one [3,17].

Proposition 13. *Let G be a right l-group. An element $u \in G^+$ is a strong order unit if and only if $\{x \in G^+ \mid x \leqslant u\} = \{x \in G^+ \mid x \preccurlyeq u\}$ and this set generates G^+.*

Proof. Let u be a strong order unit. Assume that $1 \leqslant x \leqslant u$. Then $ax = u$ for some $a \in G^+$. Hence $x^{-1}u = u^{-1}au \in G^+$, which shows that $x \preccurlyeq u$. By symmetry, this shows that $\{x \in G^+ \mid x \leqslant u\} = \{x \in G^+ \mid x \preccurlyeq u\}$. For any $a \in G^+$, there is an $n \in \mathbb{N}$ with $a \leqslant u^n$. Since u^{-1} is normal, this implies that $u^{-1}a \leqslant a \wedge u^{n-1}$. As u is normal, we obtain $a \leqslant u(a \wedge u^{n-1})$. Hence $x := a(a \wedge u^{n-1})^{-1}$ belongs to the set $X := \{x \in G^+ \mid x \leqslant u\}$, and $a = x(a \wedge u^{n-1})$. By induction, this shows that X generates the monoid G^+.

Conversely, assume that $X := \{x \in G^+ \mid x \leqslant u\} = \{x \in G^+ \mid x \preccurlyeq u\}$ generates G^+. For any $x \in X$, this implies that there exist $x^l, x^r \in X$ with $u = x^l x = x x^r$. Hence $ux = x^{ll}x^l x = x^{ll}u$ and $xu = xx^r x^{rr} = ux^{rr}$. Since X generates G^+, this shows that for any $a \in G^+$ there exist $b, c \in G^+$ with $ua = bu$ and $au = uc$. Hence $uG^+ = G^+u$, and thus u is normal. If $a \in G^+$ satisfies $a \leqslant u^n$, and $x \in X$, then $ax \leqslant u^n x = yu^n$ for some $y \in X$. Hence $ax \leqslant y^l y u^n = u^{n+1}$. By induction, this shows that u is a strong order unit. $\square$

As an immediate consequence, we get

Corollary. *A Gaussian group with finitely many atoms is a Garside group if and only if it has a strong order unit.*

Example 2. Let G be a right l-group with a strong order unit u. Consider the interval $X = [u^{-1}, 1] = \{x \in G \mid u^{-1} \leqslant x \leqslant 1\}$ in G. Since u is normal, every $a \in G$ satisfies $u^{-1} \leqslant a \Leftrightarrow au \in G^+ \Leftrightarrow a \in G^+u^{-1} = u^{-1}G^+ \Leftrightarrow ua \in G^+ \Leftrightarrow u^{-1} \preccurlyeq a$. For $x \in X$, Eqs. (13) give $\bar{x} = u^{-1}x^{-1} \wedge 1 = u^{-1}x^{-1}$ and $\widetilde{x} = (ux \vee 1)^{-1}$. Since $u^{-1} \preccurlyeq x$, we have $x^{-1} \leqslant u$, which yields $1 \leqslant ux$. Hence $\widetilde{x} = (ux)^{-1} = x^{-1}u^{-1}$. Therefore, the maps $x \mapsto \bar{x}$ and $x \mapsto \widetilde{x}$ are inverse to each other.

For $x, y \in X$, we have $u^{-1}x^{-1} \in G^+u^{-1}$, which gives $u^{-1}x \leqslant u^{-1} \leqslant y$. Thus $u^{-1} \leqslant x \to y \leqslant 1$. Similarly, $u^{-1} \leqslant y \leqslant x^{-1}y$ gives $u^{-1} \preccurlyeq x^{-1}y$. Hence $y^{-1}x \vee 1 \leqslant u$, and thus $u^{-1} \preccurlyeq (y^{-1}x \vee 1)^{-1} = x \rightsquigarrow y$. This shows that X is closed with respect to $\to$ and $\rightsquigarrow$. So $(X; \to)$ and $(X; \rightsquigarrow)$ are L-algebras with simultaneous logical unit 1 and common zero element u^{-1}. Furthermore, $x \to \overline{x \rightsquigarrow y} = x \to u^{-1}(y^{-1}x \vee 1) = u^{-1}(y^{-1} \vee x^{-1}) \wedge 1$ and $x \rightsquigarrow \widetilde{x \to y} = x \rightsquigarrow (yx^{-1} \wedge 1)^{-1}u^{-1} = \big(u(yx^{-1} \wedge 1)x \vee 1\big)^{-1} = \big(u(y \wedge x) \vee 1\big)^{-1}$. By symmetry, this proves Eqs. (31). Finally, using the map (29), Eq. (32) can be written as $\tau(x \to y) = \tau(x) \to \tau(y)$. This follows since τ is an automorphism of the monoid $S(X)$. Thus X is a right brick.

Conversely, every right brick can be represented as an interval:

Theorem 3. *Let X be a right brick. Up to isomorphism, there exists a unique right l-group G with a strong order unit u such that $X \cong [u^{-1}, 1] := \{x \in G^- \mid u^{-1} \leqslant x\}$.*

Proof. We regard X as an L-subalgebra of the self-similar closure $S(X)$ of $(X; \to)$. Eq. (32) with $x = 0$ yields $\bar{\widetilde{y}} = \widetilde{\bar{y}}$. Thus

$$\bar{\bar{x}} = x = \widetilde{\widetilde{x}} \tag{33}$$

holds for all $x \in X$. Since $\bar{x}x = (x \to 0)x = x \wedge 0 = 0$, this implies that

$$\bar{x}x = x\widetilde{x} = 0. \tag{34}$$

Next we prove the equivalences

$$\widetilde{x} \leqslant \widetilde{y} \iff y \preccurlyeq x \iff \bar{x} \leqslant \bar{y} \iff y \rightsquigarrow x = 1 \tag{35}$$

for $x, y \in X$. Assume that $\widetilde{x} \leqslant \widetilde{y}$. Then $\widetilde{x} = a\widetilde{y}$ for some $a \in S(X)$. Hence $y0 = y\widetilde{y}\widetilde{\widetilde{y}} = x\widetilde{x}\widetilde{\widetilde{y}} = xa\widetilde{y}\widetilde{\widetilde{y}} = xa0$. Thus $y = xa$, that is, $y \preccurlyeq x$. Now assume that $y = xa$. Then Eq. (28) gives $\bar{x}y = 0a = \tau(a)0 = \tau(a)\bar{y}y \leqslant \bar{y}y$, and thus $\bar{x} \leqslant \bar{y}$. Next assume that $\bar{x} \leqslant \bar{y}$. The second equation of (31) gives $x = \bar{x} \rightsquigarrow 0 = \bar{x} \rightsquigarrow \widetilde{\bar{x} \to \bar{y}} = \bar{y} \rightsquigarrow \widetilde{\bar{y} \to \bar{x}}$. Hence

$y \rightsquigarrow x = (\bar{y} \rightsquigarrow 0) \rightsquigarrow (\bar{y} \rightsquigarrow \widetilde{\bar{y} \to \bar{x}}) = (0 \rightsquigarrow \bar{y}) \rightsquigarrow (0 \rightsquigarrow \widetilde{\bar{y} \to \bar{x}}) = 1 \rightsquigarrow 1 = 1$. Finally, assume that $y \rightsquigarrow x = 1$. By the first equation of (31), this gives $x \to \overline{x \rightsquigarrow y} = \bar{y}$. So $\bar{x}x = 0 \leqslant \overline{x \rightsquigarrow y}$ yields $\bar{x} \leqslant x \to \overline{x \rightsquigarrow y} = \bar{y}$. Therefore, Eq. (32) yields $\widetilde{x} \leqslant \widetilde{y}$, which establishes (35).

In particular, (35) shows that the partial order (25) restricts to the partial order of $(X; \rightsquigarrow)$. Suppose that there exists an element $a \in S(X) \smallsetminus X$ with $0 \leqslant a$. Then $a = bxy$ for some $b \in S(X)$ and $x, y \in X$ with $xy \notin X$. Since $0 \leqslant a \leqslant xy$, Proposition 9 gives $\bar{y} \leqslant x$, and (35) implies that $\widetilde{x} \preccurlyeq y$. Hence $0 = x\widetilde{x} \preccurlyeq xy$, and thus $xy \in X$, a contradiction. Since $0 = x\widetilde{x} \preccurlyeq x$ for all $x \in X$, the implications (26) yield

$$X = \{a \in S(X) \mid a \geqslant 0\} = \{a \in S(X) \mid a \succcurlyeq 0\}. \tag{36}$$

In particular, this implies that X is $\wedge$-closed. By Eqs. (33) and Proposition 11, the map (29) is bijective.

Now we extend the operation $x \rightsquigarrow y$ to arbitrary $x, y \in S(X) = \bigcup_{n=1}^{\infty} X^n$. By Proposition 12, every element of $S(X)$ admits a unique normal decomposition. For $x \in X$ and $a \in S(X)$ with normal decomposition $a = x_1 \cdots x_n$ and $n \geqslant 2$, we set $x \rightsquigarrow 1 := 1$ and

$$x \rightsquigarrow a := (x \rightsquigarrow x_1)\big((x_1 \rightsquigarrow x) \rightsquigarrow b\big), \tag{37}$$

where $b := x_2 \cdots x_n$. Note that Eq. (37) remains valid for $n = 1$. Furthermore, Eq. (37) implies that

$$1 \rightsquigarrow a = a$$

for all $a \in S(X)$. To complete the definition of $\rightsquigarrow$ on $S(X)$, assume that $a, c \in S(X)$ such that a has a normal decomposition $a = x_1 \cdots x_n$ with $n \geqslant 2$, and $b := x_2 \cdots x_n$. Then we set

$$a \rightsquigarrow c := b \rightsquigarrow (x_1 \rightsquigarrow c).$$

Again, this remains valid for $n = 1$, and

$$a \rightsquigarrow 1 = 1$$

holds for all $a \in S(X)$. Assume that $x, y \in X$ such that $xy \in X$. Then Proposition 9 gives $\bar{y} \leqslant x$. So the second equation of (31) yields $x \rightsquigarrow xy = x \rightsquigarrow \widetilde{\overline{xy}} = x \rightsquigarrow \widetilde{x \to \bar{y}} = \bar{y} \rightsquigarrow \widetilde{\bar{y} \to x} = \widetilde{\bar{y}} = y$. So we get

$$x \rightsquigarrow xy = y \tag{38}$$

for $x, y \in X$ with $xy \in X$. For $z \in X$ this gives $xy \rightsquigarrow z = (xy \rightsquigarrow x) \rightsquigarrow (xy \rightsquigarrow z) = (x \rightsquigarrow xy) \rightsquigarrow (x \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z)$, and thus

$$xy \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z) \tag{39}$$

for $x, y, z \in X$ with $xy \in X$. Since $(x \rightsquigarrow 0) \rightsquigarrow (x \rightsquigarrow y) = (0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y) = 1$ holds for arbitrary $x, y \in X$, the equivalences (35) give $\overline{x \rightsquigarrow y} \leqslant x$. Thus Proposition 9 with Eqs. (36) yields $x(x \rightsquigarrow y) \in X$. By the first equation of (31), we obtain $\overline{x(x \rightsquigarrow y)} = x \rightarrow \overline{x \rightsquigarrow y} = y \rightarrow \overline{y \rightsquigarrow x} = \overline{y(y \rightsquigarrow x)}$. Hence

$$x(x \rightsquigarrow y) = y(y \rightsquigarrow x) \in X \tag{40}$$

for all $x, y \in X$.

Next we prove

$$x \rightsquigarrow yb = (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow b\big) \tag{41}$$

for arbitrary $x, y \in X$ and $b \in S(X)$. We proceed by induction. Assume first that $yb \in X$. Then (36) implies that $b \in X$. By Eqs. (40) and (38), we obtain

$$\begin{aligned} x \rightsquigarrow yb &= (x \rightsquigarrow yb)\big((yb \rightsquigarrow x) \rightsquigarrow 1\big) = (x \rightsquigarrow yb)\big((yb \rightsquigarrow x) \rightsquigarrow (yb \rightsquigarrow y)\big) \\ &= (x \rightsquigarrow yb)\big((x \rightsquigarrow yb) \rightsquigarrow (x \rightsquigarrow y)\big) = (x \rightsquigarrow y)\big((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow yb)\big) \\ &= (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow yb)\big) = (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow b\big). \end{aligned}$$

Now suppose that Eq. (41) has been verified for $yb \in X^n$, where $n \geqslant 1$. Assume that $yb \in X^{n+1}$, and consider normal decompositions $yb = y_0 \cdots y_n$ and $b = z_0 \cdots z_m$. With $c := y_1 \cdots y_n$ and $d := z_1 \cdots z_m$, Proposition 12 implies that $y_0 = yz$ for some $z \in X$. Thus $b = zc = z_0 d$, which yields $z_0 = zt$ for some $t \in X$, and $c = td$. Since $c \in X^n$, Eq. (39) and the inductive hypothesis yield

$$\begin{aligned} x \rightsquigarrow yb = x \rightsquigarrow y_0 c &= (x \rightsquigarrow y_0)\big((y_0 \rightsquigarrow x) \rightsquigarrow c\big) \\ &= (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow z\big)\big((z \rightsquigarrow (y \rightsquigarrow x)) \rightsquigarrow c\big) \\ &= (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow z\big)\big((z \rightsquigarrow (y \rightsquigarrow x)) \rightsquigarrow t\big)\big((t \rightsquigarrow (z \rightsquigarrow (y \rightsquigarrow x))) \rightsquigarrow d\big) \\ &= (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow zt\big)\big((t \rightsquigarrow (z \rightsquigarrow (y \rightsquigarrow x))) \rightsquigarrow d\big) \\ &= (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow z_0\big)\big((z_0 \rightsquigarrow (y \rightsquigarrow x)) \rightsquigarrow d\big) \\ &= (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow z_0 d\big) = (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow b\big), \end{aligned}$$

which establishes Eq. (41).

Using Eq. (41), we prove

$$xa \rightsquigarrow b = a \rightsquigarrow (x \rightsquigarrow b) \tag{42}$$

for arbitrary $x \in X$ and $a, b \in S(X)$. Assume first that $xa \in X$. Let $b = z_1 \cdots z_n$ be a normal decomposition with $e := z_2 \cdots z_n$. By Eq. (39), we only have to deal with the case $n \geqslant 2$. By induction, we can assume that Eq. (42) holds for $xa \in X$ and e instead of b. Then $(z_1 \rightsquigarrow x)((x \rightsquigarrow z_1) \rightsquigarrow a) = z_1 \rightsquigarrow xa \in X$. Therefore, Eqs. (37), (39), and (41) yield

$$\begin{aligned} xa \rightsquigarrow b &= (xa \rightsquigarrow z_1)\big((z_1 \rightsquigarrow xa) \rightsquigarrow e\big) = (xa \rightsquigarrow z_1)\big((z_1 \rightsquigarrow x)((x \rightsquigarrow z_1) \rightsquigarrow a) \rightsquigarrow e\big) \\ &= \big(a \rightsquigarrow (x \rightsquigarrow z_1)\big)\big(((x \rightsquigarrow z_1) \rightsquigarrow a) \rightsquigarrow ((z_1 \rightsquigarrow x) \rightsquigarrow e)\big) \\ &= a \rightsquigarrow (x \rightsquigarrow z_1)\big((z_1 \rightsquigarrow x) \rightsquigarrow e\big) = a \rightsquigarrow (x \rightsquigarrow z_1 e) = a \rightsquigarrow (x \rightsquigarrow b). \end{aligned}$$

Proceeding by induction, suppose that Eq. (42) has been verified for $xa \in X^n$ and arbitrary $b \in S(X)$. Assume that $xa \in X^{n+1}$, and let $xa = x_0 \cdots x_n$ and $a = y_0 \cdots y_m$ be normal decompositions with $n, m \geqslant 1$. Then $x_0 = xy$ for some $y \in X$. With $c := x_1 \cdots x_n$ and $d := y_1 \cdots y_m$, this gives $a = y_0 d = yc$. Hence $y_0 = yz$ for some $z \in X$, which yields $zd = c \in X^n$. Thus, by the inductive hypothesis, we get

$$\begin{aligned} xa \rightsquigarrow b = x_0 c \rightsquigarrow b = c \rightsquigarrow (x_0 \rightsquigarrow b) = c \rightsquigarrow \big(y \rightsquigarrow (x \rightsquigarrow b)\big) = d \rightsquigarrow \big(z \rightsquigarrow (y \rightsquigarrow (x \rightsquigarrow b))\big) \\ = d \rightsquigarrow \big(y_0 \rightsquigarrow (x \rightsquigarrow b)\big) = y_0 d \rightsquigarrow (x \rightsquigarrow b) = a \rightsquigarrow (x \rightsquigarrow b). \end{aligned}$$

From Eq. (42), we easily obtain

$$ab \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c) \tag{43}$$

for all $a, b, c \in S(X)$. For $a \in X$, this is just Eq. (42). Assume that Eq. (43) has been verified for $a \in X^n$. For $x \in X$ and $a \in X^n$, we then have $(xa)b \rightsquigarrow c = ab \rightsquigarrow (x \rightsquigarrow c) = b \rightsquigarrow \big(a \rightsquigarrow (x \rightsquigarrow c)\big) = b \rightsquigarrow (xa \rightsquigarrow c)$, which proves Eq. (43).

Now we extend Eq. (41) to all $x \in S(X)$. Assume that

$$a \rightsquigarrow yb = (a \rightsquigarrow y)\big((y \rightsquigarrow a) \rightsquigarrow b\big) \tag{44}$$

has been verified for all $y \in X$ and $b \in S(X)$, and $a \in X^n$. For $x \in X$ and $a \in X^n$, Eq. (43) gives

$$\begin{aligned} xa \rightsquigarrow yb &= a \rightsquigarrow (x \rightsquigarrow yb) = a \rightsquigarrow (x \rightsquigarrow y)\big((y \rightsquigarrow x) \rightsquigarrow b\big) \\ &= \big(a \rightsquigarrow (x \rightsquigarrow y)\big)\big(((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow ((y \rightsquigarrow x) \rightsquigarrow b)\big) \\ &= (xa \rightsquigarrow y)\big((y \rightsquigarrow x)((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow b\big) = (xa \rightsquigarrow y)\big((y \rightsquigarrow xa) \rightsquigarrow b\big), \end{aligned}$$

which establishes Eq. (44).

Next we prove

$$a(a \rightsquigarrow b) = b(b \rightsquigarrow a) \tag{45}$$

for $a, b \in S(X)$. By Eq. (40), this holds for $x, y \in X$. Assume that Eq. (45) has been shown for $b \in X$ and $a \in X^n$. For such a, b and $x \in X$, Eqs. (43) and (44), and the inductive hypothesis give $xa(xa \rightsquigarrow b) = xa(a \rightsquigarrow (x \rightsquigarrow b)) = x(x \rightsquigarrow b)\big((x \rightsquigarrow b) \rightsquigarrow a\big) = b(b \rightsquigarrow x)\big((x \rightsquigarrow b) \rightsquigarrow a\big) = b(b \rightsquigarrow xa)$. Thus Eq. (45) holds for all $a \in S(X)$ and $b \in X$. Assume that Eq. (45) has been verified for $a \in S(X)$ and $b \in X^n$. For such a, b and $y \in X$, Eqs. (43) and (44) then imply that $a(a \rightsquigarrow yb) = a(a \rightsquigarrow y)\big((y \rightsquigarrow a) \rightsquigarrow b\big) = y(y \rightsquigarrow a)\big((y \rightsquigarrow a) \rightsquigarrow b\big) = yb\big(b \rightsquigarrow (y \rightsquigarrow a)\big) = yb(yb \rightsquigarrow a)$. This proves Eq. (45).

To show that $a \rightsquigarrow a = 1$ holds for all $a \in S(X)$, assume that this has been proved for a particular $a \in S(X)$. Then Eqs. (43) and (44) give $xa \rightsquigarrow xa = a \rightsquigarrow (x \rightsquigarrow xa) = a \rightsquigarrow (x \rightsquigarrow x)\big((x \rightsquigarrow x) \rightsquigarrow a\big) = a \rightsquigarrow a = 1$ for all $x \in X$. With Eqs. (43) and (45), this shows that $(S(X); \rightsquigarrow)$ is a right hoop. To show that $(S(X); \rightsquigarrow)$ is self-similar, we have to verify

$$a \rightsquigarrow ab = b \tag{46}$$

in $S(X)$. For $a \in X$, this follows by Eq. (44). Assume that (46) has been verified for a particular $a \in S(X)$. For any $x \in X$, this implies that $xa \rightsquigarrow (xa)b = a \rightsquigarrow (x \rightsquigarrow xab) = a \rightsquigarrow ab = b$. Thus $(S(X); \rightsquigarrow)$ is a self-similar right hoop.

So we have proved that $S(X)$ is a right l-cone. By Theorem 1, the group of fractions is a right l-group G with $G^- = S(X)$. By the dual of Proposition 13, 0^{-1} is a strong order unit of G. The proof is complete. $\square$

Remarks. 1. Theorem 3 implies that every right brick $(X; \to, \rightsquigarrow)$ has a mirror image $(X; \rightsquigarrow, \to)$ which is a right brick with the opposite multiplication in $S(X)$. Therefore, Eq. (32) in Definition 5 has a symmetric counterpart

$$\overline{\tilde{x} \rightsquigarrow \tilde{y}} = \widetilde{\bar{x} \rightsquigarrow \bar{y}}$$

which holds in every right brick.

2. By Theorem 3, every right brick is a lattice. Eqs. (13) show that the meet and join can be represented as

$$x \wedge y = ((x \to y) \to \bar{x})^{\sim}, \qquad x \vee y = \tilde{x} \to \overline{\tilde{x} \rightsquigarrow \tilde{y}}.$$

3. For any right brick X, each interval $[0^n, 1]$ in the corresponding right l-group G is again a right brick. The direct limit of the intervals $[0^n, 1]$ is the negative cone of G.

7. Discrete L-algebras and block labellings

In Section 4, solutions of the quantum Yang–Baxter equation have been associated to modular noetherian right l-groups with duality. By Proposition 7, the underlying lattice of such a right l-group G is distributive. If G is a Garside group, the set $X(G^-)$ of atoms is finite, and G is of I-type ([48], Theorem 2.2; cf. [62], Theorem 1). So the structure of G is encoded in the hypercube generated by the cycle set $X(G^-)$.

Our next aim is to show that the concept of “I-type” admits a vast extension to a class of right l-groups G containing all modular Garside groups. Instead of a hypercube, the interval $[\bigwedge X(G^-), 1]$ is then a (dual) geometric lattice, so that G is completely determined by the corresponding geometry. In particular, this will imply that every finite dimensional projective space over a skew-field with a “non-degenerate” labelling (see Section 8) corresponds to a modular right l-group G of generalized “I-type”. The construction of G from the translates of $[\bigwedge X(G^-), 1]$ can be viewed as an “S-pasted sum” in the sense of Herrmann [46].

Let L be a lower semimodular lattice with ascending chain condition. (In what follows, it will be convenient to work with lower instead of upper semimodular lattices.) Recall that $i \in L$ is *meet-irreducible* if $i \neq 1$ and $i = a \wedge b$ implies that $i = a$ or $i = b$. Thus every $a \in L$ is a finite meet of meet-irreducibles. For any $a \in L$, the maximal chains in the interval $[a, 1]$ are of the same length $r(a) \in \mathbb{N} \cup \{\infty\}$, the *rank* of a. The elements of rank 1 are called *points*, and the elements of rank 2 are the *lines* of L. The set of points will be denoted by $X(L)$.

If L has a smallest element 0, then $\dim L := r(0) - 1$ is called the *dimension* of L. If every meet-irreducible $i \in L$ is a point, L^{op} is said to be a *geometric lattice* (see [44]). Every geometric lattice admits a unique decomposition into indecomposable geometric lattices, which can be characterized by the property that every line contains at least three points. Modular indecomposable geometric lattices of dimension $\geqslant 3$ are equivalent to projective spaces over a skew-field [44].

Definition 7. Let L be a lower semimodular lattice with ascending chain condition. Consider the $\wedge$-sublattice

$$\Lambda(L) := \{x_1 \wedge \cdots \wedge x_n \mid x_1, \ldots, x_n \in X(L)\}.$$

The sublattice $\mathrm{Rad}(L) := \{a \in L \mid \exists\, b \in \Lambda(L) \colon b \leqslant a\}$ will be called the *radical* of L. We call a right l-group G is *geometric* if G is noetherian and lower semimodular such that $\mathrm{Rad}(G^-) = \Lambda(G^-)$. The dimension $\dim G := \dim \Lambda(G^-)$ will be called the *dimension* of G.

Note that for $a < b$ in $\Lambda(G^-)$, the lower semimodularity of G implies that b covers a in $\Lambda(G^-)$ if and only if b covers a in G. Hence $\Lambda(G^-)$ is a lower semimodular lattice, and thus $\Lambda(G^-)^{\mathrm{op}}$ is geometric.

Proposition 14. *Let G be a lower semimodular noetherian right l-group. Then $\Lambda(G^-)^{\mathrm{op}}$ is a geometric lattice. If G is modular, G is geometric in the sense of Definition 7.*

Proof. The first assertion has already been proved. Thus assume that G is modular. For a given $a \in \Lambda(G^-)$, let p be join-irreducible in $[a, 1]$, and let p_0 be the single element in $p^- \cap [a, 1]$. The lower semimodularity implies that $x \geqslant p_0$ for any $x \in X(G^-) \cap [a, 1]$.

Hence $p_0 = a$. Since every $b \in [a, 1]$ is a join of join-irreducibles, this implies that $b = c_1 \vee \cdots \vee c_n$ for suitable $c_i \in [a, 1]$ with $a \in c_i^-$. Now the dual argument shows that the meet-irreducibles of $[a, 1]$ belong to $X(G^-)$. Whence $[a, 1] \subset \Lambda(G^-)$. $\square$

The relationship to Theorem 3 is given by

Proposition 15. *Let G be a modular noetherian right l-group. If $\Delta := \bigwedge X(G^-)$ exists in G, then Δ^{-1} is a strong order unit of G.*

Proof. For $x, y \in X(G^-)$ with $x \neq y$, we have $\Delta x \leqslant (x \to y)x = x \wedge y$. Hence $\Delta x \leqslant \Delta$, and thus $\Delta x \Delta^{-1} \leqslant 1$. If $c \in G$ covers $x\Delta$, then either $c = \Delta$ or $c \vee \Delta$ covers Δ. Suppose that $c \nleqslant 1$. By the modularity of G, the lattice $[c \vee \Delta, c \vee 1]$ is isomorphic to $[\Delta, 1]$. So there are elements $a_1, \dots, a_n \in G$ covered by $c \vee 1$ such that $a_1 \wedge \cdots \wedge a_n \wedge 1 = \Delta$. Right multiplication by $(c \vee 1)^{-1}$ would then carry $a_1, \dots, a_n, 1$ into lower neighbors of 1, and their meet would be less than Δ, which is impossible. Hence $x\Delta = \Delta y$ for some $y \in X(G^-)$, and thus Δ is normal. Furthermore, an easy induction shows that any $a \in G^-$ satisfies $\Delta^n \leqslant a$ for a sufficiently large $n \in \mathbb{N}$. $\square$

Next we show that modularity follows if $X(G^-)$ is finite.

Proposition 16. *Let G be a noetherian right l-group with $X(G^-)$ finite. With $\Delta := \bigwedge X(G^-)$, the following are equivalent.*

(a) *The interval $[\Delta, 1]$ is lower semimodular.*
(b) *$\widetilde{X}(G^-)$ is closed with respect to the operation $\to$.*
(c) *G is modular.*

A group G with these equivalent properties is a Garside group.

Proof. (a) $\Rightarrow$ (b): For $x, y \in X(G^-)$ with $x \neq y$, the interval $[x \wedge y, x]$ is of length 1. Hence $x \wedge y = (x \to y)x$ yields $x \to y \in X(G^-)$, which proves (b).

(b) $\Rightarrow$ (c): By Proposition 5, G is lower semimodular. Since finite meets exist in $\Lambda(G^-)$, it follows that $\Lambda(G^-)$ is a lattice. If $a, b \in \Lambda(G^-)$ and a covers b in $\Lambda(G^-)$, there is an atom $x \in X(G^-)$ with $x \wedge a = b$. So a covers b in G. Hence $\Lambda(G^-)$ is lower semimodular, and thus $\Lambda(G^-)^{\mathrm{op}}$ is a geometric lattice. Let H denote the set of elements $h \in G$ which cover Δ. The theorem of Basterfield and Kelly [2] (cf. [45], Theorem 1) implies that $|X(G^-)| \leqslant |H \cap \Lambda(G^-)| \leqslant |H|$. For any $h \in H$, the equation $(h \to \Delta)h = h \wedge \Delta = \Delta$ shows that $h \to \Delta \in X(G^-)$. Furthermore, Eq. (14) yields $(h \to \Delta) \rightsquigarrow \Delta = h \vee \Delta = h$. Therefore, we get $|X(G^-)| = |H|$. By Greene's theorem ([45], Theorem 2), this implies that $\Lambda(G^-)$ is modular and $H \subset \Lambda(G^-)$. Hence G^{op} satisfies (a), and thus G is modular by Proposition 5.

Finally, assume that (a)–(c) hold. By the Remark after Proposition 5, G is Gaussian. Thus G is Garside by Proposition 15 and the corollary of Proposition 13. □

Corollary 1. *Every distributive noetherian right l-group is geometric. If $\Delta := \bigwedge X(G^-)$ exists, G is a Garside group.*

Proof. The first statement follows by Proposition 14. If Δ exists, the interval $[\Delta, 1]$ is finite. By Proposition 16, G is Garside. □

Corollary 2. *The structure group G_X of a finite cycle set X is geometric.*

Proof. By Theorem 2, G_X is a modular noetherian right l-group with duality. Hence G_X is distributive by Proposition 7. So the assertion follows by Corollary 1. □

Definition 8. We call an L-algebra X *discrete* if every element $x < 1$ is maximal.

By Proposition 5, any lower semimodular noetherian right l-group G gives rise to a discrete L-algebra $\big(\widetilde{X}(G^-); \rightarrow\big)$. In the following definition, we use the notation

$$\downarrow\! x := \{y \in \Omega \mid y \leqslant x\}$$

for elements x of a partially ordered set Ω.

Definition 9. Let L be a lower semimodular lattice with $\mathrm{Rad}(L) = L$ and ascending chain condition. We define a *block assignment* of L to be a collection of injective lattice homomorphisms $e_x\colon \downarrow\! x \to L$ for each $x \in X(L)$ such that $e_x(x) = 1$ and $e_{e_x(x\wedge y)} e_x(a) = e_{e_y(x\wedge y)} e_y(a)$ for distinct $x, y \in X(L)$ and $a \leqslant x \wedge y$. A lattice L with a block assignment will be called a *block*.

For a discrete L-algebra X, consider the self-similar closure $S(X)$ (see [63], Theorem 3) with its partial ordering of Proposition 2. Thus $a \in S(X)$ covers $b \in S(X)$ if and only if $b = xa$ for some $x \in S^1(X) := X \smallsetminus \{1\}$. Therefore, $S(X)$ has a natural partition

$$S(X) = \bigsqcup_{n\in\mathbb{N}} S^n(X)$$

with $S^n(X) := \{x_1 \cdots x_n \mid x_1, \dots, x_n \in S^1(X)\}$. In particular, $S(X)$ satisfies the ascending chain condition. Hence $S(X)$ is a lattice.

Proposition 17. *Up to isomorphism, there is a one-to-one correspondence between discrete L-algebras and blocks.*

Proof. Let X be a discrete L-algebra. For $a \in S(X)$ and distinct $x, y \in S^1(X)$, Eq. (11) yields $xa \wedge ya = (x \wedge y)a = (x \rightarrow y)xa = (y \rightarrow x)ya$. By [28], Lemma 3.3, this implies that the lattice $S(X)$ is lower semimodular. We show that $B := \mathrm{Rad}(S(X))$ is a block. For $x \in S^1(X)$ and $a \in B \cap {\downarrow}x$ we define $e_x(a) := x \to a$. Since $(x \to a)x = a$, the map e_x is injective. Moreover, for any $a \in B \cap {\downarrow}x$, the interval $[a, x]$ is mapped onto $[e_x(a), 1]$. In fact, if $e_x(a) \leqslant b$, then $a \to bx = \big((x \to a) \to b\big)(a \to x) = 1$ and $e_x(bx) = x \to bx = b$. By [63], Proposition 4, this implies that $e_x\colon B \cap {\downarrow}x \to B$ is a lattice homomorphism with $e_x(x) = 1$. For distinct $x, y \in S^1(X)$ and $a \leqslant x \wedge y$, we have $e_x(x \wedge y) = x \to (x \to y)x = x \to y$. Hence $e_{e_x(x\wedge y)}e_x(a) = (x \to y) \to (x \to a) = (y \to x) \to (y \to a) = e_{e_y(x\wedge y)}e_y(a)$, which proves the claim.

Conversely, let B be a block with block assignment $e_x\colon {\downarrow}x \to B$ for $x \in X(B)$. Define a binary operation $\to$ on $X := X(B) \sqcup \{1\}$ with 1 as logical unit and $x \to y := e_x(x \wedge y)$ for distinct $x, y \in X(B)$. Then

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z) \tag{47}$$

clearly holds if x, y, z are not all distinct or one of them is equal to 1. Otherwise, $(x \to y) \to (x \to z) = e_x(x \wedge y) \to e_x(x \wedge z) = e_{e_x(x\wedge y)}\big(e_x(x \wedge y) \wedge e_x(x \wedge z)\big) = e_{e_x(x\wedge y)}e_x(x \wedge y \wedge z)$. By symmetry, this proves that X is a discrete L-algebra. If B is of the form $B = \mathrm{Rad}(S(Y))$ for a discrete L-algebra Y, then $X \cong Y$.

To show that $B \cong \mathrm{Rad}(S(X))$, we use the block assignment to construct an embedding $f\colon B \hookrightarrow S(X)$. We start with the natural embedding $X \hookrightarrow S(X)$. For any $a \in B$, let $l(a)$ be the length of the interval $[a, 1]$. If $l(a) > 1$, there exists an element $x \in X(B)$ with $a \leqslant x$. Thus $l(e_x(a)) = l(a)-1$. Inductively, we define $f(a) := f(e_x(a))x$. If $x \neq y \in B(X)$ and $a \leqslant y$, then $e_x(a) \leqslant e_x(x \wedge y) \in X(B)$. Since $e_x(x \wedge y)x = f(x \wedge y) = e_y(x \wedge y)y$, this gives $f(e_x(a))x = f(e_{e_x(x\wedge y)}e_x(a))e_x(x \wedge y)x = f(e_{e_y(x\wedge y)}e_y(a))e_y(x \wedge y)y = f(e_y(a))y$. Hence f is well defined and multiplicative. As f preserves the length of intervals, f is injective. For $a \in B$ and distinct $x, y \in X(B)$, we have $f(xa \wedge ya) = f(x \wedge y)f(a)$, where $f(x \wedge y) = e_x(x \wedge y)x = (x \to y)x = x \wedge y$. Thus f induces a lattice isomorphism $B \xrightarrow{\sim} \mathrm{Rad}(S(X))$. □

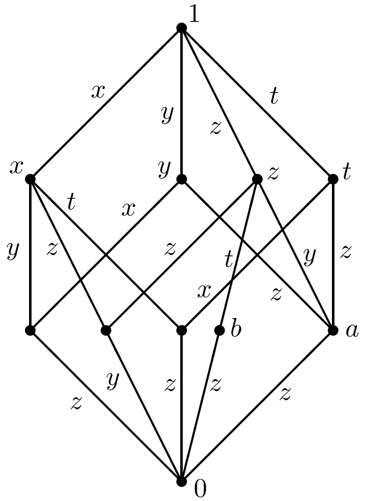
Block assignments can be conveniently described as follows.

Definition 10. Let L be a lower semimodular lattice with $\mathrm{Rad}(L) = L$ and ascending chain condition. To any pair $(a, b) \in L^2$ with $a \in b^-$ we attach a label $\lambda(a, b) \in X(L)$ in such a way that $\lambda(x, 1) = x$ for $x \in X$ and the following are satisfied:

(1) For any $b \in L$, the labels $\lambda(a, b)$ with $a \in b^-$ are distinct.
(2) If $a, b \in c^-$ and $a \neq b$, then $\lambda(a \wedge b, a) = \lambda\big(\lambda(a, c) \wedge \lambda(b, c), \lambda(a, c)\big)$.

If such a labelling λ exists, we call it a *block labelling* of L^{op}.

Example 3. $X(L) = \{x, y, z, t\}$



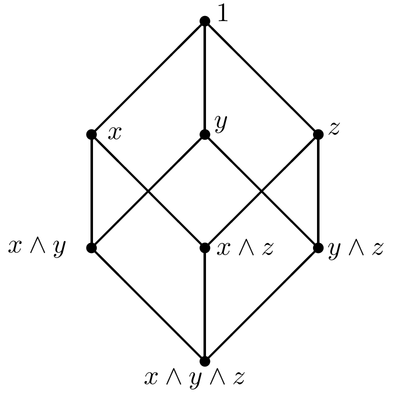
The labels give rise to a block assignment. For example, the labels $\lambda(c, z)$ with $c \in z^-$ determine the embedding $e_z\colon [0, z] \xrightarrow{\sim} [a, 1] \hookrightarrow L$. If the meet-irreducible $b \in L$ is dropped, we obtain a block labelled modular lattice which retains the whole information: From the labelling, b can be reconstructed!

Proposition 18. *Up to isomorphism, there is a one-to-one correspondence between discrete L-algebras and geometric lattices with a block labelling.*

Proof. Let X be a discrete L-algebra. We embed X into $S(X)$. For $a, b \in \Lambda(S(X))$ with $a \in b^-$, define $\lambda(a, b) := b \to a$. Then λ has values in $X(G^-)$, and it is easily checked that λ is a block labelling of $\Lambda(S(X))^{\mathrm{op}}$. Moreover, $\Lambda(S(X))^{\mathrm{op}}$ is a geometric lattice, and the block labelling recovers the L-algebra X.

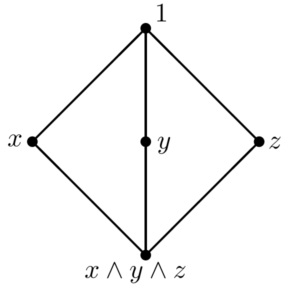
Conversely, let λ be a block labelling of a geometric lattice L^{op}. For distinct $x, y \in X(L)$, we set $x \to y := \lambda(x \wedge y, x)$. Adjoining a logical unit 1 to $X(L)$, we get a discrete L-algebra $X := X(L) \sqcup \{1\}$. To prove this, it is enough to verify Eq. (47) for distinct $x, y, z \in X(L)$. There are two cases.

<u>Case 1:</u> The $x \wedge y$, $x \wedge z$, and $y \wedge z$ are pairwise distinct.



By Definition 10, this gives $(x \to y) \to (x \to z) = \lambda(x \wedge y, x) \to \lambda(x \wedge z, x) = \lambda\big(\lambda(x \wedge y, x) \wedge \lambda(x \wedge z, x), \lambda(x \wedge y, x)\big) = \lambda(x \wedge y \wedge z, x \wedge y)$, and Eq. (47) follows by symmetry.

Case 2: $x \wedge y = y \wedge z$.



Then $(x \to y) \to (x \to z) = \lambda(x \wedge y, x) \to \lambda(x \wedge z, x) = 1$, and similarly, $(y \to x) \to (y \to z) = 1$.

To verify that $L \cong \Lambda(S(X))$, we extend λ to a function on $\{(a, b) \in L^2 \mid a \leqslant b\}$ with values in $S(X)$. For a maximal chain $a = c_0 < c_1 < \cdots < c_n = b$ we set $\lambda(a, b) := \lambda(c_0, c_1) \cdots \lambda(c_{n-1}, c_n)$. Let $a = c'_0 < c'_1 < \cdots < c'_n = b$ be another chain in L with $c'_{n-1} \neq c_{n-1}$. Then $x := \lambda(c_{n-1}, b)$ and $y := \lambda(c'_{n-1}, b)$ are distinct, and $\lambda(c_{n-1} \wedge c'_{n-1}, c_{n-1}) = x \to y$. Similarly, $\lambda(c_{n-1} \wedge c'_{n-1}, c'_{n-1}) = y \to x$. Hence $\lambda(a, c_{n-1} \wedge c'_{n-1})\lambda(c_{n-1} \wedge c'_{n-1}, c_{n-1})\lambda(c_{n-1}, b) = \lambda(a, c_{n-1} \wedge c'_{n-1})(x \wedge y)$, which shows that the extension of λ is well defined. Moreover, the construction shows that the map $f: L \to S(X)$ with $f(a) := \lambda(a, 1)$ is injective and length preserving. As in the proof of Proposition 17, it follows that f is a lattice embedding. Thus $L \cong \Lambda(S(X))$. $\square$

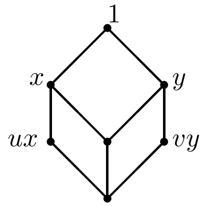
Example 3 shows that $\Lambda(S(X))$ can be smaller than $\mathrm{Rad}(S(X))$. By Proposition 14, this implies that a block labelling of a finite modular geometric lattice need not lead to a geometric right l-group.

8. Geometric Garside groups

Next we determine the discrete L-algebras for which the lattice $S(X)$ is modular.

Proposition 19. *Let X be a discrete L-algebra. The lattice $S(X)$ is modular if and only if for $x, y, u, v \in X \smallsetminus \{1\}$ with $x \neq y$ and $(x \to y) \to u = (y \to x) \to v$, there is an element $z \in X$ with $x \to z = u$ and $y \to z = v$.*

Proof. Assume that $S(X)$ is modular, and let $x, y, u, v \in X \smallsetminus \{1\}$ with $x \neq y$ and $(x \to y) \to u = (y \to x) \to v$ be given. If $(x \to y) \to u = 1$, then $x \to y = u$ and $y \to x = v$, and $z := x \wedge y$ meets the requirement. Therefore, assume that $(x \to y) \to u < 1$. Then $ux \wedge (x \wedge y) = ux \wedge y = (ux \to y)ux = \big(u \to (x \to y)\big)ux = \big((x \to y) \to u\big)(x \to y)x = \big((y \to x) \to v\big)(y \to x)y$. By symmetry, this yields $ux \wedge (x \wedge y) = vy \wedge (x \wedge y)$. So we have a subposet



of $S(X)$. As $S(X)$ is modular, $z := ux \vee vy \in S^1(X)$ satisfies $x \to z = u$ and $y \to z = v$.

Conversely, assume that the condition of the proposition holds, and let $a, b \in S(X)$ be distinct elements with $a^- \cap b^- \neq \varnothing$. Suppose that the interval $[a, a \vee b]$ is of length $l > 1$. Then there exist $c, d \in (a \vee b)^-$ with $a < c$ and $b < d$. Hence $c \wedge d > a \wedge b$. So there exists an element $e \leqslant c \wedge d$ with $a \wedge b \in e^-$. If $e < c \wedge d$, we can replace a, b by a, e or b, e or $a \vee e, b \vee e$ to get a smaller length l. Thus, by induction, we can assume that $e = c \wedge d$. Now there are $x, y, u, v \in S^1(X)$ with $c = x(a \vee b)$, $d = y(a \vee b)$, $a = uc$, and $b = vd$. Furthermore, $x \neq y$ and $c \to d = x(a \vee b) \to d = x \to \big((a \vee b) \to d\big) = x \to y$. Hence $e = (c \to d)c = (x \to y)c$. Similarly, we obtain $\big((x \to y) \to u\big)e = a \wedge b = \big((y \to x) \to v\big)e$, which yields $(x \to y) \to u = (y \to x) \to v$. So there is an element $z \in S^1(X)$ with $x \to z = u$ and $y \to z = v$. Hence $a = ux(a \vee b) \leqslant z(a \vee b)$, and similarly, $b \leqslant z(a \vee b)$, which gives $a \vee b \leqslant z(a \vee b)$, a contradiction. □

A discrete L-algebra X which satisfies the equivalent conditions of Proposition 19 will be called *modular*.

Definition 11. We call a discrete L-algebra X and the corresponding block labelling *non-degenerate* if X is modular and satisfies

(1) $\forall x, y \in S^1(X)\colon x \to y = y \to x \implies x = y$.
(2) For distinct $x, y \in S^1(X)$ there exist $u, v \in S^1(X)$ with $x = u \to v$ and $y = v \to u$.

Theorem 4. *Let X be a non-degenerate discrete L-algebra. Then $S(X)$ is a right l-cone, and its group of fractions is a modular geometric right l-group G with $\widetilde{X}(G^-) = X$.*

Proof. We show first that $S(X)$ is left cancellative. Thus let $a, b \in S(X)$ and $z \in S^1(X)$ with $za = zb$ be given. We have to show that $a = b$. If $a \vee b < 1$, there exist $c, d \in S(X)$ and $x \in S^1(X)$ with $a = cx$ and $b = dx$. Thus $zc = zd$, and a, b can be replaced by c, d. So we can assume that $a \vee b = 1$. Suppose that $a \neq b$. Then $za = zb \leqslant a \wedge b$. Hence $z \leqslant a \to b < 1$, and thus $z = a \to b = b \to a$. Consequently, $a \wedge b = (a \to b)a = za = zb$. Since $S(X)$ is modular, this implies that $a, b \in S^1(X)$. Therefore, condition (1) of Definition 11 yields $a = b$, a contradiction. Thus $S(X)$ is left cancellative, hence cancellative. By [63], Section 4, we infer that $S(X)$ embeds into the group G of left fractions.

The left Ore condition implies that any pair of elements $a, b \in G$ has a lower bound. So there are sequences $a_0, \dots, a_n$ and $b_0, \dots, b_m$ in G with $a_0 = b_0$, $a_n = a$, and $b_m = b$, such that $x_i := a_{i-1}a_i^{-1} \in S^1(X)$ and $y_j := b_{j-1}b_j^{-1} \in S^1(X)$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Assume that a and b are incomparable and $n + m$ is minimal. Then $n, m > 0$ and $a_1 \neq b_1$. We will show that a and b admit an upper bound c such that the length $l[a,c]$ of the interval $[a,c]$ is m while $l[b,c]=n$. We proceed by induction on $n+m$.

Suppose that there is no such c. By condition (2) of Definition 11, there exist $x,y\in S^1(X)$ with $x_1=x\to y$ and $y_1=y\to x$. Hence $x_1x=y_1y$, which yields $yx^{-1}a_1=y_1^{-1}x_1a_1=y_1^{-1}y_1b_1=b_1$. Thus $x^{-1}a_1$ is an upper bound of a_1 and b_1. So we can assume that $n>1$. Then the inductive hypothesis implies that a_1 and b has an upper bound d with $l[a_1,d]=m$ and $l[b,d]=1$. Similarly, there is an upper bound c of a and d with $l[a,c]=m$ and $l[d,c]=n-1$. Thus $c\geqslant a,b$ with $l[a,c]=m$ and $l[b,c]=n$.

Since $ac^{-1},bc^{-1}\in G^-$, it follows that G is a modular lattice. Whence G is a noetherian right l-group with negative cone $S(X)$, and $\widetilde{X}(G^-)=X$. By Proposition 14, G is geometric. $\square$

As a consequence, we get our main result for discrete L-algebras:

Corollary. *Up to isomorphism, there is a one-to-one correspondence between modular geometric right l-groups and non-degenerate discrete L-algebras.*

Proof. Let X be a non-degenerate discrete L-algebra. Then Theorem 4 yields a modular geometric right l-group G with $\widetilde{X}(G^-)=X$. Conversely, let G be a modular geometric right l-group. By Proposition 5, this implies that $X:=\widetilde{X}(G^-)$ is closed with respect to $\to$ and $\rightsquigarrow$. Hence $(X,\to)$ is a discrete L-algebra which is non-degenerate by Proposition 4. This completes the correspondence. $\square$

For finite discrete L-algebras, the three conditions of non-degeneracy (Definition 11) reduce to a single one:

Theorem 5. *Up to isomorphism, there exists a one-to-one correspondence between modular Garside groups and finite discrete L-algebras X satisfying*

$$\forall\, x,y\in S^1(X)\colon x\to y=y\to x\implies x=y. \tag{48}$$

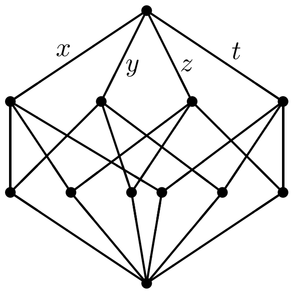
Proof. Let X be a finite discrete L-algebra satisfying (48). We show first that the implication

$$ax=ay\implies x=y \tag{49}$$

holds for all $a\in S(X)$ and $x,y\in S^1(X)$. We proceed by induction over the length of a. Suppose that $x\neq y$. Then $ax=ay\leqslant x\wedge y$ implies that $ax=ay=c(x\wedge y)$ for some $c\in S(X)$. Hence $a=c(x\to y)=c(y\to x)$, and the inductive hypothesis yields $x\to y=y\to x$, contrary to (48). As in the proof of Proposition 15, $\Delta x\leqslant\Delta$ holds for all $x\in S^1(X)$, where $\Delta:=\bigwedge X$. Thus $\Delta x=x'\Delta$ for some $x'\in S^1(X)$, which gives a map $x\mapsto x'$ that is injective by (49). Thus $x\mapsto x'$ is a bijection $S^1(X)\to S^1(X)$. In particular, this implies that for a suitable positive integer n, the element Δ^n commutes with any $x\in S^1(X)$, hence with every $a\in S(X)$.

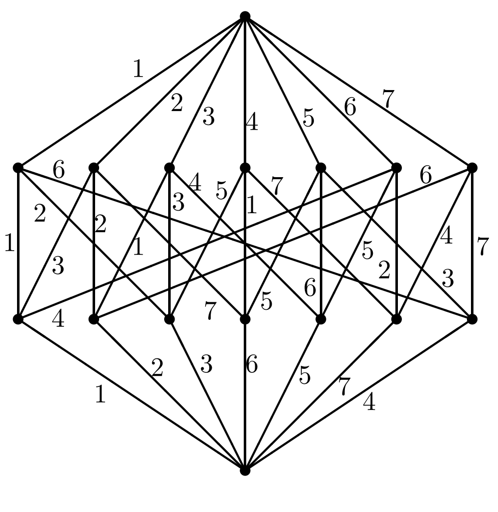
Assume that $a, b \in S(X)$ and $z \in S^1(X)$ satisfy $za = zb$. Then $\Delta a = \Delta b$. Hence $a\Delta^n = \Delta^n a = \Delta^n b = b\Delta^n$, which yields $a = b$. Thus $S(X)$ is left cancellative. So $S(X)$ embeds into its group G of left fractions. For any positive integer m, it follows that the labels of the upper neighbors of Δ^m are pairwise distinct. So the Basterfield–Kelly theorem [2] implies that the interval $[\Delta^m, 1]$ is modular for all m. Thus $S(X)$ is modular. Consequently, for any pair of distinct $x, y \in S^1(X)$, there exist $a, b \in S(X)$ with $\Delta = xa = yb$. Hence $u := (a \vee b) \to a$ and $v := (a \vee b) \to b$ are in $S^1(X)$, and $u \to v = \big(a \to (a \vee b)\big) \to (a \to b) = a \to b = a \to (a \wedge b) = a \to xa = x$, and similarly, $v \to u = y$. Thus Theorem 4 and Proposition 16 imply that G is a modular Garside group. The converse follows by the corollary of Theorem 4. $\square$

Example 4. There exist finite geometric lattices which admit no block labelling, for example, the dual of the following lattice, a truncated 4-cube:



Here $x \to y$, $x \to z$, and $x \to t$ are pairwise distinct, and the same holds if x, y, z, t are permuted. But any labelling u, v, w of x^- satisfies $u \to v = u \to w$.

Example 5. Our final example shows the lattice of the Fano plane with a non-degenerate block labelling:



Open problems. The question arises which geometric lattices admit a block labelling, and which of them admit a non-degenerate block labelling. At the moment, we have no method to construct non-degenerate block labellings systematically. It is not clear whether an infinite geometric lattice with a non-degenerate block labelling has to be modular. Neither do we know whether modularity in Definition 11 can be dropped. It would be nice to have a geometric interpretation of block labellings, especially in the case of projective geometries over a skew-field.

9. The quasi-centre of an archimedean right l-group

In the context of spherical Artin–Tits groups, the relationship between the subgroup of normal elements and the components of the Coxeter system was observed by Brieskorn and Saito [10] and Deligne [26], some years after Garside's successful treatment of the braid group case [36]. For an irreducible Coxeter matrix, they found that the subgroup of normal elements is cyclic and almost the centre of the Artin–Tits group, up to a possible involution. Picantin [57] extended the correspondence to Garside groups.

In this final section, we exhibit a large class of right l-groups G for which the normal elements form an l-subgroup of G. Since every normal element $u \in G$ satisfies $a \leqslant b \Longleftrightarrow ua \leqslant ub$ for all $a, b \in G$, the set $N(G)$ of normal elements is a partially ordered group. Following Brieskorn and Saito [10], we call $N(G)$ the *quasi-centre* of G.

Definition 12. We call a right l-group G *archimedean* if $a, b \in G^+$ with $a^n \leqslant b$ for all $n \in \mathbb{N}$ implies that $a = 1$.

For l-groups, this concept coincides with the definition of Darnel [17] which is formally weaker than that in [3]. Let us call a right l-group G *complete* if every non-empty bounded subset has a meet and join in G. Thus every noetherian right l-group is complete, and every complete right l-group is archimedean. In contrast to l-groups (see [17], Theorem 53.3), of course, an archimedean right l-group need not be commutative.

Proposition 20. *The quasi-centre $N(G)$ of an archimedean right l-group G is an l-subgroup of G.*

Proof. Let $g \in N(G)$ be given. For any $a, b \in G$, we have $a = bg \Longleftrightarrow a = g \cdot g^{-1}bg$, where $b \in G^- \Longleftrightarrow g^{-1}bg \in G^-$. Hence $a \leqslant g \Longleftrightarrow a \preccurlyeq g$ for all $a \in G$. Passing to inverses, this gives $g^{-1} \preccurlyeq a^{-1} \Longleftrightarrow g^{-1} \leqslant a^{-1}$. Since $N(G)$ is a subgroup, this implies that $g \leqslant c \Longleftrightarrow g \preccurlyeq c$ for all $c \in G$.

For $g, h \in N(G)$, we infer that $g \curlywedge h \leqslant g, h$. Hence $g \curlywedge h \leqslant g \wedge h$. Similarly, $g \wedge h \preccurlyeq g, h$, which gives $g \wedge h \preccurlyeq g \curlywedge h$. So there are elements $a, b \in G^-$ with $g \curlywedge h = a(g \wedge h)$ and $g \wedge h = (g \curlywedge h)b$. Hence $g \wedge h = a(g \wedge h)b$, and thus $g \wedge h = a^n(g \wedge h)b^n$ for all $n \in \mathbb{N}$. Assume first that $g \wedge h \in G^-$. Then $a^{-n} = (g \wedge h)b^n(g \wedge h)^{-1} \leqslant (g \wedge h)^{-1}$ for all $n \in \mathbb{N}$. Since G is archimedean, this yields $a^{-1} = 1$. So we obtain $g \curlywedge h = g \wedge h$. For any $c \in G^-$, we have

$(g\wedge h)c = gc\wedge hc = gcg^{-1}g\wedge hch^{-1}h \leqslant g\wedge h$. Hence $(g\wedge h)c(g\wedge h)^{-1} \in G^-$ for all $c \in G^-$. Similarly, $c(g \curlywedge h) \preccurlyeq g \curlywedge h$, which yields $(g \wedge h)^{-1}c(g \wedge h) = (g \curlywedge h)^{-1}c(g \curlywedge h) \in G^-$. Thus $g \wedge h \in N(G)$. Now let $g, h \in G$ be arbitrary. Then $gh^{-1} \wedge 1 \in N(G)$, which yields $g \wedge h = (gh^{-1} \wedge 1)h \in N(G)$. Furthermore, $g \curlywedge h = g(1 \curlywedge g^{-1}h) = g(1 \wedge g^{-1}h) \in N(G)$. Whence $g \vee h = (g^{-1} \curlywedge h^{-1})^{-1} \in N(G)$. Thus $N(G)$ is an l-subgroup of G. □

Corollary. *Let G be a noetherian right l-group. Then the quasi-centre $N(G)$ is a cardinal sum of infinite cyclic l-groups.*

Proof. If G is noetherian, the quasi-centre is noetherian, too. Therefore, Birkhoff's theorem ([4], Theorem 37) implies that $N(G)$ is a free abelian group with the canonical lattice structure. □

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