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# Corestriction for algebras with group action

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We define a corestriction map for equivariant Brauer groups in the sense of Fröhlich and Wall, which contain as a special case the Brauer–Clifford groups introduced by Turull. We show that this corestriction map has similar properties as the corestriction map in group cohomology (especially Galois cohomology). In particular, composing corestriction and restriction associated to a subgroup $H \leqslant G$ amounts to powering with the index $|G : H|$.



## 1. Introduction

The Brauer–Clifford group was introduced by Turull [17,18] to find correspondences between certain families of irreducible characters of finite groups, such that fields of values and Schur indices of corresponding characters are equal [cf. 16]. The character families are usually characters lying over some fixed irreducible character of a normal

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subgroup. This is the topic of *Clifford theory*. A classical result in this context is a theorem of Gallagher which states that an invariant irreducible character of a normal subgroup of a finite group extends to the whole group if and only if it extends to every Sylow subgroup [6, Theorem 21.4]. The proof uses the corestriction map of group cohomology. The aim of this paper is to provide a similar tool in the theory of the Brauer–Clifford group. This will be applied in forthcoming work in the Clifford theory over small fields [cf. 10].

Let $R$ be a commutative ring on which a group $G$ acts by ring automorphisms. As Herman and Mitra [5] have pointed out, the Brauer–Clifford group $\mathrm{BrCliff}(G,R)$ is a special case of the equivariant Brauer group introduced earlier by Fröhlich and Wall [cf. 3]. This group consists of certain equivalence classes of certain $G$-algebras over $R$. By a $G$-algebra over $R$ we mean an algebra $A$ over $R$ on which $G$ acts such that $r^g \cdot 1_A = (r \cdot 1_A)^g$ for all $r \in R$ and $g \in G$.

Let $H \leqslant G$ be a subgroup. Then (by restriction) any $G$-algebra can be viewed as an $H$-algebra. This defines a map

$$\mathrm{Res} = \mathrm{Res}^G_H \colon \mathrm{BrCliff}(G,R) \to \mathrm{BrCliff}(H,R)$$

called restriction. In this paper, we define a map

$$\mathrm{Cores} = \mathrm{Cores}^G_H \colon \mathrm{BrCliff}(H,R) \to \mathrm{BrCliff}(G,R),$$

which has analogous properties to the transfer or corestriction map from group cohomology. This map will only be defined when the index $|G:H|$ is finite, and will be called *corestriction* or *transfer* map. The main result is that if $[A]$ is an equivalence class of algebras in $\mathrm{BrCliff}(G,R)$, then we have

$$\mathrm{Cores}^G_H(\mathrm{Res}^G_H[A]) = [A]^{|G:H|}.$$

We will first show how to construct a $G$-algebra $A^{\otimes G}$ over $R$, given an arbitrary $H$-algebra $A$ over $R$. To do this, we use the well known concept of *tensor induction*. We will review this concept, and give a simple conceptual definition of it that seems not as widely known as it should be. We will then show that tensor induction respects equivalence of $G$-algebras, and thus defines a group homomorphism of Brauer–Clifford groups. Finally, in the last section we prove the main result mentioned above.

## 2. On tensor induction

Let $G$ be a group and $R$ a commutative $G$-ring. By definition, a $G$**-ring** is a ring $R$ on which the group $G$ acts by ring automorphisms. We use exponential notation $r \mapsto r^g$ to denote this action.

An $R$-module $V$ is called an $R$-module with compatible $G$-action, if $G$ acts on the abelian group $V$ such that $(vr)g = vgr^g$ for all $v \in V$, $r \in R$ and $g \in G$. We often use exponential notation, that is, we write $v^g$ instead of $vg$, in particular, when $V$ happens to be an $R$-algebra.

For the moment, we continue to use right multiplicative notation. Every $R$-module with $G$-action can be viewed as a right $RG$-module, where $RG$ denotes the **crossed product** of $R$ with $G$ (also called the **skew group ring** of $G$ over $R$). This is the set of formal sums

$$\sum_{g \in G} g r_g, \quad r_g \in R,$$

with multiplication induced by $(g_1 r_1)(g_2 r_2) = g_1 g_2 r_1^{g_2} r_2$, extended linearly. Conversely, any right module over $RG$ can be viewed as an $R$-module with $G$-action.

Let $H \leqslant G$ and let $V$ be a right $RH$-module. Since $RH \subseteq RG$, we may form the induced module $V^G = \operatorname{Ind}_H^G(V) = V \otimes_{RH} RG$. Set $\Omega = \{Hg \mid g \in G\}$ and $V_\omega = V \otimes g \subseteq V^G$ for $\omega = Hg \in \Omega$. Then $V_\omega$ depends only on the coset $\omega = Hg$, not on the specific representative $g$. Every $V_\omega$ is an $R$-submodule of $V^G$ and

$$V^G = \bigoplus_{\omega \in \Omega} V_\omega, \quad \text{with} \quad V_\omega g = V_{\omega g} \text{ for all } \omega \in \Omega.$$

The modules $V_\omega$ and $V_{\omega g}$ are isomorphic as abelian groups, but in general not as $R$-modules. Instead, the map $\varrho_{\omega,g}\colon V_\omega \to V_{\omega g}$ given by $v\varrho_{\omega,g} = vg$ has the property $(vr)\varrho_{\omega,g} = v\varrho_{\omega,g} r^g$. Following Riehm [12], we call an $R$-module $W$ a **$g$-conjugate** of the $R$-module $V$, if there is an isomorphism $\kappa\colon V \to W$ of abelian groups such that $(vr)\kappa = v\kappa r^g$ for all $v \in V$. Thus $V \otimes g$ is a $g$-conjugate of $V$.

The **tensor induced** module is, by definition, the tensor product

$$V^{\otimes G} := \bigotimes_{\omega \in \Omega} V_\omega \quad \text{(tensor product over } R\text{)}.$$

This is made into an $RG$-module by defining

$$\left( \bigotimes_{\omega \in \Omega} v_\omega \right) g = \bigotimes_{\omega g \in \Omega} v_\omega g = \bigotimes_{\omega \in \Omega} v_{\omega g^{-1}} g,$$

where $v_{\omega g^{-1}} g \in V_\omega$. Some readers may prefer to enumerate

$$\Omega = \{Hg_1, Hg_2, \dots, Hg_n\},$$

and then write

$$(v_1 \otimes \cdots \otimes v_n) g = v_{1g^{-1}} g \otimes \cdots \otimes v_{ng^{-1}} g \quad (v_i \in V_{Hg_i}).$$

It is easy to see that this extends to a well defined action of $G$ on the tensor induced module $V^{\otimes G}$, and that this action makes $V^{\otimes G}$ a module over the skew group ring $RG$.

A word on notation: The isomorphism class of $V^{\otimes G}$ depends, of course, on the subgroup $H$ and the ring $R$. If we want to emphasize the dependence on the subgroup $H$, we write $(V_H)^{\otimes G}$. If we want to emphasize the dependence on the ring $R$, we write $V^{\otimes_R G}$. Both notations can be combined.

We mention in passing that the definition of tensor induction given here is independent of the choice of a set of coset representatives. Such a definition was asked for by Kovács [9] and Pacifici [11]. Our definition (for trivial action of $G$ on $R$) appears in a paper of Knörr [7, Def. 10, attributed to the referee]. The usual definition [1, §13] yields an isomorphic module, as is not difficult to see.

The tensor induced module can be characterized by a universal property:

**2.1. Theorem.** *Let $H \leqslant G$ be groups and $V$ an $RH$-module. Let $\tau\colon V^G \to V^{\otimes G}$ be the canonical multilinear map. Let $W$ be an $RG$-module and $\beta\colon V^G \to W$ a map with the following properties:*

(i) *$\beta$ is a map of $G$-sets.*
(ii) *$\beta$ is $R$-multilinear, when $V^G$ is considered as direct product of the modules $V_\omega$ ($\omega \in \Omega = \{Hg \mid g \in G\}$).*

*Then there is a unique $RG$-module homomorphism $\varphi\colon V^{\otimes G} \to W$ making the diagram*

$$\begin{tikzcd} V^G \arrow[r, "\tau"] \arrow[dr, "\beta"'] & V^{\otimes G} \arrow[d, "\varphi"] \\ & W \end{tikzcd}$$

*commutative. The pair $(\tau, V^{\otimes G})$ is determined uniquely up to unique isomorphism by the fact that this holds for all maps $\beta\colon V^G \to W$ as above.*

**Proof.** The uniqueness of $(\tau, V^{\otimes G})$ will follow from the usual abstract nonsense argument, once we have proved existence and uniqueness of $\varphi$.

By the universal property of the tensor product $\bigotimes_{\omega\in\Omega} V_\omega$, there exists a unique $R$-module homomorphism $\varphi\colon V^{\otimes G} \to W$ making the diagram commutative, and we need to show that $\varphi$ commutes with the action of $G$. This is true on the pure tensors in $V^{\otimes G}$, since it is true for $\tau$ (by definition) and $\beta$ (by assumption). From this, the assertion follows. $\square$

The next result shows that $(\ )^{\otimes G}$ is a functor from the category of $RH$-modules to the category of $RG$-modules.

**2.2. Proposition.** *Let $\varphi: V \to W$ be a homomorphism of $RH$-modules. Then there is a unique $RG$-module homomorphism $\varphi^{\otimes G}: V^{\otimes G} \to W^{\otimes G}$, such that the diagram*

$$\begin{tikzcd} V^G \arrow[r, "\varphi\otimes 1"] \arrow[d] & W^G \arrow[d] \\ V^{\otimes G} \arrow[r, "\varphi^{\otimes G}"] & W^{\otimes G} \end{tikzcd}$$

*is commutative. (Here $V^G = V \otimes_{RH} RG$.) If $\psi: W \to U$ is another $RH$-module homomorphism, then $(\varphi\psi)^{\otimes G} = \varphi^{\otimes G}\psi^{\otimes G}$.*

**Proof.** The first assertion follows from Theorem 2.1, applied to the $\beta$ in

$$\begin{tikzcd} V^G \arrow[r, "\varphi\otimes 1"] \arrow[rr, bend right, "\beta"'] & W^G \arrow[r] & W^{\otimes G}. \end{tikzcd}$$

Of course, one can define $\varphi^{\otimes G}$ directly: Let $G = \dot\bigcup_{t\in T} Ht$ and $v_t \in V$. Then define

$$\left(\bigotimes_{t\in T}(v_t \otimes t)\right)\varphi^{\otimes G} = \bigotimes_{t\in T}(v_t\varphi \otimes t) \in \bigotimes_{t\in T}(W \otimes t) = W^{\otimes G}.$$

The second assertion follows from the uniqueness of $\varphi^{\otimes G}$, $\psi^{\otimes G}$ and $(\varphi\psi)^{\otimes G}$ in

$$\begin{tikzcd} V^G \arrow[r, "\varphi\otimes 1"] \arrow[d] & W^G \arrow[r, "\psi\otimes 1"] \arrow[d] & U^G \arrow[d] \\ V^{\otimes G} \arrow[r, "\varphi^{\otimes G}"] \arrow[rr, bend right, "(\varphi\psi)^{\otimes G}"'] & W^{\otimes G} \arrow[r, "\psi^{\otimes G}"] & U^{\otimes G}. \end{tikzcd}$$ □

**2.3. Remark.** While the involved categories are *abelian* categories, the functor $(\ )^{\otimes G}$ is not additive: In general $(\varphi_1 + \varphi_2)^{\otimes G} \neq \varphi_1^{\otimes G} + \varphi_2^{\otimes G}$.

## 3. Corestriction of $G$-algebras

As before, let $R$ be a commutative $G$-ring. A **$G$-algebra over** $R$ is a $G$-ring $A$, which is at the same time an $R$-algebra and such that $A$ is an $R$-module with compatible $G$-action. In other words, we have $(ar)^g = a^g r^g$ for all $a \in A$, $r \in R$ and $g \in G$. This can also be expressed by saying that the algebra unit $R \to \mathbf{Z}(A)$ is a homomorphism of $G$-rings. Note that now we are using exponential notation for the action of $G$ on $A$, although we can view $A$ as $RG$-module.

Let $H \leqslant G$. Given an $H$-algebra $A$ over $R$, we may form the tensor induced module $A^{\otimes G}$. Let $\Omega = \{Hg \mid g \in G\}$. Then $A^{\otimes G}$ is a tensor product of modules $A_\omega$, where $A_\omega = A \otimes g \subseteq A \otimes_{RH} RG$ for $\omega = Hg$. Since we prefer to use exponential notation in the algebra situation, we write $A^{\otimes g}$ instead of $A \otimes g$ and $a^{\otimes g}$ instead of $a \otimes g$ from now on. Each $A_\omega = A^{\otimes g}$ is in fact an algebra in its own right, via $(a^{\otimes g})(b^{\otimes g}) = (ab)^{\otimes g}$.

For each $g \in G$ and $\omega = Ht \in \Omega$, we have a ring isomorphism $A_\omega \to A_{\omega g}$ sending $a^{\otimes t}$ to $(a^{\otimes t})^g = a^{\otimes tg}$. This is not an $R$-algebra homomorphism, but we have $(ra_\omega)^g = r^g a_\omega^g$. As $R$-algebra, $A^{\otimes g}$ is a $g$-conjugate of $A$.

Since every $A_\omega$ is an $R$-algebra, the tensor induced module $A^{\otimes G}$ is an $R$-algebra. The action of $G$ on $A^{\otimes G}$ makes $A^{\otimes G}$ a $G$-algebra over $R$.

**3.1. Definition.** Let $A$ be an $H$-algebra over $R$. The **corestriction** of $A$ in $G$ is the $G$-algebra $A^{\otimes G}$. We also write $\operatorname{Cores}(A)$ or $\operatorname{Cores}_H^G(A)$ to denote this algebra.

**3.2. Remark.** This definition must not be confused with the algebra-theoretic definition of the corestriction map from the Brauer group of a field to the Brauer group of a subfield [12,2,15]. Of course, the two definitions are closely related. Let $L/F$ be a finite separable field extension and let $E \supseteq L$ be a field such that $E/F$ is Galois. Let $G = \operatorname{Gal}(E/F)$ and $H = \operatorname{Gal}(E/L)$. Then $|G : H| = |L : F|$ is finite. Let $A$ be an algebra over the field $L$. Then $A \otimes_L E$ becomes an $H$-algebra over the $H$-field $E$ by defining $(a \otimes \lambda)^h = a \otimes \lambda^h$. Let $B = (A \otimes_L E)^{\otimes G}$ in the sense of Definition 3.1. To get the corestriction map that is used in the theory of the Brauer group, one has to take $\mathbf{C}_B(G)$, the centralizer of $G$ in $B$. This is an algebra over the field $F$. Its isomorphism class does not depend on the choice of the Galois extension $E$. In particular, if $A$ is central simple over $L$, then $\mathbf{C}_B(G)$ is central simple over $F$. This defines a map $\operatorname{Br}(L) \to \operatorname{Br}(F)$, also called corestriction, and denoted by $\operatorname{cor}_{L/F}$. However, in this paper we will only deal with the corestriction of Definition 3.1.

Let $\Omega = \{Hg \mid g \in G\}$, and let $\mu_\omega \colon A_\omega \to A^{\otimes G}$ be the canonical algebra homomorphism into $A^{\otimes G}$, so

$$a_\omega \mu_\omega = 1 \otimes \cdots \otimes 1 \otimes a_\omega \otimes 1 \otimes \cdots \otimes 1,$$

where $a_\omega$ occurs at position $\omega$, of course. The action of $G$ on $A^{\otimes G}$ is uniquely determined by the property $(a_\omega \mu_\omega)^g = a_\omega^g \mu_{\omega g}$ for all $\omega \in \Omega$, $a_\omega \in A_\omega$, and $g \in G$. This, in turn, follows from $(a\mu_H)^g = a^{\otimes g}\mu_{Hg}$ for all $g \in G$ and $a \in A$ and the $G$-action property: To see this, note that for $\omega = Ht$ and $a_\omega = a^{\otimes t}$, we have

$$(a_\omega \mu_\omega)^g = (a^{\otimes t}\mu_{Ht})^g = \big((a\mu_H)^t\big)^g = (a\mu_H)^{tg} = a^{\otimes tg}\mu_{Htg} = a_\omega^g \mu_{\omega g}.$$

We have thus proved the following theorem [cf. 12, Theorem 4]:

**3.3. Theorem.** *Assume the notation introduced above. Then the action of $G$ on $A^{\otimes G}$ is uniquely determined by the fact that the diagram*

$$\begin{tikzcd} A \arrow[r, "()^{\otimes g}"] \arrow[d, "\mu_H"] & A_{Hg} \arrow[d, "\mu_{Hg}"] \\ A^{\otimes G} \arrow[r, "()^g"] & A^{\otimes G} \end{tikzcd}$$

*is commutative for all $g \in G$.*

Moreover, we have [12, Theorem 4]:

**3.4. Theorem.** *The homomorphism $\mu = \mu_H \colon A \to A^{\otimes G}$ is a homomorphism of $H$-algebras, and has the following universal property: Whenever $\psi\colon A \to B$ is an $H$-algebra homomorphism of $A$ into a $G$-algebra $B$ such that $A\psi$ and $(A\psi)^g$ commute for all $g \in G \setminus H$, then there is a unique $G$-algebra homomorphism $\alpha\colon A^{\otimes G} \to B$ making the diagram*

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[dr, "\psi"] & A^{\otimes G} \arrow[d, "\alpha"] \\ & B \end{tikzcd}$$

*commutative.*

**Proof.** That $\mu$ is a homomorphism of $H$-algebras follows from the commutativity of the diagram in Theorem 3.3 for $g \in H$.

Assume $\alpha\colon A^{\otimes G} \to B$ as in the theorem exists. Let $\omega = Ht \in \Omega$ and $a_\omega = a^{\otimes t} \in A_\omega$. Then $a_\omega \mu_\omega = (a\mu)^t$. Thus we necessarily have $a_\omega \mu_\omega \alpha = ((a\mu)^t)\alpha = (a\mu\alpha)^t = (a\psi)^t$. Let $G = \dot\bigcup_{t \in T} Ht$ and $a_t \in A$. Then $\alpha$ as in the theorem must send

$$\bigotimes_{t \in T} a_t^{\otimes t} = \prod_{t \in T} (a_t\mu)^t \qquad \text{to} \qquad \prod_{t \in T} (a_t\psi)^t.$$

Now check that this indeed defines an algebra homomorphism. $\square$

Given a homomorphism $\alpha\colon A \to B$ of $H$-algebras over $R$, the homomorphism $\alpha^{\otimes G}\colon A^{\otimes G} \to B^{\otimes G}$ defined in Proposition 2.2 is a $G$-algebra homomorphism. We denote it also by $\mathrm{Cores}(\alpha)$. In the algebra case, we can characterize $\alpha^{\otimes G}$ more elegantly by:

**3.5. Proposition.** *Let $\alpha\colon A \to B$ be a homomorphism of $H$-algebras over $R$. Then there is a unique homomorphism $\alpha^{\otimes G}\colon A^{\otimes G} \to B^{\otimes G}$ of $G$-algebras over $R$ making*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ A^{\otimes G} & \xrightarrow{\alpha^{\otimes G}} & B^{\otimes G} \end{array}$$

*commutative.*

**Proof.** This follows immediately from Theorem 3.4. □

The next two results are of course also true for modules instead of algebras.

**3.6. Proposition.** *Let $A$ and $B$ be $H$-algebras over $R$. Then*

$$A^{\otimes G} \otimes_R B^{\otimes G} \cong (A \otimes_R B)^{\otimes G}$$

*as $G$-algebras over $R$. This is a natural equivalence of functors.*

**Proof.** The isomorphism is given by the map well-defined by

$$\left(\bigotimes_{\omega\in\Omega} a_\omega\right) \otimes \left(\bigotimes_{\omega\in\Omega} b_\omega\right) \mapsto \bigotimes_{\omega\in\Omega}(a_\omega \otimes b_\omega). \qquad \square$$

(This means that the functor $(\cdot)^{\otimes G} = \mathrm{Cores}$ is actually a functor of monoidal categories.)

Let $\varphi: R \to S$ be a homomorphism of $G$-rings. If $A$ is an $H$-algebra over $R$, then $A \otimes_R S$ is an $H$-algebra over $S$.

**3.7. Proposition.** *Let $\varphi: R \to S$ be a homomorphism of $G$-rings. Then*

$$(A \otimes_R S)^{\otimes_S G} \cong (A^{\otimes_R G}) \otimes_R S$$

*naturally.*

In other words: scalar extension and corestriction commute.

**Proof.** The map

$$(A \otimes_R S) \otimes_{SH} SG \to (A \otimes_{RH} RG) \otimes_R S,$$
$$(a \otimes s)^{\otimes g} \mapsto (a^{\otimes g}) \otimes s^g,$$

is an isomorphism of $SG$-modules and sends the algebra $(A \otimes_R S)^{\otimes g}$ to the algebra $A^{\otimes g} \otimes_R S$. Thus

$$(A \otimes_R S)^{\otimes_S G} \cong \bigotimes_{t} {}_S (A^{\otimes t} \otimes_R S) \cong \left( \bigotimes_{t} {}_R A^{\otimes t} \right) \otimes_R S$$

as $S$-algebras. These isomorphisms respect the action of $G$, as is easily checked. □

## 4. Review of the Brauer–Clifford group

We recall the definition of the equivariant Brauer group by Fröhlich and Wall [3]. As Herman and Mitra [5] have pointed out, the Brauer–Clifford group as defined by Turull [17] is a special case.

Let $G$ be a group. A $G$-algebra $A$ over the commutative $G$-ring $R$ is called an **Azumaya $G$-algebra over** $R$ if it is Azumaya over $R$ as an $R$-algebra.

If $A$ and $B$ are Azumaya $G$-algebras over the $G$-ring $R$, then $A \otimes_R B$ is an Azumaya $G$-algebra over $R$.

The **Brauer–Clifford group** $\mathrm{BrCliff}(G, R)$ is the set of equivalence classes of Azumaya $G$-algebras over $R$ under an equivalence relation that will be defined below. The multiplication is induced by the tensor product $\otimes_R$ of algebras.

Let $P$ be an $RG$-module, where $RG$ denotes, as before, the skew group ring of $G$ over $R$. Then $\mathrm{End}_R P$ has a natural structure of a $G$-algebra over the $G$-ring $R$. The action of $G$ on $\mathrm{End}_R P$ is given by $p\alpha^g = ((pg^{-1})\alpha)g$ for $p \in P$ and $\alpha \in \mathrm{End}_R P$. If $P$ is a progenerator over $R$, then $\mathrm{End}_R P$ is Azumaya over $R$. A **trivial $G$-algebra over** $R$ is an algebra of the form $\mathrm{End}_R P$, where $P$ is an $RG$-module such that $P_R$ is a progenerator. Two $G$-algebras $A$ and $B$ over $R$ are called **equivalent** if there are trivial $G$-algebras $S$ and $T$ such that $A \otimes_R S \cong A \otimes_R T$ as $G$-algebras over $R$.

This definition yields the same equivalence classes as another definition using equivariant Morita equivalence [5, Proposition 9].

As mentioned before, the set of equivalence classes of Azumaya $G$-algebras over $R$ forms the **equivariant Brauer group**, or, as we call it here, the **Brauer–Clifford group** $\mathrm{BrCliff}(G, R)$.

## 5. Corestriction of equivalent $G$-algebras

We continue to assume that $G$ is a group, $H \leqslant G$ is a subgroup of finite index and $R$ is a $G$-ring.

**5.1. Lemma.** *Let $P$ be a right $RH$-module which is finitely generated projective over $R$. Then*

$$\mathrm{End}_R(P^{\otimes G}) \cong (\mathrm{End}_R P)^{\otimes G}$$

*as $G$-algebras over $R$.*

**Proof.** Write $S = \operatorname{End}_R P$. Observe that $S$ is an $H$-algebra over $R$. We have

$$P^{\otimes G} = \bigotimes_{\omega \in \Omega} P_\omega \quad \text{and} \quad S^{\otimes G} = \bigotimes_{\omega \in \Omega} S_\omega.$$

Let $t \in \omega$ and $s \in S$. We define an action of $S_\omega$ on $P_\omega$ by

$$(x \otimes t) s^{\otimes t} = (xs) \otimes t.$$

This is independent of the choice of $t \in \omega$ since

$$\begin{aligned}(x \otimes (ht)) s^{\otimes ht} = (xs) \otimes (ht) = ((xs)h) \otimes t \\ = (xhs^h) \otimes t = \big((xh) \otimes t\big)(s^h)^{\otimes t}.\end{aligned}$$

We can identify $S_\omega$ with $\operatorname{End}_R P_\omega$ via this action. Since $P_R$ is finitely generated projective, we have

$$\operatorname{End}_R(P^{\otimes G}) = \operatorname{End}_R \left( \bigotimes_{\omega \in \Omega} P_\omega \right) \cong \bigotimes_{\omega \in \Omega} \operatorname{End}_R P_\omega \cong \bigotimes_{\omega \in \Omega} S_\omega = S^{\otimes G}$$

as $R$-algebras, where $S^{\otimes G}$ acts on $P^{\otimes G}$ by

$$\left( \bigotimes_{\omega \in \Omega} x_\omega \right) \left( \bigotimes_{\omega \in \Omega} s_\omega \right) = \bigotimes_{\omega \in \Omega} (x_\omega s_\omega).$$

The isomorphism above commutes with the action of $G$, as is easy to check. This finishes the proof. □

**5.2. Proposition.** *Let $R$ be a commutative $G$-ring and $H \leqslant G$. Let $A$ and $B$ be equivalent $H$-algebras over $R$. Then $A^{\otimes G}$ and $B^{\otimes G}$ are equivalent as $G$-algebras over $R$.*

**Proof.** Let $P$ and $Q$ be $RH$-modules that are $R$-progenerators and such that $A \otimes_R \operatorname{End}_R P \cong B \otimes_R \operatorname{End}_R Q$ as $H$-algebras over $R$. Then

$$(A \otimes_R \operatorname{End}_R P)^{\otimes G} \cong (B \otimes_R \operatorname{End}_R Q)^{\otimes G}$$

as $G$-algebras over $R$ by Proposition 3.5. By Lemma 5.1 and Proposition 3.6,

$$A^{\otimes G} \otimes_R \operatorname{End}_R(P^{\otimes G}) \cong A^{\otimes G} \otimes_R (\operatorname{End}_R P)^{\otimes G} \cong (A \otimes_R \operatorname{End}_R P)^{\otimes G}$$

and similarly for $B$ and $Q$. Thus

$$A^{\otimes G} \otimes_R \operatorname{End}_R(P^{\otimes G}) \cong B^{\otimes G} \otimes_R \operatorname{End}_R(Q^{\otimes G}),$$

which shows that $A^{\otimes G}$ and $B^{\otimes G}$ are equivalent. □

**5.3. Corollary.** *Corestriction defines a group homomorphism*

$$\operatorname{Cores} = \operatorname{Cores}_H^G \colon \operatorname{BrCliff}(H, R) \to \operatorname{BrCliff}(G, R).$$

**Proof.** As $R$-algebra, $\operatorname{Cores}(A)$ is a $|G : H|$-fold tensor product of $A$ over $R$. Thus if $A$ is Azumaya over $R$, then $\operatorname{Cores}(A)$ is, too. By Proposition 5.2, Cores as mapping of Brauer–Clifford groups is well defined. By Proposition 3.6, it is a group homomorphism. □

## 6. Corestriction and restriction

In this section we prove that $\operatorname{Cores}_H^G(\operatorname{Res}_H^G[A]) = [A]^{|G:H|}$ when $A$ is an Azumaya $G$-algebra over $R$. The proof follows Tignol's proof of the same result for the Brauer group over a field [15].

Let $A$ be an Azumaya algebra over the commutative ring $R$. We need a property of the *reduced trace* $\operatorname{trd} = \operatorname{trd}_{A/R} \colon A \to R$ [8, IV.2] which is probably well known.

**6.1. Lemma.** *Let $A$ be an Azumaya algebra over the commutative ring $R$ and $g$ a ring automorphism of $A$ (which restricts to a ring automorphism of $R \cong \mathbf{Z}(A)$). Then*

$$\operatorname{trd}(a^g) = \operatorname{trd}(a)^g \quad \textit{for all } a \in A.$$

**Proof.** The reduced trace commutes with scalar extensions. This means: Let $\varphi \colon R \to S$ be a ring homomorphism (with $\varphi(1_R) = 1_S$). The homomorphism $\varphi$ makes $S$ into an $R$-module. The algebra $A \otimes_R S$ is Azumaya over $S$ and we have

$$\operatorname{trd}_{A \otimes_R S/S}(a \otimes 1) = \operatorname{trd}_{A/R}(a)^{\varphi}.$$

(This follows from the proof that the characteristic polynomial and the reduced trace are well-defined [8, Proposition IV.2.1].) We will apply this fact to $\varphi = g_{|R} \colon R \to R$. We write $A \otimes_{R,\varphi} R$ for the corresponding tensor product, to emphasize the dependence on $\varphi$.

Second, if $f \colon B \to A$ is an isomorphism of Azumaya $R$-algebras, then

$$\operatorname{trd}_{A/R}(f(b)) = \operatorname{trd}_{B/R}(b)$$

[8, Lemme IV.2.2]. We apply this to the map $f \colon B = A \otimes_{R,\varphi} R \to A$ defined by $f(a \otimes r) = a^g r$. Note that $f$ is well-defined since $ar \otimes s = a \otimes r^{\varphi} s$ in $A \otimes_{R,\varphi} R$ and so $f(ar \otimes s) = (ar)^g s = a^g r^{\varphi} s = f(a \otimes r^{\varphi} s)$. It is easy to check that $f$ is an isomorphism of $R$-algebras, the inverse is given by $a \mapsto a^{g^{-1}} \otimes 1$.

Now the composition

$$A \longrightarrow A \otimes_{R,\varphi} R \xrightarrow{f} A$$

yields the map $a \mapsto a^g$. Applying the above statements, we get

$$\begin{aligned}\operatorname{trd}_{A/R}(a^g) = \operatorname{trd}_{A/R}(f(a\otimes 1)) = \operatorname{trd}_{B/R}(a\otimes 1) &= \operatorname{trd}_{A/R}(a)^{\varphi} \\ &= \operatorname{trd}_{A/R}(a)^g\end{aligned}$$

as claimed. $\square$

We need another lemma. We write $\mathrm{S}_n$ to denote the symmetric group on $n$ letters.

**6.2. Lemma.** *Let $A$ be an Azumaya $G$-algebra over the $G$-ring $R$ and $n \in \mathbb{N}$. Then there is a group homomorphism*

$$\sigma\colon \mathrm{S}_n \to (\underbrace{A\otimes_R \cdots \otimes_R A}_{n})^*$$

*such that*

$$\sigma(\pi)^{-1}(a_1\otimes\cdots\otimes a_n)\sigma(\pi) = a_{1\pi^{-1}}\otimes\cdots\otimes a_{n\pi^{-1}}$$

*and $\sigma(\pi)^g = \sigma(\pi)$ for all $\pi \in \mathrm{S}_n$ and $g \in G$.*

**Proof.** Consider first the case where $n = 2$. Since $\operatorname{End}_R(A) \cong A\otimes_R A^{op}$, there is a unique element $t = \sum_i x_i \otimes y_i \in A\otimes_R A$ such that $\operatorname{trd}(a)1_A = \sum_i x_i a y_i$ for all $a \in A$. By a result of Goldman, this element has the properties $t^2 = 1$ and $(a\otimes b)t = t(b\otimes a)$ for all $a, b \in A$ [8, Proposition IV.4.1]. To finish the case $n = 2$, it remains to show that $t^g = t$ for $g \in G$. By Lemma 6.1, we have $\operatorname{trd}(a^g) = \operatorname{trd}(a)^g$ for $g \in G$. This means that

$$\sum_i x_i a^g y_i = \operatorname{trd}(a^g) = \operatorname{trd}(a)^g = \sum_i x_i^g a^g y_i^g$$

for all $a \in A$. Thus

$$t = \sum_i x_i \otimes y_i = \sum_i x_i^g \otimes y_i^g = t^g,$$

as desired.

Now for the general case. By what we have done already, for every pair $(i, j)$ there is an element $t_{ij} \in (A^{\otimes n})$ such that the inner automorphism induced by $t_{ij}$ switches the positions $i$ and $j$, and such that $t_{ij}$ is centralized by $G$. We first define the homomorphism $\sigma$ on the neighbor transpositions $(i, i+1)$, setting $\sigma((i, i+1)) = t_{i,i+1}$. To show that this extends to a homomorphism of the symmetric group $S_n$ into $(A^{\otimes n})^*$, we use the fact that $S_n$ is a Coxeter group generated by the neighbor transpositions with relations $\big((i, i+1)(k, k+1)\big)^{m(i,k)} = 1$, where $m(i, k)$ is the order of the corresponding element. Thus we have to check three types of relations. The first is $t_{i,i+1}^2 = 1$, which follows from the result of Goldman cited before. The second type of relation to check is $(t_{i,i+1}t_{k,k+1})^2 = 1$ whenever $\{i, i+1\} \cap \{k, k+1\} = \emptyset$. This relation is clear in view of $t_{i,i+1}^2 = 1$ and since $t_{i,i+1}$ and $t_{k,k+1}$ live in different components of $A^{\otimes n}$. Finally, we have to check that $(t_{i,i+1}t_{i+1,i+2})^3 = 1$. To do this, one can proceed as Knus and Ojanguren in their proof of $t^2 = 1$ [8, Proposition IV.4.1] and reduce the problem to the case where $A$ is a matrix ring over $R$. Then in terms of matrix units $e_{rs}$, we have $t = \sum_{r,s} e_{rs} \otimes e_{sr}$. It suffices to compute in $A \otimes A \otimes A$, where we have to check that

$$\left(\left(\sum_{r,s} e_{rs} \otimes e_{sr} \otimes 1\right)\left(\sum_{u,v} 1 \otimes e_{uv} \otimes e_{vu}\right)\right)^3 = 1 \otimes 1 \otimes 1.$$

We leave this simple computation to the reader. □

The existence of a homomorphism $\sigma: S_n \to (A^{\otimes n})^*$ was also proved by Haile [4, Lemma 1.1] and Saltman [13, Theorem 2], but we need the additional property that $\sigma(\pi)$ is centralized by the group $G$. Note that since $G$ is completely arbitrary, the lemma says that all ring automorphisms of $A$ centralize the image of $\sigma$.

**6.3. Theorem.** *Let $A$ be an Azumaya $G$-algebra over the $G$-ring $R$ and $H \leqslant G$. Then $(A_H)^{\otimes G}$ and $A^{\otimes |G:H|}$ are equivalent as $G$-algebras over $R$. In other words,*

$$\operatorname{BrCliff}(G, R) \xrightarrow{\operatorname{Res}_H^G} \operatorname{BrCliff}(H, R) \xrightarrow{\operatorname{Cores}_H^G} \operatorname{BrCliff}(G, R)$$

*sends $[A]$ to $[A]^{|G:H|}$.*

**Proof.** Write $C = (A_H)^{\otimes G}$ and $B = A^{\otimes |G:H|}$. We have to show that $B$ and $C$ are equivalent $G$-algebras. First we show that $C$ and $B$ are isomorphic as $R$-algebras. As in the construction of the tensor induced algebra, let $\Omega = \{Hg \mid g \in G\}$ and $A_\omega = A^{\otimes g} \subseteq A \otimes_{RH} RG$, when $\omega = Hg$. Every element of $A_\omega$ has the form $a^{\otimes g}$ with $a \in A$. Note that $a^{\otimes g} \mapsto a^g$ yields an isomorphism $\varphi_\omega: A_\omega \to A$ of $R$-algebras which is independent of the choice of $g \in \omega$. We can view $B$ as a tensor product of copies of $A$ indexed by $\Omega$. Define $\varphi: C \to B$ by

$$\varphi\left(\bigotimes_\omega a_\omega\right) = \bigotimes_\omega \varphi_\omega a_\omega.$$

Then $\varphi$ is a well-defined $R$-algebra isomorphism. The problem is that $\varphi$ does not commute with the $G$-actions on $C$ and $B$, respectively.

Let $\sigma$ be the group homomorphism from $\mathrm{S}_\Omega$, the group of permutations of the set $\Omega$, into the centralizer of $G$ in $B^*$ from Lemma 6.2. The action of $G$ on $\Omega$ yields a group homomorphism from $G$ into $\mathrm{S}_\Omega$. Let $\pi: G \to \mathrm{S}_\Omega \to B^*$ be the composition. Thus

$$\pi(g)^{-1}\left(\bigotimes_{\omega} b_\omega\right)\pi(g) = \bigotimes_{\omega} b_{\omega g^{-1}},$$

where $b_{\omega g^{-1}} \in A$ occurs at position $\omega$ in the last tensor. Thus

$$\begin{aligned}\varphi\left(\bigotimes_{\omega} a_\omega\right)^{\pi(g)g} &= \left(\bigotimes_{\omega} \varphi_\omega a_\omega\right)^{\pi(g)g} = \bigotimes_{\omega}\left(\varphi_{\omega g^{-1}}(a_{\omega g^{-1}})\right)^g \\ &= \bigotimes_{\omega}\varphi_\omega\left((a_{\omega g^{-1}})^g\right) = \varphi\left(\bigotimes_{\omega}(a_{\omega g^{-1}})^g\right) \\ &= \varphi\left((\bigotimes_{\omega} a_\omega)^g\right).\end{aligned}$$

Thus $\varphi(c^g) = \varphi(c)^{\pi(g)g}$ for all $c \in C$.

For $b \in B$ and $g \in G$, set $b \star g = b^g\pi(g)$. This defines a new action of $G$ on $B$:

$$(b\star g)\star h = (b^g\pi(g))^h\pi(h) = b^{gh}\pi(g)^h\pi(h) = b^{gh}\pi(gh) = b\star(gh),$$

where we have used that $\pi$ is a group homomorphism and that $h \in G$ centralizes $\pi(g)$. Furthermore, for $r \in R$ we have $(br)\star g = (b\star g)r^g$. Thus $B$ is a right $RG$-module via the star action.

Since $B$ is Azumaya over $R$, the natural homomorphism $B^{\mathrm{op}} \otimes_R B \to \operatorname{End}_R B$ is an isomorphism. Thus the map

$$\varepsilon\colon B^{\mathrm{op}} \otimes_R C \to \operatorname{End}_R B, \quad x(b\otimes c)^\varepsilon = bx\varphi(c) \quad (x \in B,\ b \in B^{\mathrm{op}},\ c \in C),$$

is an isomorphism of $R$-algebras. We claim that $\varepsilon$ is an isomorphism of $G$-algebras, where the $G$-algebra structure of $\operatorname{End}_R B$ is that induced by the $RG$-module structure of $B$ defined by the star action. Namely, we have

$$x(b^g \otimes c^g)^\varepsilon = b^g x\varphi(c^g) = b^g x\varphi(c)^{\pi(g)g}$$

and

$$\begin{aligned}x\left((b\otimes c)^\varepsilon\right)^g = \left((x\star g^{-1})(b\otimes c)^\varepsilon\right)\star g &= \left(bx^{g^{-1}}\pi(g^{-1})\varphi(c)\right)^g\pi(g) \\ &= b^g x\varphi(c)^{\pi(g)g}.\end{aligned}$$

The claim follows. Thus $B^{\mathrm{op}} \otimes_R C$ is isomorphic to the trivial $G$-algebra $\operatorname{End}_R B$. It follows that $B$ and $C$ are equivalent, as was to be shown. □

**6.4. Corollary.** *Let $R$ be a commutative $G$-ring, $G$ a finite group. Let $p$ be a prime and $P$ a Sylow $p$-subgroup of $G$. Then the $p$-torsion part* $\operatorname{BrCliff}(G,R)_p$ *of the Brauer–Clifford group is isomorphic to a subgroup of* $\operatorname{BrCliff}(P,R)$.

**Proof.** The map $\mathrm{Res}^G_P\colon \mathrm{BrCliff}(G,R) \to \mathrm{BrCliff}(P,R)$ is injective when restricted to the $p$-torsion part, since $\mathrm{Cores} \circ \mathrm{Res}$ is, by Theorem 6.3. □

**6.5. Corollary.** *Let $A$ be an Azumaya $G$-algebra over the commutative $G$-ring $R$, where $G$ is finite. Write $[A]$ for the equivalence class of $A$ in $\mathrm{BrCliff}(G,R)$. The following are equivalent:*

(i) $[A] = 1$
(ii) $[\mathrm{Res}^G_P(A)] = 1$ *in* $\mathrm{BrCliff}(P,R)$ *for all Sylow subgroups $P$ of $G$.*
(iii) *For every prime $p$ there is a Sylow $p$-subgroup $P$ such that* $[\mathrm{Res}^G_P(A)] = 1$ *in* $\mathrm{BrCliff}(P,R)$.

The next corollary is of course known, it also follows from the fact that the kernel of $\mathrm{BrCliff}(G,R) \to \mathrm{Br}(R)$ is isomorphic to a cohomology group $H^2(G, \mathcal{C}_R)$ for a certain abelian group $\mathcal{C}_R$ [3, Theorem 4.1].

**6.6. Corollary.** *Let $G$ be a finite group and $R$ a $G$-ring. Then the Brauer–Clifford group* $\mathrm{BrCliff}(G,R)$ *is torsion.*

**Proof.** We have a natural homomorphism $\mathrm{Res}\colon \mathrm{BrCliff}(G,R) \to \mathrm{Br}(R)$. The Brauer group $\mathrm{Br}(R)$ is torsion, as is well known [8,13]. So if $[A]^n = 1$ in $\mathrm{Br}(R)$, then $[A]^{n|G|} = 1$ in $\mathrm{BrCliff}(G,R)$. □

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