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Graphs associated to conjugacy classes of normal subgroups in finite groups

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Let G be a finite group and let N be a normal subgroup of G. We attach to N two graphs $\Gamma_G(N)$ and $\Gamma_G^*(N)$ related to the conjugacy classes of G contained in N and to the set of primes dividing the sizes of these classes, respectively. These graphs are subgraphs of the ordinary ones associated to the conjugacy classes of G, $\Gamma(G)$ and $\Gamma^*(G)$, which have been widely studied by several authors. We prove that the number of connected components of both graphs is at most 2, we determine the best upper bounds for the diameters and characterize the structure of N when these graphs are disconnected.

1. Introduction

Let G be a finite group and let N be a normal subgroup of G. For each element $x \in N$, the G-conjugacy class is $x^G = \{x^g \mid g \in G\}$. We will denote by $\mathrm{Con}_G(N)$ the set of conjugacy classes in G of elements of N. The elements in $\mathrm{Con}_G(N)$ are unions of

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conjugacy classes of N, and it turns out that every G-class size is a multiple of an N-class size. Recent results have showed that the G-class sizes still have a strong influence on the structure of N in spite of the fact that there may exist primes dividing the G-class sizes which, however, do not divide the order of N.

In 1990, E.A. Bertram, M. Herzog and A. Mann introduced in [3] the graph $\Gamma(G)$ associated to the sizes of the ordinary conjugacy classes of G, and later, in [4] the best bound of the diameter of this graph was attained. Our aim is to study the properties of the following subgraph of $\Gamma(G)$ regarding the G-conjugacy classes contained in N and to obtain structural properties of N in the disconnected case.

Definition 1.1. Let G be a finite group and let N be a normal subgroup in G. We define the graph $\Gamma_G(N)$ in the following way: the set of vertices is the set of non-central elements of $\mathrm{Con}_G(N)$, and two vertices x^G and y^G are joined by an edge if and only if $|x^G|$ and $|y^G|$ have a common prime divisor.

Notice that $\Gamma(N)$ is not a subgraph of $\Gamma_G(N)$ because the set of vertices of $\Gamma(N)$ needs not be included within the set of vertices of $\Gamma_G(N)$. Moreover, we remark that although $\Gamma_G(N)$ is subgraph of $\Gamma(G)$, the fact that the number of connected components and the diameter of $\Gamma(G)$ are bounded does not directly imply that the corresponding for $\Gamma_G(N)$ have to be bounded too. However, we show that both numbers, denoted by $n(\Gamma_G(N))$ and $d(\Gamma_G(N))$, are actually bounded. It is easy to check that the bounds in Theorems A and B are the best possible bounds.

Theorem A. *Let G be a finite group and let N be a normal subgroup of G. Then $n(\Gamma_G(N)) \leq 2$.*

We want to remark that there is no relation between the connectivity of $\Gamma_G(N)$ and $\Gamma(N)$. For instance, $\Gamma(N)$ can be disconnected while $\Gamma_G(N)$ is not. We can use the semilinear affine group $\Gamma(p^n)$ for appropriate p and n in order to see this. Recall that if $GF(p^n)$ is the finite field of p^n elements, then the multiplicative group $H = GF(p^n)^*$ is cyclic of order $p^n - 1$ and acts on the elementary abelian (additive) p-group of $GF(p^n)$, say K. This action is Frobenius, so the corresponding semidirect product KH is a Frobenius group with abelian kernel and complement. Moreover, α, defined by $x^\alpha = x^p$ for all $x \in K$, is an automorphism of K of order n in such a way that $H\langle\alpha\rangle \leq \mathrm{Aut}(K)$. Then $\Gamma(p^n)$ is defined as the semidirect product $K(H\langle\alpha\rangle)$. Now, take $n = 2$ and let S be a cyclic subgroup of H of order $s = 3$ (so we are assuming that 3 divides $p^2 - 1$). We have that $N := KS$ is normal in G and is also a Frobenius group with abelian kernel and complement. Hence $\Gamma(N)$ is disconnected by Theorem 2 of [3]. However, there are exactly two non-trivial G-classes in N consisting in the p^2-1 elements of $K \setminus \{1\}$, and the $(|S|-1)|K| = 2p^2$ elements of $NS \setminus K$, respectively. Therefore, $\Gamma_G(N)$ is connected.

Theorem B. *Let G be a finite group and let N be a normal subgroup of G.*

1. *If $n(\Gamma_G(N)) = 1$, then $d(\Gamma_G(N)) \le 3$.*
2. *If $n(\Gamma_G(N)) = 2$, then each connected component is a complete graph.*

We notice that the diameters of $\Gamma_G(N)$ and $\Gamma(N)$ are not either related. For instance, let P be an extraspecial group of order p^3 with $p \neq 2$. If we take $G = P \times S_3$ and $N = P \times A_3$, we have that $\Gamma(N)$ is a complete graph (all nontrivial N-classes have size p) while $\Gamma_G(N)$ has diameter 2, since the nontrivial G-classes of N have size 2, p or $2p$.

In 1995, S. Dolfi introduced in [5] the dual graph $\Gamma^*(G)$ (it was also independently studied in [1]) associated to the primes that divide the sizes of the conjugacy classes of G. In a similar way, we define the following subgraph of $\Gamma^*(G)$.

Definition 1.2. Let G be a finite group and let N be a normal subgroup in G. We define the "dual" graph of $\Gamma_G(N)$, denoted by $\Gamma^*_G(N)$, as follows: the vertices are those primes which divide the size of some class in $\mathrm{Con}_G(N)$, and two vertices p and q are joined by an edge if there exists $C \in \mathrm{Con}_G(N)$ such that pq divides $|C|$.

We also provide the best bounds for the number of components of $\Gamma^*_G(N)$ and for its diameter. We note again that these bounds cannot be obtained from the only fact that $\Gamma^*_G(N)$ is a subgraph of $\Gamma^*(G)$.

Theorem C. *If G is a finite group and $N \trianglelefteq G$, then $n(\Gamma^*_G(N)) \le 2$ and $n(\Gamma^*_G(N)) = n(\Gamma_G(N))$.*

Theorem D. *Let G be a finite group and $N \trianglelefteq G$.*

1. *If $n(\Gamma^*_G(N)) = 1$, then $d(\Gamma^*_G(N)) \le 3$.*
2. *If $n(\Gamma^*_G(N)) = 2$, then each connected component is a complete graph.*

We give a characterization of a normal subgroup N whose graph $\Gamma_G(N)$, or equivalently $\Gamma^*_G(N)$, is disconnected. We recall that a group G is said to be quasi-Frobenius if $G/\mathbf{Z}(G)$ is a Frobenius group. In this case, the inverse image in G of the kernel and complement of $G/\mathbf{Z}(G)$ are called the kernel and complement of G, respectively.

Theorem E. *Let G be a finite group and $N \trianglelefteq G$. If $\Gamma_G(N)$ has two connected components, then either N is quasi-Frobenius with abelian kernel and complement, or $N = P \times A$ where P is a p-group and $A \leqslant \mathbf{Z}(G)$.*

We point out that our proofs of Theorems B, D and E are different and independent of the proofs of the respective theorems concerning $\Gamma(G)$ and $\Gamma^*(G)$. In addition, Theorem E extends Corollary B of [2], which analyzes (with a completely different approach) the particular case in which N has exactly two coprime G-class sizes bigger than 1.

All groups considered will be finite and, if A is a group or a set, $\pi(A)$ denotes the set of primes dividing $|A|$.

2. Number of connected components of $\Gamma_G(N)$ and $\Gamma^*_G(N)$

In this section we prove Theorems A and C in an easy way by using the following lemma, that is basic for our development. The distance in both graphs will be denoted by d.

Lemma 2.1. *Let G be a finite group and $N \trianglelefteq G$. Let $B = b^G$ and $C = c^G$ be non-central elements in $\mathrm{Con}_G(N)$. If $(|B|, |C|) = 1$. Then*

1. $\mathbf{C}_G(b)\mathbf{C}_G(c) = G$.
2. *$BC = CB$ is a non-central element of $\mathrm{Con}_G(N)$ and $|BC|$ divides $|B||C|$.*
3. *Suppose that $d(B, C) \geq 3$ and $|B| < |C|$. Then $|BC| = |C|$ and $CBB^{-1} = C$. Furthermore, $C\langle BB^{-1}\rangle = C$, $\langle BB^{-1}\rangle \subseteq \langle CC^{-1}\rangle$ and $|\langle BB^{-1}\rangle|$ divides $|C|$.*

Proof. For 1, 2 and the first part of 3, it is enough to mimic the proofs of Lemmas 1 and 2 of [3], by taking into account that the product of two classes of $\mathrm{Con}_G(N)$ is contained in N again. The properties $C\langle BB^{-1}\rangle = C$ and $\langle BB^{-1}\rangle \subseteq \langle CC^{-1}\rangle$ are elementary. The fact that $|\langle BB^{-1}\rangle|$ divides $|C|$ follows from the fact that C is a normal subset that can be written as the union of right coclasses of the normal subgroup $\langle BB^{-1}\rangle$. □

Proof of Theorem A. Suppose that $\Gamma_G(N)$ has at least three connected components and take three non-central classes $B = b^G$, $C = c^G$ and $D = d^G$ in $\mathrm{Con}_G(N)$, each of which belongs to a different connected component. Certainly, any two of them have coprime size. Moreover, we can assume without loss of generality that $|B| < |C| < |D|$. By applying Lemma 2.1, we get that $|\langle BB^{-1}\rangle|$ divides both $|D|$ and $|C|$. Then, $(|C|, |D|) > 1$, which is a contradiction. □

Proof of Theorem C. Suppose that $n(\Gamma^*_G(N)) \geq 3$. We take three primes r, s and l each of which belongs to a different connected component, and let B, C and D be elements of $\mathrm{Con}_G(N)$ such that r divides $|B|$, s divides $|C|$ and l divides $|D|$. Without loss of generality we suppose that $|D| < |C| < |B|$. We have $d(B, D) \geq 3$ and $d(B, C) \geq 3$ and by applying Lemma 2.1, we obtain that $|\langle DD^{-1}\rangle|$ divides $|B|$ and $|C|$, but this leads to a contradiction, because $|B|$ and $|C|$ would have a common prime divisor. This proves that $n(\Gamma^*_G(N)) \leq 2$.

Suppose now that $n(\Gamma_G(N)) = 1$ and $n(\Gamma^*_G(N)) = 2$. Let r and s be primes such that each of them belongs to a distinct connected component of $\Gamma^*_G(N)$. Then there exist $B_r, B_s \in \Gamma_G(N)$ such that r divides $|B_r|$ and s divides $|B_s|$. Let us consider the following path in $\Gamma_G(N)$ that joins B_r and B_s, which exists because $n(\Gamma_G(N)) = 1$:

$$B_r \overset{p_1}{\longleftrightarrow} B_1 \overset{p_2}{\longleftrightarrow} B_2 \overset{p_3}{\longleftrightarrow} \cdots \overset{p_s}{\longleftrightarrow} B_s$$

where p_i is a prime dividing $|B_i|$. This provides a contradiction, because r and s are connected in $\Gamma^*_G(N)$ by the following path:

$$r \overset{B_r}{\longleftrightarrow} p_1 \overset{B_1}{\longleftrightarrow} p_2 \overset{B_2}{\longleftrightarrow} \cdots \overset{B_{s-1}}{\longleftrightarrow} p_s \overset{B_s}{\longleftrightarrow} s$$

So, we have proved that $n(\Gamma_G(N)) = 1$ implies that $n(\Gamma^*_G(N)) = 1$. Now, if $n(\Gamma_G(N)) = 2$ and $n(\Gamma^*_G(N)) = 1$ we can get a contradiction by arguing in a similar way. This shows that $n(\Gamma_G(N)) = n(\Gamma^*_G(N))$. $\square$

3. Diameter of $\Gamma_G(N)$

The following two lemmas, one for the disconnected case and the other for the connected case, summarize important structural properties of a normal subgroup N concerning the graph $\Gamma_G(N)$, which will be used for determining the diameters of $\Gamma_G(N)$ and $\Gamma^*_G(N)$. We start with the disconnected case.

Lemma 3.1. *Let G be a finite group and let N be a normal subgroup of G. Suppose that $n(\Gamma_G(N)) = 2$ and let X_1 and X_2 be the connected components of $\Gamma_G(N)$. Let B_0 be a non-central element of $\mathrm{Con}_G(N)$ of maximal size and assume that $B_0 \in X_2$. We define*

$$S = \langle C \mid C \in X_1\rangle \quad \textit{and} \quad T = \langle CC^{-1} \mid C \in X_1\rangle.$$

Then

1. *S is a normal subgroup of G and every element in S, either is central, or its G-conjugacy class is in X_1.*
2. *If C is a G-conjugacy class of N out of S, then $|T|$ divides $|C|$.*
3. *$T = [S, G]$ is normal in G and $T \leq \mathbf{Z}(S)$.*
4. *$\mathbf{Z}(G) \cap N \subseteq S$ and $\pi(S/(\mathbf{Z}(G) \cap N)) \subseteq \pi(T) \subseteq \pi(B_0)$. Moreover, S is abelian.*
5. *Let $b^G = B \in X_1$. Then $\mathbf{C}_G(b)/S$ is a q-group for some prime $q \in \pi(B_0)$.*

Proof. 1. The fact that S is normal in G is elementary. Let $C \in X_2$ and $B \in X_1$. We know that BC is a G-conjugacy class of $\mathrm{Con}_G(N)$ of maximal size between $|B|$ and $|C|$ by Lemma 2.1. Assume that $|BC| = |B|$. By Lemma 2.1 again, it follows that $|\langle CC^{-1}\rangle|$ divides $|B|$ and that $\langle CC^{-1}\rangle \subseteq \langle BB^{-1}\rangle$. On the other hand, $|B_0B| = |B_0|$ again by Lemma 2.1, and also $|\langle BB^{-1}\rangle|$ divides $|B_0|$. From these facts, we deduce that $(|B|, |B_0|) > 1$, which is a contradiction. Thus, $|BC| = |C|$ for all $C \in X_2$ and $B \in X_1$. Furthermore, we have proved that the size of every class in X_1 is less than the size of any class in X_2.

Now, take $C \in X_2$ and let A be the union of all G-conjugacy classes of size $|C|$ in S and assume that $A \neq \emptyset$. By the above paragraph, we have that if $B \in X_1$, then $BA \subseteq A$. Hence, $SA = A$, and consequently, since A is a normal subset, $|S|$ divides $|A|$. This is not possible because $A \subseteq S - \{1\}$. This contradiction shows that $A = \emptyset$, that is, S does not contain any class of size $|C|$. Therefore, since S is normal in G, then S does not contain elements whose classes are in X_2.

2. Let $B \in X_1$. As we have proved in (1), $|B| < |C|$ and then, by Lemma 2.1, we have $C\langle BB^{-1}\rangle = C$, and as a consequence, $CT = C$. Therefore, $|T|$ divides $|C|$, as wanted.

3. By definition, it is clear that $T = [S, G]$ and so, it is a normal subgroup of G. Let us prove that $T \leq \mathbf{Z}(S)$. In fact, if $B = b^G \in X_1$, then $(|T|, |G : \mathbf{C}_G(b)|) = (|T|, |B|) = 1$, because $|T|$ divides every class size in X_2 by (2). Now, since $|T : \mathbf{C}_T(b)|$ divides $(|T|, |B|) = 1$, we deduce that $T = \mathbf{C}_T(b)$. As the classes in X_1 generate S, we conclude that T is central in S.

4. Let $z \in \mathbf{Z}(G) \cap N$ and let $B = b^G \in X_1$. Note that $b^G z = (bz)^G$. Moreover $bz \in N$, because both elements lie in N. As $|(bz)^G| = |Bz| = |B|$, then $bz \in S$ and so $z \in S$. This proves that $\mathbf{Z}(G) \cap N \subseteq S$. Since $T = [S, G]$, then $[S/T, G/T] = 1$ and $S/T \subseteq \mathbf{Z}(G/T)$. In particular, S/T is abelian and as $T \leq \mathbf{Z}(S)$ by (3), then S is nilpotent. We can write $S = R \times Z$ where Z is the largest Hall subgroup of S which is contained in $\mathbf{Z}(G)$. Let p be a prime divisor of $|R|$ and let P be a Sylow p-subgroup of R. It is clear that $P \trianglelefteq G$ and $T = [S, G] = [R, G] \geqslant [P, G] > 1$. Hence p divides $|T|$ and by applying (1) and (2), $|T|$ divides $|B_0|$. Therefore, $\pi(R) \subseteq \pi(T) \subseteq \pi(B_0)$. On the other hand, it is elementary that $\pi(S/(\mathbf{Z}(G) \cap N)) \subseteq \pi(R)$, and the first part of the step is proved. We show now that $R \leq \mathbf{Z}(S)$. In fact, let $b^G = B \in X_1$. Since $(|B|, |B_0|) = 1$, we obtain in particular, $(|B|, |R|) = 1$. Thus, $|R : \mathbf{C}_R(b)| = 1$ since this index trivially divides $|R|$ and $|B|$ because $R \trianglelefteq G$. This means that $R = \mathbf{C}_R(b)$ for every generating element b of S. So, R is contained in $\mathbf{Z}(S)$ as wanted, and S is abelian.

5. By considering the primary decomposition of b, it is clear that we can write $b = b_q b_{q'}$ where b_q and $b_{q'}$ are the q-part and the q'-part of b, where q is a prime such that $b_q \notin \mathbf{Z}(G) \cap N$. Hence, $q \in \pi(B_0)$ by (4). Furthermore, it is elementary that $\mathbf{C}_G(b) \subseteq \mathbf{C}_G(b_q)$, and as a result, $|(b_q)^G|$ divides $|B|$. We claim that any element $xS \in \mathbf{C}_G(b)/S$ is a q-element. For any $x \in \mathbf{C}_G(b)$, write $x = x_q x_{q'}$ (it is possible $x_q = 1$). It is obvious that x_q and $x_{q'}$ belong to $\mathbf{C}_G(b)$. We consider $a = b_q x_{q'}$ and observe that $\mathbf{C}_G(a) = \mathbf{C}_G(b_q) \cap \mathbf{C}_G(x_{q'}) \subseteq \mathbf{C}_G(b_q)$, so $|(b_q)^G|$ divides $|a^G|$. Since $(b_q)^G \in X_1$, this forces that $a^G \in X_1$, and we conclude that $x_{q'} \in S$, that is, xS is a q-element, as wanted. This shows that $\mathbf{C}_G(b)/S$ is a q-group. □

Lemma 3.2. *Let G be a finite group and $N \trianglelefteq G$ with $\Gamma_G(N)$ connected. Let B_0 be a G-conjugacy class of $\mathrm{Con}_G(N)$ of maximal size. Let*

$$M = \langle D \mid D \in \mathrm{Con}_G(N) \text{ and } d(B_0, D) \geq 2\rangle,$$

$$K = \langle D^{-1}D \mid D \in \mathrm{Con}_G(N) \text{ and } d(B_0, D) \geq 2\rangle.$$

Then

1. *M and K are normal subgroups of G. Furthermore, $K = [M, G]$ and $K \leq \mathbf{Z}(M)$.*
2. *$\mathbf{Z}(G) \cap N \subseteq M$ and $\pi(M/(\mathbf{Z}(G) \cap N)) \subseteq \pi(K) \subseteq \pi(B_0)$. Furthermore, M is abelian.*

Proof. 1. By definition, we easily see that M and K are normal subgroups of G and $K = [M, G]$. Let us prove that $K \leq \mathbf{Z}(M)$. If $C = c^G \in \mathrm{Con}_G(N)$ satisfies that $d(B_0, C) \geq 2$, in particular we have $(|B_0|, |C|) = 1$ and then, $|B_0| = |B_0 C|$. Moreover, by Lemma 2.1, $B_0 C C^{-1} = B_0$ and as a result $|K|$ divides $|B_0|$. Therefore, $(|K|, |C|) = 1$. However, we have that $|K : \mathbf{C}_K(c)|$ divides $(|K|, |C|)$ and thus, $K = \mathbf{C}_K(c)$, which implies that $K \leq \mathbf{Z}(M)$.

2. We prove that $\mathbf{Z}(G) \cap N \subseteq M$. Let $z \in \mathbf{Z}(G) \cap N$ and let $C = c^G \in \mathrm{Con}_G(N)$ such that $d(B_0, C) = 2$. Notice that $c^G z = (cz)^G$. As $|(cz)^G| = |c^G|$, then $d(B_0, (cz)^G) = 2$. Thus, $cz \in M$ and $\mathbf{Z}(G) \cap N \subseteq M$. Since $K = [M, G]$, then $M/K \leq \mathbf{Z}(G/K)$ and since $K \leq \mathbf{Z}(M)$ by (1), we obtain that M is nilpotent. We can write $M = R \times Z$ where Z is the largest Hall subgroup of M that is contained in $\mathbf{Z}(G)$. Let q be a prime divisor of $|R|$ and let Q be the Sylow q-subgroup of R. Then $Q \trianglelefteq G$ and $K = [M, G] \geqslant [R, G] \geqslant [Q, G] > 1$. So, q divides $|K|$ and $\pi(R) \subseteq \pi(K)$. In the proof of (1), we have seen that $\pi(K) \subseteq \pi(B_0)$. Then, $\pi(R) \subseteq \pi(B_0)$. Furthermore, it is elementary that $\pi(M/(\mathbf{Z}(G) \cap N)) \subseteq \pi(K)$ and so, the first part of the step is proved. On the other hand, given a generating class $B = b^G$ of M, we know that $d(B, B_0) \geq 2$. In particular, we have $(|B|, |B_0|) = 1$ and hence $(|Q|, |B|) = 1$, where Q is the above Sylow q-subgroup. Since $|Q : \mathbf{C}_Q(b)|$ divides $(|Q|, |B|) = 1$, we have $\mathbf{C}_G(b) = Q$ and $Q \leq \mathbf{Z}(M)$. Thus, $R \leq \mathbf{Z}(M)$ and M is abelian. $\square$

The following consequence, which has interest on its own, is the key to bound the diameter of $\Gamma_G(N)$ in the connected case.

Theorem 3.3. *Let G be a finite group and N a normal subgroup of G and suppose that $\Gamma_G(N)$ is connected. Let B_0 be a non-central conjugacy class of $\mathrm{Con}_G(N)$ with maximal size. Then $d(B, B_0) \leq 2$ for every non-central $B \in \mathrm{Con}_G(N)$.*

Proof. Suppose that the theorem is false. Let $B = b^G \in \mathrm{Con}_G(N)$ such that $d(B_0, B) = 3$ and let

$$B_0 \longleftrightarrow B_1 \longleftrightarrow B_2 \longleftrightarrow B$$

be a shortest chain linking B and B_0 of length 3. By considering the primary decomposition of b, we write $b = b_q b_{q'}$ where b_q and $b_{q'}$ are the q-part and the q'-part of b, and q is a prime such that $b_q \notin \mathbf{Z}(G) \cap N$. Hence, $q \in \pi(B_0)$ by Lemma 3.2(2). Also, the fact that $\mathbf{C}_G(b) \subseteq \mathbf{C}_G(b_q)$ implies that $|(b_q)^G|$ divides $|B|$ and then any class which is connected to $(b_q)^G$ must be connected to B. This means that $d((b_q)^G, B_0)) \geq 3$.

Let M be a subgroup defined in Lemma 3.2. We claim that any element $x \in \mathbf{C}_G(b_q) \setminus M$ satisfies that xM is a q-element. Write $x = x_q x_{q'}$ and suppose that $x_{q'} \notin M$. Set $a = x_{q'} b_q$ and notice that $a \notin M$. By the definition of M, we have $d(a^G, B_0) \leq 1$ and since $\mathbf{C}_G(a) = \mathbf{C}_G(x_{q'}) \cap \mathbf{C}_G(b_q)$, it follows that $|(b_q)^G|$ divides $|a^G|$. These facts show that $d((b_q)^G, B_0)) \leq 2$, a contradiction. Therefore, $x_{q'} \in M$ and xM is a q-element. In conclusion, $\mathbf{C}_G(b_q)/M$ is a q-group. Now, observe that $|B_2|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |G : \mathbf{C}_G(b_q)||\mathbf{C}_G(b_q) : M||M : \mathbf{Z}(G) \cap N|.$$

Also, we know by Lemma 3.2(2) that $\pi(M/\mathbf{Z}(G) \cap N) \subseteq \pi(B_0)$ and we have seen in the above paragraph that $\mathbf{C}_G(b_q)/M$ is a q-group for some $q \in \pi(B_0)$. Consequently, $|B_2|$ must divide $|(b_q)^G|$ (which divides $|B|$), because $(|B_2|, |B_0|) = 1$. This is a contradiction, since B_1 and B would be joined by an edge. $\square$

Proof of Theorem B. 1. Suppose that D_1 and D_2 are classes of $\mathrm{Con}_G(N)$ such that $d(D_1, D_2) = 4$. Let B_0 be a class of maximal size in $\mathrm{Con}_G(N)$. By Theorem 3.3 we know that $d(B_0, D_i) \leq 2$ for $i = 1, 2$. We can suppose then that $d(B_0, D_i) = 2$ for $i = 1, 2$. Furthermore, without loss of generality, $|D_1| > |D_2|$. Then, by Lemma 2.1 it is true that $|\langle D_2 D_2^{-1} \rangle|$ divides $|D_1|$. In addition, $B_0 D_2$ is a conjugacy class of $\mathrm{Con}_G(N)$ and $|B_0 D_2| = |B_0|$ by Lemma 2.1(2) and by the maximality of B_0. It follows that $B_0 D_2 D_2^{-1} = B_0$ and $|\langle D_2 D_2^{-1} \rangle|$ divides $|B_0|$. Therefore, B_0 and D_1 are joined by an edge, which is a contradiction. This proves that $d(\Gamma_G(N)) \leq 3$.

2. Let $B_1 = b_1^G$ and $B_2 = b_2^G$ in X_1. Notice that $b_1, b_2 \in S$ where S is the subgroup defined in Lemma 3.1. By applying the properties of that result, we know that $|b_2^G|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |G : \mathbf{C}_G(b_1)||\mathbf{C}_G(b_1) : S||S : \mathbf{Z}(G) \cap N|$$

where the primes dividing $|\mathbf{C}_G(b_1) : S|$ and $|S : \mathbf{Z}(G) \cap N|$ are in $\pi(B_0)$. So, we have that $|b_2^G|$ divides $|b_1^G|$. By arguing symmetrically we also get that $|b_1^G|$ divides $|b_2^G|$, so we conclude that all classes in X_1 have the same size. Hence, X_1 is a complete graph. Now, we prove that X_2 is also a complete graph. It is enough to consider again S and T defined in Lemma 3.1 and observe that every $C \in X_2$ is out of S and that $|T|$ divides $|C|$ by Lemma 3.1(1) and (2). $\square$

Remark 3.4. In the proof of Theorem B(2), we have seen that all G-classes of N lying in the connected component X_1 (the component which does not contain the classes of maximal size) must have the same size. Moreover, in the proof of Lemma 3.1(1) we have seen that this size is less than the size of every class in X_2.

4. Diameter of $\Gamma_G^*(N)$

Proof of Theorem D. 1. Suppose that there exist two primes r and s in $\Gamma_G^*(N)$ such that $d(r,s) = 4$ and we will get a contradiction. This means that the primes r and s are connected by a path of length 4, say

$$r \xleftrightarrow{B_1} p_1 \xleftrightarrow{B_2} p_2 \xleftrightarrow{B_3} p_3 \xleftrightarrow{B_4} s$$

where $B_i \in \mathrm{Con}_G(N)$ for $i = 1, \dots, 4$ and $p_i \in \Gamma_G^*(N)$ for $i = 1, 2, 3$. By Theorem 3.3 we know that $d(B_i, B_0) \leqslant 2$ for $i = 1, \dots, 4$ where B_0 is a non-central G-conjugacy class of maximal size. Notice that $d(B_1, B_4) = 3$ and we distinguish only two possibilities:

Case 1. $d(B_0, B_1) = 2 = d(B_0, B_4)$. By symmetry, we can assume for instance that $|B_1| > |B_4|$. Since $d(B_1, B_4) = 3$, by Lemma 2.1 we have that $|\langle B_4 B_4^{-1}\rangle|$ divides $|B_1|$. Moreover, $B_0 B_4$ is an element of $\mathrm{Con}_G(N)$ such that $|B_0 B_4| = |B_0|$ and by Lemma 2.1, $|\langle B_4 B_4^{-1}\rangle|$ divides $|B_0|$. Therefore, $d(B_0, B_1) = 1$, because their cardinalities have a prime common divisor. This is a contradiction.

Case 2. Either $d(B_0, B_1) = 2$ and $d(B_0, B_4) = 1$, or $d(B_0, B_1) = 1$ and $d(B_0, B_4) = 2$. Without loss we assume for instance the latter case. Let us consider the subgroup M defined in Lemma 3.2 and let $B_4 = b^G$. Since $d(B_0, B_4) = 2$, then $b \in M$ by definition. Moreover, $|B_1|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |G : \mathbf{C}_G(b)||\mathbf{C}_G(b) : M||M : \mathbf{Z}(G) \cap N|.$$

Now, notice that $r \notin \pi(B_0)$, otherwise it yields $d(r, s) \leq 3$, a contradiction, and trivially $r \notin \pi(B_4)$. Also, $\pi(M/\mathbf{Z}(G) \cap N) \subseteq \pi(B_0)$, so we have that r (which divides $|B_1|$) must divide $|\mathbf{C}_G(b) : M|$. Therefore, there exists an r-element $y \in \mathbf{C}_G(b) \setminus M$. On the other hand, $b \in M$, and by Lemma 3.2(2), the r-part of b is central in G, that is, we can assume that b is an r'-element, by replacing b by its r'-part. As y and b have coprime orders, we have

$$\mathbf{C}_G(yb) = \mathbf{C}_G(y) \cap \mathbf{C}_G(b) \subseteq \mathbf{C}_G(b).$$

Consequently, $|B_4|$ divides $|(yb)^G|$. Furthermore, since $yb \notin M$, by the definition of M we have $d((yb)^G, B_0) \leq 1$. As $d(B_0, B_1) = 1$ by hypothesis, we deduce that $d(B_1, (yb)^G) \leq 2$. Now, s divides $|B_1|$ and r divides $|(yb)^G|$, and this forces that $d(r, s) \leq 3$, which is a contradiction.

2. Let X_1 and X_2 be the connected components of $\Gamma_G(N)$ where X_2 is the component that contains the G-conjugacy class with the largest size. Let us prove first that X_1^*, X_2^* are the connected components of $\Gamma_G^*(N)$, where $X_i^* = \{p \in \pi(B) \mid B \in X_i\}$, and secondly, that X_1^* and X_2^* are complete graphs.

Let X be a connected component of $\Gamma_G(N)$ and let $r, s \in X^*$. Then there exist B_r, B_s such that r divides $|B_r|$ and s divides $|B_s|$. Let us consider one of the paths in $\Gamma_G(N)$ that joins B_r and B_s:

$$B_r \xleftrightarrow{p_1} B_1 \xleftrightarrow{p_2} B_2 \xleftrightarrow{p_3} \cdots \xleftrightarrow{p_s} B_s$$

where $B_i \in \mathrm{Con}_G(N)$ for $i = 1, \dots, s-1$ and $p_i \in \Gamma_G^*(N)$ for $i = 1, \dots, s$. So, r and s are connected in $\Gamma_G^*(N)$ in the following way:

$$r \xleftrightarrow{B_r} p_1 \xleftrightarrow{B_1} p_2 \xleftrightarrow{B_2} \cdots \xleftrightarrow{B_{s-1}} p_s \xleftrightarrow{B_s} s$$

Therefore, X^* is contained in a connected component Y of $\Gamma_G^*(N)$. Now, we take $q \in Y$, which is connected by an edge to some $r \in X^*$. Then there exists $B \in \mathrm{Con}_G(N)$ such that qr divides $|B|$. It follows that $B \in X$ and $q \in X^*$. Thus, $X^* = Y$ and X^* is a connected component of $\Gamma_G^*(N)$ as wanted.

By Remark 3.4, all classes in X_1 have the same size, which trivially implies that X_1^* is a complete graph. Let us show that X_2^* is a complete graph too. Suppose that B_0 is a conjugacy class with maximal size, which lies in X_2, and let $B_1 = b_1^G \in X_1$. Then, the subgroup S defined in Lemma 3.1 is abelian, and $S \subseteq \mathbf{C}_G(b_1)$. Now, if $p \in X_2^*$, there exists $D \in X_2$ such that p divides $|D|$. Notice that $|D|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |B_1||\mathbf{C}_G(b_1) : S||S : \mathbf{Z}(G) \cap N|,$$

and by Lemma 3.1(4) and (5), we know that $|\mathbf{C}_G(b_1) : S|$ is a q-power with $q \in \pi(B_0)$ and $\pi(S/(\mathbf{Z}(G) \cap N)) \subseteq \pi(B_0)$. It follows that $\pi(D) \subseteq \pi(B_0)$. Therefore, all primes in X_2^* are in $\pi(B_0)$ and so, X_2^* trivially is a complete graph. $\square$

5. Structure of N in the disconnected case

Proof of Theorem E. Suppose that X_1 and X_2 are the two connected components of $\Gamma_G(N)$, where X_2 is the one containing the classes of maximal size. Let S be the subgroup defined in Lemma 3.1.

Step 1: If $S \leq \mathbf{Z}(N)$, then $N = P \times A$ with $A \leq \mathbf{Z}(G) \cap N$ and P a p-group.

We can choose a p-element x and a q-element y of N, for some primes p and q, such that $x^G \in X_1$ and $y^G \in X_2$. If $p = q$ for every election of x and y, it is clear that $N = P \times A$ with $A \leq \mathbf{Z}(G) \cap N$. Assume then that $p \neq q$. Since $x \in S \leq \mathbf{Z}(N)$, we obtain $N/S = \mathbf{C}_N(x)/S$ and this group has prime power order by Lemma 3.1(5). As a consequence, N is nilpotent and $[x, y] = 1$. As x and y have coprime order, $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ and so, $|(xy)^G|$ divides $|x^G|$ and $|y^G|$, a contradiction. This proves the step.

Notice that we can assume that $S\mathbf{Z}(N) < N$, because if $S\mathbf{Z}(N) = N$ then N is abelian and $S \leq \mathbf{Z}(N)$. For the rest of the proof, we assume $\mathbf{Z}(N) < S\mathbf{Z}(N) < N$ and we will prove that N is quasi-Frobenius with abelian kernel and complement. We divide the proof into several steps. Let us denote $\pi = \{p \text{ prime } \mid p \text{ divides } |B| \text{ with } B \in X_1\}$.

Step 2: N has a normal π-complement and abelian Hall π-subgroups.

Let us prove that N is p-nilpotent and has abelian Sylow p-subgroups for every $p \in \pi$. Let $a \in N \setminus S$, then obviously $a^G \in X_2$ and $|a^G|$ is a π'-number. If we take $P \in \mathrm{Syl}_p(N)$, then there exists $g \in N$ such that $P^g \subseteq \mathbf{C}_N(a)$, that is, $a \in \mathbf{C}_N(P^g) = \mathbf{C}_N(P)^g$. Thus, we can write

$$N = S \cup \bigcup_{g \in N} \mathbf{C}_N(P)^g$$

and, by counting elements, it follows that

$$|N| \leq (|S| - 1) + |N : \mathbf{N}_N(\mathbf{C}_N(P))|(|\mathbf{C}_N(P)| - 1) + 1.$$

Hence

$$1 \leq \frac{|S|}{|N|} + \frac{|\mathbf{C}_N(P)|}{|\mathbf{N}_N(\mathbf{C}_N(P))|} - \frac{1}{|\mathbf{N}_N(\mathbf{C}_N(P))|}.$$

However, if $\mathbf{C}_N(P) < \mathbf{N}_N(\mathbf{C}_N(P))$, as we are assuming that $S < N$, we have

$$1 \leq \frac{1}{2} + \frac{1}{2} - \frac{1}{|\mathbf{N}_N(\mathbf{C}_N(P))|},$$

which is a contradiction. This implies that $\mathbf{C}_N(P) = \mathbf{N}_N(\mathbf{C}_N(P))$, and in particular,

$$P \leq \mathbf{N}_N(P) \leq \mathbf{N}_N(\mathbf{C}_N(P)) \leq \mathbf{C}_N(P),$$

so $\mathbf{C}_N(P) = \mathbf{N}_N(P)$ and P is abelian. By Burnside's p-nilpotency criterion (see for instance 17.9 of [7]), we get that N is p-nilpotent for every $p \in \pi$ and so, N has normal π-complement. In particular, N is π-separable and there exists a Hall π-subgroup H of N. By reasoning with H similarly as with P, we obtain $\mathbf{C}_N(H) = \mathbf{N}_N(H)$ and so, H is abelian too. The step is finished.

Let $K/\mathbf{Z}(N)$ be the normal π-complement of $N/\mathbf{Z}(N)$. By applying Lemma 3.1(4), we get that $S\mathbf{Z}(N)/\mathbf{Z}(N)$ is a normal π'-subgroup of $N/\mathbf{Z}(N)$, so $S \leq K$.

Step 3: $K = \mathbf{C}_N(x)$ for every $x \in S \setminus \mathbf{Z}(G) \cap N$ and $S \leq \mathbf{Z}(K)$.

Let $x \in S \setminus \mathbf{Z}(G) \cap N$. Then $x^G \in X_1$ by Lemma 3.1(1) and $\mathbf{C}_G(x)/S$ is a π'-group by Lemma 3.1(5). Since $|x^N|$ is a π-number, we obtain that $\mathbf{C}_N(x)/\mathbf{Z}(N)$ is a Hall

π'-subgroup of $N/\mathbf{Z}(N)$. Thus, $K = \mathbf{C}_N(x)$ for every $x \in S \setminus \mathbf{Z}(G) \cap N$ and, in particular, $S \leq \mathbf{Z}(K)$.

Step 4: $K = S$.

Let H be an abelian Hall π-subgroup of N. We have seen in the proof of Step 2 that

$$N = S \cup \bigcup_{g \in N} \mathbf{C}_N(H)^g,$$

which trivially implies that

$$N = \bigcup_{g \in N} S\mathbf{C}_N(H)^g.$$

This forces that $N = \mathbf{C}_N(H)S$ and consequently, $HS \trianglelefteq N$. Suppose that $S < K$ and we will get a contradiction. Let $a \in K \setminus S$, then $a^G \in X_2$ by Lemma 3.1(1), so $|a^G|$ is a π'-number and as a result, $a \in \mathbf{C}_K(H^g) = \mathbf{C}_K(H)^g$ for some $g \in N$. Moreover, $S \leq \mathbf{Z}(K)$ by Step 3, so we have the following equalities

$$\mathbf{C}_K(H^g) = \mathbf{C}_K(H^g S) = \mathbf{C}_K(HS) = \mathbf{C}_K(H).$$

Thus, $a \in \mathbf{C}_K(H)$ for every $a \in K \setminus S$ and we conclude that $K = \langle K \setminus S \rangle \subseteq \mathbf{C}_N(H)$. As H is abelian and $N = HK$, we have $H \leq \mathbf{Z}(N) \leq K$. This implies that $N = K$ and then, $S \leq \mathbf{Z}(N)$ by Step 3, which contradicts the assumption made after Step 1.

Step 5: N is quasi-Frobenius with abelian kernel and complement.

Let $\overline{N} = N/\mathbf{Z}(N)$ and let $\overline{K} = K/\mathbf{Z}(N)$. If $\overline{K} = \mathbf{C}_{\overline{N}}(\bar{x})$ for all $\bar{x} \in \overline{K} \setminus \{1\}$, this is equivalent to the fact that N is quasi-Frobenius with abelian kernel K and abelian complement H. Suppose then that $\overline{K} < \mathbf{C}_{\overline{N}}(\bar{x})$ for some $\bar{x} \in \overline{K} \setminus \{1\}$. Also, we can suppose that $o(\bar{x})$ is an r-number for some prime $r \in \pi'$. Now, let $\bar{y} \in \mathbf{C}_{\overline{N}}(\bar{x}) \setminus \overline{K}$ such that $o(\bar{y})$ is a q-number for some $q \in \pi$. We can suppose without loss of generality that $o(y)$ is a q-number. Notice that $[y, x] \in \mathbf{Z}(N)$ because $\bar{y} \in \mathbf{C}_{\overline{N}}(\bar{x})$ and, since $(o(x), o(y)) = 1$, it easily follows that $[x, y] = 1$. Then $y \in \mathbf{C}_N(x) = K$ and $\bar{y} \in \overline{K}$, which is a contradiction. $\square$

Example 5.1. We show that the converse of the above theorem is false. It is known that the special linear group $H = SL(2, 5)$ acts Frobeniusly on $K \cong \mathbb{Z}_{11} \times \mathbb{Z}_{11}$. As a consequence, the action of any subgroup of H on K is also Frobenius. We consider, in particular, a Sylow 5-subgroup P of H and $\mathbf{N}_H(P)$ acting Frobeniusly on K. We define the semidirect product $N := KP$, which is trivially a normal subgroup of $G := K\mathbf{N}_H(P)$. Thus, N is a Frobenius group with abelian kernel and complement. In fact, N decomposes into the following disjoint union

$$N = \{1\} \cup (K \setminus \{1\}) \cup (\bigcup_{k \in K} P^k \setminus \{1\}),$$

and $K \setminus \{1\}$ is partitioned into N-classes of cardinality 5, whereas the elements of $\bigcup_{k\in K}(P^k \setminus \{1\})$ are decomposed into N-classes of cardinality 121. Therefore, the set of class sizes of N is $\{1, 5, 121\}$. Now, let us compute the G-class sizes of N. As G is a Frobenius group with kernel K and complement $\mathbf{N}_H(P)$, it follows that K is decomposed exactly into the trivial class and G-classes of size $|\mathbf{N}_H(P)| = 20$. That is, the N-classes contained in $K \setminus \{1\}$ are grouped 4 by 4 to form G-classes. And on the other hand, the four N-conjugacy classes contained in $\bigcup_{k\in K} P^k \setminus \{1\}$ of size 121, are grouped in pairs and become two G-conjugacy classes of size 121×2. Then the set of G-class sizes of N is $\{1, 20, 242\}$ and $\Gamma_G(N)$ is a connected graph.

Example 5.2. The following example shows that the case in which N is a p-group in Theorem E actually occurs. Let G be the group of the library of the small groups of GAP [6] with number $\mathrm{Id}(324, 8)$ and with the presentation

$$\langle x, y, z \mid x^3 = y^4 = z^9 = 1, [x, y] = 1, z^y = z^{-1}, z^2 = xzxzx = x^{-1}zx^{-1}zx^{-1}\rangle.$$

By using GAP, one can check that G has an abelian normal subgroup $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and the set of G-class sizes of N is $\{1, 2, 3\}$, so $\Gamma_G(N)$ is disconnected.

Open question. The referee proposed us the following question: whether it is possible to obtain any information on the structure of G from the graph $\Gamma_G(N)$ or not. We believe that in general $\Gamma_G(N)$ may provide few information of G, although possibly one could get further information on the action of G on N. In fact, $G/\mathbf{C}_G(N)$ is always immerged in $\mathrm{Aut}(N)$. For giving an easy example, we consider the case in which $\Gamma_G(N)$ is just one vertex, as it happens with $G = S_3$ and $N = A_3$. Now, take N any p-elementary abelian group of order p^s and let us consider the action of $G = \mathrm{Hol}(N)$ on N. As a result of the fact that $\mathrm{Aut}(N)$ acts transitively on $N \setminus \{1\}$, it follows that $\Gamma_G(N)$ consists only in one vertex, whilst $\mathrm{Aut}(N) \cong \mathrm{GL}(s, p)$ and so G might have a extremely more complex structure.

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