Journal of Algebra 472 (2017) 546–572

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Computing resolutions of quotient singularities ☆

Maria Donten-Bury [a,b], Simon Keicher [c,*]

[a] *University of Warsaw, Institute of Mathematics, Banacha 2, 02-097 Warszawa, Poland*
[b] *Institut für Mathematik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany*
[c] *Departamento de Matematica, Facultad de Ciencias Fisicas y Matematicas, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

ARTICLE INFO

Article history:
Received 4 April 2016
Available online 10 November 2016
Communicated by Seth Sullivant

MSC:
14E15
14Q10
14Q15
14C20
14L24

Keywords:
Cox ring
Resolution of singularities
Quotient singularity
Mori dream space

ABSTRACT

Let $G \subseteq \mathrm{GL}(n)$ be a finite group without pseudo-reflections. We present an algorithm to compute and verify a candidate for the Cox ring of a resolution $X \to \mathbb{C}^n/G$, which is based just on the geometry of the singularity $\mathbb{C}^n/G$, without further knowledge of its resolutions. We explain the use of our implementation of the algorithms in `Singular`. As an application, we determine the Cox rings of resolutions $X \to \mathbb{C}^3/G$ for all $G \subseteq \mathrm{GL}(3)$ with the aforementioned property and of order $|G| \leq 12$. We also provide examples in dimension 4.

☆ This work was completed while the first author held a Dahlem Research School Postdoctoral Fellowship at Freie Universität Berlin. The first author was partially supported by a Polish National Science Centre project 2013/11/D/ST1/02580 and the second author was supported by Proyecto FONDECYT Postdoctorado N. 3160016.

* Corresponding author.
E-mail addresses: m.donten@mimuw.edu.pl (M. Donten-Bury), keicher@mail.mathematik.uni-tuebingen.de (S. Keicher).

http://dx.doi.org/10.1016/j.jalgebra.2016.10.042
0021-8693/

1. Introduction

We consider quotient spaces $\mathbb{C}^n/G = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]^G$, where G is a finite group acting linearly on $\mathbb{C}^n$. By Noether's theorem the ring of invariants of G is finitely generated, hence such quotients are (complex) algebraic varieties. They are usually singular and in such a case are called *quotient singularities*. In their construction, geometry meets finite group theory and there have been many attempts of extending this relation to resolutions of $\mathbb{C}^n/G$. An example, probably the most important one, of describing the geometric structure of crepant (which in this case means that the canonical divisor is linearly trivial) resolutions of $\mathbb{C}^n/G$ in terms of algebraic properties of the group G is the McKay correspondence; for an introduction see, e.g., [24]. Though proved in several cases (see section 4.2), in general it reveals how much there is still to learn about this class of singularities and their resolutions.

In this paper, we study quotient singularities $X_0 := \mathbb{C}^n/G$ and their resolutions in terms of the *Cox ring*

$$\mathcal{R}(X_0) \;=\; \bigoplus_{\mathrm{Cl}(X)} \Gamma(X_0, \mathcal{O}(D)),$$

see [1] for details on the construction. Cox rings have already been successfully used by various authors to study resolutions of quotient singularities, in particular for symplectic quotients in [12,10], described by a generating set in a simpler ring, and via an algorithmic approach based on toric ambient modifications in [15]. In this article we generalize the methods used in [15]. While finishing a draft of this paper, we also found out about Yamagishi's work [28], which extends [12,10]. However, our approach is different from [28] and our methods are not restricted to the class of crepant resolutions. In particular, as explained below, we do not try to construct the generators of the Cox ring directly from the group structure data, but we obtain it via toric ambient modifications, chosen in an intermediate step based on tropical geometry.

Our first contribution, presented in Section 2, is an algorithm to compute and verify a candidate for the Cox ring of a resolution $X \to X_0$ without requiring further knowledge of X_0. More precisely, Algorithm 2.5 computes $\mathcal{R}(X_0)$ and Algorithm 2.8 then computes a candidate for the Cox ring of a resolution $X \to X_0$ and verifies the result. Here, the main tool are toric ambient modifications as in [14,16]: we embed X_0 into an affine toric variety Z_0 and compute a resolution $Z \to Z_0$. The proper transform $X \to X_0$ then is the desired candidate for a resolution which we can verify algorithmically. Note that the choice of the toric resolution involves a tropical step; this is in the spirit of Tevelev and Teissier [25,26]. Algorithm 2.8 is a variant of the algorithm [22] given for Mori dream spaces, which, in turn, is based on [1,18] where the algorithm has been shown to work in the setting of complete rational complexity one T-varieties. See also [17] for the case of affine $\mathbb{C}^*$-surfaces. Our algorithms are implemented in a library for the Open Source computer algebra system `Singular` [8]; we explain its use in Section 3 by examples.

Our second contribution concerns resolutions of three-dimensional quotient singularities; it is presented in Section 4. We first classify in Proposition 4.2 the finite, non-abelian groups $G \subseteq \mathrm{GL}(3)$ with $|G| \leq 12$ and without pseudo-reflections (see Remark 2.2). When G is abelian, the quotient singularity is toric; since the algorithm of constructing a (toric) resolution and the structure of the Cox ring is known in this case, we do not consider it. Using our algorithms from Section 2, we first present the Cox rings of all singularities $\mathbb{C}^3/G$ on the list, see Proposition 4.3 and then their resolutions, see Theorem 4.5. Then we discuss certain properties of the obtained resolutions in order to understand what can be expected from the output of the algorithm in general. In particular, we check whether the resolutions for subgroups of SL(3) on the list are crepant. In Subsection 4.3, we then give a modified algorithm to produce smaller resolutions. We apply it to obtain two more crepant resolutions.

Finally, in Section 5, we apply our methods to two four-dimensional examples. Note that in dimension 4 less is known about resolutions of quotient singularities, hence computational experiments are even more valuable than in dimension 3. We provide the Cox rings of $X_0 = \mathbb{C}^4/G$ for chosen representations of $G = S_3$ and $G = D_8$ and compute the Cox rings of modifications $X \to X_0$. We are able to retrieve crepant resolutions that were also found in [10], see Proposition 5.3 and 5.4.

2. Algorithmic resolution

In this section, we describe the algorithm to compute the Cox ring of a resolution of a quotient singularity $X_0 := \mathbb{C}^n/G$ where G is a finite group. It is divided into two parts: the first one, Algorithm 2.5, returns a presentation of the Cox ring of X_0 and the second one, Algorithm 2.8, computes the Cox ring of a candidate for a resolution $X \to X_0$ and then verifies the choice of X. The latter algorithm is a variant of [1, Thm 3.4.4.9] and [22, Alg. 2.4.8]. It produces such a candidate X using a tropical step.

2.1. The setting and Cox rings

We work with representations of G, usually assuming that they are faithful. In this case we will often identify the representation with its image in $\mathrm{GL}(n)$, i.e., consider G as a matrix group.

Definition 2.1. A matrix group $G \subseteq \mathrm{GL}(n)$ is *small* if it does not have *pseudo-reflections*; these are $A \in G \subseteq \mathrm{GL}(n)$ of finite order such that the subspace of fixed points $(\mathbb{C}^n)^A$ is a hyperplane. We also say that a representation of an abstract group G is small, if its image in $\mathrm{GL}(n)$ is small.

Remark 2.2. Note that the assumption on the representation of G not having pseudo-reflections is not restrictive: let $H \subseteq G$ be the normal subgroup generated by all pseudo-reflections. Then

$$\mathbb{C}^n/G \;\cong\; (\mathbb{C}^n/H)/(G/H) \;\cong\; \mathbb{C}^n/(G/H)$$

since $\mathbb{C}^n/H$ is smooth by the Chevalley–Shephard–Todd theorem. In particular, the singularity for G is the same as the one for the smaller group G/H.

As outlined in the introduction, our central tool for the study of both X_0 and its resolutions is the Cox ring. In this paper, we are particularly interested in the case where the Cox ring and the class group are finitely generated; one then calls the underlying variety a *Mori dream space*. Given such a Mori dream space X, the Cox ring $\mathcal{R}(X)$ together with a choice of a collection of overlapping polyhedral cones in $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ determines X up to isomorphism. If X is a surface, $\mathcal{R}(X)$ even fully encodes X. The following construction summarizes this situation and shows how X can be retrieved as a quotient of an open subset of $\mathrm{Spec}(\mathcal{R}(X))$. We refer to the book [1, Sect. 3] for details.

To this end, recall that a variety X is an *A_2-variety* if each two points admit a common affine open neighborhood. Examples include the class of quasiprojective varieties. We call X an *A_2-maximal* variety if X is an A_2-variety and there is no open embedding $X \subsetneq Y$ into an A_2-variety Y and $\mathrm{codim}_Y(Y \setminus X) \geq 2$.

Construction 2.3. (See [1, Sect. 3I] and [14].) Let X be a normal, irreducible A_2-maximal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$ such that both $K := \mathrm{Cl}(X)$ and $R := \mathcal{R}(X)$ are finitely generated. Then X is the good quotient of an open subset $\widehat{X}$ of $\mathrm{Spec}(R)$

$$X \xleftarrow[/\!/ H]{} \widehat{X} \;\subseteq\; \overline{X} \;:=\; \mathrm{Spec}(R) \;\subseteq\; \mathbb{C}^r$$

by the action of the characteristic quasitorus $H := \mathrm{Spec}(\mathbb{C}[K])$, and $\widehat{X}$ can be described combinatorially by a set Φ of pairwise overlapping polyhedral cones in $K \otimes_{\mathbb{Z}} \mathbb{Q}$. The triple $(R, \mathfrak{F}, \Phi)$, where $\mathfrak{F}$ is a generating set of R, is called a *bunched ring* and already determines X up to isomorphism.

Moreover, each variety X as in Construction 2.3 comes with an embedding into the so-called *canonical toric ambient variety*; this is a toric variety $Z \supseteq X$ with a fan $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the embedded variety X inherits several nice properties from Z; we refer to [1, Subsect. 3.2.5] for details as well as its explicit computation from the Cox ring of X.

2.2. Cox ring of the quotient singularity

We start from explaining how to compute a presentation $\mathcal{R}(X_0) = \mathbb{C}[T_1, \ldots, T_s]/I_0$ of the Cox ring of $X_0 = \mathbb{C}^n/G$, which is suitable for being an input data for computing the Cox ring of a resolution of X_0. The algorithm generalizes [15, Prop. 3.1]. First note that, by [2, Thm 3.1], $\mathcal{R}(X_0)$ is isomorphic to the ring of invariants $\mathbb{C}[S_1, \ldots, S_n]^{[G,G]}$ that comes with a grading by the (finite) group $\mathrm{Cl}(X_0) = \mathbb{X}(G')$ of characters of the

abelianization $G' := G/[G,G]$. On the geometric side, the situation becomes (cf. [15, Prop. 3.1]):

$$\begin{array}{ccc} \mathbb{C}^n/[G,G] & \subseteq & \mathbb{C}^s \\ \big\downarrow{\scriptstyle /G'} & & {\scriptstyle /G'}\big\downarrow \\ X_0 & \subseteq & \mathbb{C}^s/G' \end{array}$$

To obtain a presentation suitable for further steps, we need to find a set of G'-homogeneous generators $g_1, \ldots, g_s$ of $\mathbb{C}[S_1, \ldots, S_n]^{[G,G]}$. Thus we first compute any minimal set of generators and then turn it into a set of G'-homogeneous ones. This is done separately in the sets of polynomials of a fixed (standard) degree as follows.

Construction 2.4 *(G'-homogeneous generators for $\mathbb{C}[S_1, \ldots, S_n]^{[G,G]}$).* Consider generators $\mathcal{F} = \{f_1, \ldots, f_s\}$ for $\mathbb{C}[S_1, ..., S_n]^{[G,G]}$. Let $d \in \mathbb{Z}_{\geq 0}$ be such that there is $f_i \in \mathcal{F}$ of degree d. Let V_d denote the vector space of all polynomials of degree d in $\mathbb{C}[S_1, \ldots, S_n]^{[G,G]}$. Denote by $h_{d,1}, \ldots, h_{d,s_d}$ a basis of V_d chosen such that its elements either belong to $\mathcal{F}$ or are products of elements of $\mathcal{F}$ of smaller degree.

Observe that we have an induced action of G' on V_d. Thus, there is a homomorphism (often, but not always, an injection)

$$G' \ \to \ \mathrm{GL}(s_d), \qquad G' \ni [m] \ \mapsto \ M_{d,m},$$

where $M_{d,m} \in \mathrm{GL}(s_d)$ of the automorphism of V_d given by $m \in G \subseteq \mathrm{GL}(n)$ in basis $h_{d,1}, \ldots, h_{d,s_d}$. We describe the image of G' in $\mathrm{GL}(s_d)$ by computing a generating set $\langle M_{d,1}, \ldots, M_{d,k} \rangle \subseteq \mathrm{GL}(s_d)$, which is an image of a generating set of G. Moreover, since G' is abelian, we can find a basis $\overline{h}_{d,1}, \ldots, \overline{h}_{d,s_d}$ of V_d in which all $M_{d,i}$ are simultaneously diagonalized.

Going through the possible values of $d \in \mathbb{Z}_{\geq 0}$, we collect the bases $\overline{h}_{d,1}, \ldots, \overline{h}_{d,s_d}$ and remove redundant elements; the result is a set of G'-homogeneous polynomials $g_1, \ldots, g_s$ which generate $\mathbb{C}[S_1, \ldots, S_n]^{[G,G]}$. We will write χ_{ij} for the eigenvalue of g_i with respect to $M_{d_i,j}$, where $d_i = \deg(g_i)$.

Let $g_1, \ldots, g_s$ and χ_{ij} be as in Construction 2.4. We then obtain the desired presentation $\mathcal{R}(X_0) = \mathbb{C}[T_1, \ldots, T_s]/I_0$ by computing the kernel $I_0 \subseteq \mathbb{C}[T_1, \ldots, T_s]$ of the homomorphism

$$\mathbb{C}[T_1, \ldots, T_s] \ \to \ \mathbb{C}[S_1, \ldots, S_n]^{[G,G]}, \qquad T_i \ \mapsto \ g_i.$$

This is a standard procedure, it requires eliminating variables $S_1, \ldots, S_n$ from the ideal generated by all $T_i - g_i$ in the enlarged ring $\mathbb{C}[S_1, \ldots, S_n, T_1, \ldots, T_r]$, see for example [6, Thm. 3.2] or [9, Lem. 2.1]. Finally, to install the $\mathbb{X}(G') = \mathrm{Cl}(X_0)$-grading on $\mathcal{R}(X_0)$,

it remains to define the degrees of the generators as $\deg(T_i) = (\chi_{i1}, \dots, \chi_{ik}) \in \mathbb{X}(G')$; note that we give only values on a generating set of G', which determine the character uniquely. The *degree map* of $\mathcal{R}(X_0) = \mathbb{C}[T_1, \dots, T_s]/I_0$, i.e., the homomorphism

$$\mathbb{Z}^s \to \mathrm{Cl}(X_0), \qquad e_i \mapsto \deg(T_i)$$

is defined by the *degree matrix*, the matrix with columns $\deg(T_1), \dots, \deg(T_s)$. We summarize this as follows.

Algorithm 2.5 *(Cox ring of X_0). Input:* a small group $G \subseteq \mathrm{GL}(n)$, given by a generating set.

- Compute generators $f_1, \dots, f_s$ for the invariant ring $\mathbb{C}[S_1, \dots, S_n]^{[G,G]}$.
- As in Construction 2.4, modify $f_1, \dots, f_s$ to a set $g_1, \dots, g_s$ of G'-homogeneous generators of $\mathbb{C}[S_1, \dots, S_n]^{[G,G]}$ by computing eigenvectors of the induced action of G' on subspaces of polynomials of fixed degree d, where $d \in \{\deg(f_1), \dots, \deg(f_s)\}$.
- In $\mathbb{C}[S_1, \dots, S_n, T_1, \dots, T_s]$, define the ideal $J = \langle T_i - g_i; 1 \leq i \leq s \rangle$ and compute its elimination ideal $I_0 := J \cap \mathbb{C}[T_1, \dots, T_s]$.
- Define the matrix Q_0 with the i-th column $(\chi_{i1}, \dots, \chi_{ik})$, where χ_{ij} are eigenvalues of g_i with respect to $M_{d_i,j}$, as in Construction 2.4.

Output: $I_0 \subseteq \mathbb{C}[T_1, \dots, T_s]$ and Q_0. Then $\mathbb{C}[T_1, \dots, T_s]/I_0$ is a presentation of $\mathcal{R}(X_0)$, where $X_0 = \mathbb{C}^n/G$, in terms of generators and relations with Q_0 as degree matrix.

Example 2.6. We consider the faithful representation of the quaternion group $G := Q_8$ with 8 elements whose image $G \subseteq \mathrm{GL}(2)$ is generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

where $i \in \mathbb{C}$ denotes the imaginary unit. There are no pseudo-reflections since no matrix in G has 1 as eigenvalue. We now apply the steps of Algorithm 2.5. Generators for the invariant ring $\mathbb{C}[S_1, S_2]^{[G,G]}$, where $[G, G] = \langle -\mathrm{id} \rangle$, are

$$f_1 = S_1 S_2, \qquad f_2 = S_1^2, \qquad f_3 = S_2^2.$$

Moreover, the group $G' = G/[G, G]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and acts on the linear hull $V_2 := \mathrm{lin}_{\mathbb{C}}(f_1, f_2, f_3)$; we work with the description

$$G' \cong \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle \subseteq \mathrm{GL}(3).$$

The vector space V_2 then has the G'-invariant subspaces $\mathrm{lin}_{\mathbb{C}}(f_1)$ and $\mathrm{lin}_{\mathbb{C}}(f_2, f_3)$. We have a basis (g_1, g_2, g_3) for V_2 consisting of the eigenvectors

$$g_1 := f_1, \qquad g_2 := f_2 + f_3, \qquad g_3 := -f_2 + f_3 \in V$$

where the weights $w_i := (\chi_{i1}, \chi_{i2}) \in (\mathbb{Z}/2\mathbb{Z})^2 \cong G'$ are $w_1 = (\overline{0}, \overline{1})$, $w_2 = (\overline{1}, \overline{0})$ and $w_3 = (\overline{1}, \overline{1})$. The degree map is

$$Q_0 \colon \mathbb{Z}^3 \to (\mathbb{Z}/2\mathbb{Z})^2, \qquad e_i \mapsto \begin{bmatrix} \overline{0} & \overline{1} & \overline{1} \\ \overline{1} & \overline{0} & \overline{1} \end{bmatrix} \cdot e_i$$

In the ring $\mathbb{C}[S_1, S_2, T_1, T_2, T_3]$, we consider the ideal $J := \langle T_1 - g_1, T_2 - g_2, T_3 - g_3 \rangle$ and compute its elimination ideal $I_0 := J \cap \mathbb{C}[T_1, T_2, T_3]$. We arrive at the G'-graded Cox ring with degree matrix Q_0:

$$\mathcal{R}(\mathbb{C}^2/G) \cong \mathbb{C}[T_1, T_2, T_3]/I_0, \qquad I_0 = \langle 4T_1^2 - T_2^2 + T_3^2 \rangle. \qquad \square$$

2.3. Cox ring of the resolution

We now turn to the computation of the Cox ring of a resolution using the presentation $\mathcal{R}(X_0) = \mathbb{C}[T_1, \ldots, T_s]/I_0$ of the Cox ring obtained using Algorithm 2.5. There are two basic ingredients of the algorithm: the tropical variety $\mathrm{Trop}(X_0)$ constructed from the presentation of $\mathcal{R}(X_0)$ and the algorithm for computing the Cox ring of a toric ambient modification, [16, Alg. 3.6]. The idea is to embed X_0 into its ambient affine toric variety $Z_0 = \mathbb{C}^n/G'$ and to use the tropical variety $\mathrm{Trop}(X_0)$ to determine the centers for toric ambient modifications of X_0. Then we perform a toric resolution $Z \to Z_0$, consider the proper transform $X \to X_0$, compute its Cox ring and check whether it actually corresponds to a resolution of X_0. Let us explain the details.

Step 1: the toric ambient variety. Recall that Q_0 is the degree matrix of $\mathcal{R}(X_0)$. We fix a *Gale dual* matrix for Q_0, i.e., a matrix P_0 such that its transpose P_0^* fits into the exact sequence

$$0 \longleftarrow \mathrm{Cl}(X_0) \xleftarrow{Q_0} \mathbb{Z}^s \xleftarrow{P_0^*} \ker(Q_0) \longleftarrow 0 \tag{1}$$

see, e.g., [22] for its computation. The defining fan Σ_0 of the canonical toric ambient variety Z_0 of X_0 (cf. the paragraph after Construction 2.3) can be written down explicitly:

$$\Sigma_0 = \mathrm{faces}(\sigma), \qquad \sigma := P_0(\mathbb{Q}^s_{\geq 0}) \subseteq \mathbb{Q}^n.$$

The corresponding (affine) toric ambient variety is $Z_0 = \mathbb{C}^n/G'$ with the abelianization $G' := G/[G, G]$.

Step 2: the tropical variety. The tropical variety $\mathrm{Trop}(X_0)$ lies in the same linear space $\mathbb{Q}^n$ as the fan Σ_0. Consider the affine variety $\overline{X}_0 := \mathrm{Spec}(\mathcal{R}(X_0)) \subseteq \mathbb{C}^s$.

Construction 2.7. Given a polynomial $f \in \mathbb{C}[T_1, \ldots, T_s]$, its tropical variety $\mathrm{Trop}(f) \subseteq \mathbb{Q}^s$ is the support of the codimension-one skeleton of the normal fan over the Newton polytope of f. In the previous setting, we then obtain tropical varieties

$$\mathrm{Trop}\left(\overline{X}_0 \cap (\mathbb{C}^*)^s\right) \;:=\; \bigcap_{f \in I_0} \mathrm{Trop}(f) \;\subseteq\; \mathbb{Q}^s,$$
$$\mathrm{Trop}(X_0) \;:=\; P_0\left(\mathrm{Trop}\left(\overline{X}_0 \cap (\mathbb{C}^*)^s\right)\right) \;\subseteq\; \mathbb{Q}^n.$$

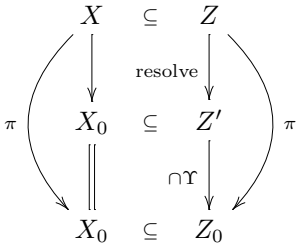
It is possible to establish a fan structure on $\mathrm{Trop}(X_0)$, see, e.g., [4,23]; we will denote this fan by Υ.

Note that in Construction 2.7, despite the infinite intersection, $\mathrm{Trop}\left(\overline{X}_0 \cap (\mathbb{C}^*)^s\right)$ can be computed using finitely many Gröbner basis computations, see [4].

Step 3: determining the modification. From the previous steps, we obtain two polyhedral fans Σ_0, Υ in $\mathbb{Q}^n$. We use the second one to determine a modification of the first one as follows; this is similar to a procedure proposed by Tevelev in a related context, see Remark 2.15. Firstly, we compute the fan

$$\Sigma' \;:=\; \{\sigma \cap \tau \mid \tau \in \Upsilon\}$$

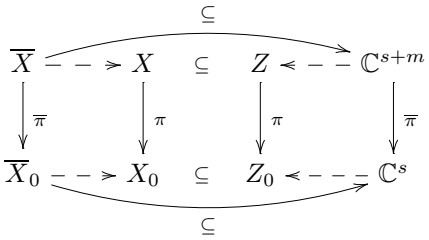
with support $|\Sigma'| = \mathrm{Trop}(X_0)$. Note that, by construction and [25, Lem. 2.2], the toric variety Z' corresponding to Σ' is still a toric ambient variety of X_0. Secondly, we resolve singular cones of Σ' by inserting new rays, compare [7, 11.1] for the procedure. This yields a regular fan Σ and a toric resolution $Z \to Z'$, where Z is the toric variety corresponding to Σ. The following diagram summarizes the situation; the map π is the composition of the toric morphism defined by the map of fans $\Sigma' \to \Sigma_0$ and the resolution, and X is the proper transform of X_0 under π (see also the next step).



For the next step, we write the primitive generators $v_1, \ldots, v_m$ of the rays added to Σ_0 in this process into the columns of the enlarged $(s+m) \times n$-matrix $P := [P_0, v_1, \ldots, v_m]$, and we compute a Gale dual matrix Q of P.

Step 4: modifying the ideal of $\mathrm{Spec}(\mathcal{R}(X_0))$. Finally, we apply [16, Alg. 3.6] to compute the Cox ring of the proper transform $X \to X_0$ under π; its main idea is as follows.

Extending the diagram from the previous step to the spectra $\overline{Y} := \mathrm{Spec}(\mathcal{R}(Y))$ of the respective Cox rings, we obtain



where a dashed arrow from some $\overline{Y}$ to the corresponding Y means that Y is the quotient of an open subset of $\overline{Y}$ by a quasitorus as in Construction 2.3. The spectrum of the Cox ring of the proper transform is the closure

$$\overline{X} \;=\; \overline{\overline{\pi}^{-1}(\overline{X}_0 \cap (\mathbb{C}^*)^s)} \;\subseteq\; \mathbb{C}^{s+m}.$$

This gives a recipe to compute $\mathcal{R}(X) = \mathcal{O}(\overline{X})$: using the presentation $\mathcal{R}(X_0) = \mathbb{C}[T_1,\ldots,T_s]/I_0$, compute the pullback $\overline{\pi}^*(I_0) \subseteq \mathbb{C}[T_1^{\pm 1},\ldots,T_{s+m}^{\pm 1}]$ and then the saturation $I := \overline{\pi}^*(I_0) : (T_1 \cdots T_{s+m})^\infty$ in the ring $\mathbb{C}[T_1,\ldots,T_{s+m}]$. By [16, Alg. 3.6], if all variables T_i define prime elements in $R := \mathbb{C}[T_1,\ldots,T_{s+m}]/I$ and $\dim(I) - \dim(\langle T_i, T_j\rangle + I) \geq 2$ for all $i \neq j$, then R is the Cox ring of X.

It then remains to check whether X is smooth – if it is, then X is a resolution of X_0, see the proof below. We also provide an algorithm for testing smoothness in the current setting, see Algorithm 2.11.

The following algorithm summarizes the process of computing and verifying a candidate for the Cox ring $\mathcal{R}(X)$ of a resolution $X \to X_0$. Note that it is close to [22, Alg. 2.4.8] which in turn is based on [1, Thm 3.4.4.9] where the complete, complexity-one-case is treated. See also [18, Ch. 3] and check [17, Sect. 3] for the case of affine $\mathbb{C}^*$-surfaces.

Algorithm 2.8 *(Candidate for the Cox ring of a resolution $X \to X_0 = \mathbb{C}^n/G$). Input:* a finite small group $G \subseteq \mathrm{GL}(n)$.

- Compute the Cox ring $\mathcal{R}(X_0) = \mathbb{C}[T_1,\ldots,T_s]/I_0$ of X_0 and the matrix Q_0 with Algorithm 2.5.
- Determine a matrix P_0 that is Gale dual to Q_0.
- Define the fan $\Sigma_0 := \mathrm{faces}(\sigma)$ with the cone $\sigma := P_0(\mathbb{Q}^n_{\geq 0}) \subseteq \mathbb{Q}^n$.
- Compute a fan Υ with support $\mathrm{Trop}(X_0)$ to obtain $\Sigma' := \{\sigma \cap \tau \mid \tau \in \Upsilon\}$.
- Stellarly subdivide Σ' at primitive vectors $v_1,\ldots,v_m \in \mathbb{Z}^n$ until we obtain a regular fan Σ.

- Compute a Gale dual Q of the enlarged matrix $P := [P_0, v_1, \ldots, v_m]$.
- Compute a presentation $R := \mathbb{C}[T_1, \ldots, T_r]/I$, where $r := s + m$, of the Cox ring of the proper transform $X \to X_0$ under π using [16, Alg. 3.6] with the `'verify'`-option, as explained above.
- Verify smoothness of the variety X determined by (R, Σ), e.g. with Algorithm 2.11.

Output: R and Q. If the verifications of the last two steps were successful, then R is the Cox ring of a resolution $X \to X_0$ and Q is the degree matrix of R.

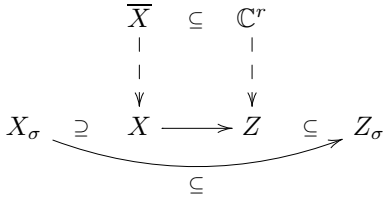
Lemma 2.9. *In the setting of Algorithm 2.8, if* $\mathrm{Spec}(R) \cap (\mathbb{C}^*)^r$ *is smooth and all variables* $T_1, \ldots, T_r$ *are prime, then* R *is normal.*

Proof. Write $\overline{X}_i$ for the localization of $\overline{X} := \mathrm{Spec}(R)$ at the product $T_1 \cdots T_i$. By assumption, $\widehat{X} := \mathrm{Spec}(R) \cap (\mathbb{C}^*)^r$ is smooth and therefore also $\overline{X}_r$ is smooth and in particular normal. Since T_r is prime, this implies normality of $\overline{X}_{r-1}$, see [1, Lem. IV.1.2.7]. Iterating this argument, $\overline{X}_0 = \mathrm{Spec}(R)$ is normal. □

Proof of Algorithm 2.8. By construction and [16, Alg. 3.6], $X \to X_0$ is a modification and R the Cox ring of X. Note that [16, Alg. 3.6] directly translates to the affine setting and the required normality test of R is provided by Lemma 2.9 if the smoothness requirement is ensured (e.g. by explicit computations). □

Remark 2.10. If the smoothness test in Algorithm 2.8 is left out, R is still the Cox ring of a modification of X_0 provided that R is normal.

We can test smoothness of X using the following straight-forward test; compare also [15]. The idea is to first test the toric charts Z_σ, where σ is a cone of the fan of the canonical toric ambient variety Z, for being smooth; it then suffices to check the smoothness of $X_\sigma := Z_\sigma \cap X$ if all Z_σ are smooth.



For the latter, we use the following ad hoc algorithm which directly looks for singularities of X that lie in $Z_\sigma \cong (\mathbb{C}^*)^l \times \mathbb{C}^k$ for suitable $k, l \in \mathbb{Z}_{\geq 0}$.

Algorithm 2.11 *(Smoothness test). Input:* a normal, irreducible A_2-maximal variety X with $\mathcal{R}(X)$ finitely generated.

- Compute the fan Σ of the canonical toric ambient variety Z of X.
- Return `false` if Σ is not regular.
- Return `true` if and only if all X_σ where $\sigma \in \Sigma$ are smooth.

Output: `true` if X is smooth, `false` otherwise.

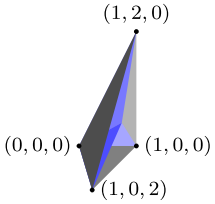
Remark 2.12. In Algorithm 2.11, the smoothness of X_σ can be checked as follows. According to [1, Constr. 3.3.1.1], we can compute a pairwise disjoint decomposition

$$X \;=\; X_1 \sqcup \ldots \sqcup X_u$$

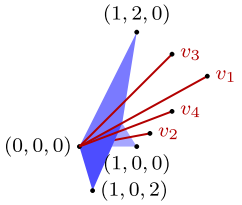
For each i, one then produces equations for X_i in $\mathcal{O}(\mathbb{C}^m \times (\mathbb{C}^*)^{n-m}) \cong \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. The test then boils down to $X_i^{\mathrm{sing}} \cap \mathbb{C}^m \times \mathbb{T}^{n-m}$ being empty or not. This can be done by a saturation computation using Gröbner bases.

Remark 2.13. Using the torus action on X and the fact that the singular locus is closed one may often reduce the number of sets X_σ for which the computations have to be performed to check the smoothness.

Example 2.14. Consider the Cox ring $\mathcal{R}(X_0) = \mathbb{C}[T_1, T_2, T_3]/I_0$ computed in Example 2.6 with its degree-map $Q\colon \mathbb{Z}^3 \to (\mathbb{Z}/2\mathbb{Z})^2$. We apply the steps of Algorithm 2.8 to X_0. The canonical toric ambient variety is the affine toric variety $Z_0 = Z_0(\sigma)$ with $\sigma \subseteq \mathbb{Q}^3$ being the polyhedral cone spanned by $(1,0,0), (1,2,0), (1,0,2)$. The following picture shows the steps for the toric resolution.



σ (gray) and Σ (blue)



Σ and new rays $\mathbb{Q}_{\geq 0} \cdot v_i$ (red)

The two-dimensional fan $\Sigma = \{\sigma\} \cap \mathrm{Trop}(X_0)$ has already the additional ray $\mathbb{Q}_{\geq 0} \cdot v_1$ where $v_1 := (3,2,2)$. The fan can be resolved by insertion of the further rays through

$$v_2 \;:=\; (2,1,2), \qquad v_3 \;:=\; (2,2,1), \qquad v_4 \;:=\; (2,1,1).$$

This yields a toric resolution $Z \to Z_0$ which then induces a resolution $X \to X_0$ as the proper transform of $Z \to Z_0$; its $\mathbb{Z}^4$-graded Cox ring $\mathcal{R}(X)$ and degree matrix are

$$\mathcal{R}(X) \;=\; \mathbb{C}[T_1, \ldots, T_7]/\langle 4T_1^2T_7 - T_6T_2^2 + T_3^2T_5 \rangle,$$

$$\begin{bmatrix} -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

One verifies that X is smooth. As X is a surface, we can verify minimality of the resolution by inspecting the self-intersection numbers of divisors corresponding to $V(\widehat{X}; T_i)$: they are $-1, -1, -1, -2, -2, -2, -2$. □

We will use Algorithms 2.5 and 2.8 in Section 4 to compute resolutions of quotient singularities. The next section explains an implementation of the algorithms of this section. We close this section with the following remark.

Remark 2.15. A result by Tevelev [25] indicates that the steps performed in Algorithm 2.8 should work, provided that we find the correct embedding $X \to Z$ and fan structure on $\mathrm{Trop}(X)$. Moreover, in [26], Tevelev shows that any embedded resolution of singularities is induced by an equivariant map of toric varieties in the following sense. Consider a birational morphism $\pi\colon Y_2 \to Y_1$ of smooth projective varieties with exceptional locus $D \subseteq Y_2$ such that the restriction of π to $Y_2 \supseteq X_2 \to X_1 \subseteq Y_1$ is a resolution of singularities. Assume that D has simple normal crossings and intersects X_2 transversally. Then there are embeddings $Y_1 \subseteq \mathbb{P}^n$ and $Y_2 \subseteq Z_2$ with a smooth toric variety Z_2 such that X_2 and Y_2 intersect the toric boundary of Z_2 transversally and π is a restriction of a toric morphism $\overline{\pi}\colon Z_2 \to Z_1$.

$$\begin{array}{ccccc} X_2 & \longrightarrow & Y_2 & \longrightarrow & Z_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 = \mathbb{P}^n \end{array}$$

Hence the main task seems to be to find the correct embeddings $X_i \subseteq Z_i$ – it would be interesting to investigate this in further work.

3. Implementation

We have implemented Algorithms 2.5 and 2.8 in the library `quotsingcox.lib` for the Open Source computer algebra system `Singular` [8]. It will be made available at [11]. We currently make use of the interface to `Normaliz` [5]. We shortly explain its use in the following two examples.

Example 3.1 *(Algorithm 2.5).* We recompute Example 2.6 using the library `quotsingcox.lib` [11]. In `Singular`, we first load the built-in library for convex geometry (implemented by Y. Ren) and our library with

```
> LIB "gfanlib.so";  LIB "quotsingcox.lib";
```

Next, we enter generators for $G \subseteq \mathrm{GL}(2)$. We first have to define a ring where not only the generators of G are defined but also all eigenvalues of $[G, G]$; $\mathbb{Q}(i)$ will work here:

```
> ring R = (0,CplxUnit),T(1..2),dp; minpoly = CplxUnit^2 + 1;
> matrix M0[2][2] = CplxUnit,0, 0,-CplxUnit;
> matrix M1[2][2] = 0,-CplxUnit,-CplxUnit,0;
> list G = M0, M1; // the group G
```

Next, we have to enter generators for the derived subgroup. For integer entries, you can also use the command `list GG = gapDerivedSubgroup(G);` which writes a file and applies `GAP` [13].

```
> matrix M2[2][2] = -1,0,0,-1;
> list GG = M2; // derived subgroup of G
```

We can then compute the Cox ring $\mathcal{R}(X_0)$ and investigate it using the following commands:

```
> def R0 = coxquot(G, GG); setring R0; R0;
// characteristic :  0
// 1 parameter :  CplxUnit
// minpoly :  (CplxUnit^2+1)
// number of vars :  3
// block 1 :  ordering dp
// :  names T(1) T(2) T(3)
// block 2 :  ordering C
> ideal I0 = Inew; I0; // Cox ring of X0 is R0/I0
I0[1]=T(1)^2-T(2)^2-4*T(3)^2
```

Generators for $\mathrm{Cl}(X_0)$ as a matrix subgroup `Gnew` of $\mathrm{GL}(3)$ and the degrees $\deg(T_i) \in \mathrm{Cl}(X_0)$ of the generators of $\mathcal{R}(X_0)$ can be displayed with `print(Gnew[1])` and `print(Gnew[2])`. This yields the matrices with columns $-e_1, -e_2, e_3$ and $-e_2, -e_1, -e_3$, respectively.

```
> Wnew; // degrees of the generators (eigenvalues)
[1]:
_[1,1]=-1
_[2,1]=-1
[2]:
_[1,1]=-1
_[2,1]=1
[3]:
_[1,1]=1
_[2,1]=-1
```

□

Example 3.2 *(Algorithm 2.8).* Building on Example 3.1, we recompute Example 2.14. The next step has to be prepared manually: one has to give the isomorphism type of `Gnew` as a product $\prod \mathbb{Z}/n_i\mathbb{Z}$. Here, `gapStructureDescription(Gnew);` can be used to

see that $n_1 = n_2 = 2$. We store the integers n_i, translate the degrees given in `Wnew` into the columns of a matrix Q_0 and compute a Gale dual matrix P_0 for Q_0:

```
> list ClX0 = 2, 2; // ZZ/2ZZ x ZZ/2ZZ
> intmat Q0[2][3] = 1,1,0,1,0,1; // degree matrix
> intmat P0 = finiteGaleDual(Q0, ClX0); P0;
1,1,1
1,2,0
0,0,2
```

We can then compute, verify and print a candidate for the Cox ring of a resolution $X \to X_0$ using Algorithm 2.8 with the following commands. We present the Cox ring as $\mathbb{C}[T_1, \dots, T_s]/I_0$; the algorithm returns the ideal I_0.

```
> def R = resolveQuotSing(I0, P0, 1); setring R; R; // 1 means verify
1st part of Verification successful:  please check primality of the variables (e.g.  with
primeVars).
// characteristic :  0
// 1 parameter :  CplxUnit
// minpoly :  (CplxUnit^2+1)
// number of vars :  7
// block 1 :  ordering dp
// :  names T(1) T(2) T(3) T(4) T(5) T(6) T(7)
// block 2 :  ordering C
> Inew; // Cox ring of X is R/Inew
Inew[1]=T(1)^2*T(5)-T(2)^2*T(6)-4*T(3)^2*T(7)
```

One can check the primality of the variables T_i using the command `primeVars(Inew);`. The $\mathbb{Z}^4$-grading is stored in the new degree matrix Q; it is obtained as a Gale dual matrix Q to the new matrix P:

```
> intmat P = L[2];
> intmat Q = gale(P); Q;
-1,-1,-1,1,0,0,0,
-2,-1,-1,0,2,0,0,
3,0,1,0,-4,2,0,
2,0,0,0,-3,1,1
```

□

4. Resolutions of 3-dimensional quotient singularities

4.1. Cox rings

In this section, we compute Cox rings of resolutions of quotient singularities $\mathbb{C}^3/G$ where G is a group of order at most 12 and discuss the results. As explained in Remark 2.2, we will consider only faithful representations of G without pseudo-reflections.

Notation 4.1. In the following, we denote by S_3 the permutation group of three elements, by D_{2n} the dihedral group of $2n$ elements, by Q_8 the quaternion group and by A_4 the

alternating group of four elements. Moreover, we write BD_3 for the *binary dihedral group*, i.e., the abstract group with 12 elements

$$\mathrm{BD}_3 \;=\; \left\langle a, x;\, a^6 = 1,\, x^2 = a^3,\, x^{-1}ax = a^{-1} \right\rangle .$$

Proposition 4.2. *Up to conjugacy in* $\mathrm{GL}(3)$*, the* 3*-dimensional faithful small representations of non-abelian groups up to order* 12 *are as follows.*

No.	G	gen.s in GL(3)
1	S_3	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
2	D_8	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
3	Q_8	$\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
4	Q_8	$\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
5	D_{10}	$\begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5^4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
6	D_{12}	$\begin{bmatrix} -\zeta_3^2 & 0 & 0 \\ 0 & -\zeta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
7	A_4	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
8	BD_3	$\begin{bmatrix} -\zeta_3^2 & 0 & 0 \\ 0 & -\zeta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
9	BD_3	$\begin{bmatrix} -\zeta_3^2 & 0 & 0 \\ 0 & -\zeta_3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & i \end{bmatrix}$.
10	BD_3	$\begin{bmatrix} -\zeta_3^2 & 0 & 0 \\ 0 & -\zeta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

In the table, $i \in \mathbb{C}$ *denotes the imaginary unit, and* $\zeta_k \in \mathbb{C}$ *is a primitive* k*-th root of unity. All listed representations are reducible except for the* A_4*-case.*

Proof. To classify these representations we use the library of groups of small order, which is a part of `GAP` [13]. We do the following steps:

(i) view the character tables to get dimensions of irreducible representations,
(ii) find all irreducible representations: for a group of small order and a given character one can either locate a representation in the literature or construct easily,
(iii) combine irreducible representations to get all faithful 3-dimensional representations of a given group (in particular, direct sums of 1-dimensional representations are not allowed since they are not faithful representations – the image is an abelian group),
(iv) eliminate representations with pseudo-reflections (see Remark 2.2),
(v) check whether obtained representations have different groups as images (if representations differ just by a permutation of conjugacy classes in G, they give the same quotient); for any two representations of the same group on the list above the sets of matrix traces are different, hence this condition is satisfied. □

Proposition 4.3. *In the setting of Proposition 4.2, the Cox rings of* $X_0 := \mathbb{C}^3/G$ *are as listed in the following table.*

No.	G	$Cl(X_0)$	Degree matrix	$\mathcal{R}(X_0)$
1	S_3	$\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}\overline{1} & \overline{1} & \overline{0} & \overline{0}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_3^3 - T_2^2 - 27T_4^2$
2	D_8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}\overline{1} & \overline{1} & \overline{0} & \overline{0}\\ \overline{1} & \overline{0} & \overline{1} & \overline{0}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_1^2 + T_2^2 - T_4^2$
3	Q_8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}\overline{1} & \overline{1} & \overline{0} & \overline{0}\\ \overline{1} & \overline{0} & \overline{1} & \overline{0}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $T_1^2 - T_2^2 + 4T_3^2$
4	Q_8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}\overline{1} & \overline{1} & \overline{1} & \overline{0}\\ \overline{1} & \overline{0} & \overline{0} & \overline{1}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $T_1^2 - T_3^2 + 4T_4^2$
5	D_{10}	$\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}\overline{1} & \overline{1} & \overline{0} & \overline{0}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_3^5 + T_2^2 - T_4^2$
6	D_{12}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}\overline{1} & \overline{1} & \overline{0} & \overline{0}\\ \overline{1} & \overline{0} & \overline{1} & \overline{0}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_4^3 + T_1^2 - T_2^2$
7	A_4	$\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}\overline{0} & \overline{0} & \overline{2} & \overline{1}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $T_1^3+T_3^3-3T_1T_3T_4+T_4^3-27T_2^2$
8	BD_3	$\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}\overline{3} & \overline{1} & \overline{2} & \overline{0}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_3^3 + T_1^2 - T_2^2$
9	BD_3	$\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}\overline{3} & \overline{1} & \overline{1} & \overline{2}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_4^3 + T_1^2 - T_3^2$
10	BD_3	$\mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix}\overline{3} & \overline{1} & \overline{2} & \overline{2}\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_4]/I$ with I gen. by $4T_4^3 + T_1^2 - T_2^2$

Proof. The listed Cox rings are obtained by applying Algorithm 2.5 to the list of representations from Proposition 4.2 using our implementation 3. □

Remark 4.4. Note that several rings on the list (without grading considered) are isomorphic. This is because the ring structure of the Cox ring of $\mathbb{C}^n/G$ is just the invariant ring $\mathbb{C}[x_1,\dots,x_n]^{[G,G]}$ and derived subgroups in cases 2, 3, 4 (D_8 and Q_8 representations) and also 6, 8, 9, 10 (D_{12} and BD_3 representations) are conjugate subgroups of GL(3).

Theorem 4.5. *In the setting of Proposition 4.3, a resolution $X \to X_0$ of the quotient $X_0 = \mathbb{C}^3/G$ has the following Cox ring, respectively.*

No.	G	$Cl(X)$	Degree matrix	$\mathcal{R}(X)$
1	S_3	$\mathbb{Z}^4$	$\begin{bmatrix}0 & -3 & -2 & -3 & 1 & 0 & 0 & 0\\ -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0\\ 1 & 0 & -1 & -1 & 0 & -2 & 1 & 0\\ 1 & 0 & -1 & -2 & 0 & -3 & 0 & 1\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_8]/I$ with I gen. by $4T_3^3T_7 - T_2^2T_6 - 27T_4^2T_8$
2	D_8	$\mathbb{Z}^4$	$\begin{bmatrix}-1 & -1 & 0 & -1 & 1 & 0 & 0 & 0\\ -1 & 0 & -1 & 0 & 0 & 2 & 0 & 0\\ -1 & -1 & 0 & -2 & 0 & 0 & 2 & 0\\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 1\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_8]/I$ with I gen. by $-4T_1^2T_6 + T_4^2T_7 - T_2^2T_8$
3	Q_8	$\mathbb{Z}^4$	$\begin{bmatrix}-1 & -1 & -1 & 0 & 1 & 0 & 0 & 0\\ -2 & -1 & -1 & 0 & 0 & 2 & 0 & 0\\ 1 & 0 & -1 & 0 & 0 & -2 & 2 & 0\\ 2 & 0 & 0 & 0 & 0 & -3 & 1 & 1\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_8]/I$ with I gen. by $T_1^2T_6 + 4T_3^2T_7 - T_2^2T_8$
4	Q_8	$\mathbb{Z}^4$	$\begin{bmatrix}-1 & 0 & -1 & -1 & 1 & 0 & 0 & 0\\ -2 & -1 & -1 & -1 & 0 & 2 & 0 & 0\\ 1 & 0 & 0 & -1 & 0 & -2 & 2 & 0\\ 2 & 1 & 0 & 0 & 0 & -3 & 1 & 1\end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_8]/I$ with I gen. by $-T_1^2T_6 - 4T_4^2T_7 + T_3^2T_8$

5	D_{10}	$\mathbb{Z}^5$	$\begin{bmatrix} 0 & -5 & -2 & -5 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 & 0 & -4 & 0 & 1 & 0 \\ 2 & 0 & -1 & -3 & 0 & -5 & 0 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_9]/I$ with I gen. by $4T_3^5T_7^3T_8 + T_2^2T_6 - T_4^2T_9$
6	D_{12}	$\mathbb{Z}^5$	$\begin{bmatrix} -3 & -3 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_9]/I$ with I gen. by $4T_4^3T_7^2T_9 + T_1^2T_6 - T_2^2T_8$
7	A_4	$\mathbb{Z}^3$	$\begin{bmatrix} 0 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_{10}]/I$ with I gen. by $T_4T_7T_9 + T_2T_6 + T_3T_5$, $T_4^2T_9 - T_2T_3 - T_6T_{10}$, $T_3T_6T_8 + T_2T_7 + T_4T_5$, $T_3^2T_8 - T_2T_4 - T_7T_{10}$, $27T_1^2T_4 - T_6^2T_8 + T_5T_7$, $27T_1^2T_3 - T_7^2T_9 + T_5T_6$, $T_6T_7T_8T_9 + 27T_1^2T_2 - T_5^2$, $T_3T_4T_8T_9 - T_2^2 + T_5T_{10}$, $T_3T_7T_8T_9 + T_4T_6T_8T_9 - 27T_1^2T_{10} + T_2T_5$
8	BD_3	$\mathbb{Z}^5$	$\begin{bmatrix} -3 & -3 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & -3 & 2 & 0 & 0 & 0 \\ -1 & -2 & -2 & 0 & 2 & -2 & 1 & 0 & 0 \\ -2 & -5 & -3 & 0 & 5 & -5 & 0 & 1 & 0 \\ -2 & -4 & -3 & 0 & 4 & -4 & 0 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_9]/I$ with I gen. by $4T_3^3T_7^2T_9 + T_1^2T_6 - T_2^2T_8$
9	BD_3	$\mathbb{Z}^8$	$\begin{bmatrix} -3 & 0 & -3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & -3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & -3 & 0 & 5 & -5 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & -3 & 0 & 4 & -4 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -3 & 0 & 4 & -4 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & -4 & 0 & 6 & -6 & 0 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & -6 & 0 & 9 & -9 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_{12}]/I$ with I gen. by $4T_4^3T_6^2T_9T_{10} + T_1^2T_7T_{12} - T_3^2T_8$
10	BD_3	$\mathbb{Z}^5$	$\begin{bmatrix} -3 & -3 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 3 & -1 & 2 & -3 & 0 & 2 & 0 & 0 \\ -2 & -5 & 2 & -3 & 5 & 0 & -5 & 1 & 0 \\ -2 & -4 & 2 & -3 & 4 & 0 & -4 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_9]/I$ with I gen. by $4T_4^3T_6^2T_9 + T_1^2T_7 - T_2^2T_8$

Proof. *Cases where $G \neq A_4$:* The Cox rings of the $X_0 = \mathbb{C}^3/G$ have been presented in Proposition 4.3. We then obtain the Cox ring of a resolution $X \to X_0$ using our implementation 3 of Algorithm 2.8. The verifications of the primality of the variables T_i are done directly computationally, the smoothness tests are done with Algorithm 2.11 as implemented in `MDSpackage` [15].

Case $G = A_4$: To obtain the Cox ring of a resolution of the quotient singularity for $G = A_4$ one needs to change the presentation of the Cox ring of $\mathbb{C}^3/G$. The one shown in Proposition 4.3 comes from the generating set of $\mathbb{C}[x,y,z]^{[G,G]}$

$$T_1 = x^2 + y^2 + z^2, \quad T_2 = xyz, \quad T_3 = x^2 + \varepsilon y^2 + \varepsilon^2 z^2, \quad T_4 = x^2 + \varepsilon^2 y^2 + \varepsilon z^2$$

composed of eigenvectors of $G/[G,G]$ action (where ε is the 3rd root of unity). For the resolution algorithm to work (compare Example 4.6), we want the generating set to satisfy [10, Condition 3.6], developed in the more general context including the present case in [28]. Roughly speaking, it determines when a generating set of $\mathcal{R}(X_0)$ is suitable for being extended to the Cox ring of its resolution. Thus we add to the generating set elements

$$T_5 = T_1^2 - T_3T_4, \qquad T_6 = T_3^2 - T_1T_4, \qquad T_7 = T_4^2 - T_1T_3,$$

which are obstructions for this condition for the original generating set. We obtain a presentation $\mathcal{R}(X_0) = \mathbb{C}[T_1, \dots, T_7]/I_0$ where generators for I_0 and the $\mathbb{Z}/3\mathbb{Z}$-grading matrix are

$$T_4T_5 + T_3T_6 + T_2T_7, \quad T_3T_5 + T_2T_6 + T_4T_7, \quad T_3^2 - T_2T_4 - T_7,$$
$$T_2T_3 - T_4^2 + T_6, \quad T_2^2 - T_3T_4 - T_5, \quad 27T_1^2 - T_2T_5 - T_4T_6 - T_3T_7,$$
$$\begin{bmatrix} \overline{0} & \overline{0} & \overline{2} & \overline{1} & \overline{0} & \overline{2} & \overline{1} \end{bmatrix}.$$

The ring in the table is obtained via toric ambient modification from the Cox ring of $\mathbb{C}^3/G$, presented as above. More precisely, we apply Algorithm 4.15 (to be explained in Subsection 4.3) with the choice of the three vectors

$$v_1 := (0,0,2,1,0,1,1), \qquad v_2 := (0,0,2,1,0,1,2) \qquad v_3 := (1,0,3,1,1,2,3).$$

All the necessary verifications, in particular the smoothness test, are successful. □

Example 4.6 *(Non-example).* Note that for $G = A_4$, if we apply Algorithm 2.8 directly to the presentation of $\mathcal{R}(X_0)$ given in case 7 of Proposition 4.3, we only obtain the Cox ring of a modification $X \to X_0$ where X is not smooth. More precisely, it computes the $\mathbb{Z}^8$-graded Cox ring and degree matrix

$$\begin{array}{c} \mathbb{C}[T_1, \dots, T_{12}]/I \text{ with } I \text{ gen. by} \\ T_3^3T_6^2T_7T_8^2T_{11} + T_4^3T_6T_7^2T_9^2T_{12} \\ - 3T_1T_3T_4T_6T_7T_8T_9T_{11}T_{12} + T_1^3T_8T_9T_{11}^2T_{12}^2 \\ - 27T_2^2T_{10} \end{array} \qquad \begin{bmatrix} -2 & -3 & -2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

□

Remark 4.7. Using these methods and its implementation, one can directly go to higher group orders $|G|$. Since the number of different representations to consider grows quickly for isomorphism types of G with higher order, we end the table at order 12.

4.2. Properties of quotients and their resolutions

We discuss certain geometric properties of the quotient singularities and their resolutions corresponding to the Cox rings from Theorem 4.5. At first, we turn to torus actions on quotient spaces: we consider the relation between the form of the Cox ring and the existence of a torus action on $\mathbb{C}^n/G$.

Proposition 4.8. *Let V be an affine space which is a direct sum of n representations of a given group G. Then V admits an action of $(\mathbb{C}^*)^n$ which commutes with the action of G.*

Proof. Let $(x_{i,1} \dots, x_{i,n_i})$ be the coordinates of the i-th representation (of dimension d_i). The following action commutes with the action of G.

$$(\mathbb{C}^*)^n \times V \to V, \qquad \begin{aligned}&(t_1,\dots,t_n)\cdot(x_{1,1}\dots,x_{1,d_1},\dots,x_{n,1}\dots,x_{n,d_n})\\ &=(t_1x_{1,1}\dots,t_1x_{1,d_1},\dots,t_nx_{n,1}\dots,t_nx_{n,d_n})\end{aligned} \qquad \square$$

Corollary 4.9. *A quotient of $\mathbb{C}^3$ by a direct sum of a 2-dimensional and a 1-dimensional representation is a T-variety of complexity one.*

Remark 4.10. All representations in the table in Proposition 4.2 except the case of A_4 are direct sums of two irreducible representations, as it can be easily seen from the generating matrices. In particular, all varieties X_0 in Proposition 4.3 except possibly for the A_4 case are T-varieties of complexity one.

Thus, the only 3-dimensional quotient singularities which are possibly not T-varieties of complexity one correspond to irreducible representations. The representation of A_4 in Proposition 4.2 is irreducible and we expect that there is no $(\mathbb{C}^*)^2$ action.

Note that although we do not give a proof that for 3-dimensional quotient singularities with the $(\mathbb{C}^*)^2$-action (as in Remark 4.10) Algorithm 2.8 will always compute the Cox ring of a resolution and that it will have just one trinomial relation, it seems natural to predict such behavior. This is because of similar results in slightly different settings. In [1, Thm 3.4.4.9] the authors construct the Cox ring for any complete T-variety of complexity one and determine its single relation. Recently, an analogous description of the Cox rings of affine $\mathbb{C}^*$-surfaces was provided in [17, Sect. 3].

Remark 4.11. By the previous discussion, it would be interesting to continue the list of results in Theorem 4.5 with irreducible, 3-dimensional representations. However, the smallest cases of such representations (one of order 21, two of order 24 and two of order 27) are at the moment computationally out of reach on our machines.

We now turn to properties of the resolutions found in Theorem 4.5. Recall that a resolution of singularities $\pi\colon X \to X_0$ is called *crepant* if $K_X = \pi^* K_{X_0}$. For surface quotient singularities for $G \subseteq \mathrm{SL}(2)$, i.e. du Val singularities, crepant resolutions are the minimal ones: the special fiber is a tree of smooth rational curves dual to a Dynkin diagram A_n, D_n, E_6, E_7 or E_8. The relation between the structure of G (its conjugacy classes or irreducible representations) and the shape of the diagram of the resolution of $\mathbb{C}/G$ was noticed by McKay. The postulated relation between the geometry of crepant resolutions of (Gorenstein) quotient singularities and the structure of the group is called the *McKay correspondence*. It has been studied in several special cases and in different formulations. In particular, it is proved in dimension 3, see e.g. [19], for symplectic singularities in dimension 4, see [20], and a weak version for any $G \subset \mathrm{SL}(n)$ (the equality of the dimension of cohomology space and the number of conjugacy classes of G) is due to Batyrev [3].

It is a natural question to ask how good the resolutions obtained using Algorithm 2.8 are and what properties to expect of them. In particular, we would like to know whether

they are crepant for $G \subset \mathrm{SL}(n)$. To test this property for the 3-dimensional results in Theorem 4.5 we can use the McKay correspondence.

Remark 4.12. Among groups listed in Proposition 4.2 the cases 1, 2, 3, 5, 6, 7 and 8 are in SL(3).

Proposition 4.13. *In Theorem 4.5, the Cox rings of resolutions of singularities in cases 1 (S_3) and 5 (D_{10}), are not the Cox rings of crepant resolutions. The resolutions in cases 2 (D_8), 3 (Q_8), 6 (D_{12}), 7 (A_4) and 8 (BD_3) are crepant.*

Proof. By [12, Lem. 2.11], the class group $\mathrm{Cl}(X)$ of a resolution $\pi\colon X \to \mathbb{C}^n/G$ of a quotient singularity is a free group and its rank m is equal to the number of irreducible components of the exceptional divisor. By the McKay correspondence in dimension $n = 3$ we know that m is the number of conjugacy classes of junior elements in G where an element $g \in G$ is *junior* if

$$\mathrm{age}(g) \;=\; 1 \qquad \text{where} \qquad \mathrm{age}(g) \;:=\; \frac{1}{r}\sum_{k=1}^{n} a_k$$

with a_k coming from the exponents of eigenvalues $e^{\frac{2\pi i a_k}{r}}$ of g, see e.g. [24]. Thus the conjugacy classes of junior elements can be determined with simple computations (e.g. using `GAP` [13]), and the results are as follows:

Case number	1	2	3	5	6	7	8
Junior classes	2	4	4	3	5	3	5

Comparing them with the rank of $\mathrm{Cl}(X)$ given in Theorem 4.5 we obtain that the resolutions in cases 1 and 5 have too many components of the exceptional divisor to be crepant.

In the remaining cases obtained resolutions are crepant: since any crepant divisor appears necessarily on any resolution (see e.g. [19, 2.3]), it is enough to show that the number of exceptional divisors is equal to the number of junior classes. □

Note that in dimension 3 there is one Cox ring corresponding to all crepant resolutions, because they are all birationally equivalent and flops preserve smoothness in dimension 3.

Remark 4.14. We can describe the geometry of all crepant resolutions for groups 2, 3, 6, 7 and 8 of Proposition 4.2 by computing GIT quotients of the spectrum of their Cox rings. To determine the GIT fan describing the variation of the quotients, we use [21].

Case number	2	3	6	7	8
Max. GIT-cones within $\mathrm{Mov}(X)$	9	1	16	5	1

Actually, the cases 3 and 8 are products of a representation of a finite group in SL(2) and a 1-dimensional trivial representation, so the resolutions will be just products of minimal resolutions of du Val singularities by $\mathbb{C}$. Hence, there is just one crepant resolution, i.e., just one chamber in the GIT fan restricted to the cone of movable divisor classes $\mathrm{Mov}(X)$.

4.3. Smaller resolutions

In Algorithm 2.8, we computed Cox rings of resolutions as proper transforms under toric resolutions. The latter was obtained by intersection of the tropical variety with the affine toric ambient variety of X_0. This may yield unnecessarily many new rays and therefore bigger resolutions. In this subsection, we shortly describe an immediate alternative version that works with subsets of these rays in a brute force manner. A similar idea was used in [15, Sect. 3]. We can then improve two resolutions of Theorem 4.5.

Algorithm 4.15. *Input:* as in the first steps of Algorithm 2.8: the Cox ring $\mathbb{C}[T_1,\dots,T_s]/I_0$ of X_0, $\sigma = P_0(\mathbb{Q}^s_{\geq 0})$, a fan Υ with support $\mathrm{Trop}(X_0)$, $\Sigma = \{\sigma\} \cap \Upsilon$.

- Resolve Σ, i.e., determine a set $\mathcal{V} \subseteq \mathbb{Q}^n$ of primitive vectors such that the stellar subdivision of Σ at $\mathcal{V}$ is regular.
- Insert into $\mathcal{V}$ primitive generators for the rays of Σ that are not rays of σ.
- For each $k = 1,\dots,|\mathcal{V}|$, do:
  - For each $v_{i_1},\dots,v_{i_k} \in \mathcal{V}$, do:
    - Perform the steps of Algorithm 2.8 starting from line 6 with $v_1,\dots,v_m$ replaced by $v_{i_1},\dots,v_{i_k}$ and Σ replaced by the stellar subdivision of σ at these vectors. Stop, if the verification is positive.

Output (if provided): the Cox ring $\mathcal{R}(X) = \mathbb{C}[T_1,\dots,T_r]/I$ of a resolution $X \to X_0$.

Proposition 4.16. *In the setting of Theorem 4.5, the following are Cox rings of crepant resolutions.*

No.	G	$Cl(X)$	Degree matrix	$\mathcal{R}(X)$
1	S_3	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_6]/I$ with I gen. by $4T_3^3T_6 - T_2^2T_5 - 27T_4^2$
5	D_{10}	$\mathbb{Z}^3$	$\begin{bmatrix} 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_7]/I$ with I gen. by $4T_3^5T_6^3T_7 + T_2^2T_5 - T_4^2$

Proof. In case 1, the fan Σ_0 of the ambient affine toric variety has the maximal cone σ generated by

$$(1,0,0,0), \quad (1,2,0,0), \quad (0,0,1,0), \quad (0,0,0,1) \ \in \mathbb{Q}^4.$$

Algorithm 4.15 then stellarly subdivides $\Sigma_0 = \mathrm{fan}(\sigma)$ at $(1,1,0,0)$ and $(1,2,1,1) \in \mathbb{Q}^4$. The verifications succeed. Similarly, in case 5, the cone σ is generated by

$$(1,0,0,0), \quad (1,2,0,0), \quad (0,0,1,0), \quad (0,0,0,1) \ \in \mathbb{Q}^4.$$

Again, Algorithm 4.15 stellarly subdivides Σ_0 at the vectors $(1,1,0,0)$, $(1,2,1,1)$ and $(2,4,1,2) \in \mathbb{Q}^4$.

Both resolutions are crepant, because the number of exceptional divisors is the same as the number of junior conjugacy classes in G, compare the end of the proof of Proposition 4.13. □

Remark 4.17. As in Remark 4.14, we compute the variation of GIT-quotients for the two cases of Proposition 4.16:

Case number	1	5
Max. GIT-cones within $\mathrm{Mov}(X)$	2	3

Remark 4.18. Applying Algorithm 4.15 to the other cases of Theorem 4.5 does not yield better resolutions in the sense of smaller Picard number or fewer generators. Note that these cases are precisely the ones which are not in SL(3).

5. Two 4-dimensional examples

In this section, we present two 4-dimensional examples. In dimension 4 much less is known about the resolutions of quotient singularities. In particular, crepant resolutions do not always exist, and the McKay correspondence has been proved just for the symplectic case by Kaledin, see [20]. Hence it is a very appropriate setting for computational experiments with constructing resolutions via Cox rings. Moreover, it is interesting also from the point of view of the Cox ring theory: while in dimension 3 all our cases except for one can be defined with a single trinomial relation, here the ring structure will be more complicated.

The groups given below are complex symplectic, hence we work here with symplectic quotient singularities. Both examples have also been treated in [10] where the Cox rings of symplectic resolutions were constructed.

Proposition 5.1. *Consider the 4-dimensional quotient singularities $\mathbb{C}^4/G$ for the two small groups $G \subseteq \mathrm{GL}(4)$*

G	gen.s in GL(4)
S_3	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$
D_8	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$

The Cox ring of the quotient space $X_0 := \mathbb{C}^n/G$, the degree matrix and the class group of X_0 are given in the table on the next page.

Proof. This is again an application of Algorithm 2.5. □

We now compute the Cox rings of a resolution for the two cases $X_0 = \mathbb{C}^4/S_3$ and $X_0 = \mathbb{C}^4/D_8$ presented in the previous proposition. In Proposition 5.3, we present a resolution for the D_8-case thereby retrieving a result of Grab and the first author [10, Prop. 5.15]; Proposition 5.4 treats the case of S_3.

Remark 5.2. In the case of $G = D_8$ in Proposition 5.1, applying Algorithm 2.8 to $X_0 = \mathbb{C}^4/G$ without changes, we obtain a modification $X \to X_0$ with $\mathrm{Cl}(X) = \mathbb{Z}^{16}$ and the Cox ring $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_{26}]/I$ with 28 generators for I. Due to this size, we could not verify smoothness on our machines. Similarly, the smoothness tests in the case $G = S_3$ were computationally infeasible.

It turns out, that after some changes in Algorithm 4.15, we obtain the Cox rings of symplectic resolutions for S_3 and D_8. Note that for a symplectic quotient singularity a crepant resolution and a symplectic resolution are the same, see [27, Thm 2.5].

Proposition 5.3. *In the case of $G = D_8$ in Proposition 5.1, applying a variant of Algorithm 2.8 to $X_0 = \mathbb{C}^4/G$, we obtain a resolution $X \to X_0$ with $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the Cox ring is $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_{12}]/I$ where generators for I and the degree matrix are*

$$\begin{aligned}
T_5T_8 + T_4T_9 - 2T_6T_{10}, &\quad T_3T_8 + T_1T_9 - T_2T_{10},\\
T_2T_5 - 2T_3T_6 + T_7T_9, &\quad 2T_1T_5 - T_2T_6 + T_7T_{10},\\
2T_3T_4 - T_2T_6 - T_7T_{10}, &\quad T_2T_4 - 2T_1T_6 - T_7T_8,\\
T_7^2T_{12} - T_2^2 + 4T_1T_3, &\quad T_6T_7T_{12} + 2T_1T_9 - T_2T_{10},\\
T_5T_7T_{12} + T_2T_9 - 2T_3T_{10}, &\quad T_4T_7T_{12} - T_2T_8 + 2T_1T_{10},\\
T_4T_5T_{12} - T_6^2T_{12} + T_8T_9 - T_{10}^2, &\quad T_7^2T_{11} + T_4T_5 - T_6^2,\\
2T_3T_7T_{11} - T_6T_9 + T_5T_{10}, &\quad T_2T_7T_{11} - T_4T_9 + T_6T_{10},\\
2T_1T_7T_{11} + T_6T_8 - T_4T_{10}, &\quad 4T_3^2T_{11} + T_5^2T_{12} - T_9^2,\\
2T_2T_3T_{11} + T_5T_6T_{12} - T_9T_{10}, &\quad 4T_1T_3T_{11} + T_4T_5T_{12} + T_8T_9 - 2T_{10}^2,\\
2T_1T_2T_{11} + T_4T_6T_{12} - T_8T_{10}, &\quad 4T_1^2T_{11} + T_4^2T_{12} - T_8^2
\end{aligned}$$

G	$Cl(X_0)$	Degree matrix	$\mathcal{R}(X_0)$
S_3	$\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_{12}]/I$ with I gen. by
			$3T_9T_{10} - T_7T_{11} + T_6T_{12}$, $3T_5T_{10} - 2T_4T_{11} + T_3T_{12}$, $3T_6T_8 + T_9T_{11} - T_7T_{12}$, $3T_4T_8 - T_2T_{11} - 2T_5T_{12}$, $3T_3T_8 - 3T_2T_{10} - T_5T_{11} - T_4T_{12}$, $T_5T_7 - 2T_4T_9 + T_1T_{12}$, $T_4T_7 - 2T_3T_9 + T_1T_{11}$, $T_2T_7 - 3T_1T_8 + 2T_5T_9$, $T_5T_6 - T_3T_9 + T_1T_{11}$, $2T_4T_6 - T_3T_7 + 3T_1T_{10}$, $T_2T_6 + T_4T_9 - T_1T_{12}$, $T_1T_5 + 6T_9T_{11} - 3T_7T_{12}$, $T_4^2 - T_3T_5 - 3T_{11}^2 + 9T_{10}T_{12}$, $T_2T_4 + T_5^2 + 9T_8T_{11} - 3T_{12}^2$, $T_1T_4 + 3T_7T_{11} - 6T_6T_{12}$, $T_2T_3 + T_4T_5 + 27T_8T_{10} - 3T_{11}T_{12}$, $T_1T_3 + 9T_7T_{10} - 6T_6T_{11}$, $T_1T_2 + 9T_7T_8 - 6T_9T_{12}$, $T_1^2 + 3T_7^2 - 12T_6T_9$, $4T_9^3 - T_2^2 - 27T_8^2$, $2T_7T_9^2 + T_2T_5 - 9T_8T_{12}$, $4T_6T_9^2 - T_5^2 - 3T_{12}^2$, $2T_1T_9^2 - 9T_5T_8 - 3T_2T_{12}$, $T_7^2T_9 - 4T_6T_9^2 - 9T_8T_{11} + 3T_{12}^2$, $2T_6T_7T_9 - T_4T_5 - 3T_{11}T_{12}$, $T_1T_7T_9 - 3T_2T_{11} - 3T_5T_{12}$, $4T_6^2T_9 - T_3T_5 - 6T_{11}^2 + 9T_{10}T_{12}$, $2T_1T_6T_9 + 3T_5T_{11} - 3T_4T_{12}$, $T_7^3 - 4T_6T_7T_9 - 27T_8T_{10} + 3T_{11}T_{12}$, $T_6T_7^2 - 4T_6^2T_9 + 3T_{11}^2 - 9T_{10}T_{12}$, $T_1T_7^2 - 4T_1T_6T_9 - 9T_3T_8 - 3T_5T_{11} + 6T_4T_{12}$, $2T_6^2T_7 - T_3T_4 - 9T_{10}T_{11}$, $T_1T_6T_7 + 3T_4T_{11} - 3T_3T_{12}$, $4T_6^3 - T_3^2 - 27T_{10}^2$, $2T_1T_6^2 + 9T_4T_{10} - 3T_3T_{11}$
D_8	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{C}[T_1,\dots,T_{10}]/I$ with I gen. by
			$T_5T_8 + T_4T_9 - 2T_6T_{10}$, $T_3T_8 + T_1T_9 - T_2T_{10}$, $T_7^2 - T_8T_9 + T_{10}^2$, $T_6T_7 + 2T_1T_9 - T_2T_{10}$, $T_5T_7 + T_2T_9 - 2T_3T_{10}$, $T_4T_7 - T_2T_8 + 2T_1T_{10}$, $2T_3T_7 - T_6T_9 + T_5T_{10}$, $T_2T_7 - T_4T_9 + T_6T_{10}$, $2T_1T_7 + T_6T_8 - T_4T_{10}$, $T_4T_5 - T_6^2 + T_8T_9 - T_{10}^2$, $T_2T_5 - 2T_3T_6 + T_7T_9$, $2T_1T_5 - T_2T_6 + T_7T_{10}$, $T_3T_4 - T_1T_5 - T_7T_{10}$, $T_2T_4 - 2T_1T_6 - T_7T_8$, $4T_3^2 + T_5^2 - T_9^2$, $2T_2T_3 + T_5T_6 - T_9T_{10}$, $4T_1T_3 + T_6^2 - T_{10}^2$, $T_2^2 + T_6^2 - T_8T_9$, $2T_1T_2 + T_4T_6 - T_8T_{10}$, $4T_1^2 + T_4^2 - T_8^2$

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}.$$

After a suitable linear change of coordinates one sees that this ring is isomorphic to the Cox ring of symplectic, i.e. crepant, resolutions of considered representation of D_8 computed in [10, Sect. 5].

Proof. This is an application of Algorithm 4.15 with the following modifications (similar to [15, Sect. 3]): There are three minimal elements σ_1, σ_2, σ_3 of the set consisting of all singular $P_0(\gamma_0^*)$ such that γ_0 is an $\mathfrak{F}$-face. We form the cone-wise intersections $\Sigma_i = \{\sigma_i\} \cap \Upsilon$ and resolve the fans: $\Sigma_i' \to \Sigma_i$. Denoting by $\mathrm{primit}(\Xi^{(1)})$ the primitive generators of the rays of a fan or cone Ξ, we have

$$\bigcup_{i=1}^{3} \mathrm{primit}\left(\Sigma_i'^{(1)}\right) \setminus \mathrm{primit}\left(\sigma_i^{(1)}\right) \;=\; \left\{ \begin{smallmatrix} (2,2,1,0,0,0,1,0,0,0), \\ (2,2,2,1,1,1,1,0,0,0) \end{smallmatrix} \right\} \;=:\; \{v_1, v_2\}.$$

The remaining steps of the algorithm deliver the result. All needed verifications, in particular the smoothness test, succeed. □

Proposition 5.4. *In the case of $G = S_3$ in Proposition 5.1 we obtain a resolution $X \to X_0$ with $\mathrm{Cl}(X) = \mathbb{Z}$ and the Cox ring is $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_{13}]/I$ where generators for I and the degree matrix are*

$$\begin{aligned}
3T_9T_{10} - T_7T_{11} + T_6T_{12}, &\quad 3T_5T_{10} - 2T_4T_{11} + T_3T_{12}, \\
3T_6T_8 + T_9T_{11} - T_7T_{12}, &\quad 3T_4T_8 - T_2T_{11} - 2T_5T_{12}, \\
3T_3T_8 - 3T_2T_{10} - T_5T_{11} - T_4T_{12}, &\quad T_5T_7 - 2T_4T_9 + T_1T_{12}, \\
T_4T_7 - 2T_3T_9 + T_1T_{11}, &\quad T_2T_7 - 3T_1T_8 + 2T_5T_9, \\
T_5T_6 - T_3T_9 + T_1T_{11}, &\quad 2T_4T_6 - T_3T_7 + 3T_1T_{10}, \\
T_2T_6 + T_4T_9 - T_1T_{12}, &\quad T_1T_5T_{13} + 6T_9T_{11} - 3T_7T_{12}, \\
T_4^2T_{13} - T_3T_5T_{13} - 3T_{11}^2 + 9T_{10}T_{12}, &\quad T_2T_4T_{13} + T_5^2T_{13} + 9T_8T_{11} - 3T_{12}^2, \\
T_1T_4T_{13} + 3T_7T_{11} - 6T_6T_{12}, &\quad T_2T_3T_{13} + T_4T_5T_{13} + 27T_8T_{10} - 3T_{11}T_{12}, \\
T_1T_3T_{13} + 9T_7T_{10} - 6T_6T_{11}, &\quad T_1T_2T_{13} + 9T_7T_8 - 6T_9T_{12}, \\
T_1^2T_{13} + 3T_7^2 - 12T_6T_9, &\quad 4T_9^3 - T_2^2T_{13} - 27T_8^2, \\
2T_7T_9^2 + T_2T_5T_{13} - 9T_8T_{12}, &\quad 4T_6T_9^2 - T_5^2T_{13} - 3T_{12}^2, \\
2T_1T_9^2 - 9T_5T_8 - 3T_2T_{12}, &\quad T_7^2T_9 - 4T_6T_9^2 - 9T_8T_{11} + 3T_{12}^2, \\
2T_6T_7T_9 - T_4T_5T_{13} - 3T_{11}T_{12}, &\quad T_1T_7T_9 - 3T_2T_{11} - 3T_5T_{12}, \\
4T_6^2T_9 - T_3T_5T_{13} - 6T_{11}^2 + 9T_{10}T_{12}, &\quad 2T_1T_6T_9 + 3T_5T_{11} - 3T_4T_{12}, \\
T_1^2T_9 - 3T_2T_4 - 3T_5^2, &\quad T_7^3 - 4T_6T_7T_9 - 27T_8T_{10} + 3T_{11}T_{12}, \\
T_6T_7^2 - 4T_6^2T_9 + 3T_{11}^2 - 9T_{10}T_{12}, &\quad T_1T_7^2 - 4T_1T_6T_9 - 18T_3T_8 + 9T_2T_{10} + 9T_4T_{12}, \\
2T_6^2T_7 - T_3T_4T_{13} - 9T_{10}T_{11}, &\quad T_1T_6T_7 + 3T_4T_{11} - 3T_3T_{12}, \\
T_1^2T_7 - 3T_2T_3 - 3T_4T_5, &\quad 4T_6^3 - T_3^2T_{13} - 27T_{10}^2, \\
2T_1T_6^2 + 9T_4T_{10} - 3T_3T_{11}, &\quad T_1^2T_6 - 3T_4^2 + 3T_3T_5
\end{aligned}$$

$$\begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

After a suitable linear change of coordinates one sees that this ring is isomorphic to the Cox ring of symplectic resolution of considered representation of S_3 computed in [10, Sect. 4].

Proof. This is an application of Algorithm 4.15; one new ray is added to the fan of the toric ambient variety corresponding to the Cox ring of X_0. To show that the ring is isomorphic to the one given by a generating set in [10, Sect. 4] it is sufficient to compute the relations between these generators and perform the coordinate change to pass from one representation of S_3 to the other. Having this, the smoothness tests are not needed: by [10, Prop. 4.4] we indeed obtain the Cox ring of the symplectic resolution of X_0. □

Acknowledgments

We would like to thank Jürgen Hausen for several helpful discussions. We further thank Yue Ren and Christof Söger respectively for answering question about Singular and Normaliz. Lastly, we would like to thank the anonymous referee for his valuable comments and suggestions.

References

[1] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface, Cox Rings, Cambridge Stud. Adv. Math., vol. 144, Cambridge University Press, Cambridge, 2014.
[2] I.V. Arzhantsev, S.A. Gaĭfullin, Cox rings, semigroups, and automorphisms of affine varieties, Mat. Sb. 201 (1) (2010) 3–24.
[3] V.V. Batyrev, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, J. Eur. Math. Soc. (JEMS) 1 (1) (1999) 5–33.
[4] T. Bogart, A.N. Jensen, D. Speyer, B. Sturmfels, R.R. Thomas, Computing tropical varieties, J. Symbolic Comput. 42 (1–2) (2007) 54–73.
[5] W. Bruns, B. Ichim, T. Römer, R. Sieg, C. Söger, Normaliz. algorithms for rational cones and affine monoids, available at https://www.normaliz.uni-osnabrueck.de.
[6] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Undergrad. Texts Math., Springer-Verlag, New York, 1992.
[7] D.A. Cox, J.B. Little, H.K. Schenck, Toric Varieties, Grad. Stud. Math., vol. 124, American Mathematical Society, Providence, RI, 2011.
[8] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 4-0-3 – a computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2016.
[9] F. Di Biase, R. Urbanke, An algorithm to calculate the kernel of certain polynomial ring homomorphisms, Exp. Math. 4 (3) (1995) 227–234.
[10] M. Donten-Bury, M. Grab, Cox rings of some symplectic resolutions of quotient singularities, preprint, arXiv:1504.07463, 2015.
[11] M. Donten-Bury, S. Keicher, `quotsingcox.lib` – a Singular library to compute Cox rings of quotient singularities, http://www.mathematik.uni-tuebingen.de/~keicher/quotsingcox/, 2016.
[12] M. Donten-Bury, J.A. Wiśniewski, On 81 symplectic resolutions of a 4-dimensional quotient by a group of order 32, preprint, arXiv:1409.4204, 2014.
[13] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.5, 2014.
[14] J. Hausen, Cox rings and combinatorics. II, Mosc. Math. J. 8 (4) (2008) 711–757, 847.
[15] J. Hausen, S. Keicher, A software package for Mori dream spaces, LMS J. Comput. Math. 18 (1) (2015) 647–659.
[16] J. Hausen, S. Keicher, A. Laface, Computing Cox rings, Math. Comp. 85 (297) (2016) 467–502.

[17] J. Hausen, M. Wrobel, Non-complete rational T-varieties of complexity one, preprint, arXiv:1512.08930, 2015.
[18] E. Huggenberger, Fano Varieties with Torus Action of Complexity One, PhD thesis, Universität Tübingen, Wilhelmstr. 32, 72074 Tübingen, 2013.
[19] Y. Ito, M. Reid, The McKay correspondence for finite subgroups of SL(3, **C**), in: Higher-Dimensional Complex Varieties, Trento, 1994, de Gruyter, Berlin, 1996, pp. 221–240.
[20] D. Kaledin, McKay correspondence for symplectic quotient singularities, Invent. Math. 148 (1) (2002) 151–175.
[21] S. Keicher, Computing the GIT-fan, Internat. J. Algebra Comput. 22 (7) (2012) 1250064.
[22] S. Keicher, Algorithms for Mori Dream Spaces, PhD thesis, Universität Tübingen, 2014.
[23] D. Maclagan, B. Sturmfels, Introduction to Tropical Geometry, Grad. Stud. Math., vol. 161, American Mathematical Society, Providence, RI, 2015.
[24] M. Reid, La correspondance de McKay, in: Séminaire Bourbaki, vol. 1999/2000, Astérisque (276) (2002) 53–72.
[25] J. Tevelev, Compactifications of subvarieties of tori, Amer. J. Math. 129 (4) (2007) 1087–1104.
[26] J. Tevelev, On a question of B. Teissier, Collect. Math. 65 (1) (2014) 61–66.
[27] M. Verbitsky, Holomorphic symplectic geometry and orbifold singularities, Asian J. Math. 4 (3) (2000) 553–563.
[28] R. Yamagishi, On smoothness of minimal models of quotient singularities by finite subgroups of $SL_n(\mathbb{C})$, preprint, arXiv:1602.01572, 2016.