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Counting characters above invariant characters in solvable groups

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ABSTRACT

This paper discusses two related questions. First, given a G-invariant character θ of a normal subgroup N of a solvable group, what can we say if the number of characters of G above θ is in some sense as small as possible? Isaacs and Navarro [5] have shown that under certain assumptions about primes dividing the order of the group, one can show that G/N must have a very particular structure. Here we show that these assumptions can be weakened to obtain results about all solvable groups.

We also discuss a related question about blocks. For a prime p and a p-block B of G, we let $k(B)$ denote the number of ordinary characters in B. It is relatively easy to show that $k(B)$ is bounded below by $k(G, D)$, which is the number of conjugacy classes of G that intersect the defect group D of B. In this paper we ask what can be said if equality is achieved. We show that for p-solvable groups, if $k(B) = k(G, D)$, then B is nilpotent and thus $k(B) = |\mathrm{Irr}(D)|$. In addition, we show that this result holds for many blocks of arbitrary finite groups, including all blocks of the symmetric groups. We also extend a result on fully ramified coprime actions in [5].

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1. Introduction

We begin by mentioning a result of Isaacs and Navarro which motivates much of this paper. Recall that if π is a set of primes, we let $k_\pi(G)$ denote the number of conjugacy classes of G of elements of order divisible only by primes in π.

Theorem 1.1. *[5] Let G be a p-solvable group, and if $p = 2$, suppose that $|G|$ is not divisible by a Fermat or Mersenne prime. Let $N \lhd G$ be a p-subgroup, and suppose $\theta \in \mathrm{Irr}(N)$ is G-invariant. If $|\mathrm{Irr}(G|\theta)| = k_{p'}(G/N)$, then G has a normal Sylow p-subgroup.*

In [5] it is shown that Theorem 1.1 is false if G is not assumed to be p-solvable, though it is speculated in [5] that it may be possible to classify the exceptions.

In order to apply this result to blocks, we need to be able to switch the roles of p and p' in the statement of Theorem 1.1 so that we may apply this result to $\mathbf{O}_{p'}(G)$. We also need to remove the hypothesis on Fermat or Mersenne primes. We accomplish both of these here with the following result:

Theorem 1.2. *Let π be a set of primes and let G be a π-separable group. Let $N \lhd G$ be a π-subgroup, and suppose that θ is invariant in G and $|\mathrm{Irr}(G|\theta)| = k_{\pi'}(G/N)$. Then G has a normal Hall π-subgroup.*

Now fix a prime p. Let B be a p-block of G with defect group D, let $k(B)$ denote the number of ordinary irreducible characters in B, and let $\ell(B)$ denote the number of irreducible Brauer characters in B. For a subgroup H of G, we let $k(G, H)$ denote the number of conjugacy classes of G that intersect H. By using results of Brauer about B-elements (see below), it is relatively easy to prove that

$$k(B) \geq k(G, D).$$

We investigate what can be said if equality holds in p-solvable groups.

Theorem 1.3. *Let B be a block of the p-solvable group G with defect group D. If $k(B) = k(G, D)$, then B is nilpotent. In this case we have $\ell(B) = 1$ and $k(G, D) = k(B) = k(D)$.*

We do not know to what extent the hypothesis that G is p-solvable can be removed from Theorem 1.3. However, we are easily able to prove the following:

Theorem 1.4. *Let B be a block of the finite group G with defect group D. Suppose that either D is abelian, or B is the principal block of G. If $k(B) = k(G, D)$, then B is nilpotent, and we have $\ell(B) = 1$ and $k(G, D) = k(B) = k(D)$.*

It is likely that Theorem 1.4 is in the literature, but as our proof is easy and we will need the main idea of the proof elsewhere, we include it here. We will also show, in the final section, that Theorem 1.4 holds for all blocks of the symmetric group. Notice that in each of Theorems 1.3 and 1.4, we have the conclusion that $k(G, D) = k(D)$ for the defect group D. This is of course equivalent to saying that D controls its own fusion.

In the process of proving Theorem 1.2, we use the idea of a "fully ramified coprime action", which was used in [5]. We show that these actions are actually trivial.

Theorem 1.5. *Let H act coprimely on the group Q. Suppose there is a normal subgroup N of H that acts trivially on Q and a character $\theta \in \mathrm{Irr}(N)$ with the following property: for each character $\beta \in \mathrm{Irr}(Q)$, assume that we have θ is fully ramified in H_β. Then H acts trivially on Q.*

In the next section, we discuss some important properties of blocks of finite groups and prove Theorem 1.4. In the third section, we prove Theorem 1.5 about fully ramified coprime actions, which strengthens a result from [5]. Then, we prove a stronger version of the main result of [5]. In the penultimate section, we are then able to prove our main result on blocks that achieve the lower bound, and prove a partial converse. Finally, we briefly prove that our main result on blocks holds (somewhat vacuously) in the symmetric groups.

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2. Small blocks

We will need some block theory to prove Theorem 1.2 about ordinary characters. (Interestingly, we will then use Theorem 1.2 to prove some block theoretic results.) Recall (see [8], for instance) that for a block B of a finite group G, a B-Brauer element of B is a pair (x, b_x), where x is an element of p-power order of G and b_x is a block of $\mathbf{C}_G(x)$ such that $b_x^G = B$. (We will often abbreviate this by simply calling (x, b_x) a B-element.) Of course, G acts on the set of B-elements by conjugation, and we will be considering G-classes of B-elements. Note that (see Theorem 4.14 of [8]) if x is a p-element of G, then there exists a block b_x of $\mathbf{C}_G(x)$ that induces B if and only if x is contained in a defect group of B. However, the block b_x is not uniquely defined by x and B; given a p-element x contained in a defect group of B, there could be multiple blocks of $\mathbf{C}_G(x)$ that induce B. This observation will be key in what follows.

The following is a consequence of Brauer's second main theorem (see for instance Theorem 5.12 or Problem 5.7 of [8]):

Lemma 2.1. *Let B be a block of the finite group G. Then*

$$k(B) = \sum_{(x_i, b_{x_i})} \ell(b_{x_i}),$$

where the sum is over a set $\{(x_1, b_{x_1}), \ldots, (x_n, b_{x_n})\}$ *of representatives of conjugacy classes of* B*-elements of* G*.*

In the notation of the above lemma, suppose that (x, b_x) is a B-element, where B has defect group D. Thus $b_x^G = B$, and therefore x is contained in a conjugate of D. Thus we may choose each x_i in the above lemma to be contained in a fixed defect group D of G. Of course, as mentioned before, for a particular element x in D, there is at least one but could be multiple blocks of $\mathbf{C}_G(x)$ that induce to B, and these blocks of $\mathbf{C}_G(x)$ need not be conjugate in G. Therefore the element x could appear multiple times in the above sum. Thus the sum

$$\sum_{(x_i, b_{x_i})} \ell(b_{x_i}),$$

(where the sum is over the classes of B-elements of G) is bounded below by the number of conjugacy classes of G that intersect D nontrivially. Thus we have proven:

Theorem 2.2. *Let* B *be a block of a finite group with defect group* D*. Then*

$$k(B) \geq k(G, D).$$

Ultimately we will be interested in proving results about when equality occurs in the above theorem. We will prove a nontrivial necessary condition for equality in p-solvable groups. First, however, we may prove necessary conditions for equality for the principal block of an arbitrary finite group, or blocks of finite groups with abelian defect group.

We begin with an important lemma:

Lemma 2.3. *Let* B *be a block of the finite group* G *with defect group* D*. Suppose that* $k(B) = k(G, D)$*. Then for each element* $x \in D$*, there exists a unique block* b_x *of* $\mathbf{C}_G(x)$ *such that* $b_x^G = B$*, and for this block* b_x*, we have* $\ell(b_x) = 1$*.*

Proof. In the above argument proving the lower bound, if we have the equality $k(B) = k(G, D)$, then for each $x \in D$, there must be exactly one block b_x that induces B. Moreover, the equality forces $\ell(b_x) = 1$ for each such block. $\square$

We also need a result of Puig and Watanabe giving a sufficient condition for a block B to be nilpotent. We briefly recall the definition of nilpotence of a block, which was originally defined and studied by Broué and Puig in [2]. For a block B of a finite group G, a B-subpair is defined to be a pair (Q, b_Q), where Q is a p-subgroup of G and b_Q is a block of $Q\mathbf{C}_G(Q)$ such that $b_Q^G = B$. (Thus a B-element is just a B-subpair where the subgroup Q is cyclic.) Note that G acts by conjugation on the B-subpairs of G, and we let $\mathbf{N}_G(Q, b_Q)$ be the stabilizer of the pair (Q, b_Q) under this action. Broué and Puig defined a block B to be nilpotent if $\mathbf{N}_G(Q, b_Q)/\mathbf{C}_G(Q)$ is a p-group for every B-subpair (Q, b_Q).

This definition of course is motivated by the characterization of p-nilpotent groups by Frobenius. Broué and Puig [2] proved the following nice result:

Theorem 2.4. *Let G be a finite group and suppose B is a nilpotent block of G with defect group D. Then $\ell(B) = 1$ and $k(B) = |\mathrm{Irr}(D)|$.*

In the process of proving this, they also showed that if B is nilpotent, then $\ell(b_Q) = 1$ for each B-subpair. One of our main tools here will be a result of Puig and Watanabe [10], which is a partial converse of this last property of nilpotent blocks:

Theorem 2.5. *[10] Let B be a block of the finite group G, and suppose that the defect group D of B is abelian. Assume that $\ell(b_x) = 1$ for each B-element (x, b_x). Then B is nilpotent.*

As a consequence, we may now easily prove Theorem 1.4 from the Introduction.

Proof of Theorem 1.4. First assume the defect group of D is abelian and $k(B) = k(G, D)$. By Lemma 2.3, we have that $\ell(b_x) = 1$ for each B-element (x, b_x). Thus by Theorem 2.5, B is nilpotent, and by Theorem 2.4 we have that $\ell(B) = 1$ and $k(B) = k(G, D) = k(D)$ in this case.

Now assume that B is the principal block of G. Again, by using Lemma 2.3 we see that $\ell(b_x) = 1$ for every B-element of G. In particular, if $x = 1$, then $b_x = B$ and we have $\ell(B) = 1$. Thus by Corollary 6.13 of [8] we have that G has a normal p-complement N. In this case of course we have $\mathbf{N}_G(P)/\mathbf{C}_G(P)$ is a p-group for every p-subgroup P of G, and thus B is nilpotent and we again are done by Theorem 2.4. □

Finally, we end this section by putting some of our results into a broader context. As we have seen, our hypothesis that $k(B) = k(G, D)$ forces each p-element x to have a unique (up to conjugacy) block b_x of $\mathbf{C}_G(x)$ that induces to B. Moreover, $\ell(b_x) = 1$ for each such block. It has been conjectured by Puig that the condition that $\ell(b_Q) = 1$ for every B-subpair (Q, b_Q) (not just B-elements) is a necessary and sufficient condition for B to be nilpotent, and Watanabe [11] has shown that this condition does guarantee that B is nilpotent for solvable groups (or, in general, groups for which the Alperin weight condition is known to be true for all subgroups "involved" in G). Our main results on blocks, Theorems 1.3 and 1.4, give a sufficient condition for nilpotency in terms of the B-elements only, not all of the B-subpairs.

3. Fully ramified coprime actions

We now examine fully ramified coprime actions, which will be used in the proof of Theorem 1.2. These were also used in the proof of the main result of [5]. Our approach differs from that in [5] in that we use the block-theoretic results of the previous section to shorten and improve the proof of Corollary 3.4 of [5]. We will also use a result of

DeMeyer and Janusz [3] about groups of central type to extend the results of [5] to the case where the group that is acting is an arbitrary finite group (of order coprime to the order of the group being acted on).

We need the following definitions:

Definition 3.1. Let N be a normal subgroup of the group G, and suppose $\theta \in \mathrm{Irr}(N)$ is invariant in G. We say θ is fully ramified in G if there exists a unique character $\chi \in \mathrm{Irr}(G|\theta)$. (See [4] for more on fully ramified characters.)

We mention two important properties of fully ramified characters. Let $\theta \in \mathrm{Irr}(N)$ be invariant in G, where $N \lhd G$, and let $\chi \in \mathrm{Irr}(G)$ lie above θ. Then θ is fully ramified in G if and only if $\chi(1)/\theta(1) = \sqrt{|G : N|}$. Moreover, θ is fully ramified in G if and only if there exists a character $\psi \in \mathrm{Irr}(G|\theta)$ such that ψ vanishes off of N.

Definition 3.2. Suppose H acts coprimely on the finite group Q, and suppose $N \lhd H$ acts trivially on Q. We say the action of H on Q is a fully ramified coprime action with respect to N and $\theta \in \mathrm{Irr}(N)$ if, for every character $\beta \in \mathrm{Irr}(Q)$, we have that θ is fully ramified in H_β. If such an action exists, we say that H is a fully ramified coprime action on Q.

Note that by setting β to be the trivial character of Q, we see that θ must be invariant and fully ramified in H. We now show that if Q is abelian, then any fully ramified coprime action on Q must be trivial. We first prove this assuming the normal subgroup of H is the kernel of the action, but then extend it to any normal subgroup of H that acts trivially on Q.

Lemma 3.3. *Let H act coprimely on the abelian p-group Q, and let $K \lhd H$ be the kernel of the action. Suppose the action is fully ramified for the H-invariant character $\theta \in \mathrm{Irr}(K)$. Then H acts trivially on Q.*

Proof. Let G be the semidirect product of H acting on Q. Thus $K = \mathbf{O}_{p'}(G)$ and θ is invariant in G. Thus $\mathrm{Irr}(G|\theta)$ consists of a single block B with normal abelian defect group Q. Of course, as Q is a normal Sylow p-subgroup of G, we have $k_p(G) = k(G, Q)$, which is also equal to the number of G-orbits of characters of Q.

We count the characters of G above θ. As θ is invariant in G, we see that θ has an inflation $\widehat{\theta} \in \mathrm{Irr}(KQ)$, which is also invariant in G. Note that if $\beta \in \mathrm{Irr}(Q)$, the fact that θ is fully ramified in H_β implies that $\widehat{\theta}$ is fully ramified in $G_\beta = QH_\beta$. Thus $|\mathrm{Irr}(G|\widehat{\theta}\beta)| = |\mathrm{Irr}(G_{\widehat{\theta}\beta}|\widehat{\theta}\beta)| = 1$ by the Clifford correspondence. Therefore

$$k(B) = |\mathrm{Irr}(G|\theta)| = \sum_{\beta} |\mathrm{Irr}(G|\widehat{\theta}\beta)|,$$

which is in turn equal to the number of G orbits of characters of Q (here the sum is over a set of representatives of the G-orbits of $\mathrm{Irr}(Q)$). This is of course equal to $k_p(G) = k(G, Q)$.

Thus we now have $k(B) = k(G, Q)$, where Q is the defect group for B. Therefore by Lemma 2.3 we have that $\ell(b_x) = 1$ for each B-element (x, b_x). As Q is abelian, Theorem 2.5 gives that B is nilpotent, and Lemma 9 of [9] gives that G has a normal p-complement. Thus $H = K$ and we are done. □

Notice that the above result is purely character theoretic in nature, but the proof relied on non-trivial results from block theory. It would be nice (and is likely the case) if there were a purely character theoretic proof of Lemma 3.3 but we have not yet been able to find one.

Lemma 3.4. *Let H act coprimely on the abelian group p-group Q, and let $N \lhd H$ be such that N acts trivially on Q. Suppose the action is fully ramified for the H-invariant character $\theta \in \mathrm{Irr}(N)$. Then H acts trivially on Q.*

Proof. We work by induction on $|H/N|$. Let K be the kernel of the action of H on Q (so we have $N \leq K$), and let $\varphi \in \mathrm{Irr}(K)$ lie above θ. Of course H is the stabilizer of the trivial character in Q, and thus by assumption θ is fully ramified in H. As φ lies over θ, we have that $|\mathrm{Irr}(H|\varphi)| = 1$ (note that we may not conclude that φ is fully ramified in H, as we do not know if φ is actually invariant in H).

Let T be the stabilizer of φ in H. We claim that the hypotheses hold for the action of T on Q with the normal subgroup K of T. Thus we need to show that if $\beta \in \mathrm{Irr}(Q)$, then φ is fully ramified in T_β. Let $\gamma \in \mathrm{Irr}(T_\beta|\varphi)$. Then $\gamma^{H_\beta} \in \mathrm{Irr}(H_\beta|\varphi) \subseteq \mathrm{Irr}(H_\beta|\theta)$, and thus if there were multiple characters in $\mathrm{Irr}(T_\beta|\varphi)$, then by the Clifford correspondence there would be multiple characters in $\mathrm{Irr}(H_\beta|\theta)$, contradicting our assumption. Thus T satisfies the hypotheses of the lemma with respect to the normal subgroup K.

If $T = H$, then Lemma 3.3 applied to K and φ shows that H acts trivially on Q and we are done. Thus we may assume that $T < G$. In this case, we have by induction that T acts trivially on Q, and thus $T = K$ and φ induces irreducibly to H. As θ is invariant in H, we have that each of the H-conjugates of φ must lie over θ. Let $\delta \in \mathrm{Irr}(Q)$, and note that $K \leq H_\delta \leq H$. By assumption, θ is fully ramified in H_δ. If $H_\delta < H$, then there are multiple H_δ orbits of characters of K lying above θ, and thus there are multiple characters of H_δ lying above θ. This contradicts our assumption that θ is fully ramified in H_δ. Therefore $H_\delta = H$ for every character $\delta \in \mathrm{Irr}(Q)$. Therefore H acts trivially on Q. □

We now examine the case when we have a fully ramified coprime action on a perfect group. As we will see, these actions will also turn out to be trivial.

The following is Theorem 3.8 of [5], and it depends on the classification of finite simple groups:

Theorem 3.5. *Let H be a p-group that acts coprimely and nontrivially on the perfect group G. Then there exists an irreducible character $\beta \in \mathrm{Irr}(G)$ such that $|H/H_\beta| = p$.*

Corollary 3.6. *Suppose the group H has a fully ramified coprime action on the perfect group G, with respect to the normal subgroup N of H. If H/N is a p-group, then H acts trivially on G.*

Proof. Assume H acts nontrivially on G. Let $\theta \in \mathrm{Irr}(N)$ be such that the action is fully ramified with respect to θ. Then θ is fully ramified in H and θ is fully ramified in the stabilizer of the character β from Theorem 3.5 (applied to H/N). This contradicts Lemma 3.2 of [5]. $\square$

We are now almost able to show that if there is a fully ramified coprime action of H on Q, then the action is trivial. First, we briefly digress to discuss character triple isomorphisms and groups of central type.

Recall that if $N \lhd G$ with $\theta \in \mathrm{Irr}(N)$ invariant in G and $\widehat{N} \lhd \widehat{G}$ with $\widehat{\theta} \in \mathrm{Irr}(\widehat{N})$ invariant in $\widehat{G}$, then the triples (G, N, θ) and $(\widehat{G}, \widehat{N}, \widehat{\theta})$ are said to be "character triple isomorphic" if $G/N \cong \widehat{G}/\widehat{N}$ and the behavior of the characters of the subgroups of G containing N lying above θ is in some sense the same as the behavior of the characters of the subgroups of $\widehat{G}$ containing $\widehat{N}$ that lie above $\widehat{\theta}$. (See Chapter 11 of [4] for more details.) Moreover, if N is a π-group for some set of primes π, then an isomorphic character triple $(\widehat{G}, \widehat{N}, \widehat{\theta})$ may be chosen such that $\widehat{N}$ is a central π-subgroup of $\widehat{G}$. As is often the case, we will end up replacing our character triple (G, N, θ) with such an isomorphic triple that is easier to work with. Now suppose that G is the semidirect product of a π-group H acting coprimely on the group Q, with a fully ramified coprime action of H with respect to the normal subgroup N of G and the character $\theta \in \mathrm{Irr}(N)$. If $(\widehat{G}, \widehat{N}, \widehat{\theta})$ is an isomorphic character triple with $\widehat{N}$ a central π-subgroup of $\widehat{G}$, then it is immediate from the properties of character triple isomorphisms that there are subgroups $\widehat{H}$ and $\widehat{Q}$ of $\widehat{G}$ such that $\widehat{H}$ has a fully ramified coprime action on $\widehat{Q}$ with the character $\widehat{\theta}$ of $\widehat{N}$. Of course, in this case H acts trivially on Q if and only if $\widehat{H}$ acts trivially on $\widehat{Q}$. Thus we may (and on occasion will) use character triple replacements, which will not affect our hypotheses or conclusions.

Also, recall (see for instance [3]) that a group G is said to have central type if there is an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)^2 = |G : \mathbb{Z}(G)|$. This condition is equivalent to saying that the unique irreducible constituent of $\chi_{\mathbb{Z}(G)}$ is fully ramified in G. We say the triple (G, N, θ) is of central type if $N = \mathbb{Z}(G)$ and θ is fully ramified in G.

Compare the following to Theorem 3.5 of [5].

Theorem 3.7. *Suppose that there is a fully ramified coprime action of H on Q. Then H acts trivially on Q.*

Proof. We work by induction on $|Q| + |H : M|$, where M is the normal subgroup of H in the definition of a fully ramified coprime action. If Q contains a proper nontrivial H-invariant normal subgroup N, then the hypotheses hold for the action of H on Q/N. Thus by induction H acts trivially on Q/N.

We claim that the hypotheses also hold for the action of H on N. We do this by showing that if $\mu \in \mathrm{Irr}(N)$, then H_μ is also the stabilizer of a character $\chi \in \mathrm{Irr}(Q)$. By Theorem 13.31 of [4], we have that there exists a character $\chi \in \mathrm{Irr}(Q)$ lying over μ such that χ is H_μ-invariant, and thus $H_\mu \leq H_\chi$. By Theorem 13.27 of [4], there exists a character ν of N lying under χ such that ν is H_χ-invariant, and thus $H_\chi \leq H_\nu$. Thus $\nu = \mu^g$ for some $g \in Q$. However, H fixes g modulo N, and therefore $\mu = \nu$ and we have $H_\chi = H_\mu$, as desired.

Therefore the hypotheses hold for N, and as $N < Q$, we have by induction that H acts trivially on N. Thus $[Q, H, H] = 1$, and therefore (since $|H|$ is coprime to $|Q|$) we have that $[Q, H] = 1$, as desired.

Thus we may assume that Q has no nontrivial proper H-invariant normal subgroups. Thus $Q = Q'$ or Q is an abelian p-group. If Q is abelian, then we are done by Lemma 3.4.

Thus we may assume that $Q' = Q$. Let G be the semidirect product of H acting on Q. By assumption, there is a fully ramified action of H on Q, and thus there is a normal subgroup M of H and a character θ of M that satisfies the definition of a fully ramified coprime action. Note that M is normal in G. By replacing (G, M, θ) by an isomorphic character triple (which does not affect our hypotheses or conclusion, see the earlier discussion) we may assume that M is central in G. Let β be any character of Q. We claim that the triple (H_β, M, θ) is of central type; thus we must show $M = \mathbf{Z}(H_\beta)$. As θ is fully ramified in H_β, we have a unique character α of H_β that lies over θ. Note that α vanishes off of M, and thus $\mathbf{Z}(H_\beta) \leq \mathbf{Z}(\alpha) \leq M \leq \mathbf{Z}(H_\beta)$ and thus we have equality throughout. Thus $M = \mathbf{Z}(H_\beta)$ and the triple (H_β, M, θ) is of central type.

Now let p be a prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. By Theorem 2 of [3] applied to H_β, we have that P_β is of central type with center $M \cap P_\beta$, where we are using the irreducible character $\theta_{M \cap P}$. Thus the hypotheses hold for the normal subgroup $M \cap P$ of P, and if $|P : P \cap M| < |H : M|$, we have by induction that P centralizes Q. As this holds for every prime divisor p of $|H|$, we are done unless H/M is a p-group. However, we are done in this case by Lemma 3.6. □

We point out here that much of the argument to prove Theorem 3.7 comes from [5].

4. The proof of Theorem 1.2

We are now able to prove the improved version of the main result of [5]. Following [5], we actually prove a slightly stronger result, which immediately implies our Theorem 1.2.

Theorem 4.1. *Let G be a π-separable group and let N be a normal subgroup of G. Suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant and π-special. If $|\mathrm{Irr}(G|\theta)| = k_{\pi'}(G/N)$, then G/N has a normal Hall π-subgroup.*

Our proof follows (indeed, is essentially the same as) the proof of Theorem 4.2 in [5]. Only now, we have a stronger version of the result about fully ramified coprime actions in hand, and we will use that. Thus we will only sketch the proof and highlight the slight differences between our proof and Theorem 4.2 in [5]. With the exception of the fully ramified coprime action results, all of the results used in the proof of Theorem 4.2 depend only on the coprimeness of the groups in question, and not on the fact that π was specifically the set $\{p\}$. Our fully ramified coprime action result (Theorem 3.7) replaces the weaker version in [5].

Proof. We work by induction on $|G : N|$. By replacing (G, N, θ) with an isomorphic character triple, we may assume that N is a central π-subgroup of G (see for instance Lemma 2.1 of [7]).

Suppose first that G/N has a minimal normal subgroup M/N with π order. We may exactly copy paragraphs 2–5 of the proof of Theorem 4.2 of [5] in this case to get that G/N has a normal Hall π-subgroup. (Notice that the known properties of π-special characters replace Lemma 4.3 in [5].)

Thus we may assume that $\mathbf{O}_\pi(G/N)$ is trivial, and we let M/N be a minimal normal π'-subgroup of G/N. Arguing exactly as in paragraphs 7–13 of the proof of Theorem 4.2 in [5], we see that we can reduce this to the case that G/M is a π-group, and thus if H is a Hall π-subgroup of G, we have $H \cap M = N$ and $HM = G$. (This slightly different conclusion follows from the fact that N is a normal π-subgroup of G.) Moreover, again using the arguments in paragraphs 7–13, we get that θ is fully ramified in H_β for every character $\beta \in \mathrm{Irr}(M/N)$. Thus we may apply Theorem 3.7 to see that H acts trivially on M/N, and thus H/N is normal in G/N. $\square$

Theorem 1.2 now follows immediately by applying Theorem 4.1 to the normal π-subgroup N, noting that θ is trivially π-special in this case.

5. Nilpotent blocks

We may now prove Theorem 1.3.

Proof. Of course, the second statement follows from the first from the Broué–Puig theorem [2]. We now prove that if $k(B) = k(G, D)$, then B is nilpotent (assuming that G is p-solvable). We work by induction on $|G|$.

Let $N = \mathbf{O}_{p'}(G)$ and let $\theta \in \mathrm{Irr}(N)$ be covered by B. First assume that θ is not invariant in G, and let $T = G_\theta$ and let B' be the unique block of T that induces to B. We may of course assume that D is a defect group of T. From Theorem 2.2 we have

$$k(B) = k(B') \geq k(T, D) \geq k(G, D) = k(B).$$

Here the last equality is our assumption. Thus we have equality throughout and therefore $k(B') = k(T, D)$. By the inductive hypothesis, we have that B' is nilpotent, and by Lemma 1 of [9] we have that B is nilpotent.

Thus we may assume that θ is invariant in G. Therefore D is a full Sylow p-subgroup of G, and $k(G, D) = k_p(G) = k_p(G/N)$, where the last equality follows from, for instance, Lemma 2.3 of [5]. Our hypothesis now implies that $|\mathrm{Irr}(G|\theta)| = k(B) = k(G, D) = k_p(G/N)$. By Theorem 1.2 we have that G has a normal p-complement. Thus by [2] we have that B is nilpotent. $\square$

We have a partial converse.

Theorem 5.1. *Let G be a p-solvable group and let B be a nilpotent block of G with defect group D. If φ is the unique Brauer character in B, then there exists a subgroup H of G and a nilpotent block b of H with defect group D such that if θ is the unique Brauer character of b then $\theta^G = \varphi$ and $k(H, D) = k(b) = k(B)$.*

Proof. Let $N = \mathbf{O}_{p'}(G)$ and let $\gamma \in \mathrm{Irr}(N)$ be covered by B. Let T be the stabilizer of γ in G. If $T < G$, then let B' be the block of T that corresponds to B via the Fong–Reynolds correspondence, and working by induction on $|G|$, we are done.

Thus we may assume that γ is invariant in G. By Lemma 2 of [9] we see that N is a full p-complement of G. We take $b = B$ and note that $k(D) = k(G/N) = k_p(G)$ by Lemma 2.3 of [5]. $\square$

6. The symmetric groups

Recall that a block B has defect zero if the defect group D of B is trivial. Blocks with defect zero are trivially nilpotent, as the only B-pair is trivial. We will show that if B is a block of the symmetric group S_n with defect group D such that $k(B) = k(S_n, D)$, then B has defect zero and B is therefore nilpotent. We will assume in this section the reader is familiar with the representation theory of the symmetric group; see [6] or [1] for more details.

Theorem 6.1. *Let B be a p-block of a symmetric group S_n with defect group D. If $k(B) = k(S_n, D)$, then B has defect zero and we trivially have that B is nilpotent and $\ell(B) = 1$ and $k(B) = k(S_n, D) = k(D) = 1$.*

Proof. Suppose first that B has weight $w \geq 2$. Then $k(B)$ is known to be equal to

$$\sum_{w_1+w_2+\ldots+w_p=w} p(w_1)p(w_2)\cdots p(w_p),$$

where $p(k)$ is the partition function.

It is known that if the p-block B of S_n has weight w, then a defect group of B is a Sylow p-subgroup of $S_{pw} \leq S_n$. Thus $k(S_n, D)$ is equal to the number of classes of S_{pw} of p-power order. These are indexed by partitions of pw for which each part has p-power length. By removing the rows of length one, we see that these are in one-to-one correspondence with the partitions of $pw - pk$ with only p-power parts, as k runs from 0 to w. Thus $k(S_n, D)$ is equal to

$$\sum_{0\leq \ell \leq w} |\{\sigma \vdash p\ell | \sigma \text{ has only } p\text{-power parts}\}| .$$

As $w \geq 2$, we have

$$\sum_{0\leq \ell \leq w} |\{\sigma \vdash p\ell | \sigma \text{ has only } p\text{-power parts}\}| \leq \sum_{\ell \leq w} p(\ell),$$

which is of course strictly less than

$$\sum_{w_1+w_2+\ldots+w_p=w} p(w_1)p(w_2)\cdots p(w_p) = k(B).$$

If $w = 1$, then the defect group is abelian and we are done by Theorem 1.4.

Thus the condition $k(B) = k(S_n, D)$ holds if and only if B has defect zero. $\square$

We do not yet know if the condition that $k(B) = k(G, D)$ implies that B is nilpotent for blocks of any finite group, though we have shown here that it certainly is often the case. It seems the next thing to do would be to check the simple groups to see if a counterexample can be found.

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