Journal of Algebra 476 (2017) 48–84

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Permutability of injectors with a central socle in a finite solvable group

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ARTICLE INFO

Article history:
Received 2 August 2016
Available online 3 January 2017
Communicated by Gernot Stroth

MSC:
20D10
20D20

Keywords:
Finite solvable group theory
Fitting class
Injector
Central socle

ABSTRACT

In response to an Open Question of Doerk and Hawkes [5, IX Section 3, page 615], we shall show that if $\mathcal{Z}^\pi$ is the Fitting class formed by the finite solvable groups whose π-socle is central (where π is a set of prime numbers), then the $\mathcal{Z}^\pi$-injectors of a finite solvable group G permute with the members of a Sylow basis in G. The proof depends on the properties of certain extraspecial groups [4].

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[1] The first author is grateful to the University of Valencia for its hospitality.
[2] The third author has been supported by Proyecto MTM2014-54707-C3-1-P, Ministerio de Economía y Competitividad, Spain.

http://dx.doi.org/10.1016/j.jalgebra.2016.11.041
0021-8693/

1. Introduction

Throughout this Introduction, let H be a subgroup of a finite solvable group G, and let $\pi = \{p_1, p_2, \ldots, p_m\}$ be a set of prime numbers. We use the notation of Doerk and Hawkes [5], and as in [3], we define $\mathcal{Z}^\pi$ to be the class of finite solvable groups H such that $\mathrm{Soc}_\pi H \leq \mathbf{Z}(H)$, and as before write $\mathcal{Z}^p = \mathcal{Z}^{\{p\}}$. These classes are Fitting classes, so that any finite solvable group possesses a conjugacy class of injectors for any given such class. In [3], we described inductive methods for constructing $\mathcal{Z}^\pi$-injectors. In this work, we prove that these injectors are permutable. This means that for any such injector H there exists a *Sylow basis* Σ in G such that H permutes with every element of Σ, where a Sylow basis is a set of Sylow subgroups of G, with $|\Sigma \cap \mathrm{Syl}_p G| = 1$ for each prime number p, such that all pairs of members of Σ permute with each other [5, I(4.7)]. Doerk and Hawkes characterize the property of permutability as the one that separates manageable from unmanageable Fitting classes [5, p. 615], making the often difficult determination of permutability of a Fitting class the key to obtaining a thorough analysis of its properties. We prove:

Theorem. *If π is a set of prime numbers, and G is a finite solvable group, then the $\mathcal{Z}^\pi$-injectors of G are system permutable in G.*

Corollary. *Let $G = HK$ be a solvable semidirect product, with $K \lhd G$ and $H \cap K = 1$, and suppose U is an $\mathbf{F}_pG$-module (where p is a prime number, and $\mathbf{F}_p$ is the field of order p). Choose a Sylow p-subgroup P of H, and assume that $p \nmid |K|$. Let $\mathrm{Soc}_{\mathbf{F}_pG} U$ be the socle of U (generated by the minimal submodules). If $\mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_pG} U) = 1$ and $\mathbf{C}_G(\mathbf{C}_U(P)) = H$, then there is a Sylow basis Σ of K, such that H normalizes each subgroup in Σ.*

Proof. We can deduce the Corollary from the Theorem, using some of the results quoted in Section 2, as follows. Form the natural semidirect products

$$G_0 = GU, \quad H_0 = HU, \quad K_0 = KU, \quad P_0 = PU.$$

Then $U = \mathbf{C}_{G_0}(\mathrm{Soc}_{\mathbf{F}_pG_0} U)$, so U is the $\mathcal{Z}^p$-radical of G_0 by Lemma 2.7(c). Also P_0 is a Sylow p-subgroup of G_0 and $H_0 = \mathbf{C}_{G_0}(\mathbf{C}_U(P_0))$, so H_0 is a $\mathcal{Z}^p$-injector of G_0 by Lemma 2.7(f) (or [5, IX(4.19)]). Hence H_0 is system permutable in G_0 by the Theorem, and it follows from Lemma 2.4(d) that $H \cong H_0/U$ normalizes a Sylow basis Σ in $K \cong K_0/U$. $\square$

The lay-out of the paper is as follows: In Section 2 we state some known results and prove results on a variety of topics for later use, and in Section 3 we quote some results about extraspecial groups [4]. We begin a general study of counterexamples to permutability claims for injectors in Section 4, introducing the specific case of permutability of $\mathcal{Z}^\pi$-injectors in Section 5.

By the end of Section 5, we have established that the falsity of our Theorem would imply the existence of a prime $p \in \pi$ and an elementary abelian p-group U on which a finite solvable group $G = HK$ acts faithfully, where among the conditions that apply is that K complements H and is normal in G, and K acts irreducibly on U, while K has a normal subgroup K_∞ such that H acts irreducibly on the quotient K/K_∞. In Section 6, we establish that this K_∞ acts homogeneously on U, and in Section 7, we exploit this situation to obtain more precise information about a potential counterexample to our claim. In Section 8, we obtain the contradiction required to complete our proof.

2. Quoted results

Remark. Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 will be used in Section 4.

Notation. If r is a prime number, and G is a finite group, let $\mathrm{Syl}_r\, G$ be the set of Sylow r-subgroups of G, and write $H \text{ pr } G$ to mean that H is a *pronormal* subgroup of G [5, I(6.1)].

Lemma 2.1. *Let $G = HK$ be a finite solvable group, with $H \leq G$ and $K \lhd G$.*

(a) [2, Lemma 3(c)] *If $K = [K, H]K'$, then $K = [K, H]$.*
(b) [1, Lemma 6, Remark (ii)] *Suppose $H \cap K = 1$, and let r be a prime number. Choose $S \in \mathrm{Syl}_r\, H$, and put $H_\infty = \mathbf{O}^r(H)$ and $R = \mathbf{O}_r(K)$. If $H \text{ pr } G$, then $\mathbf{C}_R(H_\infty) \leq \mathbf{C}_R(S)$.*

Proof. (a) This is proved in the given reference.

(b) Take $M = \mathbf{C}_R(H_\infty)$ and note that HM is a semidirect product, with $M \lhd HM$. Also $H \text{ pr } G$, so $H \text{ pr } HM$ [5, I(6.3.a)], and hence $M = \mathbf{C}_M(H) \cdot [M, H]$ by the given reference. Now H_∞ centralizes M by definition, while $H = SH_\infty$, and therefore $M = \mathbf{C}_M(S) \cdot [M, S]$. Thus $[M, S] = [\mathbf{C}_M(S)[M, S], S] = [M, S, S]$. But $S[M, S]$ is an r-group, so if $[M, S] \neq 1$, then $[M, S, S] < [M, S]$ [5, A(8.3.f)]. Hence $[M, S] = 1$. □

Notation. Let Σ be a Sylow basis in a finite solvable group G [5, I(4.7)], and suppose $H \leq G$ and $N \lhd G$. Define $\Sigma \cap H = \{P \cap H : P \in \Sigma\}$ and $\Sigma N/N = \{PN/N : P \in \Sigma\}$. If $\Sigma \cap H$ is a Sylow basis in H, then Σ is said to *reduce into* H, and we write $\Sigma \searrow H$. If $HP = PH$ for all subgroups $P \in \Sigma$, then H is called Σ-*permutable.*

Lemma 2.2. *Suppose $N \lhd G$ and $N \leq H \leq G$, where G is a finite solvable group, and let Σ be a Sylow basis in G.*

(a) [5, I(6.3.c)] *Then $H \text{ pr } G$ if and only if $H/N \text{ pr } G/N$.*
(b) [5, I(4.17)] *Also $\Sigma \searrow H$ if and only if $\Sigma N/N \searrow H/N$.*
(c) *Moreover H is Σ-permutable if and only if H/N is $\Sigma N/N$-permutable.*

Proof. The statements (a) and (b) are proved in the given references.

(c) Suppose $P \in \Sigma$. If H permutes with P, then clearly H/N permutes with PN/N. Conversely if H/N permutes with PN/N, then there is a subgroup $HP/N \leq G/N$, so there is a subgroup $HP \leq G$. $\square$

Lemma 2.3. *Let Σ be a Sylow basis in a finite solvable group G.*

(a) [5, I(3.2.c)] *If $K \lhd G$, then $\Sigma \searrow K$.*
(b) (Lockett [5, I(4.22.b)]) *Suppose H, $K \leq G$ with $HK = KH$, and $P \in \Sigma$. If $\Sigma \searrow H$ and $\Sigma \searrow K$, then $\Sigma \searrow HK$ and $P \cap HK = (P \cap H)(P \cap K)$.*
(c) [5, I(5.4.b), (5.4.c) and (5.6)] *Suppose r is a prime number, and take $G_\infty = \mathbf{O}^r(G)$, $\{R_\infty\} = (\Sigma \cap G_\infty) \cap \mathrm{Syl}_r\, G_\infty$ and $S_0 = \mathbf{O}_r(\mathbf{N}_G(\Sigma))$. Then $\{S_0 R_\infty\} = \Sigma \cap \mathrm{Syl}_r\, G$.*

Proof. The statements (a) and (b) are proved in the given references.

(c) Note first that $G = \mathbf{N}_G(\Sigma)G_\infty$ by the third reference, and $\mathbf{N}_G(\Sigma) = S_0 \times \mathbf{O}_{r'}(\mathbf{N}_G(\Sigma))$ by the second reference. Moreover $\mathbf{O}_{r'}(\mathbf{N}_G(\Sigma)) \leq G_\infty$, and hence $G = S_0 G_\infty$. Finally $\Sigma \searrow S_0$ by the first reference, so we get the result. $\square$

Definition. A subgroup H of a finite solvable group G is said to be *system permutable* in G if there is a Sylow basis Σ in G such that H is Σ-permutable.

Lemma 2.4. *Let Σ be a Sylow basis in a finite solvable group G.*

(a) (Mann [5, I(6.6)]) *Suppose $H \leq G$. Then H pr G if and only if H satisfies the following condition:*
(P) *if $\Sigma \searrow H$ and $\Sigma^g \searrow H$ with $g \in G$, then $g \in \mathbf{N}_G(H)$.*
(b) [5, I(6.7)] *Suppose H pr G with $\Sigma \searrow H$. If H is system permutable in G, then H is Σ-permutable.*
(c) (Lockett [5, I(6.8)]) *Suppose H pr G with $\Sigma \searrow H$. If $\mathbf{N}_G(H) \leq L \leq G$, then $\Sigma \searrow L$.*
(d) *Suppose $G = HK$ and $N = H \cap K$, with $H \leq G$ and $K, N \lhd G$, where H pr G and $\Sigma \searrow H$. Then H is system permutable in G if and only if H normalizes the Sylow basis $(\Sigma \cap K)N/N$ in K/N.*

Proof. The statements (a) and (b) are proved in the given references.

(c) By extending $\Sigma \cap H$, we can find a Sylow basis Σ_1 in G such that $\Sigma_1 \searrow H$ and $\Sigma_1 \searrow L$ [5, I(4.16) and (4.18)]. Then $\Sigma_1 = \Sigma^g$ with $g \in G$ [5, I(4.12)], and (a) implies that $g \in \mathbf{N}_G(H) \leq L$. Hence $\Sigma \searrow L^{g^{-1}} = L$.

(d) By (b), H is system permutable if and only if H permutes with the subgroups $P \in \Sigma$. Using Lemma 2.3(a) and (b) we get $\Sigma \searrow K$, and $P = P_0 P_1$ with $P_0 = P \cap H$ and $P_1 = P \cap K$. If $HP = PH$, then H normalizes $HP \cap K = HP_1 \cap K = (H \cap K)P_1 = NP_1$. Conversely if $H \leq \mathbf{N}_G(NP_1)$, then there is a subgroup $HNP_1 = HP_0P_1 = HP$. $\square$

Notation. Let $\mathcal{F}$ be a Fitting set in a finite solvable group G [5, VIII(2.1)]. Then G has an $\mathcal{F}$*-radical* [5, VIII(2.3.b)], and $\mathcal{F}$*-injectors* [5, VIII(2.5) and (2.9)]. If $H \leq G$, we write $\mathcal{F}_H = \{S : S \in \mathcal{F},\, S \leq H\}$.

Lemma 2.5. *Let N be the $\mathcal{F}$-radical of a finite solvable group G, and choose an $\mathcal{F}$-injector H in G (where $\mathcal{F}$ is a Fitting set in G).*

(a) [5, VIII(2.11)] *Suppose $G = HK$, with $K \lhd G$ and $H \cap K = N$. If $N \leq H_1 \lhd H$, then H_1 is an $\mathcal{F}_{H_1K}$-injector in H_1K.*
(b) [5, VIII(2.13)] *If $H \leq G_1 \leq G$, then H is an $\mathcal{F}_{G_1}$-injector in G_1.*
(c) *Also $N = \bigcap_{g\in G} H^g$.*
(d) ([1, Lemmas 9 and 10], [2, Lemma 16]) *If $N < H$, then there are subgroups K and L, with $N < K \lhd G$ and $\mathbf{N}_G(H) \leq L < G$, such that $LK = G$ and $H \cap K = N$. Also $K = [K, H]N$.*

Proof. (a) The hypotheses imply that $N = H_1 \cap K$ is the unique $\mathcal{F}_K$-injector of K, and that $H_1 \in \mathcal{F}$, so this follows from the given reference.

(b) This too is proved in the given reference.

(c) If $N_0 = \bigcap_{g\in G} H^g$ is the core of H in G, then $N_0 \lhd G$ and $N_0 \leq H$. Thus N_0 is a normal $\mathcal{F}$-subgroup of G, and hence $N_0 \leq N$. Conversely it follows from the definition of an injector that $H \cap N = N$, so $N \leq H$, and hence $N \leq N_0$.

(d) Since $H > N$, we can choose a subgroup X which is minimal among the normal subgroups of G such that $H \cap X > N$. Then we take

$$S = H \cap X, \quad K = X'N, \quad L = \mathbf{N}_G(SK').$$

Now Lemma 9 in the first reference shows that H satisfies the following condition:

(Γ) if $X \leq G$ with $X \lhd HX$, then $(H \cap X)X' \lhd \mathbf{N}_G(X)$.
This implies that the following condition also holds:
(C) $SK/N = (H \cap X)X'/N \lhd \mathbf{N}_G(X)/N = G/N$.

Working in the quotient group G/N, and applying Lemma 10 in the first reference, we deduce that the subgroups K and L have the required properties. □

Remark. Lemmas 2.6, 2.7 and 2.8 will be used in Section 5.

Notation. If A is an F-algebra (where F is a field), let $\mathbf{J}(A)$ be the (Jacobson) *radical* of A [7, V(2.1)]. If U is a (right) A-module, let $\mathrm{Soc}_A U$ be the *socle* of U (generated by the minimal A-submodules of U). If further $B \subseteq A$, let $\mathrm{Ann}_U B = \{u \in U : uB = 0\}$ be the *annihilator* in U of B.

Lemma 2.6. *Let U be an FG-module, where F is a field and G is a finite group. Suppose F_1 is a finite extension of F, and put $U_1 = F_1 \otimes_F U$.*

(a) [5, B(5.2)] *If U_0 is an F-subspace of U, then $\mathbf{C}_G(F_1 \otimes_F U_0) = \mathbf{C}_G(U_0)$.*
(b) [7, V(11.9)] *Also $\mathbf{C}_{U_1}(G) = F_1 \otimes_F \mathbf{C}_U(G)$.*
(c) [8, VII(1.3.a), (1.5.a) and (1.6.b)] *Moreover*

$$\mathrm{Soc}_{FG}\, U = \mathrm{Ann}_U\, \mathbf{J}(FG), \quad \mathrm{Soc}_{F_1G}\, U_1 = \mathrm{Ann}_{U_1}(F_1 \otimes_F \mathbf{J}(FG)).$$

(d) *Suppose F_0 is a subfield of F, and note that U can be regarded as an F_0G-module. Then $\mathrm{Soc}_{F_0G}\, U = \mathrm{Soc}_{FG}\, U$.*
(e) [8, VII(1.8)] *Also $\mathrm{Soc}_{F_1G}\, U_1 = F_1 \otimes_F (\mathrm{Soc}_{FG}\, U)$.*

Proof. (a) If $\xi \in F_1$, $u_0 \in U_0$ and $g \in \mathbf{C}_G(U_0)$, then $(\xi \otimes u_0)g = \xi \otimes (u_0 g) = \xi \otimes u_0$. Thus $g \in \mathbf{C}_G(F_1 \otimes_F U_0)$, so $\mathbf{C}_G(U_0) \le \mathbf{C}_G(F_1 \otimes_F U_0)$. Conversely $U_0 = 1 \otimes_F U_0 \subseteq F_1 \otimes_F U_0$, and hence $\mathbf{C}_G(U_0) \ge \mathbf{C}_G(F_1 \otimes_F U_0)$.

(b) If $\xi \in F_1$, $u \in \mathbf{C}_U(G)$ and $g \in G$, then $(\xi \otimes u)g = \xi \otimes (ug) = \xi \otimes u$. Thus $\xi \otimes u \in \mathbf{C}_{U_1}(G)$, so $F_1 \otimes_F \mathbf{C}_U(G) \le \mathbf{C}_{U_1}(G)$. Conversely let $\{\lambda_1, \lambda_2, \ \ldots \ , \lambda_n\}$ be an F-basis of F_1, and suppose $u \in \mathbf{C}_{U_1}(G)$, with $u = \sum_{i=1}^n \lambda_i \otimes u_i$ and $u_i \in U$ $(1 \le i \le n)$. If $g \in G$, then $\sum_{i=1}^n \lambda_i \otimes u_i = u = ug = \sum_{i=1}^n \lambda_i \otimes (u_i g)$, and hence $u_i = u_i g$. Thus $u_i \in \mathbf{C}_U(G)$ $(1 \le i \le n)$, so $u \in F_1 \otimes_F \mathbf{C}_U(G)$.

(c) The last reference shows that $\mathrm{Soc}_{FG}\, U = \mathrm{Ann}_U\, \mathbf{J}(FG)$. Using the first two references, we get

$$F_1G = F_1 \otimes_F FG, \quad \mathbf{J}(F_1G) = \mathbf{J}(F_1 \otimes_F FG) = F_1 \otimes_F \mathbf{J}(FG),$$
$$\mathrm{Soc}_{F_1G}\, U_1 = \mathrm{Ann}_{U_1}\, \mathbf{J}(F_1G) = \mathrm{Ann}_{U_1}(F_1 \otimes_F \mathbf{J}(FG)).$$

(d) By (c), it suffices to show that $\mathrm{Ann}_U\, \mathbf{J}(F_0G) = \mathrm{Ann}_U\, \mathbf{J}(FG)$; this can be done by copying the proof of (a), as follows. If $u \in \mathrm{Ann}_U\, \mathbf{J}(F_0G)$, $\xi \in F$ and $a \in \mathbf{J}(F_0G)$, then $u(\xi \otimes a) = \xi \otimes ua = 0$. Thus $u \in \mathrm{Ann}_U(F \otimes_{F_0} \mathbf{J}(F_0G)) = \mathrm{Ann}_U\, \mathbf{J}(FG)$, so $\mathrm{Ann}_U\, \mathbf{J}(F_0G) \le \mathrm{Ann}_U\, \mathbf{J}(FG)$. Conversely (c) implies that $\mathbf{J}(FG) = F \otimes_{F_0} \mathbf{J}(F_0G)$, so $\mathbf{J}(F_0G) = 1 \otimes \mathbf{J}(F_0G) \subseteq \mathbf{J}(FG)$, and hence $\mathrm{Ann}_U\, \mathbf{J}(F_0G) \ge \mathrm{Ann}_U\, \mathbf{J}(FG)$.

(e) By (c), it suffices to show that $\mathrm{Ann}_{U_1}(F_1 \otimes_F \mathbf{J}(FG)) = F_1 \otimes_F \mathrm{Ann}_U\, \mathbf{J}(FG)$; this can be done by copying the proof of (b), as follows. If $\xi, \eta \in F_1$, $u \in \mathrm{Ann}_U\, \mathbf{J}(FG)$ and $a \in \mathbf{J}(FG)$, then $(\xi \otimes u)(\eta \otimes a) = (\xi\eta) \otimes (ua) = 0$. Thus $\xi \otimes u \in \mathrm{Ann}_{U_1}(F_1 \otimes_F \mathbf{J}(FG))$, so $F_1 \otimes_F (\mathrm{Ann}_U\, \mathbf{J}(FG)) \le \mathrm{Ann}_{U_1}(F_1 \otimes_F \mathbf{J}(FG))$. Conversely let $\{\lambda_1, \lambda_2, \ \ldots \ , \lambda_n\}$ be an F-basis of F_1, and suppose $u = \sum_{i=1}^n \lambda_i \otimes u_i \in \mathrm{Ann}_{U_1}(F_1 \otimes_F \mathbf{J}(FG))$, with $u_i \in U$ $(1 \le i \le n)$. If $a \in \mathbf{J}(FG)$, then $0 = u(1 \otimes a) = \sum_{i=1}^n (\lambda_i \otimes u_i)(1 \otimes a) = \sum_{i=1}^n \lambda_i \otimes (u_i a)$, and hence $u_i a = 0$ $(1 \le i \le n)$. Thus $u_i \in \mathrm{Ann}_U\, \mathbf{J}(FG)$, so $u \in F_1 \otimes_F \mathrm{Ann}_U\, \mathbf{J}(FG)$. $\square$

Notation. Let $\pi = \{p_1, p_2, \ \ldots \ , p_m\}$ be a set of prime numbers, and consider a group $U = U_1 \times U_2 \times \ \ldots \ \times U_m$. If each direct factor U_i is an elementary abelian p_i-group, then U will be called an *elementary abelian π-group*. If further $U \lhd G$ and $H \leq G$, then each subgroup U_i can be regarded as an $\mathbf{F}_{p_i}H$-module, and we define

$$\mathrm{Soc}_{\pi H}\, U = \mathrm{Soc}_{\mathbf{F}_{p_1} H}\, U_1 \times \mathrm{Soc}_{\mathbf{F}_{p_2} H}\, U_2 \times \ \ldots \ \times \mathrm{Soc}_{\mathbf{F}_{p_m} H}\, U_m.$$

Finally let $\mathrm{Soc}_\pi G$ be the *π-socle* of G (generated by the minimal normal π-subgroups of G), and let $\mathcal{Z}^\pi$ be the class of finite solvable groups H such that $\mathrm{Soc}_\pi H \leq \mathbf{Z}(H)$. If G is finite and solvable, then $\{H : H \in \mathcal{Z}^\pi,\ H \leq G\}$ is a Fitting set in G [5, IX(2.9.a)].

Lemma 2.7. *Let N be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers). Put $U = \mathrm{Soc}_\pi N$, and suppose $p \in \pi$.*

(a) [3, Lemma 1(a) and (d)] *If $N \leq H \leq G$, then $\mathrm{Soc}_\pi H = \mathrm{Soc}_{\pi H} U$.*
(b) [5, IX(2.9.a.2)] *Hence $N = \mathbf{C}_G(\mathrm{Soc}_\pi G) = \mathbf{C}_G(\mathrm{Soc}_{\pi G} U) = \mathbf{C}_G(U)$.*
(c) *Suppose $U_0 \lhd G$, where U_0 is an elementary abelian π-group. If $U_0 = \mathbf{C}_G(\mathrm{Soc}_{\pi G} U_0)$, then $U_0 = U = N$.*
(d) [3, Lemma 2(a)] *Suppose $N \leq H \leq G$ with $H \in \mathcal{Z}^\pi$. Choose $P \in \mathrm{Syl}_p H$, and put $U_p = \mathrm{Soc}_p N$. Then $H \leq \mathbf{C}_G(\mathbf{C}_{U_p}(P))$.*
(e) [3, Corollary 1] *Assume that U is a p-group. Then N is the $\mathcal{Z}^p$-radical of G with $U = \mathrm{Soc}_p N$. Moreover the $\mathcal{Z}^\pi$-injectors of G are the same as the $\mathcal{Z}^p$-injectors of G.*
(f) ([3, Lemmas 5(b) and 6(a), and Theorem 1], [5, IX(4.19)]) *Assume that U is a p-group. Let Σ be a Sylow basis in G, take $\{P\} = \Sigma \cap \mathrm{Syl}_p G$, and suppose $H \leq G$. Then H is a $\mathcal{Z}^\pi$-injector in G with $\Sigma \searrow H$ if and only if $H = \mathbf{C}_G(\mathbf{C}_U(P))$.*
(g) [3, Lemmas 8(b) and 9(b), and Theorem 3] *Suppose $H \leq G$. Then H is a $\mathcal{Z}^\pi$-injector in G if and only if the following two conditions hold:*
  (i) *$H = \mathbf{C}_G(\mathrm{Soc}_{\pi H} U)$;*
  (ii) *$p \nmid |\mathbf{C}_G(\mathbf{O}_{p'}(\mathrm{Soc}_{\pi H} U)) : H|$ $(p \in \pi)$.*

Proof. (a) The given references imply that $\mathrm{Soc}_\pi H \leq U$, from which the result follows.

(b) From the given reference, together with (a), we get $N = \mathbf{C}_G(\mathrm{Soc}_\pi G) = \mathbf{C}_G(\mathrm{Soc}_{\pi G} U) \geq \mathbf{C}_G(U) \geq N$, because $N \in \mathcal{Z}^\pi$.

(c) Clearly $\mathrm{Soc}_{\pi G} U_0 \leq \mathrm{Soc}_\pi G = \mathrm{Soc}_{\pi G} U$, so $\mathrm{Soc}_{\pi G} U \leq \mathbf{C}_G(\mathrm{Soc}_{\pi G} U_0) = U_0$. It follows that $\mathrm{Soc}_{\pi G} U = \mathrm{Soc}_{\pi G} U_0$, and using (b) we get $N = \mathbf{C}_G(\mathrm{Soc}_{\pi G} U) = \mathbf{C}_G(\mathrm{Soc}_{\pi G} U_0) = U_0$.

(d) Note that $H \in \mathcal{Z}^\pi \subseteq \mathcal{Z}^p$, so it follows from the given reference that $\mathbf{C}_{U_p}(P) = \mathrm{Soc}_p H \leq \mathbf{Z}(H)$.

(e) Using (a), we get $\mathrm{Soc}_\pi G \leq U$, and hence $\mathrm{Soc}_\pi G = \mathrm{Soc}_p G$. Then (b) implies that $N = \mathbf{C}_G(\mathrm{Soc}_\pi G) = \mathbf{C}_G(\mathrm{Soc}_p G)$ is the $\mathcal{Z}^p$-radical of G, with $U = \mathrm{Soc}_p N$. To prove the last statement, let H_π be a $\mathcal{Z}^\pi$-injector in G. Then $H_\pi \in \mathcal{Z}^\pi \subseteq \mathcal{Z}^p$, and the given reference shows that G has a $\mathcal{Z}^p$-injector H_p such that $H_\pi \leq H_p$. As before it follows from (a) that $\mathrm{Soc}_\pi H_p \leq U$, so $\mathrm{Soc}_\pi H_p = \mathrm{Soc}_p H_p \leq \mathbf{Z}(H_p)$. Thus $H_p \in \mathcal{Z}^\pi$, while H_π is maximal $\mathcal{Z}^\pi$-subgroup of G, and therefore $H_\pi = H_p$.

(f) First suppose $H = \mathbf{C}_G(\mathbf{C}_U(P))$. Then (e) shows that $U = \operatorname{Soc}_p N$ where N is the $\mathcal{Z}^p$-radical of G, and it follows from the first reference that H is a $\mathcal{Z}^p$-injector in G with $\Sigma \searrow H$. Finally (e) implies that H is also a $\mathcal{Z}^\pi$-injector in G.

Conversely suppose H is a $\mathcal{Z}^\pi$-injector in G with $\Sigma \searrow H$. Then (e) shows that H is also a $\mathcal{Z}^p$-injector in G, and Theorem 1 in the first reference implies that H contains a Sylow p-subgroup of G. Since $\Sigma \searrow H$, we get $P \leq H$, and it follows from Lemma 5 in the first reference that $H = \langle H, P\rangle \leq \mathbf{C}_G(\mathbf{C}_U(P)) \in \mathcal{Z}^p$. But H is a maximal $\mathcal{Z}^p$-subgroup of G, and therefore $H = \mathbf{C}_G(\mathbf{C}_U(P))$.

(g) As in the given reference put

$$\mathbf{K}_\pi^G(H) = \mathbf{C}_G(\operatorname{Soc}_\pi H),$$

and when $N \leq H \in \mathcal{Z}^\pi$ define $\mathbf{M}_\pi^G(H)$ as follows: for each prime number $p \in \pi$ choose a subgroup $S_p^* \in \operatorname{Syl}_p \mathbf{K}_{\pi-p}(H)$ such that $S_p^* \cap H \in \operatorname{Syl}_p H$, and take

$$H^* = \langle H, S_p^* : p \in \pi\rangle, \quad \mathbf{M}_\pi^G(H) = \mathbf{K}_\pi^G(H^*).$$

First suppose H is a $\mathcal{Z}^\pi$-injector. Theorem 3 and Lemma 8(b) in the given reference imply that $N \leq H = \mathbf{M}_\pi^G(H)$, and that H satisfies the following conditions:

(K_0) $H = \mathbf{K}_\pi^G(H)$;
(K_p) $p \nmid |\mathbf{K}_{\pi-p}^G(H) : H| \quad (p \in \pi)$.

But it follows from (a) that $\mathbf{K}_\pi^G(H) = \mathbf{C}_G(\operatorname{Soc}_{\pi H} U)$ and moreover $\mathbf{K}_{\pi-p}^G(H) = \mathbf{C}_G(\mathbf{O}_{p'}(\operatorname{Soc}_{\pi H} U))$, so ($\mathrm{K}_0$) and ($\mathrm{K}_p$) are equivalent to (i) and (ii) respectively.

Conversely, suppose H satisfies (i) and (ii). It follows from (i) and (b) that $H = \mathbf{C}_G(\operatorname{Soc}_{\pi H} U) \geq \mathbf{C}_G(U) = N$, and as before (i) and (ii) imply that (K_0) and (K_p) both hold. Also $H \in \mathcal{Z}^\pi$ by (i) and (a), so Lemmas 8(b) and 9(b) in the given reference show that $H = \mathbf{M}_\pi^G(H)$, and that H is a $\mathcal{Z}^\pi$-injector. □

Notation. Write $\mathbf{C}_n$ for the cyclic group of order n (where n is a natural number), and $\mathbf{Q}_8$ for the quaternion group of order 8.

Example. Using the notation of Lemma 2.7 with $\pi = \{3\}$, there is a group G such that $\mathbf{C}_U(P) > \mathbf{C}_{\operatorname{Soc}_3 G}(P) < \operatorname{Soc}_3 G < U$.

Proof. Take $\langle \alpha, \beta_0\rangle = \mathrm{SL}_2(3)$, with

$$\alpha = \begin{pmatrix} & 1 \\ -1 & -1 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \beta_i = \beta_0^{\alpha^i},$$
$$\langle \alpha\rangle \cong \mathbf{C}_3, \quad \langle \beta_0, \beta_1, \beta_2\rangle \cong \mathbf{Q}_8.$$

Next let ι be the 2×2 identity matrix, and take $H = \langle a, b_0 \rangle \leq \mathrm{SL}_6(3)$, with

$$a = \begin{pmatrix} & & \iota \\ \iota & 0 & \\ 0 & \iota & \end{pmatrix}, \quad b_0 = \begin{pmatrix} \beta_0 & & \\ & \beta_1 & \\ & & \beta_2 \end{pmatrix}, \quad b_i = b_0^{a^i},$$
$$\langle a \rangle \cong \mathbf{C}_3, \quad \langle b_0, b_1, b_2 \rangle \cong \mathbf{Q}_8, \quad H \cong \mathrm{SL}_2(3).$$

Let U_0 be the 2-dimensional vector space over $\mathbf{F}_3$ on which $\mathrm{SL}_2(3)$ acts, so that H acts on the space $U = U_0 \oplus U_0 \oplus U_0$. Form the natural semidirect product $G = HU$ and take $P = \langle a \rangle U \in \mathrm{Syl}_3\, G$. Then $\mathrm{Soc}_3\, G = \{(u, u\alpha, u\alpha^2) : u \in U_0\}$, and $U = \mathbf{C}_G(\mathrm{Soc}_3\, G)$ is the $\mathcal{Z}^3$-radical of G. Moreover $\mathbf{C}_U(P) = \{(u, u, u) : u \in U_0\}$, and $\mathbf{C}_{\mathrm{Soc}_3\, G}(P) = \{(u, u, u) : u \in \mathbf{F}_3(1, -1)\}$. $\square$

Lemma 2.8. [8, VII(9.19)] *Let U be an irreducible FG-module, where G is a finite group, and F is a finite field of characteristic p (and p is a prime number). If $K = \mathbf{O}^p(G)$ and U is FK-homogeneous* [5, B(3.4)], *then U is FK-irreducible.* $\square$

Remark. In Sections 6 and 7 we consider a minimal counterexample, which involves an FG-module U, where G is a finite group with a normal subgroup K_∞, and F is a splitting field for all the subgroups of G. Using a well known strategy [10, Theorems 3.5, 4.4, 7.3 and 8.4], we first (in Section 6) obtain a contradiction when U is FK_∞-inhomogeneous; the proof uses Lemma 2.9. We can then (in Section 7) apply Lemmas 2.11 and 2.12 to find a normal extraspecial subgroup $R \lhd G$; the proof also uses Lemmas 2.9 and 2.10. Finally (in Section 8) we will use the information about extraspecial groups in Section 3 to complete the proof of the Theorem.

Lemma 2.9. (Glauberman [9, (13.8)]) *Let $H = SP$ be a finite semidirect product, with $P \lhd H$, where $|S|$ and $|P|$ are coprime, and suppose H permutes a finite set Ω. If P permutes Ω transitively, then S fixes at least one element of Ω.* $\square$

Lemma 2.10. [6, (3.7.1)] *Let U be an irreducible FG-module, where G is a finite group, and F is a splitting field for all the subgroups of G. If $G = G_1 \times G_2$ is a direct product, then $U = U_1 \otimes_F U_2$ is a tensor product, where U_i is an irreducible FG_i-module ($i = 1, 2$).* $\square$

Lemma 2.11. *Let U be a module which is G-faithful and FG-irreducible, where F is a field and G is a finite group. Suppose A is an abelian subgroup of G, and assume that U is FA-homogeneous.*

(a) [5, B(9.3.b)] *Then A is cyclic.*
(b) [5, B(9.2.ii)] *If F is a splitting field for A, then $A \leq \mathbf{Z}(G)$.*

Proof. (a) This follows from the given reference.

(b) The given reference implies that A is represented by scalar matrices, from which the result follows. □

Lemma 2.12. *Let P be a finite p-group (where p is a prime number).*

(a) [7, III(7.6), I(14.9) and Aufgabe 56, page 94] *Assume that every normal abelian subgroup $A \leq P$ is cyclic. Then P satisfies one of the following conclusions:*
   (i) $P \cong \mathbf{C}_{p^n}$*, with* $|\mathrm{Aut}\, P| = (p-1)p^{n-1}$*;*
   (ii) $p = 2$ *and* $P \cong \mathbf{Q}_8$*, with* $|\mathrm{Aut}\, P| = 2^3 3$*;*
   (iii) $p = 2$ *and* $|P| = 2^n$ *with* $n \geq 4$*, where P is a generalized quaternion or dihedral or quasidihedral group, and*

$$|\mathrm{Aut}\, P| = \begin{cases} 2^{2n-3} & \text{when } P \text{ is a generalized quaternion or dihedral group,} \\ 2^{2n-4} & \text{when } P \text{ is a quasidihedral group.} \end{cases}$$

(b) (P. Hall [7, III(13.10)]) *Assume that every characteristic abelian subgroup $A \leq P$ is cyclic with $A \leq \mathbf{Z}(P)$. Then $P = P_0 \circ P_1$ is a central product, where P_0 is extraspecial (or $P_0 \cong \mathbf{C}_p$), and P_1 is cyclic.*

Proof. (a) Note that if $n \geq 3$ and $P = \langle a, b \rangle$, with defining relations $a^2 = b^{2^{n-1}} = 1$ and $b^a = b^{2^{n-2}+1}$, then $\langle a, b^2 \rangle$ is a noncyclic normal abelian subgroup of P. Hence the references imply that if P is neither cyclic nor isomorphic to $\mathbf{Q}_8$, then $p = 2$ and $P = \langle a, b \rangle$ is a generalized quaternion or dihedral or quasidihedral group, where $\langle b \rangle$ is a characteristic subgroup of index 2 in P. Then the elements $\phi \in \mathrm{Aut}\, P$ are obtained by choosing $b^\phi \in \langle b \rangle$ and $a^\phi \in P - \langle b \rangle$ such that $\langle b^\phi \rangle = \langle b \rangle$ and $\langle a^\phi \rangle \cong \langle a \rangle$.

(b) The given reference shows that if P_1 is not cyclic, then $p = 2$ and $|P_1| = 2^n$ with $n \geq 4$, where P_1 is a generalized quaternion or dihedral or quasidihedral group. Then $P' = P_1' \cong \mathbf{C}_{2^{n-2}}$ and $\mathbf{Z}(P) = \mathbf{Z}(P_1) \cong \mathbf{C}_2$, so P' is a characteristic abelian subgroup of P with $P' \not\leq \mathbf{Z}(P)$. □

3. Extraspecially irreducible groups

Remark. In this Section, we quote without proof some results about extraspecial groups [4], which will be used in Section 8.

Notation. If n is a natural number, let $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ be the additive group of integers modulo n. If also r is a prime number, let $\mathbf{F}_{r^n}$ be the Galois field of order r^n, and write $\mathbf{F}_{r^n}^+$ and $\mathbf{F}_{r^n}^\times$ for the additive and multiplicative groups of $\mathbf{F}_{r^n}$ respectively. Then $\mathbf{F}_{r^n}^+$ is elementary abelian of order r^n, and $\mathbf{F}_{r^n}^\times \cong \mathbf{C}_{r^n-1}$.

Lemma 3.1. [4, Lemma 1.1]

(a) [5, B(9.3.b) and (9.8.c)] *Let W be a module which is C-faithful and $\mathbf{F}_rC$-irreducible, where C is a finite abelian group (and r is a prime number). Then $C = \langle c \rangle \cong \mathbf{C}_n$ is cyclic with $r \nmid n$, and $\dim_{\mathbf{F}_r} W = k$ where k is the order of r modulo n.*
(b) [5, B(9.8.b)] *More explicitly, assuming the hypotheses and conclusions of (a), there exist an $\mathbf{F}_r$-isomorphism $\theta : W \to \mathbf{F}_{r^k}^+$, and an element γ which is a primitive n-th root of 1 in $\mathbf{F}_{r^k}^\times$, such that $(\xi c)^\theta = \gamma \xi^\theta$ ($\xi \in W$). Thus C permutes the set $W - 0$ semiregularly.* $\square$

Definition.

(a) Let X be a (right) FG-module, where F is a field and G is a group. Then the *dual FG-module* is the vector space $X^* = \mathrm{Hom}_F(X, F)$, with $\xi(\lambda g) = (\xi g^{-1})\lambda$ ($\xi \in X$, $\lambda \in X^*$, $g \in G$).
(b) Let Q be a finite group which acts on an extraspecial r-group R (where r is a prime number), and take $Z = \mathbf{Z}(R) = R' \cong \mathbf{C}_r$. Then R will be called *extraspecially Q-irreducible* if it satisfies the following conditions:
  (i) $[R, Q] = R$;
  (ii) $[Z, Q] = 1$;
  (iii) there is no extraspecial subgroup R_0 such that $Z < R_0 < R$ and $R_0 \lhd QR$.

Lemma 3.2. ([1, Lemma 14], [4, Lemma 1.3]) *Let Q be a finite r'-group which acts on an extraspecial r-group R (where r is a prime number). Take $Z = \langle z \rangle = \mathbf{Z}(R) = R' \cong \mathbf{C}_r$, and form the $\mathbf{F}_rQ$-module $W = R/Z$. Suppose $[R, Q] = R$ and $[Z, Q] = 1$.*

(a) *Then R can be written as a central product $R = R_1 \circ R_2 \circ \ldots \circ R_n$ of extraspecially Q-irreducible groups R_i, with $R_i' = R_i \cap R_j = Z$ and $[R_i, R_j] = 1$ when $i \neq j$.*
(b) *If R is extraspecially Q-irreducible, then W satisfies one of the following conclusions:*
  (i) *W is $\mathbf{F}_rQ$-irreducible, and if $r \neq 2$ then $R^r = 1$;*
  (ii) *$W = X_1 \oplus X_2$ where X_1 and X_2 are $\mathbf{F}_rQ$-irreducible, with $X_1 = X_2^*$, and if $D_i/Z = X_i$ then $D_i' = D_i^r = 1$ ($i = 1, 2$). Moreover if $d_i \in D_i$, with $Zd_1 = \lambda \in X_2^*$ and $Zd_2 = \xi \in X_2$, then the notation can be chosen so that $[d_2, d_1] = z^{\xi\lambda}$.* $\square$

Definition.

(a) Suppose n is an even number, and consider the group $C_\infty = \langle c_0, c_1 \rangle$ with defining relations $c_0^4 = c_1^n = 1$, $c_0^2 = c_1^{n/2}$ and $c_1^{c_0} = c_1^{-1}$. Then C_∞ will be called a *quasi-quaternion group*. Put $C_1 = \langle c_1 \rangle$, and note that $\langle c_0 \rangle \cong \mathbf{C}_4$, $C_1 \cong \mathbf{C}_n$, $C_1 \lhd C_\infty$ and $|C_\infty| = 2n$. If further $n = n_0 n_1$ where n_0 is a power of 2 and $2 \nmid n_1$, then $\langle c_0 \rangle C_1^{n_1}$ is a (generalized) quaternion group of order $2n_0$ (or cyclic of order 4 when $n_0 = 2$) and

$C_1^{n_0} \cong \mathbf{C}_{n_1}$, with $\langle c_0\rangle C_1^{n_1} \cdot C_1^{n_0} = C_\infty$ and $\langle c_0\rangle C_1^{n_1} \cap C_1^{n_0} = 1$. Moreover the element $y = c_0^2 = c_1^{n/2}$ is the unique involution in C_∞ [7, III(8.2.b)].

(b) Suppose r is an odd prime number, and k is an even number, and consider the group $B_\infty = \langle b_0, c_1\rangle$ with defining relations $b_0^{2k} = c_1^{r^{k/2}+1} = 1$, $b_0^k = c_1^{(r^{k/2}+1)/2}$ and $c_1^{b_0} = c_1^r$. Then B_∞ will be called a *hyperquaternion group*. Put $B = \langle b_0\rangle \cong \mathbf{C}_{2k}$ and $C_1 = \langle c_1\rangle \cong \mathbf{C}_{r^{k/2}+1}$, and note that $c_1^{b_0^{k/2}} = c_1^{r^{k/2}} = c_1^{-1}$, so $C_\infty = \langle b_0^{k/2}, c_1\rangle$ is a quasiquaternion group. If $2 \nmid k/2$ then $B = B^4 \times B^{k/2}$, so $B_\infty = B^4 C_\infty$ with $B^4 \cap C_\infty = 1$. On the other hand, if $2 \mid k/2$ then $r^{k/2} \equiv 1$ modulo 4, so $2 \nmid (r^{k/2}+1)/2$ and $C_1 = C_1^{(r^{k/2}+1)/2} \times C_1^2$, and therefore $B_\infty = BC_1^2$ with $B \cap C_1^2 = 1$. In both cases, the element $y = b_0^k = c_1^{(r^{k/2}+1)/2}$ is the unique involution in B_∞.

Lemma 3.3. [4, Lemma 2.1] *Suppose r is an odd prime number, and k is a natural number. Then there is a group $BC_\infty R$ with $C_\infty \lhd BC_\infty$, $R \lhd BC_\infty R$, and $B \cap C_\infty = BC_\infty \cap R = 1$, where $C_\infty = \langle c_0, c_1\rangle$ is a quasiquaternion group of order $2(r^k - 1)$, and*

$$B = \langle b\rangle \cong \mathbf{C}_k, \quad \langle c_0\rangle \cong \mathbf{C}_4, \quad C_1 = \langle c_1\rangle \cong \mathbf{C}_{r^k-1},$$
$$c_0^2 = c_1^{(r^k-1)/2}, \quad c_0^b = c_0, \quad c_1^b = c_1^r, \quad c_1^{c_0} = c_1^{-1}.$$

Also $R = D_1D_2$ is an extraspecial r-group with $Z = \mathbf{Z}(R) = R' = D_1 \cap D_2 \cong \mathbf{C}_r$, $R^r = D_i' = 1$ and $|D_i| = r^{k+1}$ $(i = 1, 2)$. Moreover if $W = R/Z$ and $X_i = D_i/Z$ are regarded as additive abelian groups, then X_1 and X_2 are modules which are BC_∞-faithful and $\mathbf{F}_rC_1$-irreducible, and

$$X_ib = X_ic_1 = X_i, \quad X_ic_0 = X_{3-i} (i = 1, 2), \quad Z = \mathbf{Z}(BC_\infty R). \qquad \square$$

Lemma 3.4. [4, Lemma 2.2] *Suppose k is a natural number. Then there is a group $BC_\infty R$ with $C_\infty \lhd BC_\infty$, $R \lhd BC_\infty R$, and $B \cap C_\infty = BC_\infty \cap R = 1$, where $C_\infty = \langle c_0, c_1\rangle$ is a dihedral group of order $2(2^k - 1)$, and*

$$B = \langle b\rangle \cong \mathbf{C}_k, \quad \langle c_0\rangle \cong \mathbf{C}_2, \quad C_1 = \langle c_1\rangle \cong \mathbf{C}_{2^k-1},$$
$$c_0^b = c_0, \quad c_1^b = c_1^2, \quad c_1^{c_0} = c_1^{-1}.$$

Also $R = D_1D_2$ is an extraspecial 2-group with $Z = \mathbf{Z}(R) = R' = D_1 \cap D_2 \cong \mathbf{C}_2$, $D_i^2 = D_i' = 1$, $|D_i| = 2^{k+1}$ $(i = 1, 2)$. Moreover if $W = R/Z$ and $X_i = D_i/Z$ are regarded as additive abelian groups, then X_1 and X_2 are BC_∞-faithful and $\mathbf{F}_rC_1$-irreducible, and

$$X_ib = X_ic_1 = X_i, \quad X_ic_0 = X_{3-i} \ (i = 1, 2), \quad Z = \mathbf{Z}(BC_\infty R). \qquad \square$$

Lemma 3.5. [4, Lemma 2.3] *Suppose r is an odd prime number, and k is an even number. Then there is a group $B_\infty R$ with $R \lhd B_\infty R$ and $B_\infty \cap R = 1$, where $B_\infty = \langle b_0, c_1\rangle$ is a* *hyperquaternion group of order* $k(r^{k/2}+1)$, *with*

$$B = \langle b_0 \rangle \cong \mathbf{C}_{2k}, \quad C_1 = \langle c_1 \rangle \cong \mathbf{C}_{r^{k/2}+1}, \quad b_0^k = c_1^{(r^{k/2}+1)/2}, \quad c_1^{b_0} = c_1^r.$$

Also R *is an extraspecial* r*-group with* $Z = \mathbf{Z}(R) = R' \cong \mathbf{C}_r$, $R^r = 1$ *and* $|R| = r^{k+1}$. *Moreover if* $W = R/Z$ *is regarded as an additive abelian group, then* W *is a module which is* B_∞*-faithful and* $\mathbf{F}_r C_1$*-irreducible, and*

$$Z = \mathbf{Z}(B_\infty R). \qquad \square$$

Lemma 3.6. [4, Lemma 2.4] *Let* k *be an even number. Then there is a group* BC_1R *with* $C_1 \lhd BC_1$, $R \lhd BC_1R$ *and* $B \cap C_1 = BC_1 \cap R = 1$, *with* $|BC_1| = k(2^{k/2}+1)$, *where*

$$B = \langle b \rangle \cong \mathbf{C}_k, \quad C_1 = \langle c_1 \rangle \cong \mathbf{C}_{2^{k/2}+1}, \quad c_1^b = c_1^2.$$

Also R *is an extraspecial* 2*-group with* $Z = \mathbf{Z}(R) = R' \cong \mathbf{C}_2$ *and* $|R| = 2^{k+1}$. *Moreover if* $W = R/Z$ *is regarded as an additive abelian group, then* W *is a module which is* BC_1*-faithful and* $\mathbf{F}_2 C_1$*-irreducible, and*

$$Z = \mathbf{Z}(BC_1R). \qquad \square$$

Lemma 3.7. [4, Lemma 3.2] *Suppose* q *and* r *are distinct prime numbers, and let* k *be the order of* r *modulo* q. *Suppose* CR *is a group with* $R \lhd CR$ *and* $C \cap R = 1$, *where* $C \cong \mathbf{C}_q$ *and* R *is a* C*-faithful extraspecial* r*-group. Put* $Z = \mathbf{Z}(R) = R' \cong \mathbf{C}_r$, *and assume that* R *is extraspecially* C*-irreducible, with* $[R,C] = R$ *and* $[Z,C] = 1$. *Put* $\Gamma = \mathrm{Aut}(CR)$, $\Theta = \mathbf{C}_\Gamma(Z)$, $\Psi = \mathbf{N}_\Theta(C)$, *and suppose* $2 \nmid k$.

(a) *The group* CR *is unique (up to isomorphism), and* R *is of type (ii) in Lemma 3.2(b) with* $|R| = r^{2k+1}$.
(b) *If* $r \neq 2$, *then* $BC_\infty \leq \Psi$, *where* $BC_\infty R$ *is the group constructed in Lemma 3.3.*
(c) *If* $r = 2$, *then* $BC_\infty \leq \Psi$, *where* $BC_\infty R$ *is the group constructed in Lemma 3.4.* $\square$

Lemma 3.8. [4, Lemma 3.4] *Suppose* q *and* r *are distinct prime numbers, and let* k *be the order of* r *modulo* q. *Suppose* CR *is a group with* $R \lhd CR$ *and* $C \cap R = 1$, *where* $C \cong \mathbf{C}_q$, *and* R *is a* C*-faithful extraspecial* r*-group. Put* $Z = \mathbf{Z}(R) = R' \cong \mathbf{C}_r$, *and assume that* R *is extraspecially* C*-irreducible, with* $[R,C] = R$ *and* $[Z,C] = 1$. *Put* $\Gamma = \mathrm{Aut}(CR)$, $\Theta = \mathbf{C}_\Gamma(Z)$, $\Psi = \mathbf{N}_\Theta(C)$, *and suppose* $2 \mid k$.

(a) *The group* CR *is unique (up to isomorphism), and* R *is of type (i) in Lemma 3.2(b) with* $|R| = r^{k+1}$.
(b) *If* $r \neq 2$, *then* $B_\infty \leq \Psi$, *where* $B_\infty R$ *is the group constructed in Lemma 3.5.*
(c) *If* $r = 2$, *then* $BC_1 \leq \Psi$, *where* BC_1R *is the group constructed in Lemma 3.6.* $\square$

Remark. With the hypotheses of Lemmas 3.7 and 3.8, first suppose $r \neq 2$. If $2 \nmid k$, then it follows from Lemmas 3.7(a) and 3.2(b,ii) that $R^r = 1$, so R is a central product of k nonabelian groups of order r^3 and exponent r. Similarly if $2 \mid k$, then Lemmas 3.8(a) and 3.2(b.i) show that $R^r = 1$, so R is a central product of $k/2$ nonabelian groups of order r^3 and exponent r.

Next suppose $r = 2$. If $2 \nmid k$, then it follows from Lemmas 3.7(b) and 3.2(b.ii) that R is a central product of k dihedral groups of order 8. Finally suppose $2 \mid k$, and put $Z = \mathbf{Z}(R) = R'$, $W = R/Z$ and $\Delta = \mathrm{Aut}\, R$, $\Lambda = \mathbf{C}_\Delta(Z)$, $\Omega = \mathbf{C}_\Lambda(W)$. Then $q \nmid 2^{k/2} - 1$ but $q \mid 2^k - 1 = (2^{k/2} - 1)(2^{k/2} + 1)$, so $q \mid 2^{k/2} + 1$, and as in the proof of [4, Lemma 3.4(c)] we get

$$|\Lambda| = 2^k 2^{(k/2)^2 - (k/2) + 1}(2^2 - 1)(2^4 - 1)\ \ldots\ (2^{k-2} - 1)(2^{k/2} + 1),$$

so $\Lambda/\Omega = \mathrm{O}_k^-(2)$. This implies that R is a central product of $(k/2) - 1$ dihedral groups of order 8 with a single quaternion group [11, Theorem 1(c)].

Hypothesis A. Suppose q and r are distinct prime numbers, and let k be the order of r modulo q. Take CR as in Lemma 3.7 if $2 \nmid k$, and as in Lemma 3.8 if $2 \mid k$, and put

$$Z = \mathbf{Z}(CR) = R' \cong \mathbf{C}_r, \quad W = R/Z,$$
$$\Gamma = \mathrm{Aut}(CR), \quad \Theta = \mathbf{C}_\Gamma(Z), \quad \Psi = \mathbf{N}_\Theta(C).$$

Lemma 3.9. [4, Lemma 4.2] *Assume Hypothesis A, and suppose $q = 2$. Then $k = 1$, $|R| = r^3$ and $W = X_1 \oplus X_2$, where the modules X_1 and X_2 are $\mathbf{F}_r C$-isomorphic to each other. Moreover $\Psi = \mathrm{SL}_2(r)$.* □

Lemma 3.10. [4, Lemma 4.3] *Assume Hypothesis A, and suppose $2 \nmid k$.*

(a) *If $q \neq 2$ and $r \neq 2$, then $\Psi = BC_\infty$ as in Lemma 3.3.*
(b) *If $r = 2$, then $\Psi = BC_\infty$ as in Lemma 3.4.* □

Lemma 3.11. [4, Lemma 4.4] *Assume Hypothesis A, and suppose $2 \mid k$.*

(a) *If $r \neq 2$, then $\Psi = B_\infty$ as in Lemma 3.5.*
(b) *If $r = 2$, then $\Psi = BC_1$ as in Lemma 3.6.* □

Lemma 3.12. [4, Lemma 5.2] *Suppose r is an odd prime number, and k is a natural number, and let $BC_\infty R$ be the group described in Lemma 3.3, with $R = D_1 D_2$ and $X_i = D_i/R'$ $(i = 1, 2)$.*

(a) *If $L \leq BC_\infty$ and $\mathbf{C}_{X_1}(L) \neq 0$, then there is an element $c \in C_1$ such that $L \leq B^c$.*
(b) *There are elements $d_0, d_1, \ldots, d_{k-1} \in D_1$ and $e_0, e_1, \ldots, e_{k-1} \in D_2$ such that R can be written as a central product $R = E_0 \circ E_1 \circ \ldots \circ E_{k-1}$, with $|E_i| = r^3$, $E_i^r = 1$,*

$E_i' = Z$ and $[E_i, E_j] = 1$ when $i \neq j$, where $E_i = \langle d_i, e_i\rangle$ and $d_i^b = d_{i+1}$, $e_i^b = e_{i+1}$ $(i \in \mathbf{Z}_k)$. $\square$

Lemma 3.13. [4, Lemma 5.3] *Suppose k is a natural number, and let $BC_\infty R$ be the group described in Lemma 3.4, with $R = D_1 D_2$ and $X_i = D_i/R'$ $(i = 1, 2)$.*

(a) *If $L \leq BC_\infty$ and $\mathbf{C}_{X_1}(L) \neq 0$, then there is an element $c \in C_1$ such that $L \leq B^c$.*
(b) *There are elements $d_0, d_1, \ldots, d_{k-1} \in D_1$ and $e_0, e_1, \ldots, e_{k-1} \in D_2$ such that R can be written as a central product $R = E_0 \circ E_1 \circ \ldots \circ E_{k-1}$, with $E_i = \langle d_i, e_i\rangle \cong \mathbf{D}_8$, $[E_i, E_j] = 1$ when $i \neq j$, and $d_i^b = d_{i+1}$, $e_i^b = e_{i+1}$ $(i \in \mathbf{Z}_k)$.* $\square$

Lemma 3.14. [4, Lemma 5.4] *Suppose r is an odd prime number, and $k = k_0 k_1$ is an even number, where k_0 is a power of 2 and $2 \nmid k_1$. Let $B_\infty R$ be the group described in Lemma 3.5.*

(a) *There are subgroups $D_1, D_2 \leq R$ with $D_1 D_2 = R$, $D_1 \cap D_2 = Z$, $D_i' = D_i^r = 1$ and $|D_i| = r^{(k/2)+1}$. Moreover if $W = R/Z$ and $X_i = D_i/Z$ are regarded as additive abelian groups, then $W = X_1 \oplus X_2$ and $X_i b^{2k_0} = X_i$ $(i = 1, 2)$.*
(b) *If $L \leq B_\infty$ and $\mathbf{C}_W(L) \neq 0$, then there is an element $c \in C_1$ such that $L \leq (B^{2k_0})^c$.*
(c) *There are elements $d_0, d_1, \ldots, d_{(k/2)-1} \in D_1$ and $e_0, e_1, \ldots, e_{(k/2)-1} \in D_2$ such that R can be written as a central product $R = E_0 \circ E_1 \circ \ldots \circ E_{(k/2)-1}$, with $|E_i| = r^3$, $E_i^r = 1$, $E_i' = Z$ and $[E_i, E_j] = 1$ when $i \neq j$, where $E_i = \langle d_i, e_i\rangle$ and $d_i^{b^{2k_0}} = d_{i+2k_0}$, $e_i^{b^{2k_0}} = e_{i+2k_0}$ $(i \in \mathbf{Z}_{k/2})$.* $\square$

Lemma 3.15. [4, Lemma 5.5] *Suppose $k = k_0 k_1$ is an even number, where k_0 is a power of 2 and $2 \nmid k_1$. Let $BC_1 R$ be the group described in Lemma 3.6.*

(a) *There are subgroups $D_1, D_2 \leq R$ with $D_1 D_2 = R$, $D_1 \cap D_2 = Z$, $D_i' = 1$ and $|D_i| = 2^{(k/2)+1}$. Moreover if $W = R/Z$ and $X_i = D_i/Z$ are regarded as additive abelian groups, then $W = X_1 \oplus X_2$ and $X_i b^{k_0} = X_i$ $(i = 1, 2)$.*
(b) *If $P \leq L \leq BC_1$ with $2 \nmid |P|$ and $\mathbf{C}_W(P) = \mathbf{C}_W(L) \neq 0$, then there is an element $c \in C_1$ such that $L \leq (B^{k_0})^c$.*
(c) *There are elements $d_0, d_1, \ldots, d_{(k/2)-1} \in D_1$ and $e_0, e_1, \ldots, e_{(k/2)-1} \in D_2$ such that R can be written as a central product $R = E_0 \circ E_1 \circ \ldots \circ E_{(k/2)-1}$, with $|E_i| = 2^3$, $E_i^2 \leq E_i' = Z$ and $[E_i, E_j] = 1$ when $i \neq j$, where $E_i = \langle d_i, e_i\rangle$ and $d_i^{b^{k_0}} = d_{i+k_0}$, $e_i^{b^{k_0}} = e_{i+k_0}$ $(i \in \mathbf{Z}_{k/2})$.* $\square$

4. Minimal groups with an injector which is not system permutable

Remark. In this section, we use Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5, and we assume the following hypothesis.

Hypothesis B. Let N be the $\mathcal{F}$-radical of a finite solvable group G_0, and suppose H_0 is an $\mathcal{F}$-injector in G_0 (where $\mathcal{F}$ is a Fitting set in G_0) [5, VIII(2.1), (2.3.b) and (2.5)]. Assume that H_0 is not system permutable in G_0, but that whenever $N \leq G_1 < G_0$, the $\mathcal{F}_{G_1}$-injectors of G_1 are system permutable in G_1. Applying Lemma 2.5(d), take subgroups K_0 and L_0, with $N < K_0 \lhd G_0$ and $\mathbf{N}_{G_0}(H_0) \leq L_0 < G_0$, such that $L_0K_0 = G_0$ and $H_0 \cap K_0 = N$. Finally choose a Sylow basis Σ_0 in G_0 such that $\Sigma_0 \searrow H_0$ and $\Sigma_0 \searrow L_0$ [5, I(4.16) and (4.18)].

Lemma 4.1. *Take G_0, H_0, K_0, N and Σ_0 as in Hypothesis B.*

(a) *Then $G_0 = H_0K_0$, with $K_0 = [K_0, H_0]N$ and $H_0 \not\leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$.*
(b) *If $N \leq H_1 < H_0$ with $H_1 \lhd H_0$, then $H_1 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$.*
(c) *If $N \leq K_1 < K_0$ with $K_1 \lhd H_0K_1$ and $\Sigma_0 \searrow K_1$, then*

$$H_0 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_1)N/N).$$

Proof. (a) Suppose $H_0K_0 < G_0$; we must show that this is impossible, and it suffices to prove that H_0 is Σ_0-permutable. Note that $\Sigma_0 \searrow K_0$ by Lemma 2.3(a), and hence $\Sigma_0 \searrow H_0K_0$ by Lemma 2.3(b). Consider a subgroup $P \in \Sigma_0 \cap \operatorname{Syl} G_0$; we must show that H_0 permutes with P. If $P_0 = P \cap L_0$ and $P_1 = P \cap H_0K_0$, then it follows from Lemma 2.4(b) that H_0 permutes with P_0 and P_1, so H_0 also permutes with P_0P_1. But $L_0 \cdot H_0K_0 = G_0$, so $P_0P_1 = P$ by Lemma 2.3(b), which completes the proof that $G_0 = H_0K_0$. Finally $K_0 = [K_0, H_0]N$ by Lemma 2.5(d), and $H_0 \not\leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$ by Lemma 2.4(d).

(b) Note that H_1 is an $\mathcal{F}_{H_1K_0}$-injector of H_1K_0 by Lemma 2.5(a), so H_1 is system permutable in H_1K_0 by Hypothesis B. Moreover $\Sigma_1 = \Sigma_0 \cap H_1K_0$ is a Sylow basis in H_1K_0 with $\Sigma_1 \searrow H_1$, so the result follows from Lemma 2.4(d).

(c) Note that H_0 is an $\mathcal{F}_{H_0K_1}$-injector of H_0K_1 by Lemma 2.5(b), so H_0 is system permutable in H_0K_1 by Hypothesis B. Moreover $\Sigma_1 = \Sigma_0 \cap H_0K_1$ is a Sylow basis in H_0K_1 with $\Sigma_1 \searrow H_0$, so again the result follows from Lemma 2.4(d). □

Lemma 4.2.

(a) *There is a prime number r such that $H_0 = SH_\infty$ and $H_\infty \lhd H_0$, where S is a cyclic r-group with $\Sigma_0 \searrow S$, and $N \leq H_\infty < H_0$. Also $H_\infty = [H_\infty, S]N$, and $H_\infty \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$.*
(b) *There is a prime number q such that $K_0 = QK_0'N$, where Q is a q-group with $\{Q\} = (\Sigma_0 \cap K_0) \cap \operatorname{Syl}_q K_0$, and the module $K_0/K_0'N$ is $\mathbf{Z}H_0$-indecomposable.*
(c) *Hence the module $V = K_0/K_0^qK_0'N$ is $\mathbf{F}_qH_0$-indecomposable. Moreover $K_0 = QK_\infty$ with $K_\infty \lhd G_0$, where $K_0^qK_0'N \leq K_\infty < K_0$ and the module K_0/K_∞ is $\mathbf{F}_qH_0$-irreducible. Also $H_0 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_\infty)N/N)$.*
(d) *If $N \leq K_1 \leq K_0$ and $K_1 \not\leq K_\infty$, with $K_1 \lhd H_0K_1$ and $\Sigma_0 \searrow K_1$, then $K_1 = K_0$.*

Proof. (a) Suppose H_1/N and H_2/N are distinct maximal normal subgroups of H_0/N. Then Lemma 4.1(b) shows that H_1 and H_2 both normalize $(\Sigma_0 \cap K_0)N/N$, and hence $H_0 = H_1H_2 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$. Because of Lemma 2.4(d), this contradicts Hypothesis B, and implies that H_0/N has a unique maximal normal subgroup. Put $H_\infty = \gamma_\infty(H_0)N$, where $\gamma_\infty(H_0)$ is the nilpotent residual of H_0 [5, II(2.3)], and note that $N \leq H_\infty < H_0$ with $H_\infty \lhd G_0$, and that H_0/H_∞ is nilpotent. Hence H_0/H_∞ is cyclic of prime power order. Since $\Sigma_0 \searrow H_0$, it follows that there is a prime number r and a cyclic r-group S such that $\Sigma_0 \searrow S$ and $H_0 = SH_\infty$. Moreover $H_\infty = [H_\infty, H_0]N = [H_\infty, SH_\infty]N = [H_\infty, S]H'_\infty N$, and therefore $H_\infty = [H_\infty, S]N$ by Lemma 2.1(a). Finally $H_\infty \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$ by Lemma 4.1(b).

(b) Suppose K_1 and K_2 are distinct maximal members of the set

$$\Lambda = \{L : N \leq L < K_0,\ L \lhd G_0\}.$$

Then Lemma 4.1(c) shows that $H_0 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_i)N/N)$ $(i = 1, 2)$, and hence H_0 also normalizes $(\Sigma_0 \cap K_1K_2)N/N = (\Sigma_0 \cap K_0)N/N$ by Lemma 2.3(b). This contradicts Lemma 4.1(a), and proves that Λ has a unique maximal member, say K_∞. Now $K'_0N \in \Lambda$ and K_0/K'_0N is abelian, so there is a prime number q such that K_0/K'_0N is a $\mathbf{Z}H_0$-indecomposable q-group, which gives the result.

(c) The first two sentences follow from (b), while the last containment is a consequence of Lemma 4.1(c).

(d) Suppose $K_1 < K_0$; we must show that this is impossible. Lemma 4.1(c) implies that $H_0 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_1)N/N)$, while $H_0 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_\infty)N/N)$ by (c). As before it follows that H_0 also normalizes $(\Sigma_0 \cap K_1K_\infty)N/N = (\Sigma_0 \cap K_0)N/N$, which contradicts Lemma 4.1(a). □

Lemma 4.3. *Choose S, H_∞ and Q, K_∞ as in Lemma 4.2, and take $Q_\infty = Q \cap K_\infty$ and $\{R\} = (\Sigma_0 \cap K_0) \cap \operatorname{Syl}_r K_0$.*

(a) *Then $K_0 = QRN$, so K_0/N is a $\{q, r\}$-group. Hence $q \neq r$, and $N < RN < K_0$.*
(b) *Also $K_\infty/N = (Q_\infty N/N) \times (RN/N)$, so $Q_\infty N \lhd G_0$ and $RN \lhd G_0$. Hence $H_0 \not\leq \mathbf{N}_{G_0}(QN)$ and $K_0^qK'_0N = Q^qQ'RN$.*
(c) *Moreover $\langle H_0, Q\rangle = G_0$.*
(d) *Hence $\mathbf{C}_{H_0}(K_0/RN) = N$.*
(e) *Also $[R, Q]N = RN$.*

Proof. (a) Take $\{Q_0\} = \Sigma_0 \cap \operatorname{Syl}_q G_0$ and $\{R_0\} = \Sigma_0 \cap \operatorname{Syl}_r G_0$, and put $K_1 = QRN$. Then Q_0R_0 is a Hall $\{q, r\}$-subgroup of G_0 with $QR = Q_0R_0 \cap K_0$. Since $S \leq R_0$, it follows that $S \leq \mathbf{N}_{G_0}(QR) \leq \mathbf{N}_{G_0}(QRN)$. Moreover Lemma 4.2(a) shows that $H_\infty \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N) \leq \mathbf{N}_{G_0}(QRN)$, and hence $H_0 = SH_\infty \leq \mathbf{N}_{G_0}(QRN) = \mathbf{N}_{G_0}(K_1)$, so $K_1 \lhd H_0K_1$. It is also clear that $K_1 \not\leq K_\infty$ and $\Sigma_0 \searrow K_1$, so $K_0 = K_1 = QRN$ by

Lemma 4.2(d). Finally the fact that $H_0 \nleq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$ implies that $q \neq r$ and $N < RN < K_0$.

(b) Suppose $T = Q_\infty$ or R, so $T \in \Sigma_0 \cap K_\infty$, and put $K_1 = \mathbf{N}_{K_0}(TN)$. Then $K_0 = K_1 K_\infty$ by Frattini's argument [7, I(7.8)], so $K_1 \nleq K_\infty$. Using Lemma 4.2(c) we get $H_0 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_\infty)N/N) \leq \mathbf{N}_{G_0}(TN)$, and hence $K_1 \lhd H_0 K_1$. Moreover T pr K_0 and $\Sigma_0 \searrow T$, with $\mathbf{N}_{K_0}(T) \leq K_1$, so $\Sigma_0 \searrow K_1$ by Lemma 2.4(c). It now follows from Lemma 4.2(d) that $K_1 = K_0$, so $TN \lhd K_0$. Thus K_0 normalizes $Q_\infty N$ and RN, which proves that $K_\infty/N = (Q_\infty N/N) \times (RN/N)$. Therefore $H_0 \leq \mathbf{N}_{G_0}(RN)$, whereas $H_0 \nleq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N)$ by Lemma 4.1(a), and hence $H_0 \nleq \mathbf{N}_{G_0}(QN)$. Finally $RN \lhd G_0$, so $K_0^q K_0' N = Q^q Q' RN$.

(c) Put $G_1 = \langle H_0, Q\rangle$ and $K_1 = G_1 \cap K_0$, and note that $K_1 \lhd G_1 = H_0 K_1$. Also $K_1 = K_1 \cap QRN = QN(K_1 \cap R)$, which implies that $K_1 \nleq K_\infty$ and $\Sigma_0 \searrow K_1$. Thus $K_1 = K_0$ by Lemma 4.2(d), so $G_1 = H_0 K_0 = G_0$.

(d) Put $H_1 = \mathbf{C}_{H_0}(K_0/RN)$ and $G_1 = \mathbf{N}_{G_0}(H_1)$, and note that $H_0 \leq G_1$. Also $K_0 = [K_0, H_0]N$ by Lemma 4.1(a), so $H_1 < H_0$, and it follows from Lemma 4.1(b) that $H_1 \leq \mathbf{N}_{G_0}((\Sigma_0 \cap K_0)N/N) \leq \mathbf{N}_{G_0}(QN)$. Hence $[H_1, Q] \leq QN \cap [H_1, K_0] \leq QN \cap RN = N \leq H_1$, so $Q \leq G_1$. Thus $G_1 \geq \langle H_0, Q\rangle = G_0$ by (c), so $H_1 \lhd G_0$. Finally $N \leq H_1 \leq \bigcap_{g \in G_0} H_0^g = N$ by Lemma 2.5(c).

(e) Put $K_1 = Q[R,Q]N = \langle Q^{K_0}\rangle N$, and note that $K_1/N = \mathbf{O}^{q'}(K_0/N)$, so $K_1 \lhd G_0$. Now $H_0 Q[R,Q]N \geq \langle H_0, Q\rangle = G_0$ by (d), and hence $[R,Q]N/N \in \mathrm{Syl}_r(K_0/N)$, so $[R,Q]N = RN$. $\square$

Lemma 4.4. *Take S, H_∞ and R as in Lemma 4.3.*

(a) *Then $\mathbf{C}_{RN/N}(H_\infty) \leq \mathbf{C}_{RN/N}(S)$.*
(b) *Also $S = \langle a\rangle$ can be chosen so that $a = a_0 d$, where $a_0 \in \mathbf{O}_r(\mathbf{N}_{G_0}(\Sigma_0))$ and $d \in R$.*

Proof. (a) Note that $H_0 \cap K_0 = N$, $H_\infty/N = \mathbf{O}^r(H_0/N)$ and $RN/N = \mathbf{O}_r(K_0/N)$. Also H_0 pr G_0 (because H_0 is an injector), so H_0/N pr G_0/N by Lemma 2.2(c). Taking $\{S_1\} = (\Sigma_0 \cap H_0) \cap \mathrm{Syl}_r H_0$, and applying Lemma 2.1(b) in the group G_0/N, we deduce that $\mathbf{C}_{RN/N}(H_\infty) \leq \mathbf{C}_{RN/N}(S_1) \leq \mathbf{C}_{RN/N}(S)$.

(b) Take $\{S_\infty\} = (\Sigma_0 \cap H_\infty) \cap \mathrm{Syl}_r H_\infty$, $R_\infty = S_\infty R$ and $S_0 = \mathbf{O}_r(\mathbf{N}_{G_0}(\Sigma_0))$. Then $\{R_\infty\} = (\Sigma_0 \cap H_\infty K_0) \cap \mathrm{Syl}_r(H_\infty K_0)$ and $H_\infty K_0 = \mathbf{O}^r(G_0)$, and it follows from Lemma 2.3(c) that $\{S_0 R_\infty\} = \Sigma_0 \cap \mathrm{Syl}_r G_0$. Hence $S \leq S_0 R_\infty = S_0 R S_\infty$, so $a = a_0 d a_\infty$, where $a_0 \in S_0$, $d \in R$ and $a_\infty \in S_\infty$. Thus $a a_\infty^{-1} = a_0 d$, and $H_0 = \langle a a_\infty^{-1}\rangle H_\infty$, so we get the result by replacing S by $\langle a a_\infty^{-1}\rangle$. $\square$

5. Minimal groups with a $\mathcal{Z}^\pi$-injector which is not system permutable

Remark. In this section we use Lemmas 2.6, 2.7 and 2.8.

Lemma 5.1. *Let U be an irreducible FG-module, where F is a finite field, and G is a finite group. Suppose $G = HK$ with $K \lhd G$ and $H \cap K = 1$, and $H = SP$ with $P \lhd H$ and $S \cap P = 1$, where P is a p-group and S is an r-group (and p and r are distinct prime numbers). Assume that $\mathbf{C}_U(P) \leq \mathbf{C}_U(S)$.*

(a) *If F is a field of characteristic p, then U is $F(PK)$-irreducible.*
(b) *Suppose U is $F(PK)$-irreducible, and let $U_1, U_2, \ldots, U_n$ be the FK-homogeneous components of U, and put $P_1 = \mathbf{N}_P(U_1)$. Then the notation can be chosen so that $\mathbf{N}_H(U_1) = SP_1$ and $\mathbf{C}_{U_1}(P_1) \leq \mathbf{C}_{U_1}(S)$.*

Proof. (a) Since $PK \lhd G$, it follows from Clifford's theorem that $U = X_1 \oplus X_2 \oplus \ldots \oplus X_m$ is a direct sum of $F(PK)$-irreducible modules X_i [7, V(17.3.a)]. Put

$$Y = X_2 \oplus X_3 \oplus \ \ldots \ \oplus X_m,$$

and let Ω be the set of $F(PK)$-submodules X such that $U = X \oplus Y$. Suppose $a \in S$ and $X \in \Omega$ with $X^a \notin \Omega$. Then the conjugates X^a and X_i^a are irreducible $F(PK)$-modules with $U = X^a \oplus X_2^a \oplus X_3^a \oplus \ \ldots \ \oplus X_m^a$, and it follows from the theorem of Krull, Remak and Schmidt that there is an index $i \neq 1$ such that $X_i^a \in \Omega$ [7, I(12.3)]. Thus $X_i \leq Y$ but $X_i^a \nleq Y$, and hence $\mathbf{C}_{X_i}(P) \neq \mathbf{C}_{X_i}(P)^a$. This contradicts the fact that $\mathbf{C}_U(P) \leq \mathbf{C}_U(S)$, and proves that $X^a \in \Omega$, so S permutes the set Ω.

Note that $U = X_1 \oplus Y$, and let $\phi : U \to X_1$ and $\psi : U \to Y$ be the projection maps. If $X \in \Omega$, then the restricted map $\phi_X : X \to X_1$ is an $F(PK)$-isomorphism, and hence there is an $F(PK)$-homomorphism $\theta = \phi_X^{-1}\psi : X_1 \to Y$. This shows that the members of Ω are the submodules $\{\xi \oplus \xi\theta : \xi \in X_1\}$ with $\theta \in \mathrm{Hom}_{F(PK)}(X_1, Y)$, so $|\Omega| = \left|\mathrm{Hom}_{F(PK)}(X_1, Y)\right|$. Now $\mathrm{Hom}_{F(PK)}(X_1, X)$ is a vector space over F, and therefore $|\Omega|$ is a power of p. But S is an r-group, and it follows that there is an element $X \in \Omega$ which is stabilized by S, so X is an $F(HK)$-submodule. Then the irreducibility of U implies that $U = X \cong X_1$, so U is $F(PK)$-irreducible.

(b) Since $K \lhd G$, Clifford's theorem shows that $U = U_1 \oplus U_2 \oplus \ \ldots \ \oplus U_n$ is the direct sum of the FK-homogeneous components U_i, which are permuted by H, and permuted transitively by P [7, V(17.3.d)]. Hence $n = p^s$ for some exponent s, and there is a transversal $\{b_1, b_2, \ \ldots \ , b_{p^s}\}$ to P_1 in P, such that $b_1 = 1$ and $U_1 b_i = U_i$ $(1 \leq i \leq p^s)$. Moreover S is an r-group, so S normalizes at least one of the submodules U_i. Then the notation can be chosen so that $S \leq \mathbf{N}_H(U_1)$, and hence $\mathbf{N}_H(U_1) = SP_1$. If $u \in \mathbf{C}_{U_1}(P_1)$ and $g \in P$, then $g = g_1 b_i$, with $g_1 \in P_1$ and $i \in \{1, 2, \ \ldots \ , p^s\}$, so $ug = ug_1 b_i = ub_i$. Thus $\{u, ub_2, ub_3, \ \ldots \ , ub_{p^s}\}$ is a P-orbit, and $\sum_{i=1}^{p^s} ub_i \in \mathbf{C}_U(P)$. Conversely if $\sum_{i=1}^{p^s} u_i \in \mathbf{C}_U(P)$ with $u_i \in U_i$ $(1 \leq i \leq p^s)$, then the set $\{u_1, u_2, \ \ldots \ , u_{p^s}\}$ is permuted by P. Hence $u_1 \in \mathbf{C}_{U_1}(P_1)$, and the argument above shows that $u_i = u_1 b_i$ $(1 \leq i \leq p^s)$. This proves that $\mathbf{C}_U(P) = \{\sum_{i=1}^{p^s} ub_i : u \in \mathbf{C}_{U_1}(P_1)\}$. Finally if $u \in \mathbf{C}_{U_1}(P_1)$, then S permutes the set $\{u, ub_2, ub_3, \ \ldots \ , ub_{p^s}\}$, and hence S fixes u, which shows that $\mathbf{C}_{U_1}(P_1) \leq \mathbf{C}_{U_1}(S)$. $\square$

Definition. Let $\pi = \{p_1, p_2, \ \ldots \ , p_m\}$ be a set of prime numbers, and consider an elementary abelian π-group $U = U_1 \oplus U_2 \oplus \ \ldots \ \oplus U_m$, such that each direct summand U_i is an F_iG-module, where F_i is a field of characteristic p_i $(1 \le i \le m)$, and G is a group. If each field F_i is a splitting field for G, then U will be said to be *πG-split.*

Lemma 5.2. *Suppose G_0 is a finite solvable group, with a $\mathcal{Z}^\pi$-injector which is not system permutable in G_0 (where π is a set of prime numbers). Then there exist an elementary abelian π-group U, and a finite solvable group G which acts on U, with a subgroup $H \le G$, such that the following three conditions hold.*

(i) *The group U is πG_1-split for all subgroups $G_1 \le G$, and $\mathbf{C}_G(\mathrm{Soc}_{\pi G}\, U) = 1$.*
(ii) *Also $H = \mathbf{C}_G(\mathrm{Soc}_{\pi H}\, U)$, and $p \nmid |\mathbf{C}_G(\mathbf{O}_{p'}(\mathrm{Soc}_{\pi H}\, U)) : H|$ $(p \in \pi)$.*
(iii) *Moreover H is not system permutable in G.*

Proof. Let N be the $\mathcal{Z}^\pi$-radical of G_0, and choose a $\mathcal{Z}^\pi$-injector H_0 of G_0, and take

$$G = G_0/N, \quad H = H_0/N.$$

Suppose $\pi = \{p_1, p_2, \ \ldots \ , p_m\}$, and for each index i, choose a finite field F_i of characteristic p_i, such that F_i is a splitting field for all the subgroups of G [5, B(5.21)]. Take

$$\begin{aligned} U_i^\circ &= \mathrm{Soc}_{p_i} N, & U_0 &= U_1^\circ \oplus U_2^\circ \oplus \ \ldots \ \oplus U_m^\circ = \mathrm{Soc}_\pi N, \\ U_i &= F_i \otimes_{\mathbf{F}_{p_i}} U_i^\circ, & U &= U_1 \oplus U_2 \oplus \ \ldots \ \oplus U_m. \end{aligned}$$

Then Lemma 2.6(a), (d) and (e) imply that

$$\begin{aligned} \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} G}\, U_i) &= \mathbf{C}_G(\mathrm{Soc}_{F_i G}(F_i \otimes_{\mathbf{F}_{p_i}} U_i^\circ)) \\ &= \mathbf{C}_G(F_i \otimes_{\mathbf{F}_{p_i}} \mathrm{Soc}_{\mathbf{F}_{p_i} G}\, U_i^\circ) = \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} G}\, U_i^\circ), \end{aligned}$$

and similarly $\mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} H}\, U_i) = \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} H}\, U_i^\circ)$. Hence

$$\begin{aligned} \mathbf{C}_G(\mathrm{Soc}_{\pi G}\, U) &= \textstyle\bigcap_{i=1}^m \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} G}\, U_i) = \bigcap_{i=1}^m \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} G}\, U_i^\circ) \\ &= \mathbf{C}_G(\mathrm{Soc}_{\pi G}\, U_0) = \mathbf{C}_{G_0/N}(\mathrm{Soc}_{\pi G_0}\, U_0) \\ &= \mathbf{C}_{G_0}(\mathrm{Soc}_{\pi G_0}\, U_0)/N. \end{aligned}$$

But $\mathbf{C}_{G_0}(\mathrm{Soc}_{\pi G_0}\, U_0) = N$ by Lemma 2.7(b), so $\mathbf{C}_G(\mathrm{Soc}_{\pi G}\, U) = N/N = 1$, which shows that the condition (i) holds. Similarly

$$\begin{aligned} \mathbf{C}_G(\mathrm{Soc}_{\pi H}\, U) &= \textstyle\bigcap_{i=1}^m \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} H}\, U_i) = \bigcap_{i=1}^m \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_i} H}\, U_i^\circ) \\ &= \mathbf{C}_G(\mathrm{Soc}_{\pi H}\, U_0) = \mathbf{C}_{G_0/N}(\mathrm{Soc}_{\pi H_0}\, U_0) \\ &= \mathbf{C}_{G_0}(\mathrm{Soc}_{\pi H_0}\, U_0)/N. \end{aligned}$$

But $\mathbf{C}_{G_0}(\mathrm{Soc}_{\pi H_0} U_0) = H_0$ by Lemma 2.7(g.i), so $\mathbf{C}_G(\mathrm{Soc}_{\pi H} U) = H_0/N = H$, as in (ii). Moreover

$$\begin{aligned} \mathbf{C}_G(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H} U)) &= \textstyle\bigcap_{j\neq i} \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_j} H} U_j) = \bigcap_{j\neq i} \mathbf{C}_G(\mathrm{Soc}_{\mathbf{F}_{p_j} H} U_j^\circ) \\ &= \mathbf{C}_G(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H} U_0)) = \mathbf{C}_{G_0/N}(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H_0} U_0)) \\ &= \mathbf{C}_{G_0}(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H_0} U_0))/N, \\ |\mathbf{C}_G(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H} U)) : H| &= |\mathbf{C}_{G_0}(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H_0} U_0))/N : H_0/N| \\ &= |\mathbf{C}_{G_0}(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H_0} U_0)) : H_0|. \end{aligned}$$

But $p_i \nmid |\mathbf{C}_{G_0}(\mathbf{O}_{p_i'}(\mathrm{Soc}_{\pi H_0} U_0)) : H_0|$ by Lemma 2.7(g.ii), so this completes the proof that the condition (ii) holds.

To prove (iii), consider a Sylow basis Σ in G, and choose a Sylow basis Σ_0 in G_0 such that $\Sigma = \Sigma_0 N/N$ [5, I(4.13.a)]. Then H_0 is not Σ_0-permutable by hypothesis, so H is not Σ-permutable by Lemma 2.2(c). □

Hypothesis C1. For the rest of this section, and in the next two sections, let G be a finite solvable group which acts on an elementary abelian π-group U (where π is a set of prime numbers). Suppose $H \leq G$, and that the three conditions in Lemma 5.2 hold. On the other hand, assume that if G_1 is a finite solvable group which acts on an elementary abelian π-group U_1, with a subgroup $H_1 \leq G_1$, such that U_1, G_1 and H_1 satisfy the conditions (i) and (ii) in Lemma 5.2, and if $|G_1| \cdot |U_1| < |G| \cdot |U|$, then H_1 is system permutable in G_1.

Lemma 5.3. *Assuming Hypothesis C1, form the natural semidirect products $G_0 = GU$ and $H_0 = HU$.*

(a) *Then U is the $\mathcal{Z}^\pi$-radical of G_0, and H_0 is a $\mathcal{Z}^\pi$-injector of G_0.*
(b) *Hence the $\mathcal{Z}^\pi$-injectors of G_0 are not system permutable in G_0, but if $U \leq G_1 < G_0$ then the $\mathcal{Z}^\pi$-injectors of G_1 are system permutable in G_1.*

Proof. (a) The condition (i) in Lemma 5.2 implies that $U = \mathbf{C}_{G_0}(\mathrm{Soc}_{\pi G_0} U)$, so U is the $\mathcal{Z}^\pi$-radical of G_0 by Lemma 2.7(c). Moreover (ii) implies that $H_0 = \mathbf{C}_{G_0}(\mathrm{Soc}_{\pi H_0} U)$, and $p \nmid |\mathbf{C}_{G_0}(\mathbf{O}_{p'}(\mathrm{Soc}_{\pi H_0} U)) : H_0|$ $(p \in \pi)$, so H_0 is a $\mathcal{Z}^\pi$-injector by Lemma 2.7(g).

(b) Let Σ_0 be a Sylow basis in G_0. Since $G_0/U \cong G$, the quotient $\Sigma_0 U/U$ corresponds to a Sylow basis Σ in G. It follows from Hypothesis C1 that H is not Σ-permutable, so H_0/U is not $\Sigma_0 U/U$-permutable, and hence H_0 is not Σ_0-permutable by Lemma 2.2(c). Because of (a), this means that the $\mathcal{Z}^\pi$-injectors of G_0 are not system permutable in G_0.

Now let N_1 be the $\mathcal{Z}^\pi$-radical of G_1, choose a $\mathcal{Z}^\pi$-injector H_1 of G_1, and take

$$G_2 = G_1/N_1, \quad H_2 = H_1/N_1, \quad U_2 = \mathrm{Soc}_\pi N_1.$$

Then $N_1 = \mathbf{C}_{G_1}(U_2)$ by Lemma 2.7(b), so G_2 acts on U_2. Also $U \leq N_1 \leq G_1 \leq G$, and using Lemma 2.7(a) we get

$$\begin{aligned}\operatorname{Soc}_\pi G_1 &= \operatorname{Soc}_{\pi G_1} U_2 = \operatorname{Soc}_{\pi G_2} U_2\\ &\leq U_2 = \operatorname{Soc}_\pi N_1 = \operatorname{Soc}_{\pi N_1} U\\ &\leq U.\end{aligned}$$

Since G and U satisfy the condition (i) in Lemma 5.2, if $p \in \pi$ then $\mathbf{O}_p(U)$ is a vector space over a finite field F, which is a splitting field for all the subgroups of G. Then $\mathbf{O}_p(U_2) = \mathbf{O}_p(\operatorname{Soc}_{\pi N_1} U) = \operatorname{Soc}_{\mathbf{F}_p N_1} \mathbf{O}_p(U) = \operatorname{Soc}_{F N_1} \mathbf{O}_p(U)$ by Lemma 2.6(d), so $\mathbf{O}_p(U_2)$ is an F-subspace of $\mathbf{O}_p(U)$. This holds for all prime numbers $p \in \pi$, so U_2 is πG_3-split for all subgroups $G_3 \leq G_2$. Moreover $\mathbf{C}_{G_2}(\operatorname{Soc}_{\pi G_2} U_2) = \mathbf{C}_{G_1}(\operatorname{Soc}_\pi G_1)/N_1 = N_1/N_1 = 1$ by Lemma 2.7(b), so G_2 and U_2 satisfy (i). Using Lemma 2.7(g) we also deduce that $H_1 = \mathbf{C}_{G_1}(\operatorname{Soc}_{\pi H_1} U_2)$ and $p \nmid |\mathbf{C}_{G_1}(\mathbf{O}_{p'}(\operatorname{Soc}_{\pi H_1} U_2)) : H_1|$ $(p \in \pi)$. Hence

$$\begin{aligned}H_2 &= \mathbf{C}_{G_1}(\operatorname{Soc}_{\pi H_1} U_2)/N_1 = \mathbf{C}_{G_2}(\operatorname{Soc}_{\pi H_2} U_2),\\ p &\nmid |\mathbf{C}_{G_1}(\mathbf{O}_{p'}(\operatorname{Soc}_{\pi H_1} U_2))/N_1 : H_1/N_1|\\ &= |\mathbf{C}_{G_2}(\mathbf{O}_{p'}(\operatorname{Soc}_{\pi H_2} U_2)) : H_2| \quad (p \in \pi),\end{aligned}$$

so H_2 and U_2 satisfy the condition (ii) in Lemma 5.2.

Since $|G_2| = |G_1/N_1| < |G|$, it follows from Hypothesis C1 that H_2 is system permutable in G_2. Let Σ_1 be a Sylow basis in G_1 with $\Sigma_1 \searrow H_1$, and consider the basis $\Sigma_2 = \Sigma_1 N_1/N_1$ in G_2. Note that H_1 pr G_1 (because H_1 is an injector), so H_2 pr G_2 by Lemma 2.2(a). Also $\Sigma_2 \searrow H_2$ by Lemma 2.2(b), and hence H_2 is Σ_2-permutable by Lemma 2.4(b). Finally H_1 is Σ_1-permutable by Lemma 2.2(b). □

Lemma 5.4. *Assume Hypothesis C1, and choose a Sylow basis Σ in G with $\Sigma \searrow H$.*

- (a) *There is a normal complement K for H in G, with $HK = G$, $H \cap K = 1$ and $K = [K, H] \lhd G$.*
- (b) *There is a prime number r such that $H = SH_\infty$, where S is a cyclic r-group and $1 < H_\infty < H$, with $\Sigma \searrow S$ and $H_\infty \lhd H$. Also $H_\infty = [H_\infty, S]$ and $H_\infty \leq \mathbf{N}_G(\Sigma \cap K)$.*
- (c) *There is a prime number q such that $K = QR$, where Q is a q-group and R is an r-group, with $\{Q\} = (\Sigma \cap K) \cap \operatorname{Syl}_q K$ and $\{R\} = (\Sigma \cap K) \cap \operatorname{Syl}_r K$. Also $q \neq r$ and $1 < R < K$.*
- (d) *The module $V = K/K^q K'$ is $\mathbf{F}_q H$-indecomposable, and there is a subgroup K_∞ with $K^q K' \leq K_\infty < K$ and $K_\infty \lhd G$, such that K/K_∞ is $\mathbf{F}_q H$-irreducible.*
- (e) *If $Q_\infty = Q \cap K_\infty$, then $K_\infty = Q_\infty \times R$. Hence Q_∞, $R \lhd G$ but $H \not\leq \mathbf{N}_G(Q)$.*
- (f) *Also $\langle H, Q\rangle = G$, $\mathbf{C}_H(K/R) = 1$ and $R = [R, Q]$.*
- (g) *Moreover $\mathbf{C}_R(H_\infty) \leq \mathbf{C}_R(S)$, and $S = \langle a \rangle$ can be chosen so that $a = a_0 d$, with $a_0 \in \mathbf{O}_r(\mathbf{N}_G(\Sigma))$ and $d \in R$.*

Proof. Form the natural semidirect products $G_0 = GU$ and $H_0 = HU$, and suppose $\pi = \{p_1, p_2, \ \ldots\ , p_m\}$. Take $\{P_i\} = \Sigma \cap \operatorname{Syl}_{p_i} G$ and $U_i = \mathbf{O}_{p_i}(U)$ $(1 \leq i \leq m)$, and define the Sylow basis Σ_0 in G_0 by taking

$$\Sigma_0 \cap \mathrm{Syl}_s G_0 = \begin{cases} \{P_i U_i\} & \text{when } s = p_i, \\ \Sigma \cap \mathrm{Syl}_s G & \text{when } s \notin \pi. \end{cases}$$

Then Lemma 5.3 implies that Hypothesis B in Section 4 is satisfied (where $\mathcal{F}$ is the set of $\mathcal{Z}^\pi$-subgroups of G_0). Now the results follow from Lemmas 4.1, 4.2, 4.3 and 4.4. □

Hypothesis C2. For the rest of this section, and the next two sections, take a Sylow basis Σ, and subgroups H, K, Q, R, S, K_∞, Q_∞, and elements a, a_0, d as in Lemma 5.4. Put $S^\circ = \langle a_0 \rangle$ and $D = \langle d \rangle$, and note that

$$\begin{aligned} G &= HK, & H \cap K &= 1, & K = [K,H] &\lhd G, \\ K &= QR, & Q \cap R &= 1, & R = [R,Q] &\lhd G, \\ K_\infty &< K, & Q_\infty = Q \cap K_\infty &\lhd G, & K_\infty = Q_\infty \times R &\lhd G, \\ S = \langle a \rangle &\le H, & S^\circ = \langle a_0 \rangle &\le \mathbf{N}_G(\Sigma), & D = \langle d \rangle &\le R, \\ & & a &= a_0 d. & & \end{aligned}$$

Also Q is a q-group and R is an r-group, where q and r are distinct prime numbers, and $\Sigma \cap K = \{1, Q, R\}$.

Lemma 5.5. *Assume Hypotheses C1 and C2, and suppose $p \in \pi$. Take $\{P\} = \Sigma \cap \mathrm{Syl}_p G$, and put $U_p = \mathbf{O}_p(U)$. Suppose U_p is an FG-module, where F is a finite field of characteristic p, and F is a splitting field for all the subgroups of G. Let U_0 be an FG-irreducible submodule of U_p, and put $H_0 = \mathbf{C}_G(\mathbf{C}_{U_0}(P))$. Assume that $0 < U_0 < U$.*

(a) *Then $H \le H_0$.*
(b) *Also $H_0 Q = Q H_0$.*
(c) *Moreover $G = H_0 Q$.*
(d) *Hence $R \le \mathbf{C}_G(U_0)$.*

Proof. (a) Lemma 5.3(a) shows that U is the $\mathcal{Z}^\pi$-radical of GU and that $HU \in \mathcal{Z}^\pi$. Also $(P \cap H)U_p \in \mathrm{Syl}_p(HU)$, and $\mathbf{C}_{U_p}((P \cap H)U_p) \ge \mathbf{C}_{U_0}(P)$. Applying Lemma 2.7(d) we deduce that $HU \le \mathbf{C}_{GU}(\mathbf{C}_{U_p}((P \cap H)U_p)) \le \mathbf{C}_{GU}(\mathbf{C}_{U_0}(P)) = H_0 U$, and hence $H = G \cap HU \le G \cap H_0 U = H_0$.

(b) Put $N = \mathbf{C}_G(U_0)$, and note that $U_0 \ge \mathbf{C}_{U_0}(P)$, so $N \le H_0$. Take

$$G_1 = G/N, \quad H_1 = H_0/N, \quad P_1 = PN/N, \quad \Sigma_1 = \Sigma N/N,$$

with $P_1 \in \Sigma_1$. Form the natural semidirect products

$$G_2 = G_1 U_0, \quad H_2 = H_1 U_0,$$

and define the Sylow basis Σ_2 in G_2 by taking

$$\Sigma_2 \cap \operatorname{Syl}_s G_2 = \begin{cases} \{P_1 U_0\} & \text{when } s = p, \\ \Sigma_1 \cap \operatorname{Syl}_s G_1 & \text{when } s \neq p. \end{cases}$$

Then $U_0 = \operatorname{Soc}_\pi G_2$ by Lemma 2.7(c), so Lemma 2.7(f) shows that H_2 is a $\mathcal{Z}^\pi$-injector in G_2 and $\Sigma_2 \searrow H_2$. But H_2/U_0 and $\Sigma_2 U_0/U_0$ correspond to H_1 and Σ_1 under the isomorphism $G_2/U_0 \cong G_1$, so

$$H_1 \text{ pr } G_1, \quad \Sigma_1 \searrow H_1, \quad H_0 \text{ pr } G, \quad \Sigma \searrow H_0,$$

using Lemma 2.2(a) and (b).

Now U_0 is FG-irreducible, so $\mathbf{C}_{G_1}(\operatorname{Soc}_{\pi G_1} U_0) = \mathbf{C}_G(U_0)/N = N/N = 1$, and hence G_1 and U_0 satisfy the condition (i) in Lemma 5.2. Applying Lemma 2.7(g) we also deduce that $H_2 = \mathbf{C}_{G_2}(\operatorname{Soc}_{\pi H_2} U_0)$ and that $s \nmid |\mathbf{C}_{G_2}(\mathbf{O}_{s'}(\operatorname{Soc}_{\pi H_2} U_0)) : H_2|$ $(s \in \pi)$. But $\operatorname{Soc}_{\pi H_2} U_0 = \operatorname{Soc}_{\pi H_1} U_0$, so $H_1 = H_2 \cap G_1 = \mathbf{C}_{G_1}(\operatorname{Soc}_{\pi H_1} U_0)$ and

$$\begin{aligned} |\mathbf{C}_{G_2}(\mathbf{O}_{s'}(\operatorname{Soc}_{\pi H_2} U_0)) : H_2| &= |\mathbf{C}_{G_1}(\mathbf{O}_{s'}(\operatorname{Soc}_{\pi H_1} U_0))U_0 : H_1 U_0| \\ &= |\mathbf{C}_{G_1}(\mathbf{O}_{s'}(\operatorname{Soc}_{\pi H_1} U_0)) : H_1|, \end{aligned}$$

and hence H_1 and U_0 satisfy the condition (ii) in Lemma 5.2. Since $|U_0| < |U|$, Hypothesis C1 implies that H_1 is system permutable in G_1, and as before, it follows from Lemma 2.4(b) that H_1 is Σ_1-permutable. Then H_0 is Σ-permutable by Lemma 2.2(c), so $H_0 Q = Q H_0$.

(c) Using (a) and (b), together with Lemma 5.4(f), we get $H_0 Q \geq \langle H, Q \rangle = G$.

(d) It follows from (c) that $|G : H_0| = |H_0 Q : H_0|$ is a power of q, while $|H_0 R : H_0|$ is a power of r. This implies that $R \leq H_0 = \mathbf{C}_G(\mathbf{C}_{U_0}(P))$, so $\mathbf{C}_{U_0}(R) \geq \mathbf{C}_{U_0}(P) \neq 0$. Since U_0 is FG-irreducible, we deduce that $\mathbf{C}_{U_0}(R) = U_0$. □

Lemma 5.6. *Take H_∞ and V as in Lemma 5.4, and put $H^\circ = S^\circ H_\infty$, $R_\infty = R^r R'$ and $V^\circ = Q/Q^q Q'$, $W = R/R^r R'$.*

(a) *There is a prime number $p \in \pi$ such that U is an irreducible FG-module, where F is a finite field of characteristic p, and F is a splitting field for all the subgroups of G.*
(b) *If $\{P\} = \Sigma \cap \operatorname{Syl}_p G$, then $H = \mathbf{C}_G(\mathbf{C}_U(P))$.*
(c) *If $P \leq H_1 < H$ with $\Sigma \searrow H_1$, then $H_1 \leq \mathbf{N}_H(Q)$.*
(d) *Hence $P \lhd H$, with $H = SP$ and $P = [P, S] \leq \mathbf{N}_H(Q)$. Also p, q and r are distinct, and $\Sigma = \{1, P, Q, RS\}$ with $1 < P < H$.*
(e) *Also $Q_\infty = Q^q Q'$, and the module V is $\mathbf{F}_q H$-irreducible, with $\mathbf{C}_H(V) = 1$.*
(f) *Moreover $\mathbf{C}_R(P) \leq \mathbf{C}_R(S)$, so $\mathbf{C}_W(P) \leq \mathbf{C}_W(S)$.*
(g) *Hence V° is $\mathbf{F}_q H^\circ$-irreducible, $\mathbf{C}_{H^\circ}(V^\circ) = 1$ and $\mathbf{C}_W(H^\circ) = \mathbf{C}_W(P)$.*

Proof. (a) If $\pi = \{p_1, p_2, \ldots, p_m\}$, then it follows from Hypothesis C1 and the condition (i) in Lemma 5.2 that $U = U_1 \oplus U_2 \oplus \ldots \oplus U_m$ is a direct sum of $F_i G$-modules U_i, where F_i is a finite field of characteristic p_i, and F_i is a splitting field for all the subgroups of G; we must therefore prove that $U = U_i$ is F_iG-irreducible for some index i. If this is not the case, then Lemma 5.5(d) implies that $R \leq \mathbf{C}_G(\mathrm{Soc}_{\pi G} U)$. But $\mathbf{C}_G(\mathrm{Soc}_{\pi G} U) = 1$ by the condition (i) in Lemma 5.2, so this contradicts Lemma 5.4(c).

(b) It follows from Lemmas 5.3(a) and 2.7(f) that $HU = \mathbf{C}_{GU}(\mathbf{C}_U(PU)) = \mathbf{C}_G(\mathbf{C}_U(P))U$, which gives the result.

(c) If $G_1 = H_1K$, then $H_1 = \mathbf{C}_{G_1}(\mathbf{C}_U(P))$, and as in Lemma 2.7(f) it follows that H_1U is a $\mathcal{Z}^\pi$-injector of G_1U. Moreover G_1U and H_1U satisfy the conditions (i) and (ii) in Lemma 2.7(g), which implies that G_1 and H_1 satisfy the conditions (i) and (ii) in Lemma 5.2. Hence H_1 is system permutable in G_1 by Hypothesis C1, so H_1 is $(\Sigma \cap G_1)$-permutable by Lemma 2.4(b), and $H_1 \leq \mathbf{N}_H(Q)$ by Lemma 2.4(d).

(d) Put $H_1 = \mathbf{N}_H(P)$, and note that $H = H_1H_\infty$ by Frattini's argument [7, I(7.8)], and $\Sigma \searrow H_1$ by Lemma 2.4(c). If $H_1 < H$, then $H_1 \leq \mathbf{N}_H(Q)$ by (c), while $H_\infty \leq \mathbf{N}_H(Q)$ by Lemma 5.4(b), so $H = H_1H_\infty \leq \mathbf{N}_H(Q)$. This contradicts Lemma 5.4(e), which proves that $H_1 = H$, so $P \lhd H$.

We now get a subgroup $H_2 = SP$. If $H_2 < H$, then (c) implies that $H_2 \leq \mathbf{N}_H(Q)$, and hence $H = H_2H_\infty \leq \mathbf{N}_H(Q)$. As before this contradicts Lemma 5.4(e), and proves that $H = H_2 = SP$. Since $H_\infty = [H_\infty, S]$, it is also clear that $H_\infty = \mathbf{O}^r(H) = P$, so $P = [P, S] \leq \mathbf{N}_H(Q)$, and $1 < P < H$ by Lemma 5.4(b). Finally $P \in \mathrm{Syl}_p\, G$, and K is a $\{q, r\}$-group, so $p \notin \{q, r\}$, while $q \neq r$ by Lemma 5.4(c). Hence $\Sigma = \{1, P, Q, RS\}$.

(e) Now H is a $\{p, r\}$-group by (d), so V is completely $\mathbf{F}_qH$-reducible by Maschke's theorem [5, A(11.5)]. Then Lemma 5.4(d) implies that V is $\mathbf{F}_qH$-irreducible, so $K_\infty = K^qK'$. Since $R \lhd G$, it also follows that $K_\infty = Q^qQ'R$ and $Q_\infty = Q^qQ'$. Finally $K/R \cong Q$ and $V \cong (K/R)/\Phi(K/R)$, and hence $\mathbf{C}_H(V) = \mathbf{C}_H(K/R)$ [7, III(3.18)]. Now Lemma 5.4(f) shows that $\mathbf{C}_H(V) = 1$.

(f) Since $P = H_\infty$, Lemma 5.4(g) shows that $\mathbf{C}_R(P) \leq \mathbf{C}_R(S)$. To prove the last equation, suppose $\xi \in \mathbf{C}_W(P)$. By the theory of coprime actions, there is an element $x \in \mathbf{C}_R(P)$ such that $\xi = R_\infty x$ [7, I(18.6)]. Then $x \in \mathbf{C}_R(P) \leq \mathbf{C}_R(S)$, and hence $\xi = R_\infty x \in \mathbf{C}_W(S)$.

(g) Note that $V^\circ = Q/Q_\infty \cong K/K_\infty = V$, and that d centralizes both V and W, so the action of a_0 on V and W is the same as the action of $a_0d = a$. The results therefore follow from (e) and (f). □

Hypothesis C3. For the rest of this section, and the next two sections, take the prime number p, the field F, the subgroups H°, P, R_∞ and the vector spaces U, V, V°, W as in Lemma 5.6. Note that

$$\begin{gathered} H^\circ = S^\circ P \leq \mathbf{N}_G(Q), \\ \begin{aligned} H &= SP, & S \cap P &= 1, & P &= [P, S] \lhd H, \\ K_\infty &= K^qK', & Q_\infty &= Q^qQ', & R_\infty &= R^rR', \\ V &= K/K_\infty, & V^\circ &= Q/Q_\infty, & W &= R/R_\infty, \\ \mathbf{C}_U(P) &\leq \mathbf{C}_U(S), & \mathbf{C}_R(P) &\leq \mathbf{C}_R(S), & \mathbf{C}_W(P) &\leq \mathbf{C}_W(S). \end{aligned} \end{gathered}$$

Also P is a p-group, Q is a q-group and RS is an r-group, where p, q and r are distinct prime numbers, and $\Sigma = \{1, P, Q, RS\}$.

Lemma 5.7. *Assume Hypotheses C1, C2 and C3.*

(a) *Then* $D \leq \mathbf{C}_R(H^\circ)$.
(b) *Also* $R = [D, Q]$.
(c) *Hence* $R \leq \langle Q, S\rangle$.
(d) *Moreover* $\mathbf{C}_W(H^\circ) \neq 0$.

Proof. (a) Note that $d = a_0^{-1}a \in \mathbf{N}_G(P)$, so $[P, D] \leq P \cap R = 1$. Hence $D \leq \mathbf{C}_R(P) \leq \mathbf{C}_R(S)$ by Lemma 5.6(f), and therefore $[D, H^\circ] = [D, S^\circ P] \leq [D, SDP] = 1$.

(b) Put $R_1 = [D, Q]$, and note that $D \leq \mathbf{N}_G(QR_1)$. Also $H^\circ \leq \mathbf{N}_G(Q)$, and it follows from (a) that $H^\circ \leq \mathbf{N}_G(R_1)$. Hence $H \leq DH^\circ \leq \mathbf{N}_G(QR_1)$, and using Lemma 5.4(f) we get $G = \langle H, Q\rangle \leq HQR_1$. Thus $R_1 \in \mathrm{Syl}_r\, K$, so $R_1 = R$.

(c) Suppose $k \in \mathbf{Z}$ and $c \in Q$. Using (a) we get $[d^k, c] = [a_0^{-k}a^k, c] = (c^{-a_0^{-k}})^{a^k} c \in \langle Q, S\rangle$, so the result follows from (b).

(d) Finally $d \notin R_\infty$ by (b), so (a) implies that the coset $R_\infty d$ is a nonzero vector in $\mathbf{C}_W(H^\circ)$. □

Lemma 5.8. *The module U is FK-irreducible.*

Proof. Note that U is FG-irreducible by Lemma 5.6(a), so it follows from Lemma 5.1(a) that U is $F(PK)$-irreducible. Applying Lemma 5.1(b), let $U_1, U_2, \ \ldots \ , U_n$ be the FK-homogeneous components of U, and choose the notation so that

$$G_1 = \mathbf{N}_G(U_1), \quad N = \mathbf{C}_{G_1}(U_1), \quad P_1 = \mathbf{N}_P(U_1), \quad H_1 = \mathbf{C}_{G_1}(\mathbf{C}_{U_1}(P_1))$$
$$U = U_1 \oplus U_2 \oplus \ \ldots \ \oplus U_n, \quad \mathbf{N}_H(U_1) = SP_1, \quad \mathbf{C}_{U_1}(S) \leq \mathbf{C}_{U_1}(P_1).$$

Then U_1 is $F(P_1K)$-irreducible by Clifford's theorem [7, V(17.3.e)], so U_1 is also FK-irreducible by Lemma 2.8. Suppose $U_1 < U$; we must show that this is impossible. Now $G_1 = \mathbf{N}_H(U_1)K = SP_1K = SRP_1Q$, which implies that $\Sigma \searrow G_1$. Also $S \leq \mathbf{C}_{G_1}(\mathbf{C}_{U_1}(P_1)) = H_1$. Moreover $U_1 \geq \mathbf{C}_{U_1}(P_1)$, so $N \leq H_1$, and we put

$$G_2 = G_1/N, \quad H_2 = H_1/N, \quad P_2 = P_1N/N,$$
$$\Sigma_1 = \Sigma \cap G_1, \quad \Sigma_2 = \Sigma_1N/N,$$

with $P_2 \in \Sigma_2$. Then $H_2 = \mathbf{C}_{G_2}(\mathbf{C}_{U_1}(P_2))$, and U_1 is a module which is G_2-faithful and FG_2-irreducible. As before Lemma 2.7(f) and (g) can be used to show that G_2, H_2 and U_1 satisfy the conditions (i) and (ii) in Lemma 5.2. Since $|U_1| < |U|$ it follows from Hypothesis C1 that H_2 is Σ_2-permutable. Then H_1 is Σ_1-permutable by Lemma 2.2(c), so there is a subgroup $H_1Q \geq \langle Q, S\rangle \geq R$ by Lemma 5.7(c). But $r \nmid |H_1Q : H_1|$, and therefore $R \leq H_1 = \mathbf{C}_{G_1}(\mathbf{C}_{U_1}(P_1))$. Hence $\mathbf{C}_U(R) \geq \mathbf{C}_{U_1}(R) \geq \mathbf{C}_{U_1}(P_1) \neq 0$, and so R centralizes U (because U is FG-irreducible). Finally $\mathbf{C}_G(U) = 1$ by the condition (i) in Lemma 5.2, so this contradicts Lemma 5.4(c). □

6. Groups in which U is FK_∞-inhomogeneous

Remark. In this section, we follow a well known strategy [10, Theorems 3.5, 4.4, 7.3 and 8.4], and consider the case when U is FK_∞-inhomogeneous. We use Lemma 2.9, and throughout the section, we assume Hypotheses C1, C2 and C3 in Section 5.

Lemma 6.1. *Suppose U is FK_∞-inhomogeneous, and let $\{c_1, c_2, \ldots, c_{q^t}\}$ be a transversal to Q_∞ in Q, with $c_1 = 1$.*

(a) *Then $\{c_1, c_2, \ldots, c_{q^t}\}$ is a transversal to K_∞ in K, and $\{c_1, c_2, \ldots, c_{q^t}\}$ is also a transversal to HK_∞ in G. Hence $V = \{v_1, v_2, \ldots, v_{q^t}\}$, with $v_i = K_\infty c_i$ $(1 \leq i < q^t)$.*
(b) *Also $U = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_{q^t}$, where each subspace Y_i is stabilized by K_∞, and $\mathbf{N}_G(Y_1) = HK_\infty$. Moreover H permutes the set $\Omega = \{Y_1, Y_2, \ldots, Y_{q^t}\}$, and $Y_i = Y_1 c_i$ $(1 \leq i \leq q^t)$. Hence the permutation action of H on V is equivalent to the action of H on Ω.*
(c) *Finally the P-orbits in V are stabilized by S.*

Proof. (a) This follows from the definitions.

(b) Clifford's theorem gives $U = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_m$, where the submodules Y_i are the FK_∞-homogeneous components of U, and the set

$$\Omega = \{Y_1, Y_2, \ldots, Y_m\}$$

is permuted by G [7, V(17,3)]. Lemma 5.8 implies that $K/K_\infty \cong QK_\infty/K_\infty$ permutes Ω transitively, while $q \nmid |H|$, so it follows from Lemma 2.9 that H normalizes at least one of the submodules Y_i. Choose the notation so that $H \leq \mathbf{N}_G(Y_1)$, and hence $HK_\infty \leq \mathbf{N}_G(Y_1)$. Now $|G : \mathbf{N}_G(Y_1)| = m \neq 1$ by hypothesis, so $\mathbf{N}_G(Y_1) < G$. But Lemma 5.6(e) shows that HK_∞ is a maximal subgroup of G, and therefore $HK_\infty = \mathbf{N}_G(Y_1)$. This proves that $m = |G : HK_\infty| = q^t$, and that $\{c_1, c_2, \ldots, c_{q^t}\}$ is a transversal to $\mathbf{N}_G(Y_1)$ in G, so the notation can be chosen so that $Y_i = Y_1 c_i$ $(1 \leq i \leq q^t)$. Moreover if $g \in H$, then $Y_1 c_i g = Y_1 g c_i^g = Y_1 c_i^g$, while $(K_\infty c_i)^g = K_\infty c_i^g$, which gives the required permutation equivalence.

(c) Let $\Omega_0, \Omega_1 \ldots, \Omega_n$ be the P-orbits in Ω, and suppose

$$\Omega_j = \{Y_{j1}, Y_{j2}, \ldots, Y_{jm_j}\} \quad (1 \leq j \leq n).$$

Then S permutes the set $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$, and because of (b), it suffices to show that S stabilizes each orbit Ω_j. Take $U_j = Y_{j1} \oplus Y_{j2} \oplus \ldots \oplus Y_{jm_j}$, and note that

$U = U_0 \oplus U_1 \oplus \ \dots \ \oplus U_n$. If $\Omega_j^a = \Omega_k$, then $U_j^a = U_k$, so $\mathbf{C}_{U_j}(P)^a = \mathbf{C}_{U_k}(P) \neq 0$. But $\mathbf{C}_{U_j}(P)^a = \mathbf{C}_{U_j}(P)$ by Lemma 5.6(b), and hence $j = k$ so a stabilizes Ω_j. $\square$

Remark. In the next two lemmas we obtain properties of H by exploiting the condition in Lemma 6.1(c), ignoring the subgroup R and the module U.

Lemma 6.2. *Assume that the P-orbits in V are stabilized by S.*

(a) *Then V is $\mathbf{F}_q P$-irreducible.*
(b) *Also V is $\mathbf{F}_q P$-primitive.*

Proof. (a) Note that V is $\mathbf{F}_q H$-irreducible by Lemma 5.6(e). If V_0 is an $\mathbf{F}_q P$-submodule of V, then the hypothesis implies that S stabilizes V_0, so $V_0 = V$.

(b) Suppose $V = Z_1 \oplus Z_2 \oplus \dots \oplus Z_{p^s}$, where the subspaces Z_i are permuted transitively by P, and put $P_1 = \mathbf{N}_P(Z_1)$; we must deduce that $s = 0$. If $s > 0$, then $|P : P_1| = p^s \neq 1$, so there is a subgroup P_∞ with $P_1 \le P_\infty \lhd P$ and $P/P_\infty \cong \mathbf{C}_p$. Choose transversals $\{b_1', b_2', \ \dots \ , b_{p^{s-1}}'\}$ to P_1 in P_∞, and $\{1, b, b^2, \ \dots \ , b^{p-1}\}$ to P_∞ in P, and take

$$V_i = (Z_1 b_1' \oplus Z_1 b_2' \oplus \ \dots \ \oplus Z_1 b_{p^{s-1}}') b^i = V_0 b^i \quad (i \in \mathbf{Z}_p)$$

Then $V = V_0 \oplus V_1 \oplus \ \dots \ \oplus V_{p-1}$, and the subspaces V_i are stabilized by P_∞, and are permuted regularly by P/P_∞. Now the subset

$$\Upsilon_1 = V_0 \cup V_1 \cup \ \dots \ \cup V_{p-1} \subseteq V$$

is stabilized by P, and is therefore also stabilized by S. Suppose $u, v \in V_i - 0$ with $ua \in V_j$ and $va \in V_k$. Then $u + v \in V_i$, so $ua + va = (u + v)a \in \Upsilon_1$, and therefore $j = k$. This proves that S permutes the set $\{V_i : i \in \mathbf{Z}_p\}$, and we can choose the notation so that S stabilizes V_0 (because $r \neq p$). It follows that S normalizes the subgroup $P_\infty = \mathbf{N}_P(V_0)$, so S acts on P/P_∞. Hence there exist an integer $h \in \mathbf{Z}_p$ and elements $g_i \in P_\infty$, such that $P_\infty b^a = P_\infty b^h$ and $b^{ia} = g_i b^{ih}$. Then

$$V_i a = V_0 b^i a = V_0 a b^{ia} = V_0 g_i b^{ih} = V_0 b^{ih} = V_{ih} \quad (i \in \mathbf{Z}_p).$$

Now $[P, S] = P$ by Lemma 5.6(d), and hence $h \neq 1$. Suppose $h \neq -1$, and put

$$\Upsilon_2 = \{u + v : u \in V_i,\ v \in V_{i+1},\ i \in \mathbf{Z}_p\} \subseteq V.$$

Note that P stabilizes Υ_2, and choose vectors $u \in V_0 - 0$ and $v \in V_1 - 0$. Then $u + v \in \Upsilon_2$, while $ua \in V_0$ and $va \in V_h$. Hence $(u + v)a \notin \Upsilon_2$ (because $h \neq \pm 1$), which contradicts the hypothesis. This proves that $h = -1$.

It follows that $r = 2$, so $p \neq 2$ and $q \neq 2$. Then P_∞ must act intransitively on $V_0 - 0$, so there are disjoint nonempty sets Γ_0, $\Delta_0 \subseteq V_0 - 0$, both stabilized by P_∞, and we take

$$\Gamma_i = \Gamma_0 b^i, \quad \Delta_i = \Delta_0 b^i, \quad \Upsilon_3 = \{u + v : u \in \Gamma_i,\, v \in \Delta_{i+1},\, i \in \mathbf{Z}_p\} \subseteq V.$$

Then P/P_∞ permutes the sets $\{\Gamma_i : i \in \mathbf{Z}_p\}$ and $\{\Delta_i : i \in \mathbf{Z}_p\}$ regularly, so P/P_∞ stabilizes Υ_3. As above, choose vectors $u \in \Gamma_0$ and $v \in \Delta_1$. Then $u + v \in \Upsilon_3$, and $ua \in \Gamma_0$, but $va \in \Delta_{p-1}$, so $(u + v)a \notin \Upsilon_3$. This contradiction completes the proof. $\square$

Lemma 6.3. *Assume that the P-orbits in V are stabilized by S.*

(a) *Then P is cyclic.*
(b) *Also $V = \{0\} \cup \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$, where for each index j, P permutes Γ_j regularly, and there is a vector $v_j' \in \Gamma_j$ such that S fixes v_j'.*

Proof. (a) Suppose P is not cyclic; we must show that this is impossible. Note that V is a module which is P-faithful and $\mathbf{F}_q P$-irreducible by Lemmas 5.6(e) and 6.2(a), and let A be an abelian normal subgroup of P. It follows from Clifford's theorem that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$, where the submodules V_i are the $\mathbf{F}_q A$-homogeneous components of V, and are permuted by P [7, V(17.3)]. But Lemma 6.2(b) shows that $m = 1$, so V is $\mathbf{F}_q A$-homogeneous. Since $\mathbf{C}_A(V) = 1$, Lemma 2.11(a) implies that A is cyclic. Thus every normal abelian subgroup of P is cyclic, so Lemma 2.12(a) shows that $p = 2$ and either $P \cong \mathbf{Q}_8$ or else $\mathrm{Aut}\, P$ is a 2-group. But if $\mathrm{Aut}\, P$ is a 2-group then $[P, S] = 1$, which contradicts Lemma 5.6(d).

We may now suppose that $P \cong \mathbf{Q}_8$, and take $\langle z \rangle = \mathbf{Z}(P) = P' \cong \mathbf{C}_2$. Then $|\mathrm{Aut}\, P| = 2^3 3$, so it follows from Lemma 5.6(d) that $r = 3$ (and that $H/\mathbf{C}_S(P) = H/\mathbf{O}_3(H) \cong \mathrm{SL}_2(3)$). Take $V_i = \{v \in V : vz = (-1)^i v\}$ $(i = 0, 1)$, and note that V_0 and V_1 are both stabilized by P, and $V = V_0 \oplus V_1$. Since V is a module which is P-faithful and $\mathbf{F}_q P$-irreducible, it follows that $V = V_1$, and $\mathbf{C}_P(v) = 1$ for all vectors $v \in V - 0$. This implies that

$$V = \{0\} \cup \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n,$$

where each subset Γ_j is permuted regularly by P. Thus $|\Gamma_j| = 8$, and S stabilizes Γ_j by hypothesis, and hence the action of H on Γ_j is equivalent to the action on the cosets of S (and $H \cong \mathrm{SL}_2(3)$). This implies that Γ_j contains two elements fixed by S (and two S-orbits of size 3). Now put $s = \dim_{\mathbf{F}_q} \mathbf{C}_V(S)$ and $t = \dim_{\mathbf{F}_q} V$, and note that $0 < s < t$. Hence $q^s = |\mathbf{C}_V(S)| = 1 + 2n$ and $q^t = |V| = 1 + 8n = 1 + 4(q^s - 1)$, and so $q^t - 4q^s + 3 = 0$. Therefore $q \mid 4q^s - q^t = 3$, so $q = 3 = r$, which contradicts Lemma 5.4(c).

(b) As in (a), note that V is a module which is P-faithful and $\mathbf{F}_q P$-irreducible by Lemmas 5.6(e) and 6.2(a), and let $\{0\}, \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be the P-orbits in V. Then P permutes each set Γ_j regularly by Lemma 3.1(b), and hence $r \nmid |\Gamma_j|$. Also S stabilizes Γ_j by hypothesis, so S fixes a vector $v_j' \in \Gamma_j$. $\square$

Lemma 6.4. *Suppose U is FK_∞-inhomogeneous, and choose the subspace Y_1 as in Lemma 6.1(b). Take the subsets $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ and the elements $v'_1, v'_2, \dots, v'_n$ as in Lemma 6.3(b), and put $M = \mathbf{C}_{HK_\infty}(Y_1)$.*

(a) *For each index j, there is an element $g_j \in \mathbf{C}_Q(a_0)$ such that $K_\infty g_j = v'_j$.*
(b) *Then $S^{g_j^{-1}} \leq M$.*
(c) *Also $H \leq M$.*
(d) *Hence $R \leq M$.*

Proof. (a) Using Lemma 6.1(a), take $v'_j = K_\infty c'_j$ with $c'_j \in Q$ $(1 \leq j \leq n)$, and note that $(K_\infty c'_j)^a = K_\infty c'_j$. Now $d \in K_\infty$ so $(K_\infty c'_j)^d = K_\infty c'_j$, and therefore $(K_\infty c'_j)^{a_0} = (K_\infty c'_j)^{ad^{-1}} = K_\infty c'_j$. Also $a_0 \in \mathbf{N}_G(Q)$, so this implies that $[a_0, c'_j] \in Q \cap K_\infty = Q_\infty$, and hence $(Q_\infty c'_j)^{a_0} = Q_\infty c'_j$. But a_0 is an r-element, while $|Q_\infty c'_j|$ is a power of q, so a_0 fixes an element $g_j \in Q_\infty c'_j = Q_\infty g_j$.

(b) Consider a vector $y_1 \in Y_1$; we must show that $y_1 a^{g_j^{-1}} = y_1$. Suppose $P = \langle b \rangle \cong \mathbf{C}_{p^s}$ as in Lemma 6.3(a), and note that $\Gamma_j = \{v'_j b^i : i \in \mathbf{Z}_{p^s}\}$ by Lemma 6.3(b). Now $v'_j = K_\infty g_j$ by (a), and we put $Y_j = Y_1 g_j$ and $y_j = y_1 g_j$. Applying Lemma 6.1(b), we get $U_j = \bigoplus_{i \in \mathbf{Z}_{p^s}} Y_j b^i \leq U$, so $u_j = \sum_{i \in \mathbf{Z}_{p^s}} y_j b^i \in \mathbf{C}_{U_j}(P) \leq \mathbf{C}_{U_j}(S)$. Moreover S stabilizes Γ_j by Lemma 6.1(c), so S permutes the set $\{Y_j b^i : i \in \mathbf{Z}_{p^s}\}$, and hence S also permutes the set $\{y_j b^i : i \in \mathbf{Z}_{p^s}\}$. Finally S fixes v'_j, so S stabilizes Y_j, and therefore S centralizes $y_j = y_1 g_j$. Thus $y_1 g_j a = y_1 g_j$, so $y_1 a^{g_j^{-1}} = y_1$.

(c) Note that $Q \neq 1$ by Lemma 5.4(c), so $V \neq 0$ and $n > 0$. From (b) we get $S^{g_1^{-1}} R \leq MR \lhd HK_\infty$. Now $S^{g_1^{-1}} R = (SR)^{g_1^{-1}} \in \mathrm{Syl}_r\, G$, and hence $S^{g_1^{-1}} R$ is conjugate to SR in HK_∞. Therefore $S \leq SR \leq MR$, and Lemma 5.6(d) implies that $P = [P, S] \leq MR$. Since MR/M is an r-group, it follows that $P \leq M = \mathbf{C}_{HK_\infty}(Y_1)$. Thus $Y_1 \leq \mathbf{C}_U(P) \leq \mathbf{C}_U(S)$, so $H \leq \mathbf{C}_{HK_\infty}(Y_1) = M$.

(d) Put $D = \langle d \rangle$, and note that $R = [D, Q]$ by Lemma 5.7(b). It therefore suffices to show that if $k \in \mathbf{Z}$ and $c \in Q$, then $[d^k, c] \in M$. Moreover $[D, Q_\infty] = 1$ by Lemma 5.4(e), so we may assume that $c \notin Q_\infty$. It follows from (a) and Lemma 6.3(b) that the set $\{1\} \cup \{g_j^{b^i} : i \in \mathbf{Z}_{p^s}, 1 \leq j \leq n\}$ is a transversal to Q_∞ in Q, and hence the set $\{1\} \cup \{g_j^{-b^i} : i \in \mathbf{Z}_{p^s}, 1 \leq j \leq n\}$ is also a transversal to Q_∞ in Q. Thus $c = z g_j^{-b^i}$ with $z \in Q_\infty$, $i \in \mathbf{Z}_{p^s}$ and $j \in \{1, 2, \dots, n\}$. Now $[D, Q_\infty] = [D, P] = 1$ by Lemmas 5.4(e) and 5.6(a), while $g_j \in \mathbf{C}_Q(a_0)$ by (a). Using these facts, we get

$$\begin{aligned}
[d^k, c] &= [d^k, z g_j^{-b^i}] = [d^k, g_j^{-b^i}] = [d^k, g_j^{-1}]^{b^i} \\
&= [a_0^{-k} a^k, g_j^{-1}]^{b^i} = [a^k, g_j^{-1}]^{b^i} = (a^{-k} a^{k g_j^{-1}})^{b^i} \\
&\in M^{b^i} = M,
\end{aligned}$$

using (b) and (c). $\square$

Lemma 6.5. *The FG-module U is FK_∞-homogeneous.*

Proof. If U is FK_∞-inhomogeneous, then Lemma 6.4(d) implies that

$$R \le \bigcap_{g\in G} M^g = \mathbf{C}_G(U).$$

But $\mathbf{C}_G(U) = 1$ by the condition (i) in Lemma 5.2, so this contradicts Lemma 5.4(c). □

7. Groups in which U is FK_∞-homogeneous

Remark. In this section we continue our basic strategy [10, Theorems 3.5, 4.4, 7.3 and 8.4], and show that R is extraspecial, using Lemmas 2.10, 2.11 and 2.12.

Lemma 7.1. *Let W be a module which is C-faithful and $\mathbf{F}_rC$-homogeneous, where $C = \langle c\rangle \cong \mathbf{C}_n$ and $r \nmid n$ (and r is a prime number). If $\omega \in W$, then the submodule $\omega(\mathbf{F}_rC)$ generated by ω is $\mathbf{F}_rC$-irreducible.*

Proof. Suppose $W = X_1 \oplus X_2 \oplus \ \dots \ \oplus X_m$, where the $\mathbf{F}_rC$-modules X_i are irreducible, and isomorphic to each other, and let k be the order of r modulo n. As in Lemma 3.1(b), there exist $\mathbf{F}_r$-isomorphisms $\theta_i : X_i \to \mathbf{F}^+_{r^k}$, and an element γ which is a primitive n-th root of 1 in $\mathbf{F}_{r^k}$, such that

$$(\omega c^j)^{\theta_i} = \gamma^j\omega^{\theta_i} \quad (\omega \in X_i,\ 1 \le i \le m,\ j \in \mathbf{Z}_n).$$

Let W_1 be an m-dimensional vector space over $\mathbf{F}_{r^k}$, and let $\{x_1, x_2, \ \dots \ , x_m\}$ be an $\mathbf{F}_{r^k}$-basis of W_1. Consider an element $\omega = \sum_{i=1}^m \omega_i \in W$ with $\omega_i \in X_i$ $(1 \le i \le m)$, and define an $\mathbf{F}_r$-isomorphism $\theta : W \to W_1$ by taking $\omega^\theta = \sum_{i=1}^m \omega_i^{\theta_i}x_i$. Suppose also that $\lambda = \sum_{j\in\mathbf{Z}_n} \alpha_j c^j \in \mathbf{F}_rC$, with $\alpha_j \in \mathbf{F}_r$ $(j \in \mathbf{Z}_n)$. Then

$$\begin{aligned}
(\omega c^j)^\theta &= \left(\textstyle\sum_{i=1}^m \omega_i c^j\right)^\theta = \textstyle\sum_{i=1}^m (\omega_i c^j)^{\theta_i}x_i \\
&= \textstyle\sum_{i=1}^m \gamma^j\omega_i^{\theta_i}x_i = \gamma^j\omega^\theta \quad (j \in \mathbf{Z}_n), \\
(\omega\lambda)^\theta &= \textstyle\sum_{j\in\mathbf{Z}_n} \alpha_j(\omega c^j)^\theta = \textstyle\sum_{j\in\mathbf{Z}_n} \alpha_j\gamma^j\omega^\theta = \mu\omega^\theta,
\end{aligned}$$

where $\mu = \sum_{j\in\mathbf{Z}_n} \alpha_j\gamma^j \in \mathbf{F}_{r^k}$. Thus $(\omega(\mathbf{F}_rC))^\theta$ is the 1-dimensional $\mathbf{F}_{r^k}$-subspace spanned by ω^θ, and $\omega(\mathbf{F}_rC)$ is $\mathbf{F}_rC$-isomorphic to X_1. □

Remark. For the rest of this section, we assume Hypotheses C1, C2 and C3 in Section 5.

Lemma 7.2. *Let Ω be the set of $\mathbf{F}_rQ$-homogeneous components of W, and let $\Omega_1, \Omega_2, \dots, \Omega_m$ be the P-orbits in Ω, with $\Omega_i = \{X_{i1}, X_{i2}, \ \dots \ , X_{in_i}\}$ $(1 \le i \le m)$.*

(a) *Each component X_{ij} is $\mathbf{F}_rQ$-irreducible $(1 \le i \le m,\ 1 \le j \le n_i)$.*
(b) *Each P-orbit Ω_i is stabilized by H° $(1 \le i \le m)$.*

(c) *Put* $L_i = \mathbf{N}_{H^\circ}(X_{i1})$ *and* $P_i = \mathbf{N}_P(X_{i1})$*. Then the notation can be chosen so that* $L_i = S^\circ P_i$ *and* $\mathbf{C}_{X_{i1}}(P_i) = \mathbf{C}_{X_{i1}}(L_i) \neq 0$ *(*$1 \le i \le m$*).*

Proof. Consider the vector $\omega = R_\infty d \in W$, and put $Z_i = X_{i1} \oplus X_{i2} \oplus \ \dots \ \oplus X_{in_i}$ $(1 \le i \le m)$. Then $W = Z_1 \oplus Z_2 \oplus \ \dots \ \oplus Z_m$, and we take

$$\begin{aligned} \omega &= \zeta_1 + \zeta_2 + \ \dots \ + \zeta_m \quad \text{with } \zeta_i \in Z_i \qquad (1 \le i \le m), \\ \zeta_i &= \xi_{i1} + \xi_{i2} + \ \dots \ + \xi_{in_i} \text{ with } \xi_{ij} \in X_{ij} \quad (1 \le j \le n_i). \end{aligned}$$

(a) Lemma 5.7(b) implies that the $\mathbf{F}_rQ$-submodule $\omega(\mathbf{F}_rQ)$ generated by ω is equal to W, and hence $\xi_{ij}(\mathbf{F}_rQ) = X_{ij}$ $(1 \le i \le m,\ 1 \le j \le n_i)$. But $[Q_\infty, R] = 1$ by Lemma 5.4(e), so Q_∞ centralizes W. Thus X_{ij} can be regarded as a homogeneous $\mathbf{F}_r(Q/Q_\infty)$-module, where Q/Q_∞ is an elementary abelian q-group. Then $Q/\mathbf{C}_Q(X_{ij})$ is cyclic by Lemma 3.1(a), and the result follows from Lemma 7.1.

(b) Note that $\omega \in \mathbf{C}_W(P)$ by Lemma 5.7(a), so $\zeta_i \in \mathbf{C}_W(P) = \mathbf{C}_W(H^\circ)$ by Lemma 5.6(g). Also S° permutes the set $\{\Omega_1, \Omega_2, \ \dots \ , \Omega_m\}$, and if $\Omega_i^{a_0} = \Omega_j$, then $\zeta_i = \zeta_i^{a_0} \in Z_i^{a_0} = Z_j$. But $\zeta_i(\mathbf{F}_rQ) = Z_i$ as in (a), so $\zeta_i \neq 0$, and hence $i = j$.

(c) Applying Lemma 5.1(b) with $K = Q$, we can choose the notation so that $L_i = S^\circ P_i$ and $\mathbf{C}_{X_{i1}}(P_i) \le \mathbf{C}_{X_{i1}}(S^\circ)$, and therefore $\mathbf{C}_{X_{i1}}(P_i) = \mathbf{C}_{X_{i1}}(L_i)$. Moreover $P_i = \mathbf{N}_P(X_{i1})$ permutes the set $\{\xi_{i1}, \xi_{i2}, \ \dots \ , \xi_{in_i}\}$, and hence ξ_{i1} is a nonzero vector in $\mathbf{C}_{X_{i1}}(P_i)$. □

Lemma 7.3. *Suppose* U *is* FK_∞*-homogeneous.*

(a) *If* $R_1 \le R$ *with* $R_1 \lhd G$*, then* U *is* FR_1*-homogeneous.*
(b) *Hence* R *is extraspecial and* $R' \le \mathbf{Z}(G)$*.*

Proof. (a) Suppose the restriction of U to K_∞ is $U_{K_\infty} = X_1 \oplus X_2 \oplus \ \dots \ \oplus X_m$, where the modules X_i are FK_∞-irreducible, and there are FK_∞-isomorphisms $\theta_i : X_1 \to X_i$ $(1 \le i \le m)$. Put $K_1 = Q_\infty R_1$, and note that $K_1 \lhd G$. It follows from Clifford's theorem that $X_1 = X_{11} \oplus X_{12} \oplus \ \dots \ \oplus X_{1n}$, where the submodules X_{1j} are the FK_1-homogeneous components of X_1, and are permuted transitively by R [7, V(17.3)]. For each index j, put $U_j = X_{1j} \oplus X_{1j}^{\theta_2} \oplus X_{1j}^{\theta_3} \oplus \ \dots \ \oplus X_{1j}^{\theta_m}$, and note that $U = U_1 \oplus U_2 \oplus \ \dots \ \oplus U_n$, where the submodules U_j are the FK_1-homogeneous components of U, and are permuted by G, and permuted transitively by R.

Since $r \nmid |PQ|$, it follows from Lemma 2.9 that PQ normalizes at least one of the submodules U_j. Choose the notation so that $PQ \le \mathbf{N}_G(U_1)$, and note that if $U_1^a = U_j$, then $\mathbf{C}_{U_1}(P)^a \le U_j$. Since S centralizes $\mathbf{C}_{U_1}(P)$, it follows that $j = 1$, so $S \le \mathbf{N}_G(U_1)$. Using Lemma 5.4(f), we get $G = \langle H, Q \rangle \le \mathbf{N}_G(U_1)$, so U_1 is an FG-submodule. But U is FG-irreducible by Lemma 5.6(a), so $U = U_1$ is FK_1-homogeneous. Finally let Y be an FK_1-irreducible submodule of U; it suffices to show that Y is FR_1-homogeneous. Now $K_1 = Q_\infty \times R_1$ by Lemma 5.4(e), so Lemma 2.10 implies that $Y = Y_1 \otimes_F Y_2$, where Y_1 is FQ_∞-irreducible and Y_2 is FR_1-irreducible. Restricting to R_1, we deduce that $Y_{R_1} = (\dim_F Y_1) \cdot Y_2$ is FR_1-homogeneous.

(b) If A is a characteristic abelian subgroup of R, then U is FA-homogeneous by (a), and F is a splitting field for A by hypothesis, so A is cyclic and $A \le \mathbf{Z}(G)$ by Lemma 2.11. It now follows from Lemma 2.12(b) that $R = R_0 \circ R_1$ is a central product, where R_0 is extraspecial (or $R_0 \cong \mathbf{C}_r$) and R_1 is cyclic. Now $G' \ge [K, H] = K \ge Q$ by Lemma 5.4(a), and $[R, Q] = R \ne 1$ by Lemma 5.4(c) and (f), and therefore $(\operatorname{Aut} R)' \ge G'/\mathbf{C}_{G'}(R) \ge Q/\mathbf{C}_Q(R) \ne 1$. Hence R is not cyclic, so R_0 is extraspecial, and $R' = R_0' \cong \mathbf{C}_r$. Finally $R_1 = \mathbf{Z}(R)$ is a characteristic abelian subgroup of R, so $R_1 \le \mathbf{Z}(G)$ as above. Thus $R_1/R' \le \mathbf{C}_{R/R'}(Q) = R'/R'$ by Lemma 5.4(f), so $R' = R_1 \le \mathbf{Z}(G)$, and $R = R_0$ is extraspecial. □

Remark. With a similar argument, using a result of Hobby [7, III(7.8.c)], it can be shown that if U is FK_∞-homogeneous, then Q is either elementary abelian or extraspecial, but we shall not need this fact.

8. Proof of the Theorem

Remark. In this section we complete the proof of the Theorem by applying the results in Section 3 to the extraspecial group R, ignoring the module U.

Hypothesis D. In the next two lemmas, let $G = HQ_0R$ be a finite group, such that $Q_0 \lhd HQ_0$, $R \lhd HQ_0R$ and $H \cap Q_0 = HQ_0 \cap R = 1$. Suppose Q_0 is an elementary abelian q-group, and R is an extraspecial r-group (where q and r are distinct prime numbers), and let k be the order of r modulo q. Put $Z = R' = \mathbf{Z}(R) \cong \mathbf{C}_r$, and assume that $[R, Q_0] = R$ and $[Z, G] = 1$. Applying Lemma 3.2(a), write $R = T_1 \circ T_2 \circ \dots \circ T_m$ as a central product of extraspecially Q_0-irreducible r-groups T_i. Put

$$\Omega = \{T_1, T_2, \dots, T_m\}, \quad L = \mathbf{N}_H(T_1), \quad Y_i = T_i/Z \ (1 \le i \le m),$$

and assume that H permutes Ω transitively.

Lemma 8.1. *Assume Hypothesis D, and suppose $2 \nmid k$. Lemma 3.7(a) implies that the groups T_i are all of type (ii) in Lemma 3.2(b), with $Y_i = X_i^* \oplus X_i$, and we assume that $\mathbf{C}_{X_1}(L) \ne 0$. Then $R = E_1 \circ E_2 \circ \dots \circ E_n$, where $|E_j| = r^3$, $E_j^r \le E_j' = Z$ and $E_j = \langle d_j, e_j \rangle$ $(1 \le j \le n)$. Moreover the set $\Delta = \{1, 2, \dots, n\}$ is permuted by H, with $d_j^g = d_{jg}$ and $e_j^g = e_{jg}$ $(j \in \Delta,\ g \in H)$.*

Proof. Note that $\mathbf{C}_{Q_0}(X_i) = \mathbf{C}_{Q_0}(X_i^*) = \mathbf{C}_{Q_0}(Y_i) = \mathbf{C}_{Q_0}(T_i)$ by the theory of coprime actions [5, A(12.3)], and $Q_0/\mathbf{C}_{Q_0}(Y_i) \cong \mathbf{C}_q$ by Lemma 3.1(a). Put

$$Q_1 = Q_0/\mathbf{C}_{Q_0}(T_1) \cong \mathbf{C}_q, \quad L_1 = L/\mathbf{C}_L(T_1).$$

If $r \neq 2$, then $q \neq 2$ by Lemmas 3.9 and 7.2(a). Then Lemma 3.7(b) implies that Q_1T_1 can be identified with a subgroup of the group C_1R in Lemma 3.3, and hence $L_1 \leq BC_\infty$ by Lemma 3.10(a). Now Lemma 3.12 shows that $T_1 = E_{11} \circ E_{12} \circ \ \ldots \ \circ E_{1k}$, with $|E_{1j}| = r^3$, $E_{1j}^r = 1$ and $E_{1j}' = Z$, $E_{1j} = \langle d_{1j}, e_{1j}\rangle$ $(1 \leq j \leq k)$. Also the set $\Gamma_1 = \{(11), (12), \ \ldots \ , (1k)\}$ is permuted semiregularly by L_1, and $d_{1j}^g = d_{(1j)g}$, $e_{1j}^g = e_{(1j)g}$ $(1 \leq j \leq k,\ g \in L)$.

Similarly if $r = 2$, then Lemma 3.7(c) shows that Q_1T_1 can be identified with a subgroup of the group C_1R in Lemma 3.4, and hence $L_1 \leq BC_\infty$ by Lemma 3.10(b). Now Lemma 3.13 shows that $T_1 = E_{11} \circ E_{12} \circ \ \ldots \ \circ E_{1k}$ with $E_{1j} = \langle d_{1j}, e_{1j}\rangle \cong \mathbf{D}_8$ $(1 \leq j \leq k)$. Also the set $\Gamma_1 = \{(11), (12), \ \ldots \ , (1k)\}$ is permuted semiregularly by L_1, and $d_{1j}^g = d_{(1j)g}$, $e_{1j}^g = e_{(1j)g}$ $(1 \leq j \leq k,\ g \in L)$.

To complete the proof, let $\{g_1, g_2, \ \ldots \ , g_m\}$ be a transversal to L in H, with $g_1 = 1$ and $T_i = T_1^{g_i}$, and put $d_{ij} = d_{1j}^{g_i}$, $e_{ij} = e_{1j}^{g_i}$, $E_{ij} = E_{1j}^{g_i}$ and $\Gamma_i = \{(i1), (i2), \ \ldots \ , (ik)\}$, $\Delta = \Gamma_1 \cup \Gamma_2 \cup \ \ldots \ \cup \Gamma_m$. Then $T_i = E_{i1} \circ E_{i2} \circ \ \ldots \ \circ E_{ik}$, and H permutes Δ, with $d_{ij}^g = d_{(ij)g}$ and $e_{ij}^g = e_{(ij)g}$ $(1 \leq i \leq m,\ 1 \leq j \leq k,\ g \in H)$. We get the result by replacing the suffices (ij) in Δ by the numbers $1, 2, \ \ldots \ , n$ (where $n = mk$). □

Lemma 8.2. *Assume Hypothesis D, and suppose* $\mathbf{C}_{Y_1}(L) \neq 0$ *and* $2 \mid k$*. If* $r = 2$ *suppose also that* $P \leq H$*, with* $2 \nmid |P|$ *and* $P_1 = \mathbf{N}_P(T_1)$*, and assume that* $\mathbf{C}_{Y_1}(P_1) = \mathbf{C}_{Y_1}(L) \neq 0$*. Then* $R = E_1 \circ E_2 \circ \ \ldots \ \circ E_n$*, where* $|E_j| = r^3$*,* $E_j^r \leq E_j' = Z$ *and* $E_j = \langle d_j, e_j\rangle$ $(1 \leq j \leq n)$*. Moreover the set* $\Delta = \{1, 2, \ \ldots \ , n\}$ *is permuted by* H*, with* $d_j^g = d_{jg}$ *and* $e_j^g = e_{jg}$ $(j \in \Delta,\ g \in H)$*.*

Proof. We can copy the proof of Lemma 8.1 as follows. Lemma 3.8(a) implies that the groups T_i are all of type (i) in Lemma 3.2(b). Moreover $\mathbf{C}_{Q_0}(Y_i) = \mathbf{C}_{Q_0}(T_i)$ by the theory of coprime actions [5, A(12.3)], and $Q_0/\mathbf{C}_{Q_0}(Y_i) \cong \mathbf{C}_q$ by Lemma 3.1(a). Put

$$Q_1 = Q_0/\mathbf{C}_{Q_0}(T_1) \cong \mathbf{C}_q, \quad L_1 = L/\mathbf{C}_L(T_1).$$

If $r \neq 2$, then Lemma 3.8(b) shows that Q_1T_1 can be identified with a subgroup of the group C_1R in Lemma 3.5, and hence $L_1 \leq B_\infty$ by Lemma 3.11(a). Since $\mathbf{C}_{Y_1}(L) \neq 0$, Lemma 3.14(b) and (c) show that $T_1 = E_{11} \circ E_{12} \circ \ \ldots \ \circ E_{1,k/2}$, with $|E_{1j}| = r^3$, $E_{1j}^r = 1$ and $E_{1j}' = Z$, $E_{1j} = \langle d_{1j}, e_{1j}\rangle$ $(1 \leq j \leq k/2)$. Also the set $\Gamma_1 = \{(11), (12), \ \ldots \ , (1, k/2)\}$ is permuted semiregularly by L_1, and $d_{1j}^g = d_{(1j)g}$, $e_{1j}^g = e_{(1j)g}$ $(1 \leq j \leq k/2,\ g \in L)$.

Similarly if $r = 2$, then Lemma 3.8(c) shows that Q_1T_1 can be identified with a subgroup of the group C_1R in Lemma 3.6, and hence $L_1 \leq BC_1$ by Lemma 3.11(b). Since $\mathbf{C}_{Y_1}(P_1) = \mathbf{C}_{Y_1}(L) \neq 0$, Lemma 3.15(b) and (c) show that $T_1 = E_{11} \circ E_{12} \circ \ \ldots \ \circ E_{1,k/2}$ with $|E_{1j}| = 2^3$, $E_{1j}^2 \leq E_{1j}' = Z$ and $E_{1j} = \langle d_{1j}, e_{1j}\rangle$ $(1 \leq j \leq k/2)$. Also the set $\Gamma_1 = \{(11), (12), \ \ldots \ , (1, k/2)\}$ is permuted semiregularly by L_1, and $d_{1j}^g = d_{(1j)g}$, $e_{1j}^g = e_{(1j)g}$ $(1 \leq j \leq k/2,\ g \in L)$.

To complete the proof, let $\{g_1, g_2, \ \ldots \ , g_m\}$ be a transversal to L in H, with $g_1 = 1$ and $T_i = T_1^{g_i}$, and put $d_{ij} = d_{1j}^{g_i}$, $e_{ij} = e_{1j}^{g_i}$, $E_{ij} = E_{1j}^{g_i}$ and $\Gamma_i = \{(i1), (i2), \ \ldots \ , (i, k/2)\}$,

$\Delta = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_m$. Then $T_i = E_{i1} \circ E_{i2} \circ \ldots \circ E_{i,k/2}$, and H permutes Δ, with $d_{ij}^g = d_{(ij)g}$ and $e_{ij}^g = e_{(ij)g}$ $(1 \leq i \leq m,\ 1 \leq j \leq k/2,\ g \in H)$. We get the result by replacing the suffices (ij) in Δ by the numbers $1, 2, \ldots, n$ (where $n = mk/2$). $\square$

Proof of the Theorem. Suppose there is a finite solvable group whose $\mathcal{Z}^\pi$-injectors are not system permutable. By Lemmas 5.2, 5.4 and 5.6, there exist a prime number $p \in \pi$ and an FG-module U, where G is a finite solvable group, and F is a finite field of characteristic p, such that F is a splitting field for all the subgroups of G, and Hypotheses C1, C2 and C3 in Section 5 are all satisfied. Then U is FK_∞-homogeneous by Lemma 6.5, and R is extraspecial with $Z = R' \leq \mathbf{Z}(G)$ by Lemma 7.3(b). Also $Q_\infty \lhd G$ by Lemma 5.4(e), and hence $H^\circ Q/Q_\infty$ acts on R, where Q/Q_∞ is an elementary abelian q-group. Applying Lemma 3.2(a), write $R = T_1 \circ T_2 \circ \ldots \circ T_m$ as a central product of extraspecially Q-irreducible r-groups T_i. Let k be the order of r modulo q, and put

$$\Omega = \{T_1, T_2, \ldots, T_m\}, \quad Q_0 = Q/Q_\infty,$$
$$W = R/Z, \quad Y_i = T_i/Z (1 \leq i \leq m).$$

First suppose $2 \nmid k$. Then Lemma 3.7(a) implies that the groups T_i are all of type (ii) in Lemma 3.2(b), with $Y_i = X_i^* \oplus X_i$, where the submodules X_i^* and X_i are $\mathbf{F}_r Q$-irreducible. It follows from Lemma 7.2(a) that $X_1^*, X_2^*, \ldots, X_m^*$ and $X_1, X_2, \ldots, X_m$ are the $\mathbf{F}_r Q$-homogeneous components of W, so they are permuted by H°, and hence H° also permutes Ω (because H° preserves the duality). Let $\Omega_1, \Omega_2, \ldots, \Omega_l$ be the H°-orbits in Ω, and write

$$\Omega_i = \{T_{i1}, T_{i2}, \ldots, T_{im_i}\}, \quad R_i = T_{i1} \circ T_{i2} \circ \ldots \circ T_{im_i},$$
$$W_i = R_i/Z, \quad L_i = \mathbf{N}_{H^\circ}(T_{i1}),$$
$$Y_{ij} = T_{ij}/Z = X_{ij}^* \oplus X_{ij} (1 \leq i \leq l,\ 1 \leq j \leq m_i).$$

Then $H^\circ Q_0 R_i$, Ω_i and L_i satisfy Hypothesis D. Moreover Lemma 7.2(c) shows that $\mathbf{C}_{X_{i1}}(L_i) \neq 0$, so the hypotheses of Lemma 8.1 also hold. We therefore get $R_i = E_{i1} \circ E_{i2} \circ \ldots \circ E_{in_i}$, where $|E_{ij}| = r^3$, $E_{ij}^r \leq E_{ij}' = Z$ and $E_{ij} = \langle d_{ij}, e_{ij} \rangle$ $(1 \leq j \leq n_i)$. Moreover the set $\Delta_i = \{(i1), (i2), \ldots, (in_i)\}$ is permuted by H°, with $d_{ij}^g = d_{(ij)g}$, $e_{ij}^g = e_{(ij)g}$ $((ij) \in \Delta_i,\ g \in H^\circ)$.

Next suppose $2 \mid k$. Then Lemma 3.8(a) implies that the groups T_i are all of type (i) in Lemma 3.2(b), so the submodules Y_i are $\mathbf{F}_r Q$-irreducible. It follows from Lemma 7.2(a) that $Y_1, Y_2, \ldots, Y_m$ are the $\mathbf{F}_r Q$-homogeneous components of W, so they are permuted by H°, and hence H° also permutes Ω. Let $\Omega_1, \Omega_2, \ldots, \Omega_l$ be the H°-orbits in Ω, and write

$$\Omega_i = \{T_{i1}, T_{i2}, \ldots, T_{im_i}\}, \quad R_i = T_{i1} \circ T_{i2} \circ \ldots \circ T_{im_i},$$
$$W_i = R_i/Z, \quad L_i = \mathbf{N}_{H^\circ}(T_{i1}), \quad P_i = \mathbf{N}_P(T_{i1}),$$
$$Y_{ij} = T_{ij}/Z (1 \leq i \leq l,\ 1 \leq j \leq m_i).$$

Then $H^\circ Q_0 R_i$, Ω_i and L_i satisfy Hypothesis D. Moreover if $r = 2$ then $p \neq 2$, and Lemma 7.2(c) shows that $\mathbf{C}_{Y_{i1}}(P_i) = \mathbf{C}_{Y_{i1}}(L_i) \neq 0$, so the hypotheses of Lemma 8.2 also hold. As before we get $R_i = E_{i1} \circ E_{i2} \circ \ \dots \ \circ E_{in_i}$, where $|E_{ij}| = r^3$, $E_{ij}^r \leq E'_{ij} = Z$ and $E_{ij} = \langle d_{ij}, e_{ij} \rangle$ $(1 \leq j \leq n_i)$. Moreover the set $\Delta_i = \{(i1), (i2), \ \dots \ , (in_i)\}$ is permuted by H°, with $d_{ij}^g = d_{(ij)g}$, $e_{ij}^g = e_{(ij)g}$ $((ij) \in \Delta_i,\ g \in H^\circ)$.

In both cases take $\Delta = \Delta_1 \cup \Delta_2 \cup \ \dots \ \cup \Delta_l$, and replace the suffices (ij) by the numbers $1, 2, \ \dots \ , n$ (where $n = n_1 + n_2 + \ \dots \ + n_l$). Then

$$R = R_1 \circ R_2 \circ \ \dots \ \circ R_l, \quad W = W_1 \oplus W_2 \oplus \ \dots \ \oplus W_l,$$
$$R = E_1 \circ E_2 \circ \ \dots \ \circ E_n, \quad |E_i| = r^3, \quad E_i^r \leq E'_i = Z,$$
$$E_i = \langle d_i, e_i \rangle, \quad d_i^g = d_{ig}, \quad e_i^g = e_{ig}\ (1 \leq i \leq n,\ g \in H^\circ).$$

Take d as in Lemma 5.7, and put $\omega = Zd$. Then Lemma 5.7(b) implies that the $\mathbf{F}_r Q$-module $\omega(\mathbf{F}_r Q)$ generated by ω is equal to W, and hence $\omega \neq 0$.

We can complete the proof by considering the action of H on R, ignoring the subgroup Q, as well as the module U. Suppose $\omega = \omega_1 + \omega_2 + \ \dots \ + \omega_l$ with $\omega_i \in W_i$, and note that $\omega_i \in \mathbf{C}_W(P)$ $(1 \leq i \leq l)$. At least one of the summands ω_i must be nonzero, and we choose the notation so that $\omega_1 \neq 0$ and $R_1 = E_1 \circ E_2 \circ \ \dots \ \circ E_{n_1}$. Put $\Lambda = \{1, 2, \ \dots \ , n_1\}$ and let $\Lambda_1, \Lambda_2, \ \dots \ , \Lambda_t$ be the P-orbits in Λ. Take $J_i = \prod_{j \in \Lambda_i} E_j$ and $M_i = J_i/Z$, and note that H° stabilizes Λ_i by Lemma 7.2(b). Then

$$R_1 = J_1 \circ J_2 \circ \ \dots \ \circ J_t, \quad W_1 = M_1 \oplus M_2 \oplus \ \dots \ \oplus M_t.$$

Continue by taking $\omega_1 = \mu_1 + \mu_2 + \ \dots \ + \mu_t$ with $\mu_j \in M_j$, and note that $\mu_j \in \mathbf{C}_W(P)$ $(1 \leq j \leq t)$. At least one of the summands μ_j must be nonzero, and we choose the notation so that $\mu_1 \neq 0$ and $\Lambda_1 = \{1, 2, \ \dots \ , s\}$. Then $\mu_1 \in J_1/Z$, so $\mu_1 = Zd'$ with $d' \in J_1 - Z$ and $d' \in \mathbf{C}_R(P)$. Take

$$d_\infty = d_1 d_2 \ \dots \ d_s, \quad e_\infty = e_1 e_2 \ \dots \ e_s, \quad E_\infty = \langle d_\infty, e_\infty \rangle Z,$$

and note that $E_\infty = \mathbf{C}_{J_1}(P)$, so $d' \in E_\infty - Z$. Suppose $[d_1, e_1] = z$, and choose elements $b_j \in P$ so that $1b_j = j$ in the action on Λ_1. Then $z \neq 1$ and $[d_j, e_j] = [d_1, e_1]^{b_j} = z^{b_j} = z$ $(1 \leq j \leq s)$. Also P permutes Λ_1 transitively, so $r \nmid s$. Hence

$$[d_\infty, e_\infty] = [d_1, e_1][d_2, e_2] \ \dots \ [d_s, e_s] = z^s \neq 1,$$

and therefore E_∞ is a nonabelian group of order r^3, with $E'_\infty = \mathbf{Z}(E_\infty) = Z$. Now $d_i^{a_0} = d_{ia_0}$ and $e_i^{a_0} = e_{ia_0}$ $(1 \leq i \leq s)$, so a_0 centralizes E_∞. Also d' is the 'component' of d in J_1, so the action of d on J_1 is the same as the action of d', and therefore

$$\mathbf{C}_{E_\infty}(S) = \mathbf{C}_{E_\infty}(a) = \mathbf{C}_{E_\infty}(a_0 d) = \mathbf{C}_{E_\infty}(d) = \mathbf{C}_{E_\infty}(d') = \langle d' \rangle Z < E_\infty.$$

On the other hand P centralizes E_∞, so this contradicts Lemma 5.6(f). $\square$

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