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Lattice extensions of Hecke algebras

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We investigate the extensions of the Hecke algebras of finite (complex) reflection groups by lattices of reflection subgroups that we introduced, for some of them, in our previous work on the Yokonuma–Hecke algebras and their connections with Artin groups. When the Hecke algebra is attached to the symmetric group, and the lattice contains all reflection subgroups, then these algebras are the diagram algebras of braids and ties of Aicardi and Juyumaya. We prove a structure theorem for these algebras, generalizing a result of Espinoza and Ryom-Hansen from the case of the symmetric group to the general case. We prove that these algebras are symmetric algebras at least when W is a Coxeter group, and in general under the trace conjecture of Broué, Malle and Michel.

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1. Introduction

Let W be finite complex reflection group, for instance a finite Coxeter group. Let B denote the braid group associated to W in the sense of Broué–Malle–Rouquier (see [8]), which in the case of a finite Coxeter group coincides with the Artin group attached to it. We denote $\pi : B \to W$ the natural projection.

The object of this paper is to introduce and analyze a family of algebras denoted $\mathcal{C}(W, \mathcal{L})$, where $\mathcal{L}$ is a finite join semi-lattice which lies inside the poset made of the full reflection subgroups of W, ordered by inclusion. Here a reflection subgroup of W is called *full* if, for any reflection in this subgroup, all the (pseudo-)reflections with the same reflecting hyperplane belong to it. The semi-lattice $\mathcal{L}$ is additionally supposed to be stable under the natural action of W on the lattice of reflection subgroups, and to contain all the cyclic (full) reflection subgroups, and the trivial subgroup as well. Such a semi-lattice will be called an *admissible* semi-lattice.

Let $\mathcal{A}$ denote the hyperplane arrangement attached to W, namely the collection of its reflecting hyperplanes. Let $\mathbf{k}$ be a commutative ring with 1, containing elements $a_{H,i}$ where $H \in \mathcal{A}$, $0 \leq i < m_H$ where m_H is the order of the cyclic subgroup of W fixing H, with the convention that $a_{H,i} = a_{w(H),i}$ for every $H \in \mathcal{A}, w \in W$ and $a_{H,0}$ is invertible inside $\mathbf{k}$. Let R denote the generic ring of Laurent polynomials with integer coefficients $\mathbf{Z}[a_{H,i}, a_{H,0}^{\pm 1}]$, with the same conventions. Our conditions on $\mathbf{k}$ mean that it is a R-algebra. We now define $\mathbf{k}$-algebras $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L})$, with the convention that $\mathcal{C}(W, \mathcal{L}) = \mathcal{C}_R(W, \mathcal{L})$.

These algebras are defined as follows. First consider the algebra $\mathbf{k}\mathcal{L}$ defined as the free $\mathbf{k}$-module with basis elements $e_\lambda, \lambda \in \mathcal{L}$, and where the multiplication is defined by $e_\lambda e_\mu = e_{\lambda \vee \mu}$. This is sometimes called the Möbius algebra of $\mathcal{L}$. Elements of $\mathcal{L}$ can be identified with the collection of reflecting hyperplanes attached to them, and we let $e_H = e_{\{H\}}$ denote the idempotent attached to the subgroup fixing $H \in \mathcal{A}$. We shall use this identification whenever it is convenient to us.

By definition W acts by automorphisms on $\mathbf{k}\mathcal{L}$, hence so does B, and one can form the semidirect product $\mathbf{k}B \ltimes \mathbf{k}\mathcal{L}$. The algebras $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L})$ are defined as the quotient of $\mathbf{k}B \ltimes \mathbf{k}\mathcal{L}$ by the two-sided ideal $\mathfrak{J}$ generated by the elements $\sigma^{m_H} - 1 - Q_s(\sigma)e_H$ where σ runs among the braided reflections of B, $s = \pi(\sigma)$ is the corresponding pseudo-reflection, $H = \mathrm{Ker}\,(s-1)$, and $Q_s(X) = \sum_{k=0}^{m_H-1} a_{H,k}X^k - 1 \in \mathbf{k}[X]$ (see section 2.3.2 for more details).

Let $\mathfrak{J}_H$ is the ideal of $\mathbf{k}B$ generated by the $\sigma^{m_H} - 1 - Q_s(\sigma) = \sigma^{m_H} - \sum_{k=0}^{m_H-1} a_{H,k}\sigma^k$. This quotient $H_k(W) = (\mathbf{k}B)/\mathfrak{J}_H$ is by definition the Hecke algebra attached to W in the sense of Broué–Malle–Rouquier, and is the usual Iwahori–Hecke algebra of W when W is a finite Coxeter group. The following preliminary result explains the title, making our algebras appear as natural extensions of the Hecke algebra $H_{\mathbf{k}}(W)$.

Proposition 1.1. *Let $\mathcal{L}$ be an admissible lattice for W. There exists a surjective algebra morphism $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L}) \to H_{\mathbf{k}}(W)$, defining a* split *extension of $H_{\mathbf{k}}(W)$.*

Proof. The natural augmentation map $\eta : \mathbf{k}\mathcal{L} \to \mathbf{k}$ defined by $e_\lambda \mapsto 1$ induces surjective morphisms of $\mathbf{k}$-algebras $\eta : \mathbf{k}B \ltimes \mathbf{k}\mathcal{L} \to \mathbf{k}B$ and $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L}) \to (\mathbf{k}B)/\mathfrak{J}_H = H_{\mathbf{k}}(W)$. The splitting comes from the fact that the assumptions on $\mathcal{L}$ imply that W belongs to $\mathcal{L}$, as the join of all the full cyclic subgroups. Then, the non-unital algebra morphism $\mathbf{k}B \to \mathcal{C}_{\mathbf{k}}(W, \mathcal{L})$ defined by $b \mapsto be_W$ is easily checked to factorize through $H_{\mathbf{k}}(W)$ and to provide a splitting. □

When X is a finite set and A is a ring, we denote by $Mat_X(A)$ the ring of $|X| \times |X|$-matrices whose entries are indexed by the elements of X. Our first result is a structure theorem of the following form, where the $\mathbf{k}\tilde{H}_{x_*}$ are slight generalizations of the Hecke algebras attached to elements of $\mathcal{L}$ and $x_* \in \mathcal{L}$ is a representative of the orbit $X \in \mathcal{L}/W$.

Theorem 1.2. *There exists an isomorphism of $\mathbf{k}$-algebras*

$$\mathcal{C}_{\mathbf{k}}(W, \mathcal{L}) \simeq \bigoplus_{X \in \mathcal{L}/W} Mat_X(\mathbf{k}\tilde{H}_{x_*}).$$

When $\mathcal{L}$ is the lattice of the reflection subgroups of a finite Coxeter group, the algebras $\mathcal{C}(W, \mathcal{L})$ were introduced in [17], under the name $\mathcal{C}_W$ and using a presentation by generators and relations, and proven to be generically semisimple. When W is the symmetric group, $\mathcal{C}_W$ coincides with the diagram algebra of braids and ties of Aicardi and Juyumaya (see [1,2,20]). Therefore the above theorem is a generalization of a theorem of Espinoza and Ryom-Hansen (see [14]), and was actually motivated by it. Note that, when W is the symmetric group, the lattice of parabolic subgroups coincides with the lattice of reflection subgroups.

We now return to the general case. We let K denote the field of fractions of R and $\bar{K}$ an algebraic closure of K. The *BMR freeness conjecture* states that $H_{\mathbf{k}}(W)$ is a free $\mathbf{k}$-module of rank $|W|$, and implies that $H_{\mathbf{k}}(W)$ is generically semisimple. Up to extending the ring of definition R to a slightly larger Laurent polynomial ring R_u, an additional conjecture of Broué–Malle–Michel, which we recall in detail in section 3, states that $H_{\mathbf{k}}(W)$ is a symmetric algebra when $\mathbf{k}$ is a R_u-algebra, with a trace enjoying some uniqueness conditions. Of course both conjectures are true when W is a finite Coxeter group.

When $\mathcal{L}$ is the lattice of parabolic subgroups of a finite complex reflection groups, the algebra $\mathcal{C}(W,\mathcal{L})$ was introduced and called $\mathcal{C}_W^p$ in [17]. It was conjectured there that $\mathcal{C}_W^p$ is a free R-module of rank $|W| \times |\mathcal{L}_p|$, where $\mathcal{L}_p$ denotes the lattice of parabolic subgroups. A consequence of the above theorem is then the following one. We denote by $W_{x_*} < W$ the stabilizer of x_*.

Theorem 1.3. *The algebra $\mathcal{C}_{\mathbf{k}}(W,\mathcal{L})$ is a free $\mathbf{k}$-module of finite rank if and only if the BMR freeness conjecture holds over $\mathbf{k}$ for every $x \in \mathcal{L}$ (this is in particular the case when W is a Coxeter group). In that case, its rank is $|W| \times |\mathcal{L}|$, and $\mathcal{C}_{\mathbf{k}}(W,\mathcal{L})$ is semisimple when $\mathbf{k}$ is an extension of K, and*

$$\mathcal{C}_{\bar{K}}(W,\mathcal{L}) \simeq \bar{K}W \ltimes \bar{K}\mathcal{L} \simeq \bigoplus_{X \in \mathcal{L}/W} Mat_X(\bar{K}W_{x_*}).$$

At the time of writing, the BMR freeness conjecture was proved for all irreducible reflection groups but the ones of Shephard–Todd types G_{17}, G_{18} and G_{19} (see [3,4,16,15,19,6,12,11,18]), and therefore the above statement was almost unconditional, and reduced the proof of conjecture 5.10 in [17] to the original BMR freeness conjecture. The last cases have been recently solved (see [23]), hence the above condition is always satisfied and the above statement is actually unconditional. Since the current proof of the conjecture is complicated and case-by-case, we prefer however to keep our statement in the present form.

We finally (conditionally) prove that these algebras are symmetric algebras. We call *strong freeness conjecture* for W the statement that $H_R(W)$ admits a basis originating from elements of B. It turns out that the proof described above of the original BMR freeness conjecture actually proves this stronger form. By contrast, the Broué–Malle–Michel trace conjecture is still largely open.

Theorem 1.4. *Assume that the strong freeness conjecture as well as the Broué–Malle–Michel trace conjecture holds for all $x \in \mathcal{L}$. This is in particular the case if W is a finite Coxeter group. Then, for any commutative R_u-algebra $\mathbf{k}$, the algebra $\mathcal{C}_{\mathbf{k}}(W,\mathcal{L})$ is a symmetric algebra.*

As an immediate corollary, we get that the diagram algebra of 'braids and ties' is a symmetric algebra as well.

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2. Structure

2.1. Semidirect extensions of group algebras by abelian algebras

In this section, we first expose fairly general results, which are basically folklore, and which are needed in the sequel. To start with, the following proposition is an explicit version of what is known in the realm of the representation theory of finite groups as Mackey–Wigner's method of "little groups" (see [21] §8.2). It can be seen as an explicit Morita equivalence (see [9] ex. 18.6). It is stated and proved in detail in [10], proposition 3.4, in the case G is finite. We explain below the additional arguments which are needed in the general case.

Proposition 2.1. *Let G be a group acting transitively (on the left) on a finite set X. Let $\mathbf{k}$ be a commutative ring with 1, and let A be the $\mathbf{k}$-algebra $G \ltimes \mathbf{k}^X$ where $\mathbf{k}^X = \oplus_{x \in X} \mathbf{k}\epsilon_x$ is endowed with the product law ($\epsilon_x \epsilon_{x'} = \delta_{x,x'}\epsilon_x$) and the action of G is induced by the one on X. Then any choice of $x_* \in X$ with stabilizer $G_0 \subseteq G$ and any choice of a "section" $\tau \colon X \to G$ such that $\tau(x).x_* = x$ for all $x \in X$, define a unique isomorphism*

$$\theta : A \longrightarrow \mathrm{Mat}_X(\mathbf{k}G_0)$$

sending each $\epsilon_x \in \mathbf{k}^X$ ($x \in X$) to $\theta(\epsilon_x) := E_{x,x}$, and each $g \in G$ to

$$\theta(g) := \sum_{x \in X} \tau(gx)^{-1} g.\tau(x) E_{gx,x}$$

(where $E_{x,y} \in \mathrm{Mat}_X(\mathbf{k})$ is the elementary matrix corresponding to $x, y \in X$).

Proof. The proof given in proposition 3.4 of [10] that θ is a surjective morphism does not use any finiteness assumption on G. It therefore remains to prove that θ is injective. We prove this directly as follows. A $\mathbf{k}$-basis of A is given by the $g\varepsilon_\alpha$ for $g \in G$ and $\alpha \in X$, and by definition

$$\theta(g\varepsilon_\alpha) = \sum_{x \in X} \tau(gx)^{-1} g\tau(x) E_{g.x,x} E_{\alpha,\alpha} = \tau(g\alpha)^{-1} g\tau(\alpha) E_{g\alpha,\alpha}.$$

It follows that a general linear combination $\sum_{g,\alpha} \lambda_{g,\alpha} g\varepsilon_\alpha$ belongs to $\mathrm{Ker}\,\theta$ iff

$$0 = \sum_{g,\alpha} \lambda_{g,\alpha} \tau(g\alpha)^{-1} g\tau(\alpha) E_{g\alpha,\alpha}$$

which means that, for all $\alpha \in X$,

$$\sum_{g \in G} \lambda_{g,\alpha} \tau(g\alpha)^{-1} g\tau(\alpha) E_{g\alpha,\alpha} = 0.$$

Let us fix such an $\alpha \in X$. For every $\beta \in X$ we have

$$0 = \sum_{g \mid g.\alpha=\beta} \lambda_{g,\alpha} \tau(g\alpha)^{-1} g \tau(\alpha) E_{g\alpha,\alpha}$$

namely

$$0 = \tau(\beta)^{-1} \left(\sum_{g \mid g.\alpha=\beta} \lambda_{g,\alpha} g \right) \tau(\alpha)$$

which implies that, for all $g \in G$, we have $\lambda_{g,\alpha} = 0$. Since this holds for every $\alpha \in X$ we get the conclusion. $\square$

Let L be a join semilattice. That is, we have a finite partially ordered set L for which there exists a least upper bound $x \vee y$ for every two $x, y \in L$. Let M be the semigroup with elements $e_\lambda, \lambda \in L$ and product law $e_\lambda e_\mu = e_{\lambda \vee \mu}$. Such a semigroup is sometimes called a band.

If L is acted upon by some group G in an order-preserving way (that is $x \leq y \Rightarrow g.x \leq g.y$ for all $x, y \in L$ and $g \in G$) then M is acted upon by G, so that we can form the algebra $\mathbf{k}M \rtimes \mathbf{k}G$. Up to exchanging meet and join, the algebra $\mathbf{k}M$ is the Möbius algebra as in [22], definition 3.9.1. We recall from [17] a G-equivariant version of the classical isomorphism $\mathbf{k}M \simeq \mathbf{k}^L$ of e.g. [22], theorem 3.9.2. Here $\mathbf{k}^L$ is the algebra of $\mathbf{k}$-valued functions on L, that is the direct product of a collection indexed by the elements of L of copies of the $\mathbf{k}$-algebra $\mathbf{k}$. As before, to $\lambda \in L$ we associate $\varepsilon_\lambda \in \mathbf{k}^L$ defined by $\varepsilon_\lambda(\lambda') = \delta_{\lambda,\lambda'}$ if $\lambda' \in L$.

Proposition 2.2. *(see [17], proposition 3.9) Let M be the band associated to a finite join semilattice L. For every commutative ring $\mathbf{k}$, the semigroup algebra $\mathbf{k}M$ is isomorphic to $\mathbf{k}^L$. If L is acted upon by some group G as above, then $\mathbf{k}M \rtimes \mathbf{k}G \simeq \mathbf{k}^L \rtimes \mathbf{k}G$, the isomorphism being given by $g \mapsto g$ for $g \in G$ and $e_\lambda \mapsto \sum_{\lambda \leq \mu} \varepsilon_\mu$.*

By decomposing L as a disjoint union of G-orbits, by combining these two results one gets that $\mathbf{k}M \rtimes \mathbf{k}G$ is isomorphic to a direct sum of $|L/G|$ matrix algebras. This will turn out to be the main result from general algebra that is needed to prove our structure theorem.

2.2. Braid groups of reflection subgroups

Let W_0 be a reflection subgroup of the reflection group W, and G a subgroup with $W_0 < G < W$ normalizing W_0. For convenience we endow $\mathbf{C}^n$ with a W-invariant unitary form.

The hyperplane complement associated to W is denoted $X = \mathbf{C}^n \setminus \bigcup \mathcal{A}$, and we let $x_0 \in X$ denote the chosen base-point, so that $B = \pi_1(X/W, x_0)$. Let $L \subset \mathbf{C}^n$ denote the fixed points set of W_0, namely the intersection of the set $\mathcal{A}_L$ of all the reflecting hyperplanes associated to the reflections in W_0. Since G normalizes W_0 we have $g(L) = L$ for all $g \in G$. We let $X_0 \subset L^\perp$ denote the hyperplane complement associated to W_0 viewed as a reflection subgroup acting on $L^\perp$. We have $X_0 = L^\perp \setminus \bigcup \mathcal{A}_L$.

Let $X^0 = \mathbf{C}^n \setminus \bigcup \mathcal{A}_L$, and x_{00} the orthogonal projection of x_0 on $L^\perp$. We write $x_0 = x_1 + x_{00}$, with $x_1 \in L$. Since $x_0 \notin \bigcup \mathcal{A}_L$ we have $x_{00} \in X_0$, and the braid groups of $W_0 < \mathrm{GL}(L^\perp)$ can be defined as $P_0 = \pi_1(X_0, x_{00})$, $B_0 = \pi_1(X_0/W_0, x_{00})$. The inclusion map $(X_0, x_{00}) \subset (X^0, x_{00} + L)$ is a W_0-equivariant deformation retract through $(z,t) \mapsto z_{L^\perp} + t z_L$ where $z_{L^\perp}$ and z_L denote the orthogonal projections of z on $L^\perp$ and L, respectively. Since $x_{00} + L$ is retractable to x_0, it follows that this inclusion provides an isomorphism $P_0 \simeq \pi_1(X^0, x_0)$ and, because of W_0-equivariance, an isomorphism $B_0 \simeq \pi_1(X^0/W_0, x_0)$.

Since W_0 is normal inside G, the projection map $X/W_0 \to X/G$ is a Galois covering, and we get a short exact sequence $1 \to \pi_1(X/W_0, x_0) \to \pi_1(X/G, x_0) \to G/W_0 \to 1$.

We consider the G-equivariant inclusion $(X, x_0) \subset (X^0, x_0)$. By standard arguments (see e.g. [13] proposition 2.2, or [5]) we know that the induced map $P = \pi_1(X, x_0) \to \pi_1(X^0, x_0)$ is surjective, and that its kernel K is normally generated by the meridians around the hyperplanes in $\mathcal{A}_L^c$. Since the following diagram is commutative

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \pi_1(X, x_0) \arrow[r] \arrow[d] & \pi_1(X^0, x_0) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K \arrow[r] & \pi_1(X/W_0, x_0) \arrow[r] \arrow[d] & \pi_1(X^0/W_0, x_0) \arrow[r] \arrow[d] & 1 \\
 & & W_0 \arrow[r, equal] \arrow[d] & W_0 \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

with the two columns and the top row being short exact sequences, it follows that the second row is exact and $K = \mathrm{Ker}\,(\pi_1(X/W_0, x_0) \to \pi_1(X^0/W_0, x_0))$. Inside $\pi_1(X/W_0, x_0)$, the collection of meridians generating K become the collection of the elements σ^{m_σ} where σ runs among the collection of (distinguished) braided reflections around the hyperplanes in $\mathcal{A}_L^c$ and m_σ is the order of their image in $W_0 \subset W$.

Since G stabilizes $\mathcal{A}_L$, the image of K under the injective map $\pi_1(X/W_0, x_0) \to \pi_1(X/G, x_0)$ is a normal subgroup of $\pi_1(X/G, x_0)$, that we still denote K. We define the *generalized braid group associated to* G and denote B_G the quotient group $\pi_1(X/G, x_0)/K$.

Let us consider the projection map $\pi : B \to W$. By the above description, B_G is the quotient of $\hat{B}_G = \pi_1(X/G, x_0) = \pi^{-1}(G)$ by K, and the short exact sequence $1 \to \pi_1(X/W_0) \to \pi_1(X/G, x_0) \to G/W_0 \to 1$ induces a short exact sequence $1 \to \pi_1(X/W_0)/K \to B_G \to G/W_0 \to 1$. Identifying $\pi_1(X/W_0)/K$ with $\pi_1(X^0/W_0, K)$ we get a short exact sequence $1 \to B_0 \to B_G \to G/W_0 \to 1$.

We now consider the central element $\boldsymbol{\pi}_0 \in P_0$ defined as the class inside $P_0 = \pi_1(X_0, x_{00})$ of the loop $\gamma_0(t) = x_{00}\exp(2i\pi t)$. By the above identifications, it is identified inside $\pi_1(X^0, x_0)$ with the path $\gamma_1 \star \gamma_0 \star \gamma_1^{-1}$, where $\gamma_1(t) = x_{00} + tx_1$ (recall that $x_0 = x_1 + x_{00}$). We prove that it remains a central element inside $B_G = \pi_1(X/G, x_0)/K$.

For this, let us consider a path $\gamma : x_0 \rightsquigarrow g.x_0$ inside X. We need to prove that the composite $\gamma^{-1} \star (g.\gamma_1 \star g.\gamma_0 \star g.\gamma_1^{-1})^{-1} \star \gamma \star (\gamma_1 \star \gamma_0 \star \gamma_1^{-1})$, which is a path $x_0 \rightsquigarrow x_0$ inside X, belongs to K. This means that its class must be 0 inside $\pi_1(X, x_0)/K = \pi_1(X^0, x_0)$. Therefore we need to prove that $\gamma \star \gamma_1 \star \gamma_0 \star \gamma_1^{-1} : x_0 \rightsquigarrow g.x_0$ is homotopic to $g.\gamma_1 \star g.\gamma_0 \star g.\gamma_1^{-1} \star \gamma$ inside X^0. For this, consider the following map $\tilde{H} : [0, 3] \times [0, 1] \to X^0$ defined, for $t, u \in [0, 1]$, by $\tilde{H}(t, u) = \gamma(u)_{L^\perp} + (1 - t)\gamma(u)_L$, $\tilde{H}(1 + t, u) = \gamma(u)_{L^\perp}\exp(2i\pi t)$, $\tilde{H}(2 + t, u) = \gamma(u)_{L^\perp} + t\gamma(u)_L$. It is not difficult to check that indeed $\tilde{H}(t, u) \in X^0$ for all t, u, and that $\tilde{H}$ is continuous. Moreover, the boundary of the rectangle $[0, 3] \times [0, 1]$ has for image the union of the two paths we are interested in. It follows that these two paths are homotopic, which proves our claim.

2.3. Proof of the structure theorem

2.3.1. Generalized Hecke algebras

We now attach to an admissible lattice $\mathcal{L}$ the following data. To each $x \in \mathcal{L}$ we attach

- the ring $R_x = \mathbf{Z}[a_{H,i}, a_{H,0}^{\pm 1}]$ where H runs among all $H \in x$, and $1 \le i \le m_H - 1$.
- the stabilizer $G_x < W$ of $x \in \mathcal{L}$ and the group $\hat{B}_x = \hat{B}_{G_x} = \pi^{-1}(G_x)$ associated to $W_0 < G_x < N_W(W_0)$, where W_0 is the full reflection subgroup associated to $x \in \mathcal{L}$.
- the group $B_x = B_{G_x}$ as in the previous section.

The generalized Hecke algebra $\tilde{H}_x$ associated to $x \in \mathcal{L}$ is then defined as the quotient of the group algebra $R_x B_x$ by the ideal generated by the Hecke relations $\sigma^{m_H} - \sum a_{H,i}\sigma^i$ for σ a braided reflection with respect to an hyperplane in x. Equivalently, it is the quotient of the group algebra $R_x\hat{B}_x$ by the relations $\sigma^{m_H} - \sum a_{H,i}\sigma^i$ for σ a braided reflection with respect to an hyperplane of x and $\sigma^{m_H} = 1$ for σ a braided reflection with respect to an hyperplane of $\mathcal{A} \setminus x$.

Now recall the short exact sequence $1 \to B_0 \to B_x \to G/W_0 \to 1$, and consider the induced injective map $R_xB_0 \to R_xB_x$. We let $\mathfrak{h}_0$ denote the ideal of R_xB_0 generated by the $\sigma^{m_H} - \sum a_{H,i}\sigma^i$ for σ a braided reflection with respect to an hyperplane of x. By definition the quotient algebra $H_0 = R_xB_0/\mathfrak{h}_0$ the usual Hecke algebra associated to W_0. We let $\mathfrak{h}_x$ the ideal of R_xB_x generated by the same elements, and choose a system $b_1, \dots, b_m$ of representatives inside B_x of $B_x/B_0 \simeq G/W_0$. Since the generating set of $\mathfrak{h}_x$ is stable under B_x-conjugation, we have $\mathfrak{h}_x = \bigoplus_{i=1}^m b_i \mathfrak{h}_0$. This implies that, as a right $R_x B_0$-module, $\tilde{H}_x = \bigoplus_{i=1}^m (b_i(R_x B_0))/(b_i \mathfrak{h}_0) = \bigoplus_{i=1}^m (b_i(R_x B_0))/(b_i \mathfrak{h}_0)$. Now, $(b_i(R_x B_0))/(b_i \mathfrak{h}_0)$ contains (the class of) b_i and is clearly a free H_0-module of rank 1. This proves that $\tilde{H}_x = R_x B_x/\mathfrak{h}_x$ is a free H_0-module of rank $|G/W_0|$. In particular, $\tilde{H}_x$ is a free R_x-module of rank $|G|$ if and only if H_0 is a free R_x-module of rank $|W_0|$. This latter assumption is exactly the BMR freeness conjecture for W_0.

2.3.2. Image of the defining ideal

Let $\mathcal{L}$ be an admissible lattice. The group B acts on $\mathcal{L}$ via the natural projection map $B \to W$. We denote $\mathfrak{J}$ the ideal of $\mathbf{k}B \ltimes \mathbf{k}\mathcal{L}$ generated by the elements $\sigma^m - 1 - Q_s(\sigma) e_H$ where

- s runs among the distinguished pseudo-reflections of W,
- σ is a braided reflection attached to it,
- $H = \mathrm{Ker}\,(s-1)$ is the fixed hyperplane, and
- $e_H \in \mathbf{k}\mathcal{L}$ is the idempotent attached to $\{H\} \in \mathcal{L}$
- m is the order of s
- $Q_s(X) = \sum_{k=0}^{m-1} a_{H,k} X^k - 1$, where $\prod_{k=1}^m (X - u_{s,i}) = X^m + \sum_{k=0}^{m-1} a_{H,k} X^k$.

Let s be a reflection, and m its order. Let $1 \leq k < m$. For any hyperplane $H \in \mathcal{A}$, we have $s(H) = H \Leftrightarrow s^k(H) = H$. It follows that, for every $x \in \mathcal{L}$, we have $s.x = x \Leftrightarrow s^k.x = x$. We consider the composite θ of the maps provided by Propositions 2.2 and 2.1

$$\mathbf{k}B \ltimes \mathbf{k}\mathcal{L} \to \mathbf{k}B \ltimes \mathbf{k}^{\mathcal{L}} \to \mathbf{k}B \ltimes \mathbf{k}^X \to Mat_X(\mathbf{k}\hat{B}_{x_*})$$

where $\hat{B}_{x_*} = \hat{B}_{G_{x_*}} = \pi^{-1}(G_{x_*})$ is the stabilizer of $x_* \in \mathcal{L}$ and X is the orbit of x_* under B (or W). We have, for all $r \in \mathbf{Z}$,

$$\theta(e_H) = \sum_{\substack{x \in X \\ H \in x}} E_{x,x} \quad \text{and} \quad \theta(\sigma^r) = \sum_{x \in X} \tau(s^r.x)^{-1} \sigma^r \tau(x) E_{s^r.x,x}.$$

Since $H \in x \Rightarrow s^r.x = x$, this implies

$$\theta(\sigma^r e_H) = \sum_{\substack{x \in X \\ H \in x}} \tau(x)^{-1} \sigma^r \tau(x) E_{x,x} \qquad \text{and} \qquad \theta(Q_s(\sigma) e_H) = \sum_{\substack{x \in X \\ H \in x}} \tau(x)^{-1} Q_s(\sigma) \tau(x) E_{x,x}$$

hence the image under θ of $\sigma^m - 1 - Q_s(\sigma) e_H$ is equal to

$$\sum_{\substack{x \in X \\ H \notin x}} \tau(x)^{-1} (\sigma^m - 1) \tau(x) E_{x,x} + \sum_{\substack{x \in X \\ H \in x}} \tau(x)^{-1} (\sigma^m - 1 - Q_s(\sigma)) \tau(x) E_{x,x}.$$

Now recall the elementary fact that, for any ring A with 1 (commutative or not), the twosided ideal of the matrix algebra $Mat_N(A)$ generated by a collection $S^\alpha, \alpha \in F$ of matrices $S^\alpha = (S^\alpha_{i,j})_{1\le i,j\le N}$ is equal to $Mat_N(I)$ where I is the twosided ideal of A generated by the $S^\alpha_{i,j}$ for $\alpha \in F$, $1 \le i,j \le N$. If follows that image of the ideal $\mathfrak{J}$ inside $Mat_X(\mathbf{k}\hat{B}_{x_*})$ is $Mat_X(\mathfrak{J}_X)$ where $\mathfrak{J}_X$ is the ideal of $\mathbf{k}\hat{B}_{x_*}$ generated by the $\sigma_x^m - 1$ for $H \notin x, x \in X$ and the $\sigma_x^m - 1 - Q_s(\sigma_x)$ for $H \in x, x \in X$, where $\sigma_x = \tau(x)^{-1}\sigma\tau(x)$. This is the same as the ideal of $\mathbf{k}\hat{B}_{x_*}$ generated by the $\sigma^m - 1$ for σ a braided reflection around some $H \notin x_*$, and the $\sigma^m - 1 - Q_s(\sigma)$ and σ for a braided reflection around some $H \in x_*$. Therefore $\mathbf{k}\hat{B}_{x_*}/\mathfrak{J}_X = \mathbf{k}\tilde{H}_x$ whence, from the isomorphism $\mathbf{k}B \ltimes \mathbf{k}\mathcal{L} \simeq \bigoplus_{X\in\mathcal{L}/W} Mat_X(\mathbf{k}\hat{B}_{x_*})$ we get the following.

Theorem 2.3. *Let $\mathcal{L}$ be an admissible lattice. Then we have an isomorphism*

$$\mathcal{C}_\mathbf{k}(W,\mathcal{L}) \simeq \bigoplus_{X\in\mathcal{L}/W} Mat_X(\mathbf{k}\tilde{H}_{x_*}).$$

The following corollary completes the proof of Theorem 1.3.

Corollary 2.4. *The algebra $\mathcal{C}_\mathbf{k}(W,\mathcal{L})$ is a free $\mathbf{k}$-module of finite rank if and only if the BMR freeness conjecture holds over $\mathbf{k}$ for every $x \in \mathcal{L}$. In that case, its rank is $|W| \times |\mathcal{L}|$, and it is generically semisimple.*

The fact that it is generically semisimple is a consequence of the fact that, under the specialization morphism $\varphi : R \to \mathbf{Q}$ defined by $a_{H,i} \mapsto 0$ if $i > 0$, $a_{H,0} \mapsto 0$, the algebra $\mathcal{C}(W,\mathcal{L}) \otimes_\varphi \mathbf{Q}$ becomes isomorphic to a semidirect product $\mathbf{Q}W \ltimes \mathbf{Q}\mathcal{L} \simeq \bigoplus_{\mathcal{L}/W} Mat_X(\mathbf{Q}W_{x_*})$, where $W_{x_*} < W$ is the stabilizer of $x_* \in \mathcal{L}$. By Maschke's theorem we get that $\mathcal{C}(W,\mathcal{L}) \otimes_\varphi \mathbf{Q}$ is semisimple, and therefore $\mathcal{C}(W,\mathcal{L})$ is generically semisimple as soon as it is a free R-module of finite rank. By Tits' deformation theorem we get that

$$\mathcal{C}_{\bar{K}}(W,\mathcal{L}) \simeq \bar{K}W \ltimes \bar{K}\mathcal{L} \simeq \bigoplus_{\mathcal{L}/W} Mat_X(\bar{K}W_{x_*}).$$

Since the BMR freeness conjecture is now proved for all irreducible reflection groups (see the introduction) this proves the following.

Corollary 2.5. *The algebra $\mathcal{C}_\mathbf{k}(W,\mathcal{L})$ is a free $\mathbf{k}$-module of rank $|W| \times |\mathcal{L}|$, and is generically semisimple.*

3. Traces

In this section, we slightly extend the ring of definition, for convenience. For W a given complex reflection group, we denote $R_u = \mathbf{Z}[u_{c,i}^{\pm 1}]$, where c runs among the conjugacy classes of distinguished pseudo-reflections, and i between 1 and the order of (a representative of) c. We consider R as a subring of R_u where $a_{H,i}, H \in \mathcal{A}$ is mapped to the $(m_H - i)$-th symmetric function in the $u_{c,k}$, where c is the conjugacy class corresponding to the distinguished pseudo-reflection with hyperplane H. We let H_u denote the Hecke algebra of W defined over R_u, that is $H_u = H(W) \otimes_R R_u$.

3.1. Reminder on canonical traces

Let W be a complex reflection group, B its braid group, $H = H_u$ its Hecke algebra, defined over the ring of definition $R_u = \mathbf{Z}[u_{c,i}^{\pm 1}]$. Let $x \mapsto \bar{x}$ the automorphism of R_u defined by $u_{c,i} \mapsto u_{c,i}^{-1}$. The group antiautomorphism $g \mapsto g^{-1}$ on B induces an antiautomorphism of $\mathbf{Z}$-algebras $\mathbf{a} : R_u B \to R_u B$ such as $\mathbf{a}(\lambda g) = \bar{\lambda} g^{-1}$ for all $\lambda \in R_u$ and $g \in B$. The Hecke ideal $\mathfrak{J}_H$ of $R_u B$ is stable by $\mathbf{a}$ hence $\mathbf{a}$ induces an automorphism of H. It has the property that, for all parabolic subalgebras H_0 of H, H_0 is $\mathbf{a}$-stable and the restriction of $\mathbf{a}$ to H_0 coincides with the antiautomorphism associated to H_0. Let $t : H \to R$ be a linear form. We assume that H admits a R_u-basis whose elements are (images of) elements of B. This is the strong freeness conjecture, which is now known for all complex reflection groups. We denote $\boldsymbol{\pi}$ the natural central element of $P = \mathrm{Ker}\,(B \twoheadrightarrow W)$. We consider the following assumptions on t.

(1) t is a symmetrizing trace on H, that is t is a linear form on H such that $t(xy) = t(yx)$ for all $x, y \in H$, and the map $x \mapsto (y \mapsto t(xy))$ defines an isomorphism between H and its dual.
(2) The trace induced on the specialization $\mathbf{C}W$ of H is the usual trace on the group algebra $\mathbf{C}W$
(3) For all $h \in H$, we have $\overline{t(\mathbf{a}(h))} t(\boldsymbol{\pi}) = t(h\boldsymbol{\pi})$.

In [7] proposition 2.2 it is proven that, if there exists a trace satisfying these assumptions, then it is unique. It is also proven there that, in case W is a Coxeter group, then the trace given by $t(T_w) = 0$ if $w \neq 1$, $t(T_1) = 1$, satisfies these assumptions.

3.2. Traces on generalized Hecke algebras

Let $\mathcal{L}$ be an admissible lattice, and $x \in \mathcal{L}$. Let W_0 denote the full reflection subgroup attached to x and H_0 the corresponding Hecke algebra. We already proved that the generalized Hecke algebra $\tilde{H}_x$ attached to x is a free H_0-module of the form $\bigoplus_{i=1}^m b_i H_0$ where the b_i are (classes inside $\tilde{H}_x$ of) representatives of $B_x/B_0 \simeq G_x/W_0$. Obviously one can assume $b_1 = 1$ hence $b_1 H_0 = H_0$. Assume that we are given a trace $t : H_0 \to R_u$ satisfying the conditions of the previous section. We extend it as a linear form $t : \tilde{H}_x \to R_u$ by $t(b_i H_0) = 0$ if $i > 1$.

Proposition 3.1. *The extended linear form $t : \tilde{H}_x \to R_u$ is a symmetrizing trace.*

Proof. In order for it to be a trace one needs to check that for all $a_1, a_2 \in H_0$ and i, j we have $t(b_i a_1 b_j a_2) = t(b_j a_2 b_i a_1)$. But clearly both terms are 0 if $b_j \notin b_i^{-1} H_0$. Therefore we need to check that $t(b_i a_1 b_i^{-1} a_2) = t(b_i^{-1} a_2 b_i a_1)$ for all i and $a_1, a_2 \in H_0$. But this means $t(b_i a_1 (b_i^{-1} a_2 b_i) b_i^{-1}) = t(b_i^{-1} a_2 b_i a_1)$. Since $a_2 \mapsto b_i^{-1} a_2 b_i$ induces a bijection of $H_0 \hookrightarrow \tilde{H}_x$ this is equivalent to saying that $t(b_i a_1 a_2 b_i^{-1}) = t(a_2 a_1)$ for all $a_1, a_2 \in H_0$. But $t(a_2 a_1) = t(a_1 a_2)$ whence we need to check that, for all i and all $a \in H_0$, we have $t(b_i a b_i^{-1}) = t(a)$. This holds true for the following reason. Let $b \in B_x$, and consider the map $a \mapsto t(bab^{-1})$. This is a trace on H_0, which satisfies obviously the conditions (1) and (2) of the previous section. It also satisfies condition (3) if we can prove that $b\boldsymbol{\pi}_0 b^{-1} = \boldsymbol{\pi}_0$ where $\boldsymbol{\pi}_0$ is the natural central element of the pure braid group P_0 of W_0. But this was proven in section 2.2 above. Therefore t is a trace on $\tilde{H}_x$. Taking a basis $e_1, \dots, e_N$ of H_0 and letting $e'_1, \dots, e'_N$ its dual basis, so that $t(e_i e'_j) = \delta_{ij}$, we get that the $b_i e_j$ form a basis for $\tilde{H}_x$, with dual basis $e'_j b_i^{-1}$. Indeed, $t(b_i e_j e'_r b_s^{-1}) = t(b_s^{-1} b_i e_j e'_r) = 0$ unless $i \neq s$, and in that case it is equal to $t(e_j e'_r) = \delta_{jr}$. Therefore t is a symmetrizing trace. □

3.3. Symmetrizing trace

We recall the following standard property of traces on matrix algebras, the proof being easy and left to the reader.

Lemma 3.2. *Let $\mathbf{k}$ be a commutative ring with 1, A a $\mathbf{k}$-algebra and $N \geq 1$. There is a 1-1 correspondence between trace forms on A and trace forms on $Mat_N(A) = Mat_N(\mathbf{k}) \otimes_{\mathbf{k}} A$, the correspondence being given by $t \mapsto tr \otimes t$, where $tr : Mat_N(\mathbf{k}) \to \mathbf{k}$ is the matrix trace. Moreover $tr \otimes t : Mat_N(\mathbf{k}) \otimes_{\mathbf{k}} A \to \mathbf{k} \otimes_{\mathbf{k}} \mathbf{k} = \mathbf{k}$ is symmetrizing if and only if t is symmetrizing.*

From the isomorphism $(\mathbf{k}B \ltimes \mathbf{k}\mathcal{L})/\mathfrak{J} \simeq \bigoplus_{X \in \mathcal{L}/W} Mat_X(\mathbf{k}\tilde{H}_{x_*})$ we are able to construct a trace form, as

$$\bigoplus_{X \in \mathcal{L}/W} t_{x_*} \otimes tr : \bigoplus_{X \in \mathcal{L}/W} Mat_X(\mathbf{k}\tilde{H}_{x_*}) = \bigoplus_{X \in \mathcal{L}/W} Mat_X(\mathbf{k}) \otimes_{\mathbf{k}} \mathbf{k}\tilde{H}_{x_*} \to \mathbf{k}$$

and by the above property it is a symmetrizing form. This proves the following.

Theorem 3.3. *Let $\mathcal{L}$ be an admissible lattice for W, and $\mathbf{k}$ a commutative R_u-algebra. If the Broué–Malle–Michel trace conjecture holds for all $x \in \mathcal{L}$, then the algebra $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L})$ is a symmetric algebra. It is in particular the case when W is a real reflection group.*

4. Main examples

We recall that a reflection subgroup W_0 of W is called *full* if, for every reflection $s \in W_0$, all the reflections with respect to the same reflecting hyperplane belong to W_0. Such a reflection subgroup is uniquely determined by the set of its reflecting hyperplanes. Of course reflection subgroups of real reflection groups and, more generally, of 2-reflection groups, are full.

Let $\mathcal{L}_\infty$ denote the poset of all full reflection subgroups, ordered by inclusion. For convenience, we prefer to consider it as a poset of subsets of $\mathcal{L}$, also ordered by inclusion.

Recall that a subset $\mathcal{L} \subset \mathcal{L}_\infty$ is called *admissible* if it is a sub-join-semilattice of $\mathcal{L}_\infty$ which satisfies the following conditions:

(1) It is W-stable.
(2) It contains all $\{H\}$, for $H \in \mathcal{A}$, as well as the trivial subgroup.

Because such an $\mathcal{L}$ always contains a minimal element (the trivial group), there is no ambiguity in the definition of the semi-lattice: the fact that $a \vee b$ exists for every two elements of $\mathcal{L}$ is in this case equivalent to saying that every finite subset of elements, including the empty one, admits a join. Moreover, since such an $\mathcal{L}$ is always finite, it is automatically a lattice. Therefore, we can equivalently talk about admissible *lattices*.

4.1. The category of admissible semi-lattices and maps

Let $\mathcal{L}$ and $\mathcal{L}'$ be two admissible semi-lattices. A map $\mathcal{L} \to \mathcal{L}'$ is called admissible if it is a W-equivariant morphism of join semi-lattices which is the identity on the cyclic and trivial reflection subgroups. The collection of admissible semi-lattices with morphisms the admissible maps forms a (small, finite) category $\mathcal{CL}_W$, and $\mathcal{C}_{\mathbf{k}}(W, \bullet)$ defines a functor from $\mathcal{CL}_W$ to the category of (associative, unital) $\mathbf{k}$-algebras. The category $\mathcal{CL}_W$ admits a terminal object that we call $\mathcal{L}_2$: it is the subset of $\mathcal{L}_\infty$ made of the trivial and cyclic reflection subgroups together with the whole group W. Obviously, for every admissible $\mathcal{L}$ there exists exactly one admissible map $\mathcal{L} \to \mathcal{L}_2$. In particular there exists exactly one admissible map $\mathcal{L}_\infty \to \mathcal{L}_2$.

More generally, define the parabolic rank of a reflection subgroup W_0 as the rank of the smallest parabolic subgroup containing W_0, or equivalently as the codimension of its set of fixed points. Then, the sub-poset $\mathcal{L}_n$ made of all reflection subgroups of parabolic rank at most n plus the whole group is an admissible semi-lattice as soon as $n \geq 2$, and there is an admissible map $\mathcal{L}_m \to \mathcal{L}_n$ when $m \geq n$ given by $W_0 \mapsto W_0$ if W_0 has parabolic rank at most n, and $W_0 \mapsto W$ if W_0 has rank at least $n+1$. This applies to $m = \infty$ as well.

4.2. The semi-lattice $\mathcal{L}_2$

The W-orbits of $\mathcal{L}_2$ are $\{\{1\}\}$, $\{W\}$ together with the $b_c = \{\{H\}; H \in c\}$ for every $c \in \mathcal{A}/W$. It is immediately checked that $\mathbf{k}\tilde{H}_1 = \mathbf{k}W$ and $\mathbf{k}\tilde{H}_{W_*} = H_{\mathbf{k}}(W)$. From Theorem 2.3 we get that

$$\mathcal{C}_{\mathbf{k}}(W, \mathcal{L}_2) = \mathbf{k}W \oplus H_{\mathbf{k}}(W) \oplus \bigoplus_{c \in \mathcal{A}/W} Mat_{|c|}(\mathbf{k}\tilde{H}_{c_*})$$

A remarkable fact about the $x = \{H\} \in \mathcal{L}$ of rank 1, for any admissible poset, is that the generalized Hecke algebras $\tilde{H}_x$ are free deformations of the group algebra $\mathbf{k}G(H)$, where $G(H) = \{w \in W \mid w(H) = H\}$, without having to invoke the BMR freeness conjecture (or, said differently, it corresponds to the trivial case (rank 1) of the BMR freeness conjecture).

4.3. The case of finite Coxeter groups

Assume that W is a real reflection group, and let (W, S) be a Coxeter system attached to it. Then B admits a presentation as an Artin group, with generators $b_s, s \in S$. The map $B \to W$ admits a natural set-theoretic section, called Tits' section, and defined by $w \mapsto b_w = b_{s_1} \dots b_{s_n}$ where $s_i \in S$ and $w = s_1 \dots s_n$ is an expression of w as a product of the generators of minimal length. The classical theory tells us that it is well-defined. We denote g_w the image of b_w inside $\mathcal{C}(W, \mathcal{L})$ under the natural R-algebra morphism $RB \to \mathcal{C}(W, \mathcal{L})$.

Since the BMR freeness conjecture is true for all reflection subgroups of W, from Theorem 1.3 we know that $\mathcal{C}(W, \mathcal{L})$ is a free R-module of rank $|W| \times |\mathcal{L}|$. More precisely, we have the following.

Proposition 4.1. *Let W be a finite Coxeter group and $\mathcal{L}$ an admissible lattice. Then $\mathcal{C}(W, \mathcal{L})$ admits for basis the elements $g_w e_L$ for $w \in W$ and $L \in \mathcal{L}$.*

Proof. Since the collection $\{g_w e_L; w \in W, L \in \mathcal{L}\}$ has the right cardinality, it is sufficient to prove that it spans the free R-module of finite rank $\mathcal{C}(W, \mathcal{L})$. For this we consider its span that we denote V; we remark that $1 \in V$, and prove that it is a left ideal of the R-algebra $\mathcal{C}(W, \mathcal{L})$. Since the $g_s, s \in S$ and $e_L, L \in \mathcal{L}$ generate $RB \ltimes R\mathcal{L}$ as an algebra, they also generate $\mathcal{C}(W, \mathcal{L})$ and therefore it is sufficient to show that $g_s.x \in V$ and $e_L.x \in V$ for x running among a spanning set of V. Setting $x = g_w e_M$ for some $w \in W, M \in \mathcal{L}$, we get $e_L g_w e_M = g_w e_{w^{-1}(L)} e_M = g_w e_{w^{-1}(L) \vee M} \in V$. Let $\ell : W \to \mathbf{N} = \mathbf{Z}_{\geq 0}$ denote the classical length function. If $\ell(sw) = \ell(w) + 1$, then $g_s x = g_s g_w e_M = g_{sw} e_M \in V$. If not, w can be written $w = sw'$ with $\ell(w') = \ell(w) - 1$. Then $g_s g_w = g_s^2 g_{w'} = g_{w'} + (u_s - 1) e_{\langle s \rangle}(1 + g_s) g_{w'} = g_{w'} + (u_s - 1) e_{\langle s \rangle} g_{w'} + (u_s - 1) e_{\langle s \rangle} g_s g_{w'} = g_{w'} + (u_s - 1) g_{w'} e_{\langle s^{w'} \rangle} + (u_s - 1) e_{\langle s \rangle} g_w$, hence $g_s g_w e_M = g_{w'} e_M + (u_s - 1) g_{w'} e_{\langle s^{w'} \rangle} e_M + (u_s - 1) e_{\langle s \rangle} g_w e_M = g_{w'} e_M + (u_s - 1) g_{w'} e_{\langle s^{w'} \rangle \vee M} + (u_s - 1) e_{\langle s \rangle} g_w e_M \in V$. This proves the claim. $\square$

This proposition implies the following corollary, which could also be directly obtained from the approach of [17] – for instance by extending the left action of $\mathcal{C}_W$ on itself to an action of $\mathcal{C}(W, \mathcal{L}_\infty)$.

Corollary 4.2. *If W is a finite Coxeter group, then $\mathcal{C}_W \simeq \mathcal{C}(W, \mathcal{L}_\infty)$.*

Proof. The elements $g_s, s \in S$ and $e_H, H \in \mathcal{A}$ clearly satisfy inside $\mathcal{C}(W, \mathcal{L}_\infty)$ the defining relations of $\mathcal{C}_W$, and from this we get an algebra morphism $\mathcal{C}_W \to \mathcal{C}(W, \mathcal{L}_\infty)$. From the above proposition and theorem 3.4 of [17] we get that it maps a basis of $\mathcal{C}_W$ to a basis of $\mathcal{C}(W, \mathcal{L}_\infty)$, and therefore it is an isomorphism. $\square$

Therefore, the construction of $\mathcal{C}(W, \mathcal{L}_\infty)$ indeed generalizes to the complex reflection group case the algebra $\mathcal{C}_W$ of a finite Coxeter group introduced in [17].

4.4. The parabolic lattice

A W-stable subposet of $\mathcal{L}_\infty$ is given by the collection $\mathcal{L}_p$ of parabolic subgroups. It can be identified with the arrangement lattice $L(\mathcal{A})$, that is the collection of all intersections of hyperplanes in $\mathcal{A}$, ordered by reverse inclusion. More precisely, there exists a map $\mathrm{Fix} : \mathcal{L} \to L(\mathcal{A})$ where $\mathrm{Fix}(x)$ is the intersection of all reflecting hyperplanes in x, and its restriction to $\mathcal{L}_p$ is a bijection.

Proposition 4.3. *For $x \in \mathcal{L}_\infty$ a reflection subgroup, let $[x] \in \mathcal{L}_p$ denote the parabolic closure of x. Then $x \mapsto [x]$ is an admissible map $\mathcal{L}_\infty \to \mathcal{L}_p$ inducing a quotient map $\mathcal{C}(W, \mathcal{L}_\infty) \to \mathcal{C}(W, \mathcal{L}_p)$.*

Proof. First note that, for every $E, F \subset W$, we have $\mathrm{Fix}(E \cup F) = \mathrm{Fix}(E) \cap \mathrm{Fix}(F)$, $\mathrm{Fix}(E) = \mathrm{Fix}(\langle E \rangle)$, and $\mathrm{Fix}(x) = \mathrm{Fix}([x])$ if x is a reflection subgroup. From this we get that, for all $x, y \in \mathcal{L}$, we have on the one hand $\mathrm{Fix}([\langle x, y \rangle]) = \mathrm{Fix}(\langle x \cup y \rangle) = \mathrm{Fix}(x \cup y) = \mathrm{Fix}(x) \cap \mathrm{Fix}(y)$, and on the other hand $\mathrm{Fix}([\langle [x] \cup [y] \rangle])\mathrm{Fix}(\langle [x] \cup [y] \rangle) = \mathrm{Fix}([x] \cup [y]) = \mathrm{Fix}([x]) \cap \mathrm{Fix}([y]) = \mathrm{Fix}(x) \cap \mathrm{Fix}(y)$. Since Fix is a bijection $\mathcal{L}_p \to L(\mathcal{A})$ this proves $[\langle x, y \rangle] = [\langle [x] \cup [y] \rangle]$, and this proves the claim, the W-invariance being obvious. $\square$

From this we recover the definition of $\mathcal{C}_W^p = \mathcal{C}(W, \mathcal{L}_p)$ given in [17] in the case of a finite Coxeter group, and extend the map $\mathcal{C}(W, \mathcal{L}_\infty) \to \mathcal{C}_W^p$ to the complex reflection group case.

4.5. Root systems

Let R be a reduced root system (in the sense of Bourbaki), W the associated real reflection group. To each $\alpha \in R$ we associate the corresponding reflection $s_\alpha = s_{-\alpha} \in W$. A root subsystem of R is by definition a subset R' of R stable under every $s_\alpha, \alpha \in R'$. The subgroup of W generated by the s_α for $\alpha \in R'$ is a reflection subgroup, and the map $R' \mapsto \langle s_\alpha, \alpha \in R'$ defines a bijection between the set $\mathcal{L}_R$ of all root subsystems and $\mathcal{L}_\infty$. The preordering induced by this bijection on $\mathcal{L}_R$ is simply the inclusion ordering. We endow $\mathcal{L}_R$ we the corresponding join semilattice structure. The cyclic reflection subgroups of W correspond to the root subsystems $\{\alpha, -\alpha\}$ for $\alpha \in R$.

We let $\mathcal{L}_c$ denote the subset of $\mathcal{L}_R$ corresponding to the *closed* subsystems, namely the $R' \in \mathcal{L}_\infty$ for which $\forall \alpha, \beta \in R' \quad \alpha + \beta \in R \Rightarrow \alpha + \beta \in R'$. Note that an intersection of closed subsystems is a closed subsystem, and that the subsystems of the form $\{\alpha, -\alpha\}$ as well as the empty subsystem are closed. We have a map $c : \mathcal{L}_R \to \mathcal{L}_c$ which associates to $R' \in \mathcal{L}_R$ its *closure*, namely the intersection of all closed subsystems containing it. It is immediately checked that c is W-equivariant and a join semilattice morphism. From this it follows that we get an admissible map $\mathcal{L}_\infty \simeq \mathcal{L}_R \to \mathcal{L}_c$.

This proves the following.

Proposition 4.4. *Let R be a reduced root system and W the associated finite Coxeter group. Under the identification $\mathcal{L}_\infty \simeq \mathcal{L}_R$, the map $c : \mathcal{L}_R \to \mathcal{L}_c$ induces a surjective morphism $\mathcal{C}(W, \mathcal{L}_\infty) \to \mathcal{C}(W, \mathcal{L}_c)$.*

This proposition proves that the algebra $\mathcal{C}(W, \mathcal{L}_c)$ is isomorphic to the algebra $\mathcal{C}_W^R$ of [17], which generically embeds into the corresponding Yokonuma–Hecke algebra. Indeed, $\mathcal{C}_W^R$ is defined as a quotient of $\mathcal{C}(W, \mathcal{L}_\infty) = \mathcal{C}_W$, and one gets immediately that the map $\mathcal{C}_W \twoheadrightarrow \mathcal{C}(W, \mathcal{L}_c)$ defined above factors through $\mathcal{C}_W \twoheadrightarrow \mathcal{C}_W^{(R)}$. The induced surjective map $\mathcal{C}_W^{(R)} \to \mathcal{C}(W, \mathcal{L}_c)$ is then checked to be injective, since the natural spanning set of $\mathcal{C}_W^{(R)}$ is mapped to a basis of $\mathcal{C}(W, \mathcal{L}_c)$. It is then immediately checked that the corresponding diagram of isomorphisms and natural projections is commutative.

$$\begin{tikzcd}
\mathcal{C}_W \arrow[rr, "\simeq"] \arrow[dr] \arrow[dd] & & \mathcal{C}(W,\mathcal{L}_\infty) \arrow[dr] \arrow[dd] & \\
& \mathcal{C}_W^{(R)} \arrow[rr, "\simeq"] \arrow[dl] & & \mathcal{C}(W,\mathcal{L}_c) \arrow[dl] \\
\mathcal{C}_W^{(p)} \arrow[rr, "\simeq"] & & \mathcal{C}(W,\mathcal{L}_p) &
\end{tikzcd}$$

4.6. A priori unrelated examples

A computer-aided exploration shows that there are other admissible lattices not originating a priori from root systems, with $\mathcal{L}_p \subset \mathcal{L} \subset \mathcal{L}_\infty$. In type A we have $\mathcal{L}_p = \mathcal{L}_\infty$, but in type D_n for $n \geq 4$ we have $\mathcal{L}_p \subsetneq \mathcal{L}_\infty$ while all root subsystems are closed. We checked for small n whether there are other admissible lattices in type D_n. This can be done as follows. First of all, one computes the W-orbits for the action on $\mathcal{L}_\infty \setminus \mathcal{L}_p$, since $\mathcal{L}_\infty \setminus \mathcal{L}$ has to be an union of them. For each such union of orbits we then test whether the obtained subset $\mathcal{L}$ satisfies the join semilattice property. In type D_4, the action of W on $\mathcal{L}_\infty \setminus \mathcal{L}_p$ is transitive (and it the orbit of a reflection subgroup of type A_1^4), so there is no intermediate admissible lattice. But in type D_5, the action has 2 orbits, one of type A_1^4 inherited from type D_4, and the other one of type $A_1 A_3$. By adding to $\mathcal{L}_p$ the orbit of type D_4 one checks by computer that the corresponding poset $\mathcal{L}$ is admissible, every two elements admitting a join. This proves that examples containing the lattice of parabolic subgroups and which are a priori not related to the theory of root systems do exist.

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