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ON MINIMAL FREE RESOLUTIONS OF SUB-PERMANENTS AND OTHER IDEALS ARISING IN COMPLEXITY THEORY

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Abstract. We compute the linear strand of the minimal free resolution of the ideal generated by $k \times k$ sub-permanents of an $n \times n$ generic matrix and of the ideal generated by square-free monomials of degree k. The latter calculation gives the full minimal free resolution by [1]. Our motivation is to lay groundwork for the use of commutative algebra in algebraic complexity theory. We also compute several Hilbert functions relevant for complexity theory.

1. Introduction

We study homological properties of two families of ideals over polynomial rings: the ideals $\mathcal{I}^{sqf;n,k} \subset \mathbb{C}[x_1, \dots, x_n]$ generated by square-free monomials of degree k in n variables and the ideals $\mathcal{I}^{perm_n,k} \subset \mathbb{C}[x_{i,j}]_{1 \le i,j \le n}$ generated by $k \times k$ sub-permanents of an $n \times n$ generic matrix. Recall that the permanent of an $m \times m$ matrix $Y = (y_{i,j})$ is the polynomial

$$perm_m(Y) = \sum_{\sigma \in \mathfrak{S}_m} y_{1,\sigma(1)} y_{2,\sigma(2)} \cdots y_{m,\sigma(m)},$$

where $\mathfrak{S}_m$ denotes the symmetric group on m elements.

We obtain our results via larger ideals $I_{1\times k}(1,n)$ (resp. $I_{1\times k}(n,n)$). The ideal $I_{1\times k}(1,n)$ is generated by all monomials of degree k in n variables. The ideal $I_{1\times k}(n,n)$ is generated by permanents of $k \times k$ matrices produced from X where repetition of rows and columns is allowed. Invariantly, $I_{1\times k}(1,n) = \oplus_{j \ge k} S^j \mathbb{C}^n$ is the ideal generated by $S^k\mathbb{C}^n$ and $I_{1\times k}(n,n) \subset Sym(\mathbb{C}^{n^2})$ is the ideal generated $S^k\mathbb{C}^n \otimes S^k\mathbb{C}^n \subset S^k(\mathbb{C}^n \otimes \mathbb{C}^n)$. The main result in each case says that the linear strand of resolution of $\mathcal{I}^{sqf;n,k}$ (resp. $\mathcal{I}^{perm_n,k}$) is the subcomplex of the linear strand of the resolution of $I_{1\times k}(1,n)$ (resp. of $I_{1\times k}(n,n)$) consisting of elements of *regular weights* (cf. §2).

Our motivation comes from complexity theory. We seek to find differences between the homological behavior of ideals generated by $k \times k$ minors (i.e., subdeterminants) of the generic matrix and the ideals generated by $k \times k$ subpermanents. The ideal generated by square-free monomials arises as the $(n-k)$-th Jacobian ideal of the monomial $x_1 x_2 \dots x_n$.

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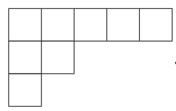
2. Preliminaries

2.1. **Representation Theory.** For proofs of the statements here, see, e.g., [3] or [9, Ch. 2]. We work exclusively over the complex numbers $\mathbb{C}$, although our results hold for an arbitrary field of characteristic 0. If W is a $\mathbb{C}$–vector space of dimension n, a choice of basis determines a maximal torus of diagonal matrices and a labeling of weights for the torus by n-tuples $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$. A weight λ is *dominant* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Irreducible representations of $GL(W)$ are in one-to-one correspondence with dominant weights λ. Let $S_\lambda W$ denote the irreducible representation associated to λ. Write $|\lambda| = \lambda_1 + \cdots + \lambda_n$ for the size of λ. A weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ is *regular* if each α_i is equal to 0 or 1. The regular weights will play an important role in stating our results.

When λ is a dominant weight with $\lambda_n \geq 0$, we say that λ is a *partition* of $r = |\lambda|$, and we write $\lambda \vdash r$. When dealing with partitions we often omit the trailing zeros. Associated to a partition is its *Young diagram* which consists of left–justified rows of boxes, with λ_i boxes in the i–th row: for example, the Young diagram associated to $\lambda = (5, 2, 1) \vdash 8$ is



The *transpose* λ' of a partition λ is obtained by transposing the corresponding Young diagram. For the example above, $\lambda' = (3, 2, 1, 1, 1)$.

Given finite dimensional $\mathbb{C}$–vector spaces F, G, the *Cauchy formulas* describe the decomposition of the symmetric and exterior powers of $F \otimes G$ into a sum of irreducible $GL(F) \times GL(G)$–representations, see, e.g., [9, Cor. 2.3.3]:

$$\begin{aligned} Sym^n(F \otimes G) &= \bigoplus_{\lambda \vdash n} S_\lambda F \otimes S_\lambda G, \\ \bigwedge^n (F \otimes G) &= \bigoplus_{\lambda \vdash n} S_\lambda F \otimes S_{\lambda'} G. \end{aligned} \tag{1}$$

Let $I_{a \times b}(m, n)$ denote the ideal generated by $S_{b^a}\mathbb{C}^m \otimes S_{b^a}\mathbb{C}^n$.

2.2. GL_m **and** $\mathfrak{S}_m$ **representations.** Let $E = \mathbb{C}^m$ equipped with its standard basis. The symmetric group $\mathfrak{S}_m$ is then contained in $GL(E)$ as the permutation matrices. Consider the irreducible representation $S_\lambda E$ where λ is a partition of m. Inside of $S_\lambda E$ we have the $\mathfrak{S}_m$-submodule spanned by the elements of regular weight $(1^m) = (1, 1, \ldots, 1)$. This submodule is denoted $[\lambda]$, the Specht module corresponding to λ. The representations $[\lambda]$ are the distinct irreducible representations of $\mathfrak{S}_m$ (see, e.g., [5]). Write

$$(S_\lambda \mathbb{C}^m)_{reg} = [\lambda].$$

Recall that for finite groups $H \subset G$, and an H-module W, $Ind_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is the induced G-module. For $n \geq m$ and $\lambda \vdash m$,

$$(S_\lambda \mathbb{C}^n)_{reg} \equiv Ind_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}([\lambda] \otimes [n - m]).$$

Introduce the notation $\tilde{\mathfrak{S}}_k = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n$, and if π is a partition of k, write $\widetilde{[\pi]} = [\pi] \times [n-k]$ for the $\tilde{\mathfrak{S}}_k$ module that is $[\pi]$ as an $\mathfrak{S}_k$-module and trivial as an $\mathfrak{S}_{n-k}$-module.

2.3. **Finite free resolutions.** Let $S = \mathbb{C}[x_1, \ldots, x_N]$ be the ring of polynomials in N variables equipped with its grading by degree. Let S_i denote its i-th graded component. Let S_+ denote the maximal ideal $S_+ = \oplus_{i>0} S_i$. Let $M = \oplus_{i \geq 0} M_i$ be a graded S-module. A complex of free graded S-modules

$$\mathbb{F} : 0 \to \mathbb{F}_N \xrightarrow{d_N} \mathbb{F}_{N-1} \to \cdots \to \mathbb{F}_1 \xrightarrow{d_1} \mathbb{F}_0$$

is a minimal free resolution of M if the only homology of $\mathbb{F}$ is $H_0(\mathbb{F}) = M$, and d_i are maps of degree 0 such that $d_i(\mathbb{F}_i) \subset S_+\mathbb{F}_{i-1}$ for all i.

Define the *graded Betti numbers* $\beta_{i,j}$ of M by

$$\mathbb{F}_i = \oplus_{j\geq 0} S(-i-j)^{\beta_{i,j}}$$

where $S(-m)$ denotes a copy of S with generator in degree m.

If M is an ideal generated in degree k, then $\mathbb{F}_i = \oplus_{j\geq k} S(-i-j)^{\beta_{i,j}}$. The linear strand of $\mathbb{F}$ is a subcomplex

$$\mathbb{F}^{lin} : 0 \to \mathbb{F}^{lin}_N \xrightarrow{d_N} \mathbb{F}^{lin}_{N-1} \to \cdots \to \mathbb{F}^{lin}_1 \xrightarrow{d_1} \mathbb{F}^{lin}_0$$

where $\mathbb{F}^{lin}_i = S(-i-k)^{\beta_{i,i+k}}$. The graded Betti numbers $\beta_{i,j}$ have the interpretation in terms of Tor functors

$$\beta_{i,j} = dim_{\mathbb{C}} Tor^S_i(S/S_+, M)_{i+j}$$

where the subscript denotes the homogeneous component. This means that $\beta_{i,j}$ can be calculated as

$$Tor^S_i(S/S_+, M)_{i+j} = H_i(\mathbb{C}(x_1, \dots, x_N; M))_{i+j} \tag{2}$$

where $\mathbb{C}(x_1, \dots, x_N; M)$ is the *Koszul complex* defined by $\mathbb{C}(x_1, \dots, x_N; M)_i = \bigwedge^i \mathbb{C}^N \otimes_S M$ and the differential $d : \bigwedge^i \mathbb{C}^N \otimes_S M \to \bigwedge^{i-1} \mathbb{C}^N \otimes_S M$ by

$$d(e_{j_1} \wedge \cdots \wedge e_{j_i} \otimes m) = \sum_{u=1}^{i} (-1)^{u+1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_u} \wedge \cdots \wedge e_{j_i} \otimes x_{j_u} m.$$

3. The linear strands of the minimal free resolutions of the ideals $\mathcal{I}^{perm_n,k}$

3.1. **The resolution of $I_{1\times k}(n,n)$.** The ideal $I_{1\times k}(n,n)$ is the ideal generated by $S_k E \otimes S_k F \subset S^k(E\otimes F)$, where $E, F \simeq \mathbb{C}^n$. The minimal free resolution of this ideal is known (see, e.g., [7], where it is denoted by $I_{1\times k}$). The linear components of this resolution are generated by

$$\underline{\mathbb{F}}^{lin}_j = \bigoplus_{a+b=j} S_{(k+b,1^a)} E. \tag{3}$$

So the j-th linear term is $\mathbb{F}^{lin}_j = \underline{\mathbb{F}}^{lin}_j \otimes S_{(k+a,1^b)} F \otimes S(-k-j)$.

Since the resolution is $GL(E) \times GL(F)$-equivariant, each module in the complex has a double weight decomposition induced by the restricted action of pairs of diagonal matrices.

3.2. **The main result.** We work over $S = \mathbb{C}[x_{i,j}]_{1\leq i,j\leq n} = Sym(E \otimes F)$.

Define a sub-complex $\mathbb{H}^{lin}$ of the complex $\mathbb{F}^{lin}$ given by (3) by setting $\underline{\mathbb{H}}^{lin}_j$ to be the subspace of $\underline{\mathbb{F}}^{lin}_j$ spanned by the basis elements of regular content. Note that $\mathbb{H}^{lin}$ is indeed a sub-complex of $\mathbb{F}^{lin}$. Let $E_j \subset E$, $F_j \subset F$ denote the span of the first j basis vectors.

Theorem 3.1. *When $k > 1$, the complex $\mathbb{H}^{lin}$ is the linear part of the minimal free resolution of the ideal $\mathcal{I}^{perm_n,k}$. Moreover,* $\dim \underline{\mathbb{H}}_j = \binom{n}{\kappa+j}^2 \binom{2(\kappa+j-1)}{j}$.

As an $\mathfrak{S}_n \times \mathfrak{S}_n$-module,

$$\underline{\mathbb{H}}^{lin}_j = Ind^{\mathfrak{S}_n\times\mathfrak{S}_n}_{\tilde{\mathfrak{S}}_{\kappa+j}\times\tilde{\mathfrak{S}}_{\kappa+j}} \Big(\bigoplus_{a+b=j} [\widetilde{\kappa+b, 1^a}]_{E_{\kappa+j}} \otimes [\widetilde{\kappa+a, 1^b}]_{F_{\kappa+j}} \Big). \tag{4}$$

Proof. Consider the complex $\mathbb{F}^{lin}$ giving the linear strand of the ideal $I_{1\times k}(n,n)$. The term $\mathbb{F}_0$ consists of the generators of $I_{1\times k}(n,n)$, the space $S_kE\otimes S_kF$.

Inside $S_kE\otimes S_kF$ is the ideal generated by the sub-permanents which consists of the subspace of regular weights. Note that the set of regular vectors in any $E^{\otimes m}\otimes F^{\otimes m}$ (where we assume $m\le n$ in order for the set of such vectors to be nonempty) spans a $\mathfrak{S}_E\times\mathfrak{S}_F$-submodule.

The linear strand of the j-the term in the minimal free resolution of the ideal $\mathcal{I}^{perm_n,k}$ is also a $\mathfrak{S}_E\times\mathfrak{S}_F$-submodule of $\mathbb{F}_j$. We claim this sub-module is generated by the span of the regular vectors. In what follows $p(i_1,\dots,i_k;j_1,\dots,j_k)$ denotes the sub-permanent formed from rows $i_1,\dots i_k$ and columns $j_1,\dots,j_k$.

We work by induction, the case $j=0$ was discussed above. Assume the result has been proven up to homological degree $j-1$ and consider the homological degree j and homogeneous degree $k+j$. The generators of the j-th module in the linear strand of the resolution of $\mathcal{I}^{perm_n,k}$ have to be contained in linear part of $\mathbb{H}^{lin}_{j-1}$, so all its weights are either regular, or such that one of the row indices i_α is 2, and/or one of the column indices j_β is 2, and all other p_u,q_u are zero or 1. Call such a weight *sub-regular*. It remains to show that no linear syzygy with a sub-regular weight can appear. To do this we show that no sub-regular weight vector in $(\mathbb{F}_j)_{subreg}$ maps to zero in $(\mathbb{H}_{j-1})\cdot(E\otimes F)$.

First consider the case where both the E and F weights are sub-regular, then (because the space is a $\mathfrak{S}_E\times\mathfrak{S}_F$-module), the weight $(2,1,\cdots,1,0,\cdots,0)\times(2,1,\cdots,1,0,\cdots,0)$ must appear in the syzygy. The only way for this to appear is to have a term divisible by $x_{1,1}$. But, since $x_{1,1}$ is not a zero-divisor in $Sym(V)$, such a term cannot map to zero because our syzygy is a syzygy of degree zero multiplied by $x_{1,1}$. But by minimality no such syzygy exists.

Finally consider the case where there is a vector of weight $(2,1^{j+k-2})\times(1^{j+k})$ appearing. Here it is more convenient to look at the calculation of the free resolution using the Koszul complex. Such a syzygy would give a Koszul cycle with summands of the form

$$z=\sum_t a_t e_{a_{1,t},b_{1,t}}\wedge\cdots\wedge e_{a_{j,t},b_{j,t}}\otimes p(I_t;J_t) \tag{5}$$

where $p(I_t;J_t)$ are subpermanents formed from distinct rows and columns and $a_t\in\mathbb{C}$. The total weight is $(2,1^{k+j-2})\times(1^{k+j})$ and the Koszul differential $d(z)$ is zero. Consider this differential. The coefficients of all the basis elements $e_{a_1,b_1}\wedge\cdots\wedge e_{a_{j-1},b_{j-1}}$ of $d(z)$ have to be zero. There are three kinds of basis elements: the indices $a_1,\dots,a_{j-1}$ can contain number 1 twice, once or not contain 1 at all. Consider the basis elements not containing 1, say the element $e_{2,1}\wedge\cdots\wedge e_{j,j-1}$. The only elements that can appear on the right hand side of the tensor product in $d(z)$ are the elements $x_{1,s}p(1,j+1,j+2,j+k-1;j,j+1,j+2,\dots,\hat{s},\cdots,j+k)$, for $s=j,j+1,\dots,j+k$.

Lemma 3.2. *Let $k>1$. The elements $x_{1,s}p(1,j+1,j+2,j+k-1;j,j+1,j+2,\dots,\hat{s},\cdots,j+k)$, for $s=j,j+1,\dots,j+k$ are linearly independent in S.*

Proof. After re-labeling, the lemma amounts to showing the polynomials $x_{1,1}P_1,\dots,x_{1,k+1}P_{k+1}$ are linearly independent, where P_i is the permanent of the matrix obtained by removing the i-th column of

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,k+1}\\ x_{2,1} & x_{2,2} & \dots & x_{2,k+1}\\ \dots & \dots & \dots & \dots\\ x_{k,1} & x_{k,2} & \dots & x_{k,k+1}\end{pmatrix}.$$

Say that

$$\sum_{s=1}^{k+1} b_s x_{1,s}P_s=0 \tag{6}$$

for some scalars b_s. We need to show that all b_s are zero. By symmetry it suffices to show that $b_1 = b_2 = b_3 = 0$. So set $x_{1,s} = 0$ for $s > 3$. Using the Laplace expansion of permanents along the first row, and writing $P_{i,j}$ for the permanent obtained by removing row 1 and columns i, j we can rewrite (6) as

$$b_1 x_{1,1}(x_{1,2}P_{1,2} + x_{1,3}P_{1,3}) + b_2 x_{1,2}(x_{1,1}P_{1,2} + x_{1,3}P_{1,3}) + b_3(x_{1,1}P_{1,3} + x_{1,2}P_{2,3}) = 0$$

which gives $b_i + b_j = 0$ for $1 \le i < j \le 3$, so $b_1 = b_2 = b_3 = 0$. □

Lemma 3.2 implies that in all the summands in z of (5) with $a_t \neq 0$, both appearances of the index 1 have to occur among $a_{1,t}, \cdots, a_{j,t}$. Now consider the coefficient of the basis element where 1 occurs among $a_1, \ldots, a_{j-1}$, say, $e_{1,1} \wedge e_{2,2} \wedge \cdots \wedge e_{j-1,j-1}$ in $d(z)$. We obtain a linear combination of the elements $x_{1,s} p(j, j+1, \cdots, j+k-1; j, j+1, \cdots, \hat{s}, \cdots, j+k)$ for $s = j, j+1, \ldots, j+k$. But these elements are trivially linearly independent (all monomials occurring in them are different) so all coefficients a_t are zero.

The rest of Theorem 3.1 follows because if π is a partition of $\kappa + j$ then the weight $(1, \cdots, 1)$ subspace of $S_\pi E_{\kappa+j}$, considered as an $\mathfrak{S}_{E_{\kappa+j}}$-module, is $[\pi]$ (see, e.g., [5]), and the space of regular vectors in $S_\pi E \otimes S_\mu F$ is $Ind_{\tilde{\mathfrak{S}}_{E_{\kappa+j}} \times \tilde{\mathfrak{S}}_{F_{\kappa+j}}}^{\mathfrak{S}_E \times \mathfrak{S}_F} \widetilde{[\pi]}_E \otimes \widetilde{[\mu]}_F$. In the formula for the dimension of $\underline{\mathbb{H}}_j$, the factor $\binom{n}{k+j}^2$ is explained by inducing. The dimensions of hook Specht modules are binomial coefficients, dim $[x, 1^y] = \binom{x+y-1}{y}$ So we need to prove that

$$\sum_{a+b=j} \binom{k+j-1}{a}\binom{k+j-1}{b} = \binom{2(k+j-1)}{j}$$

This has a combinatorial explanation. Given $2(k+j-1)$ balls, $k+j-1$ white and $k+j-1$ black, both sides of the equation calculate number of choices of $k+j-1$ of them: the left side partitions into how many white (a) and how many black (b) are chosen. □

Remark 3.3. For small n and κ, computer computations show no additional first syzygies on the $\kappa \times \kappa$ sub-permanents of a generic $n \times n$ matrix (besides the linear syzygies) in degree less than the degree of the Koszul relations 2κ. For example, for $\kappa = 3$ and $n = 5$, there are 100 cubic generators for the ideal and 5200 minimal first syzygies of degree six. There can be at most $\binom{100}{2} = 4950$ Koszul syzygies, so there must be additional non-Koszul first syzygies.

4. The minimal free resolutions of the ideals $\mathcal{I}^{sqf;n,k}$

4.1. **The resolutions of $I_{1\times k}(1,n)$.** Next we consider the case $E = \mathbb{C}$, $F = \mathbb{C}^n$. In this case $S = Sym(F)$ and the ideal $I_{1\times k}(1,n)$ is just the ideal generated by all monomials of degree k. The resolution of this ideal is well-known, see, e.g., [7] or [2].

The whole resolution is linear and $GL(F)$-equivariant, and its k-th term is

$$\mathbb{F}_j = S_{(k,1^j)} F \otimes S(-k-j). \tag{7}$$

4.2. **The resolution of $\mathcal{I}^{sqf;n,k}$.** We work over $S = \mathbb{C}[x_1, \ldots, x_n] = Sym(F)$.

Define a subcomplex $\mathbb{H}^{lin}$ of the complex $\mathbb{F}^{lin}$ given by (7) by setting $\underline{\mathbb{H}}_j^{lin}$ to be the subspace of $\underline{\mathbb{F}}_j^{lin}$ spanned by the basis elements of regular weight. Note that

$$\mathbb{H}^{lin} = \oplus_j \underline{\mathbb{H}}_j^{lin} \otimes S(-k-j)$$

is indeed a subcomplex of $\mathbb{F}^{lin}$.

Theorem 4.1. *The complex $\mathbb{H}^{lin}$ is the linear part of the minimal free resolution of the ideal $\mathcal{I}^{sqf;n,k}$. We have the $\mathfrak{S}_n$-module decomposition*

$$\underline{\mathbb{H}}_j^{lin} = Ind_{\tilde{\mathfrak{S}}_{\kappa+j}}^{\mathfrak{S}_n} \widetilde{[\kappa, 1^j]}, \tag{8}$$

which has dimension $\binom{\kappa+j-2}{j-1}\binom{n}{\kappa+j}$.

Proof. We want to show that $\mathbb{H}^{lin}$ is the linear strand of the resolution of $\mathcal{I}^{sqf;n,k}$. We proceed by induction on j. The case $j = 0$ is clear because the generators of $\mathcal{I}^{sqf;n,k}$ are precisely the generators of $I_{1\times k}(1,n)$ with regular weights. Assume we proved the result for $j-1$ and consider the j-th module in the resolution. As with the subpermanent case, it is enough to consider the elements of a subregular weight as linear relations between elements of regular weight are either of regular or subregular weight.

Consider the syzygies in homological dimension j and in homogeneous degree $k+j$ in term of cycles in the Koszul complex $\mathbb{C}(x_1,\dots,x_n;\mathcal{I}^{sqf;n,k})$. These will be cycles of the form

$$z = \sum_t a_t e_{a_{1,t}} \wedge \ldots \wedge e_{a_{j,t}} \otimes x_{u_{1,t}} x_{u_{2,t}} \ldots x_{u_{k,t}}$$

where the total weight is $(2, 1^{k+j-2})$ and all monomials $x_{u_{1,t}} x_{u_{2,t}} \ldots x_{u_{k,t}}$ are of regular weights, and a_t are scalars. In each summand with $a_t \neq 0$ we are forced to have one 1 among $a_{i,t}$ and one 1 among $u_{i,t}$. So we can assume that in each summand with $a_t \neq 0$ we have $a_{1,t} = 1$ and $u_{1,t} = 1$. We have $d(z) = 0$. But, looking at the coefficient of $d(z)$ with respect to the basis vector $e_{a_{2,t}} \wedge \ldots \wedge e_{a_{j,t}}$ we see that its coefficient is just $a_t x_1 x_{u_{1,t}} x_{u_{2,t}} \ldots x_{u_{k,t}}$ which forces a_t to be zero. The dimension formula follows as in Theorem 3.1. □

Remark 4.2. The easiest way to see that the resolution of the ideal $\mathcal{I}^{sqf;n,k}$ is linear and to see the ranks of the modules is to observe that for the $k \times n$ matrix

$$M(A,X) = \begin{pmatrix} a_{1,1}x_1 & \ldots & a_{1,n}x_n \\ \ldots & \ldots & \ldots \\ a_{k,1}x_1 & \ldots & a_{k,n}x_n \end{pmatrix}$$

where

$$A = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots \\ a_{k,1} & \ldots & a_{k,n} \end{pmatrix}$$

is a matrix of scalars with all maximal minors non-zero, the ideal of maximal minors of the matrix $M(A,X)$ is just $\mathcal{I}^{sqf;n,k}$, so the resolution in question is an Eagon-Northcott complex ([2] or [9, 6.1.6]). After this paper was submitted, a characteristic free description of the resolution of $\mathcal{I}^{sqf;n,k}$, with explicit differentials appeared in [4].

5. Additional results

For $P \in S^d\mathbb{C}^N$, let $I^{P,k} \subset S^k\mathbb{C}^N$ denote the ideal generated by the partial derivatives of P of order $d-k$.

5.1. Size two subpermanents.

Theorem 5.1. *Let $I_t^{\mathrm{perm}_n,2}$ denote the degree t component of the ideal generated by the size two sub-permanents of an $n \times n$ matrix, so $\dim I_2^{\mathrm{perm}_n,2} = \binom{n}{2}^2$. Then*

$$\dim \mathbb{C}[x_{i,j}]/I_t^{perm_n,2} = \binom{n^2+t-1}{t} - [\binom{n}{t}^2 + n^2 + (t-1)(\binom{n^2}{2} - \binom{n}{2}^2) + 2\binom{t-1}{2}(\binom{n}{2}^2 + n\binom{n}{3}) + 2n\sum_{j=3}^{t-1}\binom{t-1}{j}\binom{n}{j+1}],$$

Where recall that $\binom{n}{t} = 0$ for $t > n$, in which case the formula is $\dim S^t\mathbb{C}^{n^2}$ minus the value of the Hilbert polynomial at t.

First, the Hilbert polynomial:

Theorem 5.2. *For the ideal $\mathcal{I}^{\mathrm{perm}_n,2}$ of 2×2 permanents of an $n \times n$ matrix, the Hilbert polynomial of $Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}$ is*

$$\sum_{i=0}^{n} f_i \binom{t-1}{i}, \tag{9}$$

where f_i is the i^{th} entry in the vector

$$\left[n^2, \binom{n^2}{2} - \binom{n}{2}^2, 2\binom{n}{2}^2 + 2n\binom{n}{3}, 2n\binom{n}{4}, 2n\binom{n}{5}, \ldots, 2n\binom{n}{n}\right].$$

Proof. [6, Thm. 3.2] gives a Gröbner basis for $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$, the radical of $\mathcal{I}^{\mathrm{perm}_n,2}$, and by [6, Thm. 3.3], $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}$ has finite length, so vanishes in high degree. The Hilbert polynomial only measures dimension asymptotically, so

$$HP(Sym(V)/\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}, t) = HP(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t).$$

By [6], for any diagonal term order, the Gröbner basis for $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$ is given by quadrics of the form

$$x_{ij}x_{kl} + x_{kj}x_{il} \text{ with } i < k,\ j < l,$$

and five sets of cubic monomials

$$\begin{array}{lcc} x_{i_1j_1}x_{i_1j_2}x_{i_2j_3} & i_1 > i_2 & j_1 < j_2 < j_3 \\ x_{i_1j_1}x_{i_2j_2}x_{i_2j_3} & i_1 > i_2 & j_1 < j_2 < j_3 \\ x_{i_1j_1}x_{i_2j_1}x_{i_3j_2} & i_1 < i_2 < i_3 & j_1 > j_2 \\ x_{i_1j_1}x_{i_2j_2}x_{i_3j_2} & i_1 < i_2 < i_3 & j_1 > j_2 \\ x_{i_1j_1}x_{i_2j_2}x_{i_3j_3} & i_1 < i_2 < i_3 & j_1 > j_2 > j_3. \end{array} \tag{10}$$

The key observation is that all the cubic monomials are square-free, as are the initial terms of the quadrics. Thus the initial ideal of $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$ is a square-free monomial ideal and corresponds to the Stanley-Reisner ideal of a simplicial complex Δ. By [8, Lemma 5.2.5], the Hilbert polynomial is as in Equation (9), where f_i is the number of i-dimensional faces of Δ. As the vertex set of Δ corresponds to all lattice points (i,j) with $1 \le i,j \le n$, it is immediate that $f_0 = n^2$.

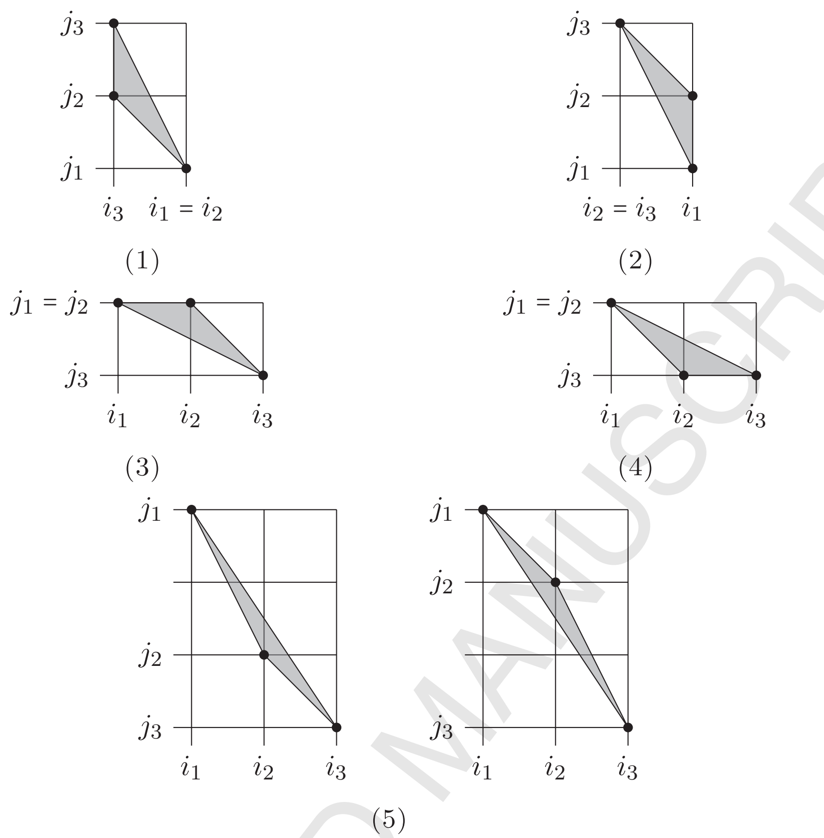


FIGURE 1. Non-triangles of Δ from Equation (10)

Since $x_{ij}x_{kl}$ is a non-face if $i < k$, $j < l$, no edge connects a southwest lattice point to a northeast lattice point. Hence, the edges of Δ consist of all pairs $(i,j),(k,l)$ with $i \geq k$ and $j \geq l$, of which there are $\binom{n^2}{2} - \binom{n}{2}^2$.

Next, consider the triangles of Δ. Equation (10) says there are no triangles in Δ of the types in Figure 1. Also, there are no triangles which contain an edge connecting vertices at positions (i,j) and (k,l) with $i < k$,$j < l$. Thus, the only triangles in Δ are right triangles, but with hypotenuse sloping from northwest to southeast. For a lattice point v at position (d,e) there are exactly $(d-1)(e-1)$ right triangles having v as their unique north-most vertex. In the rightmost column n, there are no such triangles, in the next to last column $n-1$ there are $(n-1)+(n-2)+\cdots = \binom{n}{2}$ such triangles. Continuing this way yields a total count of

$$(n-1)\binom{n}{2} + (n-2)\binom{n}{2} + \cdots 2\binom{n}{2} + \binom{n}{2} = \binom{n}{2}^2$$

such right triangles, and taking into account the right triangles for which v is the unique south-most vertex doubles this number.

However, this count neglects thin triangles–those which have all vertices in the same row or column. Since the number of thin triangles is $2n\binom{n}{3}$, the final count for the triangles of Δ is

$$2\binom{n}{2}^2 + 2n\binom{n}{3}.$$

For tetrahedra, the conditions of Equation (10) imply that there can only be thin tetrahedra, and an easy count gives $2n\binom{n}{4}$ such. The same holds for higher dimensional simplices, and concludes the proof. □

Corollary 5.3. *For the ideal $\mathcal{I}^{\mathrm{perm}_n,2}$ of 2×2 permanents of an $n\times n$ matrix, the Hilbert function of $Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}$ is, when $t\geq 3$,*

$$HF(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2},t)=\binom{n}{t}^2+HP(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2},t), \tag{11}$$

and it equals the Hilbert polynomial for $t>n$.

Proof. The Hilbert function of $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}$ in degree t is $\binom{n}{t}^2$ by [6, Thm. 3.3]. The result follows by combining Equation 9 with the short exact sequence

$$0\longrightarrow\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}\longrightarrow Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}\longrightarrow Sym(V)/\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}\longrightarrow 0,$$

and additivity of the Hilbert function. □

For the purposes of comparing with other ideals, we rephrase this as:

Corollary 5.4. $\dim I_2^{\mathrm{perm}_n,2}=\binom{n}{2}^2$. *For $t\geq 3$:*

$$\begin{aligned}\dim I_t^{perm_n,2}=&\binom{n^2+t-1}{t}-[\binom{n}{t}^2+n^2+(t-1)(\binom{n^2}{2}-\binom{n}{2}^2)+2\binom{t-1}{2}(\binom{n}{2}^2+n\binom{n}{3})\\&+2n\sum_{j=3}^{t-1}\binom{t-1}{j}\binom{n}{j+1}].\end{aligned}$$

5.2. **Hilbert functions for ideals of square-free monomials.** Although these can be deduced from our resolutions, we present the Hilbert functions and polynomials for the ideals generated by square-free monomials.

Proposition 5.5. *The Hilbert function of $\mathcal{I}^{x_1\cdots x_n,\kappa}$ in degree $\kappa+t$ is*

$$\dim\mathcal{I}_{\kappa+t}^{(x_1\cdots x_n),\kappa}=\sum_{j=0}^{n-\kappa}\binom{n}{\kappa-j}\binom{\kappa+t-1}{\kappa+j-1} \tag{12}$$

Proof. The ideal in degree $d=t+\kappa$ has a basis of the distinct monomials of degree d containing at least $n-k$ distinct indices. When we divide such a basis vector by $x_1\cdots x_n$ the denominator will have degree at most κ. For each $i\leq\kappa$, the space of possible numerators with a denominator of degree i that is fixed, has dimension $\dim S^{d-n+i}\mathbb{C}^{n-i}$, and there are $\binom{n}{i}$ possible denominators. Summing over i gives the result. □

For the Hilbert function of the coordinate ring, we have the following expression:

Proposition 5.6. *The Hilbert function of $Sym(\mathbb{C}^n)/\mathcal{I}^{x_1\cdots x_n,\kappa}$ in degree t is*

$$\dim(Sym(\mathbb{C}^n)/\mathcal{I}^{x_1\cdots x_n,\kappa})_t=\sum_{j=0}^{n-\kappa-2}\binom{n}{j+1}\binom{t-1}{j}, \tag{13}$$

if $t\geq n-\kappa-1$, and $\binom{n+t-1}{n-1}$ if $t<n-\kappa-1$.

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