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Tonoli's Calabi–Yau threefolds revisited

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ABSTRACT

We find a simple construction of Tonoli's examples of Calabi–Yau threefolds in complex $\mathbb{P}^6$. We prove that the rank of the Picard group of elements of one of these families is at least 2.

1. Introduction

From the famous Serre construction, we know that a codimension 2 submanifold $X \subset \mathbb{P}^n$ that is subcanonical (i.e. $\omega_X \simeq \mathcal{O}_X(l)$ for some $l \in \mathbb{Z}$) can be seen as the zero locus of a section of a rank two vector bundle E. In particular, in that case, we have an exact sequence

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$$0 \to \mathcal{O}_{\mathbb{P}^n} \to E \to \mathcal{I}_X(c_1(E)) \to 0.$$

There is a similar construction in codimension 3. By answering Okonek's question (see [16]), Walter showed in [21] (see also [6]) that if $n-3$ is not divisible by 4 then each locally Gorenstein subcanonical subscheme X of codimension 3 in $\mathbb{P}^n$ admits a Pfaffian resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2s-t) \to E^*(-s-t) \xrightarrow{\varphi} E(-s) \to \mathcal{I}_X \to 0,$$

where the vector bundle E is of rank $2r+1$ and $s = c_1(E) + rt$. Moreover, in that case

$$\omega_X = \mathcal{O}_X(t + 2s - n - 1). \tag{1.1}$$

In such a situation, we shall say that E defines X through the Pfaffian construction, although X is in fact defined by an additional choice of a skewsymmetric map $\varphi : E^*(-s-t) \to E(-s)$ or equivalently by a choice of a section $\sigma \in (\bigwedge^2 E)(t)$. When such a σ is specified we shall write $\mathrm{Pf}(\sigma)$ for the Pfaffian variety defined by σ.

The Pfaffian construction was applied in [2] to construct canonically embedded surfaces of general type in $\mathbb{P}^5$, and in [19] and [18] to construct Calabi–Yau threefolds in $\mathbb{P}^6$. The latter examples will be referred to as Tonoli Calabi–Yau threefolds. The resulting examples of both papers have degrees $12 \le d \le 17$. In particular, in the case of degree 17 the authors of [19] and [18] discover three distinct families of Calabi–Yau threefolds.

Pfaffian Calabi–Yau threefolds, being one of the simplest Calabi–Yau threefolds which are not described as complete intersections in toric varieties, are good testing examples for the mirror symmetry conjectures. The simplest of them, i.e. those which are arithmetically Cohen–Macaulay (or equivalently in this case those of degree $d \le 14$), have already been studied with partial success from this point of view (see [17,1,8]). An interesting phenomenon occurs for those examples: the Picard–Fuchs equation admits two points of maximal unipotent monodromy. Those points correspond to mirror partners with equivalent derived categories. On the other hand, there is not a single representative in the huge database [20] of Picard–Fuchs operators whose invariants would match the invariants of any non-arithmetically Cohen–Macaulay Pfaffian Calabi–Yau threefold (i.e. of degree $d \ge 15$). Motivated by this, we have tried to understand the geometry of these examples.

The construction of Tonoli families of degree 17 Calabi–Yau manifolds in $\mathbb{P}^6$ can be summarized as follows. Let W_3, P_7 be two vector spaces of dimension 3 and 7 respectively. Then $\mathbb{P}(W_3 \otimes P_7)$ contains a natural subvariety Seg consisting of classes of simple tensors. Seg is the image of the Segre embedding of $\mathbb{P}(W_3) \times \mathbb{P}(P_7)$ into $\mathbb{P}(W_3 \otimes P_7)$. In particular, we have a map $\pi : \mathrm{Seg} \to \mathbb{P}(W_3)$. For any point $\mathbf{P}$ in the Grassmannian $G(16, W_3 \otimes P_7)$, we shall write $L_{\mathbf{P}}$ for the corresponding linear space of dimension 15 in $\mathbb{P}(W_3 \otimes P_7)$. Observe that the projection $\pi|_{L_{\mathbf{P}} \cap \mathrm{Seg}} \to \mathbb{P}^2$ has fibers which are linear spaces of dimension ≥ 1 and the generic fiber is a $\mathbb{P}^1$. Consider

$$\tilde{\mathcal{M}}_k = \{\mathbf{P} \in G(16, W_3 \otimes P_7) \mid \pi|_{L_{\mathbf{P}} \cap \mathrm{Seg}} \text{ has exactly } k \text{ distinct } \mathbb{P}^2\text{-fibers and no fiber of bigger dimension}\}. \tag{1.2}$$

Observe now that a general point $\mathbf{P} \in G(16, W_3 \otimes P_7)$ defines a unique vector bundle of rank 13 on $\mathbb{P}(P_7)$ in the following way. If $\mathbf{P} \in G(16, W_3 \otimes P_7)$ then there is a natural map $L_{\mathbf{P}} \otimes P_7^* \to W_3$ defined up to composition with an automorphism of $L_{\mathbf{P}}$ that defines a map

$$\lambda_{\mathbf{P}} : L_{\mathbf{P}} \otimes \mathcal{O}_{\mathbb{P}(P_7)} \to W_3 \otimes \mathcal{O}_{\mathbb{P}(P_7)}(1).$$

If this map is surjective (which happens for general $\mathbf{P} \in G(16, W_3 \otimes P_7)$), its kernel is a vector bundle that we shall call $E_{\mathbf{P}}$ or $E_{\lambda_{\mathbf{P}}}$. Tonoli proves that, for each $k = 8, 9, 11$, the family of bundles $E_{\mathbf{P}}$ parameterized by $\mathbf{P}$ in a non-empty open subset of $\tilde{\mathcal{M}}_k$ defines a family of Pfaffian Calabi–Yau threefolds of degree 17. For a formal construction of this family we refer to Section 3.

The main result of this paper is a simple construction of Tonoli families of Calabi–Yau threefolds of degree $d = 17$. The main ingredient of this construction is the following theorem.

Theorem 1.1. *For $k \in \{8, 9, 11\}$, with the above notation, let $\mathcal{B}_k$ be the set of all $\mathbf{P} \in G(16, W_3 \otimes P_7)$ such that:*

(1) *$k = 11$ and $L_{\mathbf{P}}$ contains the graph $\Gamma_{v_1} \subset \mathrm{Seg} \subset \mathbb{P}(W_3 \otimes P_7)$ of a linear embedding $v_1 : \mathbb{P}(W_3) \to \mathbb{P}(P_7)$;*
(2) *$k = 9$ and $L_{\mathbf{P}}$ contains the graph $\Gamma_{v_2} \subset \mathrm{Seg} \subset \mathbb{P}(W_3 \otimes P_7)$ of a second Veronese embedding:*

$$v_2 : \mathbb{P}(W_3) \to \mathbb{P}(P_7);$$

(3) *$k = 8$ and $L_{\mathbf{P}}$ contains the closure of the graph Γ_{v_3} of a birational map $v_3 : \mathbb{P}(W_3) \to \mathbb{P}(P_7)$ defined by a system of cubics passing through some point.*

Then $\mathcal{B}_k \cap \tilde{\mathcal{M}}_k$ contains an open and dense subset of both $\mathcal{B}_k$ and $\tilde{\mathcal{M}}_k$.

We also discuss in Remark 5.7 what happens in the case $k = 10$; note that $\tilde{\mathcal{M}}_{10} \neq \emptyset$, but our construction does not give rise to a codimension 3 submanifold. The case $k \leq 7$ is discussed in Remark 5.6; note that for $k = 7$ the construction leads to a Gorenstein threefold that is not smooth. Moreover, for $k \leq 6$ we clearly have $\tilde{\mathcal{M}}_k \neq \emptyset$ but in this case again the construction does not lead to codimension 3 submanifolds.

Theorem 1.1 puts Tonoli's construction in a geometrical context which makes it easier to work with Tonoli examples. In particular, an explicit construction that works in characteristic 0 (cf. [19, §4]) can be implemented in Macaulay 2; we provide in [12] the

Table 1
Vector bundles defining del Pezzo surfaces.

Degree	Vector bundle defining projected del Pezzo surfaces in $\mathbb{P}^5$
3	$\mathcal{O}_{\mathbb{P}^5}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^5}(1)$
4	$2\mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}(1)$
5	$5\mathcal{O}_{\mathbb{P}^5}$
6	$\Omega^1_{\mathbb{P}^5}(1) \oplus 2\mathcal{O}_{\mathbb{P}^5}$
7	$\ker(\psi)$, where $\psi\colon 11\mathcal{O}_{\mathbb{P}^5} \to 2\mathcal{O}_{\mathbb{P}^5}(1)$ is a general map
8	$\ker(\psi)$, where $\psi\colon 14\mathcal{O}_{\mathbb{P}^5} \to 3\mathcal{O}_{\mathbb{P}^5}(1)$ is a special map with more syzygies
9	$\ker(\psi)$, where $\psi\colon 17\mathcal{O}_{\mathbb{P}^5} \to 4\mathcal{O}_{\mathbb{P}^5}(1)$ is a special map with special syzygies

Table 2
Vector bundles defining Tonoli Calabi–Yau threefolds.

Degree	Vector bundle defining the Tonoli examples of Calabi–Yau threefolds
12	$\mathcal{O}_{\mathbb{P}^6}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^6} \oplus 2\mathcal{O}_{\mathbb{P}^6}(1)$
13	$4\mathcal{O}_{\mathbb{P}^6} \oplus \mathcal{O}_{\mathbb{P}^6}(1)$
14	$7\mathcal{O}_{\mathbb{P}^6}$
15	$\Omega^1_{\mathbb{P}^6}(1) \oplus 3\mathcal{O}_{\mathbb{P}^6}$
16	$\ker(\psi)$, where $\psi\colon 13\mathcal{O}_{\mathbb{P}^6} \to 2\mathcal{O}_{\mathbb{P}^6}(1)$ is a general map
17	$\ker(\psi)$, where $\psi\colon 16\mathcal{O}_{\mathbb{P}^6} \to 3\mathcal{O}_{\mathbb{P}^6}(1)$ is a special map with more syzygies

necessary scripts. Moreover, the structure of the moduli space of those examples can be described (the unirationality of the families becomes clear). We also use our construction to recompute the dimensions of the Tonoli families of Calabi–Yau threefolds in $\mathbb{P}^6$, and point out an error in Tonoli's computation. We prove that a Tonoli Calabi–Yau threefold of degree $d = 17$ corresponding to $k = 11$ has Picard group of rank ≥ 2.

Corollary 1.2. *There exists a Calabi–Yau threefold of degree* 17 *in* $\mathbb{P}^6$ *with Picard group of rank* ≥ 2.

We conjecture that the relevant Tonoli family of Calabi–Yau threefolds of degree $d = 17$ corresponding to $k = 11$ is locally complete, which implies in particular that the Hodge numbers of Calabi–Yau threefolds in this family are $h^{11} = 2$ and $h^{12} = 24$ (see Corollary 5.17).

A second result of this paper is the description of del Pezzo surfaces in $\mathbb{P}^5$ (embedded by a subsystem of the anti-canonical bundle) in terms of Pfaffians of vector bundles and the observation of an analogy between these descriptions and descriptions of Tonoli Calabi–Yau threefolds in $\mathbb{P}^6$. In Tables 1 and 2 we present the vector bundles corresponding to these descriptions.

Observe that a Tonoli Calabi–Yau threefold of degree 12 is just a complete intersection of type $(2, 2, 3)$ and is naturally described by the Pfaffians of $\mathcal{O}_{\mathbb{P}^6} \oplus 2\mathcal{O}_{\mathbb{P}^6}(1)$. We have changed this vector bundle to an equivalent one (see the proof of [11, Lem. 3.4]) in order to make the analogy more transparent. Keeping this in mind, we associate to any del Pezzo surface $D \subset \mathbb{P}^5$ a vector bundle E_D from Table 1 defining it and in the same way to any Calabi–Yau threefold X a bundle F_X from Table 2 defining it. Observe that the bundle E_D (resp. F_X) is uniquely determined by the degree $\deg(D)$ (resp. $\deg(X)$) when $\deg(D) \leq 7$ (resp. $\deg(X) \leq 16$). For del Pezzo surfaces of degree 8 and Tonoli Calabi–Yau threefolds of degree 17, we in fact have $E_D = \mathrm{Syz}^1(HR(D))(-2)$ and $F_X = \mathrm{Syz}^1(HR(X))(-3)$, where Syz^1 denotes the sheafification of the first syzygy module of a given module, and HR is the Hartshorne–Rao module of a given variety (i.e. $HR(X) = \bigoplus_{j\in\mathbb{Z}} H^1(I_X(j))$).

The analogy can now be formalized by the following theorem.

Theorem 1.3. *Let $X \subset \mathbb{P}^6$ be a general element of a family of Tonoli Calabi–Yau threefolds and let F_X be as above. Then there exists a map $F_X \to 2O_{\mathbb{P}^6}$ whose kernel E restricted to any $\mathbb{P}^5$ defines a del Pezzo surface of degree* $\deg(X) - 9$. *Conversely, for a general del Pezzo surface $D \subset \mathbb{P}^5$ of degree* $\deg(D) \leq 8$ *with associated bundle E_D there exists an extension E'_D of the bundle E_D to $\mathbb{P}^6$ such that a general bundle F'_d fitting into a short exact sequence*

$$0 \to E'_D \to F'_D \to 2\mathcal{O}_{\mathbb{P}^6} \to 0$$

defines a Calabi–Yau threefold in $\mathbb{P}^6$ of degree $\deg(D) + 9$.

This observation, rather straightforward for degree $\deg(X) \leq 16$ and $\deg(D) \leq 7$ (see Proposition 4.5), is nontrivial for $\deg(X) = 17$ and $\deg(D) = 8$ (see Section 7). Our proof in the latter case uses Theorem 1.1.

Taking one step further, we conjecture an upper bound on the degree of Calabi–Yau threefolds in $\mathbb{P}^6$. More precisely, by analogy to the case of del Pezzo surfaces, we expect that there are no smooth Calabi–Yau threefolds of degree $d \geq 19$ in $\mathbb{P}^6$. Finally, we speculate about the possibility of constructing a degree 18 Calabi–Yau threefold with description analogous to the one of a del Pezzo surface of degree 9.

The structure of the paper is the following. In Section 2, we recall some basic facts from the theory of Pfaffians and provide some preliminary results. In particular, we describe a method to compute the dimensions of families of manifolds obtained as Pfaffian varieties associated to families of vector bundles. In Section 3 we recall Tonoli constructions in a slightly more general context and provide tools for the study of the resulting families. In particular we show how to compute the dimensions of these families. In Section 4, we quickly go through the constructions of del Pezzo surfaces of degree $d_D \leq 7$ and Calabi–Yau threefolds of degree $d_X \leq 16$. We observe that they are strictly related. In particular, Theorem 1.3 takes a stronger form in these cases. In Section 5, we provide various descriptions of the sets $\mathcal{M}_k$, compute their dimensions and prove Theorem 1.1. We apply these results to the study of Tonoli families of Calabi–Yau threefolds of degree 17. In particular, we compute the dimensions of these families.

In Section 6, we describe anti-canonically embedded del Pezzo surfaces of degree 8 in $\mathbb{P}^5$ in terms of Pfaffian varieties. In Section 7 we complete the discussion of the analogy between del Pezzo surfaces and Tonoli Calabi–Yau threefolds and finish the proof of Theorem 1.3.

In Section 8, we make the conjecture that 18 is the highest degree of a Calabi–Yau threefold in $\mathbb{P}^6$.

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2. Preliminaries

In this section, we recall some basic facts of the theory of Pfaffians that will be needed in our construction. Let $X \subset \mathbb{P}^n$ be a Calabi–Yau threefold in $\mathbb{P}^6$ or a del Pezzo surface in $\mathbb{P}^5$. Then, by [21], the variety X can be described as a Pfaffian variety associated to some vector bundle E_X of rank $2r+1$ on $\mathbb{P}^n$. Consider the related Pfaffian resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2s-t) \to E_X^*(-s-t) \xrightarrow{\varphi} E_X(-s) \to \mathcal{I}_X \to 0,$$

with $s = c_1(E_X) + rt$. Observe that by changing E_X to a suitable twist of it we may assume $t = 1$. Moreover, by Formula (1.1) for the canonical class, $s = 3$ for X being a Calabi–Yau threefold and $s = 2$ for X being a del Pezzo surface.

By [21] (or more precisely by an observation of Decker and Schreyer [4, §5] based on [21]), under the assumption $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$, the bundle E_X is, up to a possible direct sum of line bundles, the twist by $\mathcal{O}_{\mathbb{P}^n}(-s)$ of the sheafification of the first syzygy module $\mathrm{Syz}^1(M)$ of the Hartshorne–Rao module $M = \bigoplus_{j\in\mathbb{Z}} H^1(I_X(j))$.

In our constructions, we shall deal only with varieties satisfying a series of additional assumptions, which are believed to be satisfied by a general element of a Hilbert scheme of subcanonical codimension 3 varieties. For this reason, in this preliminary section, we shall also make these assumptions. In particular, we shall assume that the submanifolds under study satisfy the so-called maximal rank assumption stating that the restriction maps

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i)) \to H^0(X, \mathcal{O}_X(i))$$

are of maximal rank for $i \in \mathbb{N}$. Moreover, the Hartshorne–Rao module will be assumed to be generated in its smallest degree. Its shift by s, which we shall usually denote by M and call the shifted Hartshorne–Rao module, will then be generated in degree -1.

These assumptions restrict our attention to varieties whose shifted Hartshorne–Rao module M has a presentation

$$p\mathfrak{S}_{\mathbb{P}^n} \to q\mathfrak{S}_{\mathbb{P}^n}(1) \to M \to 0, \tag{2.1}$$

with $\mathfrak{S}_{\mathbb{P}^n}$ being the coordinate ring of $\mathbb{P}^n$. In this case, M is determined by the map in the presentation, which itself is given by a matrix of linear forms on $\mathbb{P}^n$. If the matrix M is general enough, then M is determined up to isomorphism by the associated embedding

$$\mathbb{P}^{p-1} \subset \langle \mathbb{P}^n \times \mathbb{P}^{q-1} \rangle = \mathbb{P}^{7q-1}, \tag{2.2}$$

where $\langle \mathbb{P}^n \times \mathbb{P}^{q-1} \rangle$ denotes the linear span of the image of the Segre embedding. In the above context, the idea of our constructions will be to find a suitable family of projective subspaces $\mathbb{P}^{p-1} \subset \langle \mathbb{P}^n \times \mathbb{P}^{q-1} \rangle$ determining some family of modules. The latter by [4, §5] should give rise to a family of vector bundles which, by further choices, induces a family of Pfaffian manifolds associated to these bundles.

The first obstacle in the above is that in the remark of Decker and Schreyer the direct sum of line bundles is not determined uniquely. This problem is solved in the case of Calabi–Yau threefolds by the following lemma, which together with Formula (1.1) for the canonical class of a Pfaffian variety will give us the desired uniqueness in the cases studied.

Lemma 2.1. *Let M be the shifted Hartshorne–Rao module of a Calabi–Yau threefold X of degree d in $\mathbb{P}^6$. Assume that M has a minimal presentation*

$$p\mathfrak{S}_{\mathbb{P}^6} \to q\mathfrak{S}_{\mathbb{P}^6}(1) \to M \to 0$$

with $p, q \in \mathbb{N}$ and $p - q \geq 3$. Then X is defined as a Pfaffian variety $\mathrm{Pf}(\sigma)$ for some $\sigma \in H^0((\bigwedge^2 E)(1))$, where $E = Syz^1(M) \oplus \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^6}(a_i)$ for some k and some $a_1, \ldots, a_k \geq 0$.

Proof. Since, by definition, M has only finitely many nonzero graded components, the sheaf $\mathrm{Syz}^1(M)$ is a vector bundle of rank $p - q$ obtained as the kernel of the sheafified map in the presentation of M, i.e., we have

$$0 \to \mathrm{Syz}^1(M) \to p\mathcal{O}_{\mathbb{P}^6} \to q\mathcal{O}_{\mathbb{P}^6}(1) \to 0. \tag{2.3}$$

We know that $E = \mathrm{Syz}^1(M) \oplus \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^6}(a_i)$ for some k and a_i. The only thing we need to prove is that $a_i \geq 0$. In the long cohomology exact sequence associated to (2.3) the map $H^0(p\mathcal{O}_{\mathbb{P}^6}) \to H^0(q\mathcal{O}_{\mathbb{P}^6}(1))$ is the map in the presentation of M. We infer that $\mathrm{Syz}^1(M)$ has no section and $c_1(\mathrm{Syz}^1(M)) = -q$. By analogous reasoning to [11, Lemma 3.5] and by Formula (1.1) for the canonical class we conclude that the number of negative a_i's, if nonzero, must be smaller than $3 - (p - q)$. The latter is non-positive by assumption so all $a_i \geq 0$. $\square$

Remark 2.2. A similar lemma is true for anti-canonically embedded del Pezzo surfaces in $\mathbb{P}^5$ and $p - q \geq 2$.

Since we are sometimes working in parallel with threefolds and with their surface sections, it is good to keep in mind the following lemma relating the Hartshorne–Rao module of a variety to the Hartshorne–Rao module of its hyperplane section:

Lemma 2.3. *Let $Y \subset \mathbb{P}^n$ be a variety satisfying $h^2(\mathcal{I}_{Y|\mathbb{P}^n}(j)) = 0$ for any $j \in \mathbb{Z}$. Let M be the shifted Hartshorne–Rao module of Y. Assume that M has a presentation*

$$p\mathfrak{S}_{\mathbb{P}^n}(-1) \xrightarrow{m} q\mathfrak{S}_{\mathbb{P}^n} \to M \to 0,$$

given by a matrix m of linear entries in the coordinate ring $\mathfrak{S}_{\mathbb{P}^n}$ of $\mathbb{P}^n$. Let H be a hyperplane defined by a linear equation $h = 0$. Then the shifted Hartshorne–Rao module M' of $X \cap H$ has a presentation

$$p\mathfrak{S}_H(-1) \xrightarrow{m'} q\mathfrak{S}_H \to M' \to 0,$$

with $\mathfrak{S}_H = \mathfrak{S}_{\mathbb{P}^n}/_{\langle h\rangle} = \mathfrak{S}_{\mathbb{P}^{n-1}}$ the coordinate ring of the hyperplane H, and m' the image of m via the projection map $\mathfrak{S}_{\mathbb{P}^n} \to \mathfrak{S}_{\mathbb{P}^n}/_{\langle h\rangle}$.

Proof. For each j, we have the exact sequence

$$0 \to I_{Y|\mathbb{P}^n}(j-1) \xrightarrow{\lambda} I_{Y|\mathbb{P}^n}(j) \to I_{(Y\cap H)|H}(j) \to 0,$$

where λ is given by multiplication by h. From the associated cohomology sequence in each degree and the assumed vanishing $h^2(I_{Y|\mathbb{P}^n}(j)) = 0$, we obtain $M' = M/(hM)$, and the presentation follows. $\square$

3. Constructions of families of Pfaffian varieties

In this section we shall outline the main constructions of families of Pfaffian varieties used throughout the paper. In particular, we put in a formal context the Tonoli construction of families of Calabi–Yau threefolds. We aim at constructing flat families $\mathbb{P}^n \times S \supset \mathcal{X} \to S$ of subcanonical submanifolds of some projective space $\mathbb{P}^n$.

3.1. Families given by a fixed vector bundle

The first construction of a family of Pfaffian varieties that comes to mind is to consider a vector bundle E of rank $2r + 1$ such that $h^0(\bigwedge^2 E(1)) > 0$ and consider all Pfaffian varieties of the form $\mathrm{Pf}(\sigma)$ with $\sigma \in H^0(\bigwedge^2 E(1))$. More precisely, consider the open subset $U \subset H^0(\bigwedge^2 E(1))$ defined by

$$U = \{\sigma \in H^0((\bigwedge^2 E)(1)) \mid \mathrm{codim}(D^{2r-2}(\sigma)) = 3\}.$$

The family of Pfaffian varieties parameterized by U is defined as follows. Let $\pi_1 : U \times \mathbb{P}^n \to U$ and $\pi_2 : U \times \mathbb{P}^n \to \mathbb{P}^n$ be the natural projections. Consider the bundle $\pi_2^* \bigwedge^2 E(1)$ and its evaluation section

$$\theta : U \times \mathbb{P}^n \ni (\sigma, x) \mapsto \sigma(x) \in (\bigwedge^2 E)(1).$$

Then $\mathcal{X} = \mathrm{Pf}(\theta) \subset U \times \mathbb{P}^n$ is a codimension 3 subvariety and $\pi_1|_{\mathcal{X}} : \mathcal{X} \to U$ is a flat morphism. Indeed, every fiber is a codimension 3 variety with the same Hilbert polynomial computed by the Pfaffian resolution involving the same bundle. In this way, we construct a family of Pfaffian varieties. This method is sufficient for the construction of locally complete families of Pfaffian Calabi–Yau threefolds of degree ≤ 16.

Example 3.1. Note that the Tonoli families of Calabi–Yau threefolds of degree ≤ 16 are all obtained via the above construction using bundles from the table in the introduction.

We shall now describe how to compute the dimensions of such families of Pfaffian submanifolds, i.e. the dimension of the image $\mathcal{D}$ of the forgetful map

$$\phi \colon U \to \mathrm{Hilb}_{\mathcal{X}_\sigma | \mathbb{P}^n},$$

where $\mathrm{Hilb}_{\mathcal{X}_\sigma|\mathbb{P}^n}$ is the Hilbert scheme containing $\pi_2(\pi_1|_{\mathcal{X}}^{-1}(\sigma))$ for chosen $\sigma \in U$.

Proposition 3.2. *The dimension of the family of varieties obtained as Pfaffian varieties associated to a bundle* $E = \ker(p\mathcal{O}_{\mathbb{P}^n} \to q\mathcal{O}_{\mathbb{P}^n}(1)) \oplus \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^6}(a_i)$ *for* $a_i \geq 0$ *is* $h^0(\bigwedge^2 E(1)) - \dim \mathrm{Hom}(E, E)$.

Before proving Proposition 3.2 let us formulate a preparatory result.

Lemma 3.3. *Let* Y *be a smooth variety obtained as a Pfaffian variety associated to a bundle* E *on* $\mathbb{P}^n$*. Then we have the following exact sequence:*

$$0 \to E^*(-s-t) \to E(-s) \oplus (S^2 E^*)(-t) \to E \otimes E^* \to (\bigwedge^2 E)(t) \to \mathcal{N}_{Y|\mathbb{P}^n} \to 0, \quad (3.1)$$

where $\mathcal{N}_{Y|\mathbb{P}^n}$ *is the normal bundle of* Y *in* $\mathbb{P}^n$*.*

Proof. First arguing as in [13, Prop. 2.4] we deduce that $\mathcal{N}_{Y|\mathbb{P}^n} = \mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{I}_Y) = \bigwedge^2 \mathcal{I}_Y(2s+t)$. Then from [22] we obtain the free resolution of the sheaf $\bigwedge^2 \mathcal{I}_Y$. $\square$

Proof of Proposition 3.2. Let us keep the notation preceding Proposition 3.2. Let moreover $X = \pi_2(\pi_1|_{\mathcal{X}}^{-1}(\sigma))$ for a fixed general $\sigma \in U$. Then the map $H^0((\bigwedge^2 E)(1)) \to H^0(\mathcal{N}_{X|\mathbb{P}^n})$ in Lemma 3.3 is interpreted as the tangent map to the forgetful map ϕ at σ. We want to prove that the dimension of the image of ϕ is $h^0((\bigwedge^2 E)(t)) - \dim \mathrm{Hom}(E, E)$. It is enough to prove that the rank of this tangent map at the general point σ is

$h^0((\bigwedge^2 E)(t)) - \dim \mathrm{Hom}(E,E)$. Splitting the long exact sequence (3.1) into short ones, we get

$$0 \to F \to (\bigwedge^2 E)(t) \to \mathcal{N}_{X|\mathbb{P}^n} \to 0,$$
$$0 \to G \to E \otimes E^* \to F \to 0,$$
$$0 \to E^*(-s-t) \to E(-s) \oplus (S^2E^*)(-t) \to G \to 0,$$

for some bundles F, G on $\mathbb{P}^n$.

Moreover, from the long cohomology sequences of the exact sequence

$$0 \to E \to p\mathcal{O}_{\mathbb{P}^n} \oplus \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^6}(a_i) \to q\mathcal{O}_{\mathbb{P}^n}(1) \to 0, \tag{3.2}$$

its twists, twisted duals, and the resulting resolution of $(S^2E^*)(-t)$ obtained from [22]:

$$0 \to \binom{q}{2}\mathcal{O}_{\mathbb{P}^n}(-t-2) \to q\left(p\mathcal{O}_{\mathbb{P}^n} \oplus \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^6}(-a_i)\right)(-t-1)$$
$$\to S^2\left(p\mathcal{O}_{\mathbb{P}^n} \oplus \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^6}(-a_i)\right)(-t) \to (S^2E^*)(-t) \to 0$$

we deduce that $h^0(G) = h^1(G) = 0$. It follows that $h^0(F) = h^0(E \otimes E^*) = \dim \mathrm{Hom}(E,E)$. Since $h^0(F)$ is the kernel of the tangent map to ϕ at s, we deduce that the rank of this tangent map at σ is $h^0((\bigwedge^2 E)(t)) - \dim \mathrm{Hom}(E,E)$, which gives the assertion. □

3.2. Families of Pfaffians defined by a family of vector bundles

For our purposes, in particular for the description of families of degree 17 Calabi–Yau threefolds in $\mathbb{P}^6$ as well as del Pezzo surfaces of degree 8 in $\mathbb{P}^5$, we shall need a more general construction than the one proposed in Subsection 3.1. In this construction the bundle defining the Pfaffian varieties will be allowed to change. We proceed as follows.

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n \times B$ for some smooth affine variety B. Let us denote by $\pi : \mathbb{P}^n \times B \to \mathbb{P}^n$ the natural projection. For all $\beta \in B$ denote the restricted bundle $\mathcal{E}|_{\mathbb{P}^n \times \{\beta\}}$ by $\mathcal{E}_\beta$. Moreover, by abuse of notation, write $(\bigwedge^2 \mathcal{E})(1)$ for $\bigwedge^2 \mathcal{E} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Assume now that for some $U \subset \Sigma \subset H^0((\bigwedge^2 \mathcal{E})(1))$, with U an open subset of a subvector space Σ, we know that for all $\sigma \in U$ and all $\beta \in B$ the restriction $\sigma|_{\mathbb{P}^n \times \{\beta\}}$ defines a Pfaffian variety (i.e. of codimension 3). Let

$$\theta : U \times \mathbb{P}^n \times B \ni (\sigma, x, \beta) \mapsto \sigma(x, \beta) \in (\bigwedge^2 \mathcal{E})(1)$$

be the evaluation section of $\pi^*_{\mathbb{P}^n \times B}((\bigwedge^2 \mathcal{E})(1))$ with

$$\pi_{\mathbb{P}^n \times B} : U \times \mathbb{P}^n \times B \to \mathbb{P}^n \times B$$

the natural projection and let

$$\mathcal{X} = \mathrm{Pf}(\theta) \subset U \times \mathbb{P}^n \times B$$

be its Pfaffian locus. Finally, denote by $\pi_{U,B}$ the natural projection $U \times \mathbb{P}^n \times B \to U \times B$.

Lemma 3.4. *With the notation above, $\pi_{U,B}|_{\mathcal{X}}$ is a flat morphism.*

Proof. The only thing we need to check is the equality of the Hilbert polynomials of each fiber. That follows from the Pfaffian exact sequence and the fact that all $\mathcal{E}_\beta$ have the same Hilbert polynomial, since they are restrictions of $\mathcal{E}$ which is flat over $\mathbb{P}^n$, being locally free over $\mathbb{P}^n \times B$. □

Let us now make use of Lemma 3.4 in the context of the paper.

3.3. Tonoli construction

Consider vector spaces V, W and P of dimension p, q, $n+1$ respectively. We have

$$V \otimes W \otimes P = \mathrm{Hom}(V^\vee, W) \otimes P.$$

It follows that each element $m \in V \otimes W \otimes P$ induces a map

$$\varphi_m \colon V^\vee \times \mathcal{O}_{\mathbb{P}(P)} \to W \times \mathcal{O}_{\mathbb{P}(P)}(1),$$

which globally gives

$$\varphi \colon V^\vee \times \mathcal{O}_{V\otimes W \otimes P \times \mathbb{P}(P)} \to W \times \pi^*(\mathcal{O}_{\mathbb{P}(P)}(1)),$$

where $\pi : V \otimes W \otimes P \times \mathbb{P}(P) \to \mathbb{P}(P)$ is the natural projection. Let $B \subset V \otimes W \otimes P$ be the open subset given by $B = \{m \in V \otimes W \otimes P \mid \varphi_m \text{ is surjective}\}$. Then $\mathcal{E}_B := (\ker \varphi)|_{B \times \mathbb{P}(P)}$ is a vector bundle.

For any k let now $B_k \subset B$ be an algebraic subset of B such that for each $b \in B_k$ we have

$$h^0(\bigwedge^2 \mathcal{E}_B|_{\{b\}\times\mathbb{P}(P)}(1)) = k.$$

Let $\mathcal{E}_k = \mathcal{E}|_{B_k \times \mathbb{P}(P)}$. Then by Grauert semicontinuity [7, III Cor. 12.9] there is an open subset $B'_k \subset B_k$ and a k-dimensional subspace $\Sigma_k \subset H^0(\bigwedge^2 \mathcal{E}_k|_{B'_k \times \mathbb{P}(P)}(1))$ such that the natural map

$$H^0(\bigwedge^2 \mathcal{E}_k|_{B'_k \times \mathbb{P}(P)}(1)) \supset \Sigma_k \to H^0(\bigwedge^2 \mathcal{E}_k|_{\{b\}\times\mathbb{P}(P)}(1))$$

is an isomorphism for each $b \in B'_k$.

Finally, if we know that for some b in B'_k there exists a section $\sigma \in \Sigma_k \subset H^0(\bigwedge^2 \mathcal{E}_k|_{\{b\}\times\mathbb{P}(P)}(1))$ such that $\mathrm{Pf}(\sigma)$ is a smooth codimension 3 submanifold in $\mathbb{P}(P)$ then by further restricting ourselves to an open subset $B''_k \subset B_k$ and to an open subset U of $\Sigma_k \subset H^0(\bigwedge^2 \mathcal{E}_k|_{B_k\times\mathbb{P}(P)}(1))$, we may apply Lemma 3.4 giving rise to a flat family $\mathcal{T}_{(B_k,U,p,q,n)}$ of smooth codimension 3 submanifolds in $\mathbb{P}^n$. Note that in this way $\mathcal{T}_{(B_k,p,q,n)}$ is a smooth family over an open subset $B''_k \times U \subset B_k \times \Sigma_k \subset B_k \times H^0(\bigwedge^2 \mathcal{E}_k|_{B_k\times\mathbb{P}(P)}(1))$.

Definition 3.5. A family $\mathcal{T}_{(B_k,p,q,n)}$ obtained as above will be called a Tonoli family of Pfaffian manifolds. Maximal Tonoli families for given k, p, q, n will be denoted $\mathcal{T}_{k,p,q,n}$.

Example 3.6. The three families of Calabi–Yau threefolds of degree 17 in $\mathbb{P}^6$ constructed by Tonoli in [19] are examples of Tonoli families of Pfaffian manifolds of type $\mathcal{T}_{(k,16,3,6)}$. Indeed, we choose $V = V_{16}$, $W = W_3$, $P = P_7$ three vector spaces of dimensions indicated by the subscripts. We observe that we have a rational map $\Psi : V_{16}^\vee \otimes W_3 \otimes P_7 \to G(16, W_3 \otimes P_7)$ and consider $B_k = \Psi^{-1}(\tilde{\mathcal{M}}_k)$ where $\tilde{\mathcal{M}}_k$ is given by Equation (1.2) (note that in particular $\tilde{\mathcal{M}}_k$ is irreducible). We then observe that the isomorphism class of the resulting $(\mathcal{E}_k)_\beta$ depends only on $\Psi(\beta)$. If $\Psi(\beta) = \mathbf{P}$ we shall denote $(\mathcal{E}_k)_\beta$ by $E_{\mathbf{P}}$. It is then proven in [19] that $h^0(\bigwedge^2 E_{\mathbf{P}}(1)) = k$ for general $\mathbf{P} \in \tilde{\mathcal{M}}_k$. The idea of the argument is as follows: each special $\mathbb{P}^2$ fiber produces a section of $\bigwedge^2 E_{\mathbf{P}}(1)$; we then check by a Macaulay 2 computation that these sections are independent and generate $h^0(\bigwedge^2 E_{\mathbf{P}}(1))$ for a specific randomly chosen $\mathbf{P} \in \mathcal{M}_k$ (see for example [12] for methods to perform such a check) and conclude by semicontinuity. By passing to open subsets we obtain a Tonoli family of Pfaffian manifolds $\mathcal{T}_{(\Psi^{-1}(\tilde{\mathcal{M}}_k),16,3,6)}$ which are the Calabi–Yau threefolds of degree 17 defined in [19].

We shall now present a method of computing the dimension of such Tonoli families of Pfaffian submanifolds, i.e. the dimension of the image $\mathcal{D}_{B_k,p,q,n}$ of the forgetful map

$$\varphi : B''_k \times U \to \mathrm{Hilb}_{X_{(b,\sigma)}|\mathbb{P}^6},$$

where $X_{(b,\sigma)}$ is the fiber of the family over the point $(b,\sigma) \in B''_k \times U$ and $\mathrm{Hilb}_{X_{b,\sigma}|\mathbb{P}^6}$ is the Hilbert scheme containing $X_{b,\sigma}$ for all $(b,\sigma) \in B_k \times U$.

Proposition 3.7. *Let $\mathcal{T}_{(B_k,16,3,6)}$ be a Tonoli family of Pfaffian manifolds and let $\mathcal{D}_{(B_k,16,3,6)}$ be the image of this family under the forgetful map φ to the Hilbert scheme as above. Then keeping the notation from the Tonoli construction above,*

$$\dim \mathcal{D}_{(B_k,16,3,6)} = \dim B_k + k - 16^2 - 3^2.$$

Proof. Since the dimension of the domain of the forgetful map φ is $\dim B_k + k$, in order to compute the dimension of the image $\mathcal{D}_{(B_k,16,3,6)}$, we need to compute the dimension of the fiber of φ.

Observe that in our case $E_{(b,\sigma)} = Syz^1(X_{(b,\sigma)})$ for $(b,\sigma) \in B_k'' \times U$. It follows that if $\varphi((b_1,\sigma_1)) = \varphi((b_2,\sigma_2))$, then there exists an isomorphism $\alpha : E_{b_1} \simeq E_{b_2}$. We know, moreover, that every such isomorphism α lifts to resolutions:

$$\begin{tikzcd} 0 \arrow[r] & E_{b_1} \arrow[r] \arrow[d, "\alpha"'] & V_{16} \otimes \mathcal{O}_{\mathbb{P}(V_7)} \arrow[r, "b_1"'] \arrow[d, "A"'] & V_3 \otimes \mathcal{O}_{\mathbb{P}(V_7)} \arrow[r] \arrow[d, "B"'] & 0 \\ 0 \arrow[r] & E_{b_2} \arrow[r] & V_{16} \otimes \mathcal{O}_{\mathbb{P}(V_7)} \arrow[r, "b_2"'] & V_3 \otimes \mathcal{O}_{\mathbb{P}(V_7)} \arrow[r] & 0 \end{tikzcd}$$

Now, from each fiber $\varphi^{-1}(\varphi((b_1,\sigma_1))$ we have a map $\pi_1 : \varphi^{-1}(\varphi((b_1,\sigma_1)) \to U$. By the above, the dimension of the image of π_1 is equal to $\dim \mathrm{GL}(V) + \dim \mathrm{GL}(W) - \dim \mathrm{Aut}(E,E)$, whereas the dimension of the fiber of π_1 is computed by Proposition 3.2 to be $\dim \mathrm{Aut}(E,E)$. It follows that the dimension of the fiber of φ is $\dim \mathrm{GL}(V) + \dim \mathrm{GL}(W) = p^2 + q^2$, which ends the proof. □

Remark 3.8. More generally we can show that the dimension of the Tonoli family

$$\dim \mathcal{D}_{(B_k,p,q,n)} = \dim B_k + k - p^2 - q^2.$$

4. Del Pezzo surfaces of degree ≤ 7 and Calabi–Yau threefolds of degree ≤ 16

In this section we describe anti-canonically embedded del Pezzo surfaces of degree $d \leq 7$ in $\mathbb{P}^5$ in terms of Pfaffians of vector bundles. Let us first make some general remarks on del Pezzo surfaces embedded in $\mathbb{P}^5$ via a subsystem of the anti-canonical class.

4.1. Del Pezzo surfaces in $\mathbb{P}^5$

Recall that an anti-canonical model of a smooth del Pezzo surface of degree ≥ 3 is a smooth surface of degree n in $\mathbb{P}^n$ for $3 \leq n \leq 9$.

Consider anti-canonical embeddings of these surfaces in $\mathbb{P}^5$. More precisely, for $3 \leq n \leq 7$, we consider varieties obtained as the image of the anti-canonical embedding of the del Pezzo surface of degree n composed with a general linear map $\mathbb{P}^n \to \mathbb{P}^5$. For $n = 8$, we have two del Pezzo surfaces $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^1$. So we have two types of del Pezzo surfaces of degree 8 in $\mathbb{P}^5$.

Let now D be a del Pezzo surface of degree n in $\mathbb{P}^5$ as above. Then D is clearly subcanonical, so by the theorem of Walter it admits Pfaffian resolutions, which we shall study.

It follows from the Kodaira vanishing theorem and the Serre duality that $H^i(\mathcal{I}_D) = 0$ for $i > 1$. This implies that the bundle E in the Pfaffian resolution of our del Pezzo surface D is the sheafification of the module $Syz^1(\bigoplus_{k\in\mathbb{Z}} H^1(\mathcal{I}_D(k)))$ over the coordinate ring of $\mathbb{P}^5$ plus a possible direct sum of line bundles (see Lemma 2.1).

Lemma 4.1. *The Hilbert function of the Hartshorne–Rao module of a del Pezzo surface $D \subset \mathbb{P}^5$ of degree n is 0 for $n \leq 5$ and for $n \in \{6, 7, 8, 9\}$ takes the following values starting from grade 0: $(0, 1, 0, \dots)$, $(0, 2, 1, 0, \dots)$, $(0, 3, 4, 0, \dots)$, $(0, 4, 7, 0, \dots)$ respectively. Moreover, these del Pezzo surfaces satisfy the maximal rank assumption.*

Proof. We first check the maximal rank assumption by checking a random example and concluding by semicontinuity as in [9, Lemma 5.1]. The values of the Hilbert function are then computed from the Riemann–Roch theorem as in [19]. $\square$

4.2. Constructions of degree ≤ 7 del Pezzo surfaces

We can now get a description of a general del Pezzo surface of degree $n \leq 7$ in $\mathbb{P}^5$.

Corollary 4.2. *A general del Pezzo surface of degree $n \leq 7$ in $\mathbb{P}^5$ is described as a Pfaffian variety associated to the bundle:*

(1) $\mathcal{O}_{\mathbb{P}^5}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^5}(1)$ *for* $n = 3$,
(2) $2\mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}(1)$ *for* $n = 4$,
(3) $5\mathcal{O}_{\mathbb{P}^5}$ *for* $n = 5$,
(4) $\Omega^1_{\mathbb{P}^5}(1) \oplus 2\mathcal{O}_{\mathbb{P}^5}$ *for* $n = 6$,
(5) $\ker(\psi)$ *for* $n = 7$, *where* $\psi\colon 11\mathcal{O}_{\mathbb{P}^5} \to 2\mathcal{O}_{\mathbb{P}^5}(1)$ *is a general map.*

Proof. From Lemma 4.1 we know the bundles up to a direct sum of line bundles. We next use the results of [11, Section 3] and proceed analogously. $\square$

4.3. Analogy with Tonoli Calabi–Yau threefolds of degree ≤ 16

Recall that Tonoli families of Calabi–Yau threefolds of degree $k \leq 16$ are obtained by the construction described in Section 3.1 applied to the vector bundles on $\mathbb{P}^6$ characterized in Table 2. Comparing the vector bundles appearing in the Pfaffian constructions of del Pezzo surfaces and Tonoli Calabi–Yau threefolds, we observe that the description of a general del Pezzo surface of degree d in $\mathbb{P}^5$ is similar to the description of a general Tonoli Calabi–Yau threefold of degree $d + 9$ in $\mathbb{P}^6$. The relation is partially explained by the following.

Proposition 4.3. *Let E and F be vector bundles on $\mathbb{P}^5$ and on $\mathbb{P}^6$ respectively, related by the exact sequence*

$$0 \to E \to F|_{\mathbb{P}^5} \to 2\mathcal{O}_{\mathbb{P}^5} \to 0.$$

Assume moreover that both bundles define smooth codimension 3 varieties $X \subset \mathbb{P}^6$ and $D \subset \mathbb{P}^5$. Then X is a Calabi–Yau threefold of degree d if and only if D is a del Pezzo surface of degree $d - 9$.

Proof. This follows from Formula (1.1) for the canonical class implying that $r + c_1(E) = 2$, and from the following formula for the degree of a Pfaffian variety defined by a vector bundle in terms of Chern classes of the vector bundle.

Lemma 4.4 *(See [16]). If E is a vector bundle of rank $2r+1$ on $\mathbb{P}^n$ and $s \in H^0(\bigwedge^2 E(1))$ a general section that defines, via the Pfaffian construction, a variety Y of codimension 3. Then*

$$\deg(Y) = rc_1^2(E) + c_1(E)c_2(E) + (r^2+r)c_1(E) + c_2(E) - c_3(E) + \frac{r(2r+1)(2r+2)}{12}.$$

In particular $deg(Y) - deg(D) = (r + c_1(E) + 1)^2$.

Proof. The proof is based on a computation using the Hirzebruch–Riemann–Roch theorem, the restriction of the Pfaffian sequence to a general $\mathbb{P}^3$, and the fact that the degree of a set of distinct points is equal to the Euler characteristic of its structure sheaf. □

□

Let us make the analogy more precise by proving Theorem 1.3 for $d \leq 7$. Let D be a del Pezzo surface of degree d in $\mathbb{P}^5$, and E_D be the vector bundle on $\mathbb{P}^5$ defining D through the Pfaffian construction. Consider a Tonoli Calabi–Yau threefold X of degree $d+9$ and its associated bundle F_X by the Pfaffian construction.

Observe that, for $d \leq 7$, the bundles E_D and F_X are determined by d up to a sum of rank 2 bundles of the form $\mathcal{O}(-i) \oplus \mathcal{O}(i-1)$ (see the proof [11, Lem. 3.4]). For our purpose, we choose the bundles from Tables 1 and 2 and denote them E_d and F_d respectively.

Proposition 4.5. *For $d \leq 7$, the bundle E_d is obtained as the cokernel of a general surjective map $F_d|_{\mathbb{P}^5} \to 2\mathcal{O}_{\mathbb{P}^5}$. Moreover, the bundle E_d admits an extension E'_d to $\mathbb{P}^6$ such that F_d is a general bundle fitting into a short exact sequence*

$$0 \to E'_d \to F_d \to 2\mathcal{O}_{\mathbb{P}^6} \to 0.$$

Proof. For each of the bundles F_d for $d \leq 7$ we compute the restriction to a general $\mathbb{P}^5$. We get

- $F_3|_{\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^5}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^5} \oplus 2\mathcal{O}_{\mathbb{P}^5}(1)$,
- $F_4|_{\mathbb{P}^5} = 4\mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}(1)$,
- $F_5|_{\mathbb{P}^5} = 7\mathcal{O}_{\mathbb{P}^5}$,
- $F_6|_{\mathbb{P}^5} = \Omega^1_{\mathbb{P}^5}(1) \oplus 4\mathcal{O}_{\mathbb{P}^5}$,
- $F_7|_{\mathbb{P}^5} = 2\Omega^1_{\mathbb{P}^5}(1) \oplus \mathcal{O}_{\mathbb{P}^5} = \ker(\psi')$ for $\psi' \colon 13\mathcal{O}_{\mathbb{P}^5} \to 2\mathcal{O}_{\mathbb{P}^5}(1)$ a general map.

Note that F_7 as defined above is uniquely determined up to isomorphism. It is now easy to check the first part of the proposition. For the second part we take for E'_d one of the following:

- $E'_3 = \mathcal{O}_{\mathbb{P}^6}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^6}(1)$,
- $E'_4 = 2\mathcal{O}_{\mathbb{P}^6} \oplus \mathcal{O}_{\mathbb{P}^6}(1)$,
- $E'_5 = 5\mathcal{O}_{\mathbb{P}^6}$,
- $E'_6 = \Omega^1_{\mathbb{P}^6}(1) \oplus \mathcal{O}_{\mathbb{P}^6}$,
- $E'_7 = \ker(\psi'')$ for $\psi''\colon 11\mathcal{O}_{\mathbb{P}^6} \to 2\mathcal{O}_{\mathbb{P}^6}(1)$ a general map.

It is clear that $E'_d|_{\mathbb{P}^5} = E_d$ for a general $\mathbb{P}^5 \subset \mathbb{P}^6$ and we conclude by observing that for each d there is an exact sequence

$$0 \to E'_d \to F_d \to 2\mathcal{O}_{\mathbb{P}^6} \to 0,$$

and F_d is always the general element fitting in the exact sequence. Indeed, for $d \le 6$ we have $\mathrm{Ext}^1(2\mathcal{O}_{\mathbb{P}^6}, E'_d) = 0$ and $F_d = E'_d \oplus 2\mathcal{O}_{\mathbb{P}^6}$, whereas for $d = 7$ any bundle F appearing in the exact sequence

$$0 \to E'_7 \to F \to 2\mathcal{O}_{\mathbb{P}^6} \to 0$$

is the kernel of some map $\theta\colon 13\mathcal{O}_{\mathbb{P}^6} \to 2\mathcal{O}_{\mathbb{P}^6}(1)$, hence F_7 is general among them. $\square$

Remark 4.6. The bundles $F_d|_{\mathbb{P}^5}$ and E'_d for $d \le 7$ define through the Pfaffian construction general type surfaces in their canonical embedding in $\mathbb{P}^5$ and del Pezzo threefolds in their half-anti-canonical embeddings in $\mathbb{P}^6$ respectively.

Corollary 4.7. *Theorem 1.3 holds for del Pezzo surfaces of degree* ≤ 7 *and Tonoli Calabi–Yau threefolds of degree* ≤ 16.

Remark 4.8. In view of Propositions 4.3 and 4.5, it is natural to construct Calabi–Yau threefolds and del Pezzo surfaces in pairs. In particular, having E or F one can try to reconstruct the other. The only thing missing and, in fact, the most important thing from the point of view of the cases with pairs of degrees $(8, 17)$ and $(9, 18)$ is the existence of a section of the newly constructed $\bigwedge^2 F(1)$ and $\bigwedge^2 E(1)$ defining a smooth Pfaffian variety (in particular of codimension 3).

In the next sections, we shall study this phenomenon for del Pezzo surfaces of degree 8 and Calabi–Yau threefolds of degree 17.

5. Constructions of Tonoli revisited—degree 17 Calabi–Yau threefolds

In this section our aim is to find a geometric interpretation for Tonoli constructions of Calabi–Yau threefolds of degree 17 in $\mathbb{P}^6$. By Example 3.6, these appear as Tonoli families of Pfaffian submanifolds of the form $\mathcal{T}_{B_k,6,16,3}$, where $B_k = \Psi^{-1}(\tilde{\mathcal{M}}_k)$ and $\Psi : V_{16}^* \otimes W_3 \otimes P_7 \to G(16, W_3 \otimes P_7)$ is the natural map associating to a matrix of linear forms the span of its columns. Let us keep this notation throughout the section.

The reinterpretation of Tonoli constructions of Calabi–Yau threefolds of degree 17 in $\mathbb{P}^6$, claimed in the introduction, relies on finding good geometric constructions of the sets $\tilde{\mathcal{M}}_k \subset G(16, W_3 \otimes P_7)$.

Observe first that to any $\mathbf{P} \in G(16, W_3 \otimes P_7)$ we can associate a unique $\mathcal{G}_{\mathbf{P}} = \operatorname{coker} \lambda_{\mathbf{P}}$ on $\mathbb{P}(W_3)$, where $\lambda_{\mathbf{P}} : 5\mathcal{O}_{\mathbb{P}(W_3)}(-1) \to P_7 \otimes \mathcal{O}_{\mathbb{P}(W_3)}$ is any embedding induced by the five linear equations defining $\mathbb{P}^{15} \subset \mathbb{P}(W_3 \otimes W_7)$ (note that $\mathcal{G}_{\mathbf{P}}$ is independent of the choice of $\lambda_{\mathbf{P}}$). Note that $\mathcal{G}_P$ is generically locally free of rank one and not locally free at the points where the rank of $\lambda_{\mathbb{P}}$ drops. Now, the condition $\mathbf{P} \in \tilde{\mathcal{M}}_k$ is equivalent to the condition that the projectivization $\mathbb{P}(\mathcal{G}_{\mathbf{P}})$ has exactly k special fibers isomorphic to $\mathbb{P}^2$.

Now, defining $C_{\mathbf{P}}$ by the exact sequence

$$0 \to \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to \mathcal{G}_{\mathbf{P}} \to C_{\mathbf{P}} \to 0,$$

we can consider

$$\mathcal{M}_k = \{\mathbf{P} \in G(16, W_3 \otimes P_7) \mid \mathcal{C}_{\mathbf{P}} \text{ is the structure sheaf of a scheme of length } k\}.$$

Since $\tilde{\mathcal{M}}_k \subset \mathcal{M}_k$ is a Zariski open subset, we shall look for good descriptions of $\mathcal{M}_k$.

Let us first characterize each $\mathbf{P} \in \mathcal{M}_k$, for $k = 8, 9, 11$, by studying the sheaves $\mathcal{G}_{\mathbf{P}}$ corresponding to its elements.

Proposition 5.1. *If $\mathbf{P} \in G(16, W_3 \otimes P_7)$ is a $\mathbb{P}^{15}$ in $\mathbb{P}(V_3 \otimes V_7)$ then $\mathbf{P} \in \mathcal{M}_k$ if and only if $\mathcal{G}_{\mathbf{P}}^{\vee\vee}$ is a rank two vector bundle isomorphic to:*

(1) $T_{\mathbb{P}^2}(1)$,
(2) $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$,
(3) $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(4)$,

for $k = 8$, 9 or 11, respectively.

Proof. Let us first assume that $\mathcal{G}_{\mathbf{P}}^{\vee\vee}$ is one of the above vector bundles for some $k \in \{8, 9, 11\}$. Then we have an exact sequence induced by $\lambda_{\mathbf{P}}$:

$$0 \to 5\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\lambda_{\mathbf{P}}} 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to C_{\mathbf{P}} \to 0.$$

By computing Chern classes it follows that $\operatorname{rank}(\mathcal{C}_{\mathbf{P}}) = 0$, $c_1(\mathcal{C}_{\mathbf{P}}) = 0$ and $c_2(\mathcal{C}_{\mathbf{P}}) = k$, hence $\mathcal{C}_{\mathbf{P}}$ is the structure sheaf of a scheme of length k.

For the other direction let $\mathbf{P} \in \mathcal{M}_k$. Then we have an exact sequence

$$0 \to 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to C_{\mathbf{P}} \to 0$$

with $C_{\mathbf{P}}$ the structure sheaf of a scheme of length k. From that sequence we deduce that $c_1(\mathcal{G}_{\mathbf{P}}^{\vee\vee}) = 5H$, where H is the class of a line in $\mathbb{P}^2$ and $c_2(\mathcal{G}_{\mathbf{P}}^{\vee\vee}) = (15-k)\mathrm{pt}$ where pt is the class of a point in $\mathbb{P}^2$. Moreover, by the Bertini theorem, there exists a global section of $\mathcal{G}_{\mathbf{P}}^{\vee\vee}$ vanishing in codimension 2. This gives rise to an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to I_{Z_{\mathbf{P}}}(5) \to 0$$

where $Z_{\mathbf{P}}$ is a locally complete intersection scheme of length $15-k$ on $\mathbb{P}^2$. It follows that $Z_{\mathbf{P}}$ satisfies the Cayley–Bacharach property for quadrics (we learned this method from [5]), which means in our cases:

(1) if $k = 8$ then $Z_{\mathbf{P}}$ is a locally complete intersection scheme of length 7 with no subscheme of length 6 contained in a conic;
(2) if $k = 9$ then $Z_{\mathbf{P}}$ is a locally complete intersection scheme of length 6 contained in a conic;
(3) if $k = 11$ then $Z_{\mathbf{P}}$ is a locally complete intersection scheme of length 4 contained in a line.

We deduce that

- if $k = 9$ then $Z_{\mathbf{P}}$ is a complete intersection of a quadric and a cubic, hence $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$;
- if $k = 11$ then $Z_{\mathbf{P}}$ is a complete intersection of a line and a quadric, hence $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(4)$.

It remains to handle the case $k = 8$. We have

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to I_{Z_{\mathbf{P}}}(5) \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^2}(-2)$ we obtain $h^0(\mathcal{G}_{\mathbf{P}}^{\vee\vee}(-2)) = 3$ and $\mathcal{G}_{\mathbf{P}}^{\vee\vee}(-2)$ is generated by these three sections up to a codimension 2 subset. It follows that there is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee}(-2) \to I_p(1) \to 0, \tag{5.1}$$

with $p \in \mathbb{P}^2$. This implies that for $k = 8$ we have $\mathcal{G}_{\mathbf{P}}^{\vee\vee}(-2) = T_{\mathbb{P}^2}(-1)$. $\square$

Corollary 5.2. *In the notation of Proposition 5.1, if $\mathbf{P} \in \mathcal{M}_k$ then there exists a map $\beta_{\mathbf{P}} : 7\mathcal{O}_{\mathbb{P}^2} \to L_k$ surjective outside possibly a set of codimension at least 2, such that $\beta_{\mathbf{P}} \circ \lambda_{\mathbf{P}} = 0$ and where L_k is one of the following sheaves:*

(1) $L_{11} = \mathcal{O}_{\mathbb{P}^2}(1)$,
(2) $L_9 = \mathcal{O}_{\mathbb{P}^2}(2)$,
(3) $L_8 = \mathcal{I}_p(3)$ *for some* $p \in \mathbb{P}^2$.

Proof. Let $\mathbf{P} \in \mathcal{M}_k$. Then by Proposition 5.1 we have an exact sequence

$$0 \to 5\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\lambda_{\mathbf{P}}} 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to \mathcal{C}_{\mathbf{P}} \to 0$$

and

(1) if $k = 8$ then $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = T_{\mathbb{P}^2}(1)$,
(2) if $k = 9$ then $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$,
(3) if $k = 11$ then $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(4)$.

In each case we have a surjective map

$$\sigma_{\mathbf{P}} : \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to L_k.$$

Indeed this is clear for $k = 9, 11$, whereas for $k = 8$ it follows from the exact sequence (5.1). Since the map $7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee}$ is surjective outside codimension 2, so is its composition with $\sigma_{\mathbf{P}}$, giving rise to the desired $\beta_{\mathbf{P}}$. $\square$

Remark 5.3. Note that we have a map $\sigma_{\mathbf{P}} : T_{\mathbb{P}^2}(1) \to \mathcal{I}_p(3)$ for every $p \in \mathbb{P}^2$.

Corollary 5.4. *Let $k \in \{9, 11\}$. If there exists a surjective map $\beta_{\mathbf{P}} : 7\mathcal{O}_{\mathbb{P}^2} \to L_k$ such that $\beta_{\mathbf{P}} \circ \lambda_{\mathbf{P}} = 0$, then $\mathbf{P} \in \mathcal{M}_k$.*

Proof. Consider $\beta_{\mathbf{P}} : 7\mathcal{O}_{\mathbb{P}^2} \to L_k$ and $\lambda_{\mathbf{P}} : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}$ such that $\beta_{\mathbf{P}} \circ \lambda_{\mathbf{P}} = 0$. It follows that we have a surjective map $\gamma_{\mathbf{P}} : \mathcal{G}_{\mathbf{P}} = \operatorname{coker} \lambda_{\mathbf{P}} \to L_k$. Now since L_k is an injective sheaf for $k = 9, 11$, we get in these cases a surjective map $\gamma'_{\mathbf{P}} : \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to L_k$. Its kernel is then a line bundle with known Chern class, hence the line bundle

$$H_k = \begin{cases} \mathcal{O}_{\mathbb{P}^2}(4) \text{ for } k = 11, \\ \mathcal{O}_{\mathbb{P}^2}(3) \text{ for } k = 9. \end{cases}$$

We can apply Corollary 5.1 $\square$

Corollary 5.5. *If there exists a surjective map $\beta_{\mathbf{P}} : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_p(3)$ for some $p \in \mathbb{P}^2$ such that $\beta_{\mathbf{P}} \circ \lambda_{\mathbf{P}} = 0$ then we have two possibilities:*

(1) *$\lambda_{\mathbf{P}}$ is degenerate at p and $\mathbf{P} \in \mathcal{M}_9$,*
(2) *$\lambda_{\mathbf{P}}$ is not degenerate at p and $\mathbf{P} \in \mathcal{M}_8$.*

Moreover for a general map $\beta_{\mathbf{P}} : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_p(3)$ a general $\lambda_{\mathbf{P}}$ satisfying $\beta_{\mathbf{P}} \circ \lambda_{\mathbf{P}} = 0$ is not degenerate at p.

Proof. As in the proof of Corollary 5.4, the map $\beta_{\mathbf{P}}$ induces a map $\gamma_{\mathbf{P}} : \mathcal{G}_{\mathbf{P}} = \operatorname{coker} \lambda_{\mathbf{P}} \to L_8 = \mathcal{I}_p(3)$. Now, $\gamma_{\mathbf{P}}$ extends to a map $\gamma'_{\mathbf{P}} : \mathcal{G}_{\mathbf{P}}^{\vee\vee} \to \mathcal{O}_{\mathbb{P}^2}(3)$. The latter is either surjective or not. If it is surjective we have $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$, implying $\mathbf{P} \in \mathcal{M}_9$. When $\gamma'_{\mathbf{P}}$ is not surjective then $\gamma'_{\mathbf{P}}$ maps onto $\mathcal{I}_p(3)$. Now, $\ker \gamma'_{\mathbf{P}}$ is a line bundle with first Chern class of degree 2, thus $\mathcal{O}_{\mathbb{P}^2}(2)$. It follows that $\mathcal{G}_{\mathbf{P}}^{\vee\vee} = T_{\mathbb{P}^2}(1)$, which in its turn implies by Proposition 5.1 that $\mathbf{P} \in \mathcal{M}_8$. Finally, $\gamma'_{\mathbf{P}}$ is surjective if and only if p is in the support of the sheaf $\mathcal{C}_{\mathbf{P}} = \operatorname{coker}(\mathcal{G}_{\mathbf{P}} \to \mathcal{G}_{\mathbf{P}}^{\vee\vee})$. The latter is equivalent to $\lambda_{\mathbf{P}}$ being degenerate at p. $\square$

Remark 5.6. In order to characterize geometrically $\mathcal{M}_k$ for $k \leq 7$ one can use the same approach as above. For example if $k = 7$ the exact sequence

$$0 \to 5\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\lambda_7} 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_7^{\vee\vee} \to C_7 \to 0$$

shows that $\mathcal{G}_7^{\vee\vee}$ is a rank 2 vector bundle with $c_1(\mathcal{G}_7^{\vee\vee}) = -5H$ and $c_2(\mathcal{G}_7^{\vee\vee}) = 8$ such that $\mathcal{G}_7^{\vee\vee}$ admits a section vanishing in a 0-dimensional scheme Z_7 of length 8 satisfying the Cayley–Bacharach property for quadrics, i.e. no subscheme Z_7 of length 7 lies on a conic. We get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}^{\vee\vee} \to I_{Z_7}(5) \to 0$$

and deduce that $\mathcal{G}^{\vee\vee}(-2)$ has a 3-dimensional space of sections whose general element vanishes in codimension 2. This gives the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}^{\vee\vee}(-2) \to I_{K_2}(1) \to 0$$

for some scheme K_2 of length 2. After tensoring by $\mathcal{O}_{\mathbb{P}^2}(2)$ we get

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{G}^{\vee\vee} \to I_{K_2}(3) \to 0.$$

The surjection in that sequence is used to obtain the map $\beta_7 : 7\mathcal{O}_{\mathbb{P}^2} \to I_{K_2}(3)$ such that $\beta_7 \circ \lambda_7 = 0$. Everything being general, the map β_7 will be a surjection. For the inverse direction starting from a surjection $\beta_7 : 7\mathcal{O}_{\mathbb{P}^2} \to I_{K_2}(3)$ a general map $\lambda_7 : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}$ satisfying $\beta_7 \circ \lambda_7 = 0$ will correspond to an element in $\mathcal{M}_7$.

Remark 5.7. Note that $\mathcal{M}_{10} \neq \emptyset$. In fact, elements of $\mathcal{M}_{10}$ correspond to maps $\lambda : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}$ such that there exists a commutative diagram

$$\begin{tikzcd}
5\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "\lambda"] & 7\mathcal{O}_{\mathbb{P}^2} \arrow[d] \\
4\mathcal{O}_{\mathbb{P}^2(-1)} \arrow[u] \arrow[r, "0"] & 2\mathcal{O}_{\mathbb{P}^2}
\end{tikzcd}$$

with vertical arrows being embeddings. Indeed, the degeneracy locus of a general such λ is equal to the degeneracy locus of the restricted map $4\mathcal{O}_{\mathbb{P}^2}(-1) \to 5\mathcal{O}_{\mathbb{P}^2}$ which is a scheme of codimension 2 and degree 10. To prove that general elements of $\mathcal{M}_{10}$ arise in this way we proceed similarly to the cases $k = 7, 8, 9, 11$, i.e. if $\mathbf{P} \in \mathcal{M}_{10}$ then we have an exact sequence

$$0 \to 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}^{\vee\vee} \to \mathcal{C}_{10} \to 0$$

and it follows that a general section of $\mathcal{G}^{\vee\vee}$ vanishes in a subscheme Z_5 of dimension 0 and length 5 satisfying the Cayley–Bacharach property, hence contained in a line. We hence have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}^{\vee\vee} \to \mathcal{I}_{Z_5}(5) \to 0.$$

But $\mathcal{I}_{Z_5}(5)$ has the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(4) \to \mathcal{I}_{Z_5}(5) \to 0.$$

The map $\mathcal{G}^{\vee\vee} \to \mathcal{I}_{Z_5}(5)$ inducing a map $\mathcal{G} \to \mathcal{I}_{Z_5}(5)$ gives rise to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 5\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "\lambda"] \arrow[d] & 7\mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2(-1)} \arrow[r] & \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(4) \arrow[r] & \mathcal{I}_{Z_5}(5) \arrow[r] & 0
\end{tikzcd}$$

This induces

$$\begin{tikzcd}
5\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, "\lambda"] & 7\mathcal{O}_{\mathbb{P}^2} \arrow[d] \\
4\mathcal{O}_{\mathbb{P}^2(-1)} \arrow[u] \arrow[r, "0"] & \mathcal{O}_{\mathbb{P}^2}
\end{tikzcd}$$

Now, in case λ is general the degeneracy locus of λ is equal to the degeneracy locus of the restricted map

$$4\mathcal{O}_{\mathbb{P}^2}(-1) \to 6\mathcal{O}_{\mathbb{P}^2}$$

the latter by an analogous argument is a scheme of dimension 0 and length 10 only if it factorizes through a map

$$4\mathcal{O}_{\mathbb{P}^2}(-1) \to 5\mathcal{O}_{\mathbb{P}^2},$$

which implies our general form of elements of $\mathcal{M}_{10}$.

Remark 5.8. Observe that, a priori, for each k the set $\mathcal{M}_k$ may have several components. Theorem 1.1 concerns the components for which the map $\beta_{\mathbf{P}}$ is general.

Remark 5.9. For $k = 11$ each element $\mathbf{P} \in \mathcal{M}_k$ is of one of two types. One type has $\beta_{\mathbf{P}} : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)$ surjective. The second type has $\beta_{\mathbf{P}}$ factorizing through the ideal of a point. In the latter case $\ker \beta_{\mathbf{P}} = 5\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. Since $\lambda_{\mathbf{P}}$ factorizes through the embedding $\ker \beta_{\mathbf{P}} \to 7\mathcal{O}_{\mathbb{P}^2}$, it must decompose as $\lambda_{\mathbf{P}} = \lambda_1 \oplus \lambda_2$ with $\lambda_1 : 4\mathcal{O}_{\mathbb{P}^2}(-1) \to 5\mathcal{O}_{\mathbb{P}^2}$ and $\lambda_2 : \mathcal{O}_{\mathbb{P}^2}(-1) \to 2\mathcal{O}_{\mathbb{P}^2}$. Such maps $\lambda_{\mathbf{P}}$ do not however give rise to families of Calabi–Yau threefolds because the degeneracy locus of no skew-symmetric map $E^{\vee}_{\lambda_{\mathbf{P}}}(-1) \to E_{\lambda_{\mathbf{P}}}$ is of expected codimension.

Remark 5.10. In the family of Calabi–Yau threefolds of degree 17 with $k = 9$ we can identify a subfamily obtained in the following way. Consider

$$\varphi : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(3)$$

defined by Pfaffians of a skew-symmetric matrix. The syzygies of this map recover the skew symmetric map

$$\theta : 7\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}.$$

Considering the general map

$$\iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}(-2) \oplus 7\mathcal{O}_{\mathbb{P}^2}(-1)$$

we find that

$$\theta \circ \iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}$$

defines a Calabi–Yau threefold of degree 17 with $k = 9$ (i.e. via $\lambda_{\mathbf{P}} = \theta \circ \iota$). However, one can check that in this way we can get only special Calabi–Yau threefolds with $k = 9$.

5.1. Proof of Theorem 1.1

Let $\mathbf{P} \in \mathcal{B}_k \subset G(16, V_3 \otimes V_7)$, i.e. one of the following holds:

(1) $k = 11$ and $L_{\mathbf{P}}$ contains the graph $\Gamma_{v_1} \subset \mathrm{Seg}$ of a linear embedding $v_1 : \mathbb{P}^2 \to \mathbb{P}^6$;
(2) $k = 9$ and $L_{\mathbf{P}}$ contains the graph $\Gamma_{v_2} \subset \mathrm{Seg}$ of a second Veronese embedding $v_2 : \mathbb{P}^2 \to \mathbb{P}^6$;
(3) $k = 8$ and $L_{\mathbf{P}}$ contains the closure of the graph Γ_{v_3} of a birational map $v_3 : \mathbb{P}^2 \to \mathbb{P}^6$ defined by a system of cubics passing through some point.

Then clearly the assumptions of Corollaries 5.5 and 5.4 also hold. Hence, $\mathbf{P} \in \mathcal{M}_k$ by these corollaries. Moreover, Corollary 5.2 implies that $\mathcal{B}_k$ is Zariski open in $\mathcal{M}_k$. Using Macaulay 2 (with [12]), in each case, among all $\mathbf{P} \in \mathcal{B}_k$ we can find elements of $\tilde{\mathcal{M}}_k$. We conclude that $\mathcal{B}_k \cap \mathcal{M}_k$ is a nonempty open subset of $\mathcal{M}_k$. □

5.2. Another construction

Note that we also have an alternative way to describe $\mathbf{P} \in \mathcal{M}_k$. For $k = 8, 9, 11$, consider the projectivization of the bundle $\mathcal{T}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$ or $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(4)$ respectively. The bundle is embedded in $\mathbb{P}^2 \times \mathbb{P}^l \subset \mathbb{P}^{3l+2}$ with $l = 14, 15, 17$ respectively. For each of these cases, consider the projection $\mathbb{P}^2 \times \mathbb{P}^l \to \mathbb{P}^2 \times \mathbb{P}^6$ from the spaces spanned by $\mathbb{P}^2 \times \mathbb{P}^{k-1}$, where $\mathbb{P}^{k-1}$ is spanned by k general points on the image of the projectivization of the bundle on $\mathbb{P}^l$. We obtain in this way a $\mathbb{P}^{15} \subset \mathbb{P}^{20}$ containing k special fibers, which are the proper transforms of the points from which we projected. From Proposition 5.1 a $\mathbb{P}^{15}$ in $\mathcal{M}_k$ must appear as the span of the projection of the corresponding bundle. The reason why the projection must be performed from points lying on the projectivization is the appearance of special fibers.

5.3. Dimensions of $\mathcal{M}_k$ (cf. [19, Prop. 3.5])

For each $k = 8, 9, 11$ we compute the dimension of $\mathcal{B}_k$ from Theorem 1.1, which is an open subset of an irreducible closed subvariety of $G(16, W_3 \otimes P_7)$.

Lemma 5.11. *The dimension of the space $\mathcal{B}_k$ for $k = 8,\ 9,\ 11$ is given by:*

$$\dim \mathcal{B}_k = \begin{cases} 72 \text{ for } k = 8, \\ 71 \text{ for } k = 9, \\ 70 \text{ for } k = 11. \end{cases}$$

Proof. We consider each case separately:

(1) The dimension of the space parameterizing graphs of linear embeddings $\mathbb{P}^2 \to \mathbb{P}^6$ is 20. The graph of each linear embedding spans a $\mathbb{P}^5$. The Grassmannian parameterizing all $\mathbb{P}^{15}$ containing a fixed $\mathbb{P}^5$ is of dimension 50. Since a general $\mathbb{P} \in \mathcal{B}_{11}$ contains only one graph of a linear map, the dimension of $\mathcal{B}_{11}$ is 70.

(2) The dimension of the space of quadratic embeddings $\mathbb{P}^2 \to \mathbb{P}^6$ is 41. The graph of each such embedding spans a $\mathbb{P}^9$. Hence the dimension of the family of $\mathbb{P}^{15}$ containing the graph of a fixed Veronese embedding is 30. Now, since a general $\mathbb{P} \in \mathcal{B}_9$ contains only one graph of a Veronese embedding, the dimension of $\mathcal{B}_9$ is 71.

(3) For a chosen point $p \in \mathbb{P}^2$ we have a 62-dimensional family of graphs. A general graph spans a $\mathbb{P}^{13}$, hence we have a 10-dimensional family of $\mathbb{P}^{15}$'s for each of them. Since for a fixed $p \in \mathbb{P}^2$ there is a unique graph, we conclude that $\mathcal{B}_8$ is of dimension 72. □

5.4. Dimension of Tonoli families

The aim of this subsection is to estimate the Hodge numbers of Tonoli Calabi–Yau threefolds of degree 17 by proving the following theorem:

Theorem 5.12. *Let X^k_{17} be a Tonoli Calabi–Yau threefold defined as a general Pfaffian variety associated to a vector bundle $E_{\mathbf{P}}$ for some general $\mathbf{P} \in \mathcal{B}_k$, where $\mathcal{B}_k$ is as in Theorem 1.1 and $k \in \{8, 9, 11\}$. Then*

$$h^{1,2}(X^k_{17}) \geq \dim \mathcal{B}_k - 57 + k.$$

Remark 5.13. We have not been able to compute the exact value of $h^{1,2}$, but we expect that we have equality in the inequality above.

We shall need some preliminary results. Let $X \subset \mathbb{P}^6$ be a Calabi–Yau threefold. Denote by $\mathcal{H}_X$ the component of the Hilbert scheme $Hilb_{X|\mathbb{P}^6}$ containing X. The tangent space to $\mathcal{H}_X$ at X is naturally identified to $H^0(\mathcal{N}_{X|\mathbb{P}^6})$. Consider the following map locally around $X \in \mathcal{H}_X$:

$$\mathcal{H}_X \xrightarrow{\pi} Def(X),$$

where $Def(X)$ is the local deformation space with tangent space $H^1(T_X)$.

Lemma 5.14. *Let $X \subset \mathbb{P}^6$ be a Calabi–Yau threefold. Then the natural map of tangent spaces $\tau \colon H^0(\mathcal{N}_{X|\mathbb{P}^6}) \to H^1(T_X)$ is a surjection. In particular the local deformations of X can be embedded into $\mathbb{P}^6$.*

Proof. The statement follows from the long exact cohomology sequence constructed from

$$0 \to T_X \to T_{\mathbb{P}^6}|_X \to \mathcal{N}_{X|\mathbb{P}^6} \to 0$$

and from the Euler sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to T_{\mathbb{P}^6}|_X \to 0.$$

We also infer that $H^0(T_{\mathbb{P}^6}|_X) = H^0(T_{\mathbb{P}^6})$ is the kernel of τ. □

Let us now introduce some notation fitting with the notation in Section 3. If X is a Tonoli Calabi–Yau threefold we shall denote by $\mathcal{T}_X$ the component containing X of the appropriate Tonoli family of Pfaffian varieties, and $\mathcal{D}_X$ will stand for the image of the Tonoli family $\mathcal{T}_X$ under the forgetful map to $Hilb_{X|\mathbb{P}^n}$. In particular $\mathcal{D}_X \subset \mathcal{H}_X$.

Proposition 5.15. *Let $X \subset \mathbb{P}^6$ be a Tonoli Calabi–Yau threefold. Then $h^{1,2}(X) \geq \dim \mathcal{H}_X - 48 \geq \dim \mathcal{D}_X - 48$.*

Proof. It follows from the proof of Lemma 5.14 that $h^{1,2}(X) = \dim Def(X) \geq \dim \mathcal{H}_X - h^0(T_{\mathbb{P}^6}) \geq \dim \mathcal{D}_X - 48$. $\square$

We can now prove Theorem 5.12.

Proof of Theorem 5.12. The theorem is a direct consequence of Propositions 5.15 and 3.7. $\square$

Corollary 5.16. *The deformation families of the constructed Tonoli Calabi–Yau threefolds are of dimensions* ≥ 23, 23, 24 *for* $k = 8, 9, 11$ *respectively.*

Proof. We apply Theorem 5.12 and Lemma 5.11. $\square$

Corollary 5.17. *The rank of the Picard group of the family of Tonoli Calabi–Yau threefolds of degree* 17 *with* $k = 11$ *is not smaller than* 2*.*

Proof. Since the degree of each Calabi–Yau threefold X in the family is 17, by the double point formula ([15, p. 467]) we get $2(h^{1,1}(X) - h^{1,2}(X)) = -44$. But by Theorem 5.12 we have $h^{1,2}(X) \geq 24$. Thus $h^{1,1}(X) \geq 2$. $\square$

Remark 5.18. We believe that the Picard number of Tonoli Calabi–Yau threefolds of degree 17 with $k = 11$ is in fact equal to 2 and that in the cases with $k = 9, 8$ it is equal to 1, but we cannot prove it at the moment. It would be interesting to study the example with $k = 11$ from the point of view of rationality of the rays of the Kähler cone as in [14].

Remark 5.19. The Tonoli Calabi–Yau threefold of degree 17 with $k = 11$ constructed above shows that the Barth–Lefschetz theorem cannot be generalized to subcanonical threefolds in $\mathbb{P}^6$. Another example of this phenomenon is the del Pezzo threefold of degree 7 projected to $\mathbb{P}^6$. It is obtained as the projection to $\mathbb{P}^6$ of the second Veronese embedding of $\mathbb{P}^3$ from a $\mathbb{P}^2$ intersecting it in one point.

6. Descriptions of del Pezzo surfaces of degree 8 in $\mathbb{P}^5$

We shall now describe the Pfaffian resolutions of del Pezzo surfaces of degree $d = 8$. Our approach will be parallel to the case of Tonoli Calabi–Yau threefolds. We shall look for modules M with Hilbert function $(3, 4, 0, \dots)$ for gradation starting from -1 which admit a minimal resolution

$$14\mathfrak{S}_{\mathbb{P}^5} \to 3\mathfrak{S}_{\mathbb{P}^5}(1) \to M \to 0,$$

where $\mathfrak{S}_{\mathbb{P}^5}$ is the homogeneous coordinate ring of $\mathbb{P}^5$. Observe that for such modules $c_1(Syz^1(M)) = -3$ and $\mathrm{rk}(Syz^1(M)) = 11$. It follows by Formula (1.1) for the canonical class and the formula for the degree (see Lemma 4.4) that if $\bigwedge^2(Syz^1(M))(1)$ admits a section defining a smooth Pfaffian variety D, then D must be a del Pezzo surface of degree 8 in its anti-canonical embedding (composed with a projection). Moreover, by Remark 2.2, if the shifted Hartshorne–Rao module of a smooth surface D is isomorphic to some M as above then D is defined by a Pfaffian variety associated to $Syz^1(M)$. To such a minimal presentation of M one associates an embedding

$$\mathbb{P}^{13} \to \mathbb{P}^{17} = \langle \mathbb{P}^2 \times \mathbb{P}^5 \rangle.$$

In this case, the intersection $\mathbb{P}^{13} \cap (\mathbb{P}^2 \times \mathbb{P}^5) = \mathbb{P}(\mathcal{G})$ can be seen as the projectivization of a sheaf $\mathcal{G}$ on $\mathbb{P}^2$ given by the cokernel of the embedding

$$4\mathcal{O}_{\mathbb{P}^2}(-1) \to 6\mathcal{O}_{\mathbb{P}^2}$$

corresponding to the four linear equations defining the $\mathbb{P}^{13}$.

We can now adapt the notation from Section 5 to the case of del Pezzo surfaces. $\mathcal{M}_k^D = \{\mathbb{P} \in G(14, 18) \mid \mathrm{coker}(\mathcal{G} \to \mathcal{G}^{\vee\vee})$ is the structure sheaf of a scheme of length $k\}$.

Proposition 6.1. *If $\mathbf{P} \in G(14, 18)$ is a $\mathbb{P}^{15}$ in $\mathbb{P}(V_3 \otimes V_6)$ then $\mathbf{P} \in \mathcal{M}_k^D$ if and only if $\mathcal{G}_{\mathbf{P}}^{\vee\vee}$ is a rank two vector bundle isomorphic to:*

(1) $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$,
(2) $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$,

for $k = 6$, 7 respectively.

Proof. The proof is completely analogous to the proof of Proposition 5.1. $\square$

Now, still analogously to the case of Tonoli Calabi–Yau threefolds, we deduce the following corollaries.

Corollary 6.2. *In the notation of Proposition 6.1, if $\mathbf{P} \in \mathcal{M}_k^D$ then there exists a map $\beta_{\mathbf{P}}^D : 7\mathcal{O}_{\mathbb{P}^2} \to L_k^D$ surjective outside possibly a set of codimension at least 2, such that $\beta_{\mathbf{P}}^D \circ \lambda_{\mathbf{P}}^D = 0$ and L_k^D is one of the following sheaves:*

(1) $L_7^D = \mathcal{O}_{\mathbb{P}^2}(1)$,
(2) $L_6^D = \mathcal{O}_{\mathbb{P}^2}(2)$.

Corollary 6.3. *Let $k \in \{6, 7\}$. If there exists a surjective map $\beta_{\mathbf{P}}^D : 7\mathcal{O}_{\mathbb{P}^2} \to L_k^D$ such that $\beta_{\mathbf{P}}^D \circ \lambda_{\mathbf{P}} = 0$, then $\mathbf{P} \in \mathcal{M}_k$.*

Proposition 6.4. *The Tonoli families $\mathcal{T}_{k,14,3,5}$ of Pfaffian varieties for $k = 6, 7$ are mapped via the forgetful map to open subsets in the two components of the Hilbert scheme of* *del Pezzo surfaces of degree 8 in $\mathbb{P}^5$ representing the two types $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ respectively.*

Proof. By the formulas for degree and canonical class and by semicontinuity it is clear that in this construction we obtain a family of del Pezzo surfaces. We then use Proposition 3.7 to compute the dimension of the family of constructed surfaces inside the Hilbert scheme and compare it with the dimension of the Hilbert scheme of del Pezzo surfaces. We obtain that our family gives a component of the Hilbert scheme. We finish the proof by checking one example in each case using [12]. □

Remark 6.5. Geometrically, to construct a vector bundle defining a general del Pezzo surface of type $\mathbb{P}^1 \times \mathbb{P}^1$, one considers a general $\mathbb{P}^{13} \subset \mathbb{P}^{17}$ containing the graph of a second Veronese embedding of the projective plane in $\mathbb{P}^2 \times \mathbb{P}^5 \subset \mathbb{P}^{17}$. To construct a vector bundle defining a general del Pezzo surface of type $\mathbb{F}_1$, one considers a general $\mathbb{P}^{13} \subset \mathbb{P}^{17}$ containing the graph of a linear embedding $\mathbb{P}^2 \to \mathbb{P}^5$ in $\mathbb{P}^2 \times \mathbb{P}^5 \subset \mathbb{P}^{17}$.

Remark 6.6. Observe that a general $\mathbb{P}^{13} \subset \mathbb{P}^{17}$ containing the graph of the second Veronese embedding in $\mathbb{P}^2 \times \mathbb{P}^5 \subset \mathbb{P}^{17}$ contains a one-parameter family of such graphs.

7. The analogy in degrees $(8, 17)$

Let us now finish the proof that the constructions of del Pezzo surfaces and Calabi–Yau threefolds of codimension 3 are related.

Proof of Theorem 1.3. It remains to handle the case of del Pezzo surface of degree $d_D = 8$ and Tonoli Calabi–Yau threefold of degree $d_X = 17$. On one side we have two families, on the other three. Let us start with a general del Pezzo surface of degree 8. Its shifted Hartshorne–Rao module defining the bundle E_D corresponds to a subspace of dimension 13 contained in $\mathbb{P}^{17} = \langle \mathbb{P}^2 \times \mathbb{P}^5 \rangle$ such that the intersection $\mathbb{P}^{13} \cap (\mathbb{P}^2 \times \mathbb{P}^5)$ contains either the graph of a linear map $\mathbb{P}^2 \to \mathbb{P}^5$ or the graph of a second Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$. Such a subspace is clearly the projection of a space $\mathbb{P}^{13} \subset \mathbb{P}^{20} \supset \mathbb{P}^2 \times \mathbb{P}^6$ with the analogous property, i.e. $\mathbb{P}^{13} \cap (\mathbb{P}^2 \times \mathbb{P}^6)$ contains either the graph of a linear map $\mathbb{P}^2 \to \mathbb{P}^6$ or the graph of a second Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^6$. The general such choice of extension defines a bundle E'_D on $\mathbb{P}^6$. A general extension between this bundle and $2\mathcal{O}_{\mathbb{P}^6}$ corresponds to a space $\mathbb{P}^{15} \subset \mathbb{P}^{20}$ containing the $\mathbb{P}^{13}$, i.e., $\mathbb{P}^{15} \cap (\mathbb{P}^2 \times \mathbb{P}^6)$ contains either the graph of a linear map $\mathbb{P}^2 \to \mathbb{P}^6$ or the graph of a second Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^6$. In particular, it corresponds to an element of $\mathcal{B}_9$ or $\mathcal{B}_{11}$ (notation as in Theorem 1.1). To prove that the corresponding bundle defines a Calabi–Yau threefold, we observe that a general element of $\mathcal{B}_9$ or $\mathcal{B}_{11}$ arises in this way. Indeed, for a general $\mathbf{P} \in \mathcal{B}_9$ or $\mathcal{B}_{11}$ the corresponding $L_{\mathbf{P}}$ contains the graph of a Veronese or linear embedding $\mathbb{P}^2 \to \mathbb{P}^6$ each of which spans a space of dimension smaller than 13. Hence $L_{\mathbf{P}}$ contains a $\mathbb{P}^{13}$ also containing these graphs. Finally the image of this $\mathbb{P}^{13}$ via a general projection

$$\mathbb{P}^2 \times \mathbb{P}^6 \subset \mathbb{P}^{20} \to \mathbb{P}^{17} \supset \mathbb{P}^2 \times \mathbb{P}^5$$

induced by a projection $\mathbb{P}^6 \to \mathbb{P}^5$ is a $\mathbb{P}^{13}$ defining a point in $\mathcal{M}_6^D$ or in $\mathcal{M}_7^D$. The proposition is hence proven for Calabi–Yau threefolds with $k = 9$ or 11.

Let us now consider the case of Calabi–Yau threefolds of degree 17 with $k = 8$. In this case we consider a $\mathbb{P}^{15}$ such that $\mathcal{G}^{\vee\vee} = T_{\mathbb{P}^2}(1)$; the latter admits a 2-dimensional family of surjections onto $I_p(3)$ parameterized by $p \in \mathbb{P}^2$. The appropriate composite map defines a $\mathbb{P}^{13}$ spanned by the graph of a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ defined by a system of cubics passing through a point. We claim that the projection of this $\mathbb{P}^{13} \subset \langle \mathbb{P}^2 \times \mathbb{P}^6 \rangle$ onto $\mathbb{P}^{17} = \langle \mathbb{P}^2 \times \mathbb{P}^5 \rangle$ is a general element of $\mathcal{M}_6^D$ and hence defines a del Pezzo surface D_8^1. Indeed, we just observe that the projected $\mathbb{P}^{13}$ is associated to a map $4\mathcal{O}_{\mathbb{P}^2}(-1) \to 6\mathcal{O}_{\mathbb{P}^2}$ whose cokernel admits a surjection on $I_p(3)$. We then compute the Chern classes of this cokernel and deduce that we are in $\mathcal{M}_6^D$. $\square$

Remark 7.1. In Theorem 1.3, we relate Calabi–Yau threefolds to del Pezzo surfaces or more precisely appropriate vector bundles defining these varieties via the Pfaffian construction in two steps, passing through a vector bundle E'_D on $\mathbb{P}^6$. One might wonder if there is a variety given by Pfaffians of this bundle. By Formula (1.1) and degree formulas such a variety if smooth would be a Fano threefold of index 2 and degree $d - 9$ in $\mathbb{P}^6$. And indeed for Calabi–Yau threefolds of degree $d \leq 16$ the bundle E'_D defines a family of such smooth Fano threefolds. For $d = 17$ the situation is different. The only Fano threefolds of index 2 and degree 8 in $\mathbb{P}^6$ are projections of second Veronese embeddings of $\mathbb{P}^3$. Now, using our methods, one can easily check that such a second Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^6$ is associated to a $\mathbb{P}^{13}$ corresponding to a skew-symmetric map θ as in Remark 5.10. The restriction of the associated bundle to a general $\mathbb{P}^5$ defines a del Pezzo surface of type $\mathbb{F}_1$, whereas a general extension bundle with $2\mathcal{O}_{\mathbb{P}^6}$ corresponding to a $\mathbb{P}^{15}$ containing our $\mathbb{P}^{13}$ defines a Calabi–Yau threefold from the special family discussed in Remark 5.10. We hence do not recover the whole family of Tonoli Calabi–Yau threefolds of degree 17 with $k = 9$. In the general case for any $k = 8, 9, 11$ the Pfaffians associated to a general section $s \in H^0(\bigwedge^2 E'_D)$ do not define a variety of expected codimension. Only after restricting to a general $\mathbb{P}^5$, do appropriate sections appear.

8. Problems

Assuming that the relation observed in Theorem 1.3 between Calabi–Yau threefolds in $\mathbb{P}^6$ and del Pezzo surfaces in $\mathbb{P}^5$ follows from a more general phenomenon it is natural to make the following conjecture.

Problem 8.1. *There are no Calabi–Yau threefolds of degree $d \geq 19$ in $\mathbb{P}^6$.*

The most interesting case to be studied at the moment is the case of Calabi–Yau threefolds of degree 18. Since there is a del Pezzo surface of degree 9, we can try to use it to construct Calabi–Yau threefolds of degree 18 in $\mathbb{P}^6$ or canonical surfaces of degree 18 in $\mathbb{P}^5$. Indeed, in [10], using the construction of del Pezzo surfaces of degree 9, we construct a family of canonical surfaces of degree 18 in $\mathbb{P}^5$. We have so far been unable to find the description of a general such surface and unable to prove the existence of Calabi–Yau threefolds of degree 18 in $\mathbb{P}^6$. It seems that, to solve any of these two problems, the key is to find a geometric classification of all bundles on $\mathbb{P}^5$ which define, by the Pfaffian construction, del Pezzo surfaces of degree 9 in $\mathbb{P}^5$. We plan to address this problem in a subsequent paper. Note that F. Catanese [3] has recently constructed a surface of general type with irregularity 0 canonically embedded in $\mathbb{P}^5$ as a surface of degree 24. However, we can show that this surface is not a hyperplane section of a Calabi–Yau threefold in $\mathbb{P}^6$ since it is contained in a quadric (cf. [11]).

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