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## Braid actions on quantum toroidal superalgebras

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## ABSTRACT

We prove that the quantum toroidal algebras  $\mathcal{E}_{\mathbf{s}}$  associated with different root systems  $\mathbf{s}$  of  $\mathfrak{gl}_{m|n}$  type are isomorphic. We also show the existence of Miki automorphism of  $\mathcal{E}_{\mathbf{s}}$ , which exchanges the vertical and horizontal subalgebras.

To obtain these results, we establish an action of the toroidal braid group on the direct sum  $\bigoplus_{\mathbf{s}} \mathcal{E}_{\mathbf{s}}$  of all such algebras.

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## 1. Introduction

In this paper, we continue our study of the quantum toroidal algebras associated with the Lie superalgebra  $\mathfrak{gl}_{m|n}$  initiated in [2].

The root systems of  $\mathfrak{sl}_{m|n}$  are parameterized by sequences  $\mathbf{s} = (s_1, \dots, s_{m+n})$ , where  $s_i = \pm 1$ , and 1 occurs  $m$  times,  $-1$  occurs  $n$  times. We denote the algebra  $\mathfrak{sl}_{m|n}$  given in parity  $\mathbf{s}$  by  $\mathfrak{sl}_{\mathbf{s}}$ . In [2], we introduced the quantum toroidal algebra  $\mathcal{E}_{m|n}$  corresponding to the standard parity  $(1, \dots, 1, -1, \dots, -1)$ . In this paper, we define and study the quantum toroidal algebra  $\mathcal{E}_{\mathbf{s}}$  associated with an arbitrary parity  $\mathbf{s}$ , see Definition 4.1.

The idea for the definition is already described in [2]: we require  $\mathcal{E}_{\mathbf{s}}$  to have vertical subalgebra  $U_q^{ver} \widehat{\mathfrak{sl}_{\mathbf{s}}}$ , given in the current generators, and the horizontal subalgebra

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$U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ , given in the Chevalley generators, both in parity  $\mathbf{s}$ . In addition, we want our construction to be invariant under rotations  $\widehat{\tau}$  of the Dynkin diagram which connects  $\mathcal{E}_{\mathbf{s}}$  with  $\mathcal{E}_{\tau\mathbf{s}}$ , where  $\tau\mathbf{s} = (s_2, \dots, s_{m+n}, s_1)$ . The algebra  $\mathcal{E}_{\mathbf{s}}$  depends on parameters  $q_1, q_2, q_3$ , subject to  $q_1 q_2 q_3 = 1$  with  $q^2 = q_2$ .

It is natural to expect that all algebras  $\mathcal{E}_{\mathbf{s}}$  should be isomorphic. In this paper, we prove that this is indeed so, see Corollary 5.2. Similar statements are well known, see [12] for the case of quantum affine superalgebras, [10] for the case of super Yangians.

For the proof, we establish an action of the toroidal braid group  $\widehat{\mathcal{B}}_{m+n}$  on  $\mathcal{E}_{\bullet} = \bigoplus_{\mathbf{s}} \mathcal{E}_{\mathbf{s}}$ . The group  $\widehat{\mathcal{B}}_{m+n}$  is generated by  $\widehat{T}_i, \widehat{\mathcal{Y}}_j, \widehat{\tau}$ ,  $i = 1, \dots, m+n-1$ ,  $j = 0, \dots, m+n-1$ . We already have  $\widehat{\tau}$ . The automorphisms  $\widehat{\mathcal{Y}}_j$  are given by explicit formulas, see (4.28), (5.17). The main issue is the definition of  $\widehat{T}_1$ . We follow the logic of [8]. We have an action of the extended affine braid group  $\mathcal{B}_{m+n}$  on  $U_q \widehat{\mathfrak{sl}}_{\bullet} = \bigoplus_{\mathbf{s}} U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$ , see [12]. It turns out that the maps  $T_{1,\mathbf{s}} : U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow U_q^{ver} \widehat{\mathfrak{sl}}_{\sigma_1 \mathbf{s}}$  and  $T_{1,\mathbf{s}} : U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow U_q^{hor} \widehat{\mathfrak{sl}}_{\sigma_1 \mathbf{s}}$  agree on the common part  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} = U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}} \cap U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and give rise to the map of the whole algebra  $\widehat{T}_{1,\mathbf{s}} : \mathcal{E}_{\mathbf{s}} \rightarrow \mathcal{E}_{\sigma_1 \mathbf{s}}$ , where  $\sigma_1 \mathbf{s} = (s_2, s_1, s_3, \dots, s_{m+n})$ . The action of  $\widehat{T}_1$  is not completely explicit, and we use various algebraic properties to check that it is well defined. In particular, using  $\widehat{\tau}$ , we are able to reduce the checking to computations in the vertical subalgebra, for which we can use the results of [12].

As a byproduct, we also obtain the Miki automorphism, see Theorem 5.9, which is central to the study of quantum toroidal algebras in the even case, see [8], [6]. The Miki automorphism is the highly non-explicit automorphism which maps vertical and horizontal subalgebras to each other. Note that the isomorphism from  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  in current realization to  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  in Chevalley realization is already not explicit. The Miki automorphism originates in the well known Fourier transform  $\Phi$  for toroidal braid group, see Lemma 5.6, which maps commutative generators  $\widehat{\mathcal{Y}}_i \in \widehat{\mathcal{B}}_{m+n}$  to Knizhnik-Zamolodchikov elements. The construction is as follows.

Let  $A \subset \widehat{\mathcal{B}}_{m+n}$  be the subgroup generated by  $\widehat{\mathcal{Y}}_i$  and  $\Phi(\widehat{\mathcal{Y}}_i)$ . As mentioned above, the vertical and horizontal algebras share a copy of the finite type quantum algebra  $U_q \widehat{\mathfrak{sl}}_{m|n}$ . Then  $A(U_q \widehat{\mathfrak{sl}}_{\mathbf{s}})$  generates  $\mathcal{E}_{\mathbf{s}}$ . Indeed,  $\widehat{\mathcal{Y}}_i$  acting on  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  generates the vertical subalgebra, while  $\Phi(\widehat{\mathcal{Y}}_i)$  acting on  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  generates the horizontal subalgebra, see Lemma 5.7.

Then, by definition, the Miki automorphism  $\psi$  maps

$$\psi(bg) = \Phi(b)g \quad (g \in U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}, b \in A).$$

In other words, when acting on  $\mathcal{E}_{\mathbf{s}}$ , the Fourier transform of the toroidal braid group is given as conjugation by the Miki automorphism, see Proposition 5.11. From the very construction,  $\psi(U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}) = U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$ . We also have  $\psi(U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}) = U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  since it is known that  $\Phi^2(\widehat{\mathcal{Y}}_i)$  is in the algebra generated by  $\widehat{\mathcal{Y}}_i$ .

In this paper, we construct the action of toroidal braid group and the Miki automorphism for the case  $m \neq n$  and  $m+n > 3$ .

The paper is organized as follows. In Section 2, we recall the definition of the quantum affine superalgebra  $U_q \widehat{\mathfrak{sl}}_{m|n}$  with any choice of parity. In Section 3, we recall the action of the extended affine braid group on  $U_q \widehat{\mathfrak{sl}}_{\bullet} = \oplus_{\mathbf{s}} U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  given in [12]. In Section 4, we introduce the quantum toroidal algebra  $\mathcal{E}_{\mathbf{s}}$  associated with  $\mathfrak{gl}_{m|n}$  for any choice of parity and give a few properties. In Section 5, we construct an action of the toroidal braid group on  $\mathcal{E}_{\bullet} = \oplus_{\mathbf{s}} \mathcal{E}_{\mathbf{s}}$  and the Miki automorphism. In Appendix A, we give an evaluation map using an action of the braid group of  $\mathfrak{sl}_{m+n}$  on a completion of  $U_q \widehat{\mathfrak{sl}}_{\bullet}$ .

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## 2. Quantum affine superalgebra $U_q \widehat{\mathfrak{sl}}_{m|n}$

In this section, we review definitions of the quantum affine algebra corresponding to the superalgebra  $\mathfrak{sl}_{m|n}$  and set up our notation.

### 2.1. Parities and root systems

We work over the field  $\mathbb{C}$ .

A superalgebra is a  $\mathbb{Z}_2$ -graded algebra  $A = A_0 \oplus A_1$ . Elements of  $A_0$  are called even and elements of  $A_1$  odd. We denote the parity of an element  $v \in A_i$  by  $|v| = i$ ,  $i \in \mathbb{Z}_2$ .

Fix  $m, n \in \mathbb{Z}_{\geq 0}$ , such that  $m \neq n$  and  $N = m + n \geq 3$ . We always consider various indices modulo  $N$ .

We consider the Lie superalgebras  $\mathfrak{sl}_{m|n}$  and  $\widehat{\mathfrak{sl}}_{m|n}$ . The set of Dynkin nodes are  $I = \{1, 2, \dots, N-1\}$  and  $\hat{I} = \{0, 1, \dots, N-1\}$ , respectively. It is well-known that there are different choices of the root system which lead to different Cartan matrices and different Dynkin diagrams. Such choices are parameterized by  $N$ -tuples of  $\pm 1$  with exactly  $m$  positive coordinates. Set

$$\mathcal{S}_{m|n} = \{(s_1, \dots, s_N) \mid s_i \in \{-1, 1\}, \#\{i \mid s_i = 1\} = m\}.$$

An element  $\mathbf{s} = (s_1, \dots, s_N) \in \mathcal{S}_{m|n}$  is called a *parity sequence*. The parity sequence of the form  $\mathbf{s} = (1, \dots, 1, -1, \dots, -1)$  is called the *standard parity sequence*.

Given a parity sequence  $\mathbf{s} \in \mathcal{S}_{m|n}$ , we have the Cartan matrix  $A^{\mathbf{s}} = (A_{i,j}^{\mathbf{s}})_{i,j \in I}$  and the affine Cartan matrix  $\hat{A}^{\mathbf{s}} = (A_{i,j}^{\mathbf{s}})_{i,j \in \hat{I}}$ , where

$$A_{i,j}^{\mathbf{s}} = (s_i + s_{i+1})\delta_{i,j} - s_i\delta_{i,j+1} - s_j\delta_{i+1,j} \quad (i, j \in \hat{I}). \quad (2.1)$$

Note that  $|\det A^{\mathbf{s}}| = |m - n|$  and  $\det \hat{A}^{\mathbf{s}} = 0$ .

Denote by  $\mathfrak{sl}_{\mathbf{s}}$  the superalgebra corresponding to Cartan matrix  $A^{\mathbf{s}}$ . Denote by  $\widehat{\mathfrak{sl}}_{\mathbf{s}}$  the superalgebra corresponding to Cartan matrix  $\hat{A}^{\mathbf{s}}$ . The superalgebras  $\mathfrak{sl}_{\mathbf{s}}$  are all isomorphic to  $\mathfrak{sl}_{m|n}$  and the superalgebras  $\widehat{\mathfrak{sl}}_{\mathbf{s}}$  to  $\widehat{\mathfrak{sl}}_{m|n}$ . By abuse of notation we often omit the suffix  $\mathbf{s}$  if the parity sequence is clear from the context.

Let  $P_{\mathbf{s}}$  be the integral lattice with basis  $\varepsilon_i$ ,  $i \in \hat{I}$ , and bilinear form given by

$$\langle \varepsilon_i | \varepsilon_j \rangle = s_i \delta_{i,j} \quad (i, j \in \hat{I}).$$

Let  $\Delta_{\mathbf{s}} = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} | i \in I\}$  be the set of simple roots of  $\mathfrak{sl}_{\mathbf{s}}$ , and let  $Q_{\mathbf{s}} = \oplus_{i \in I} \mathbb{Z} \alpha_i$  be the root lattice of  $\mathfrak{sl}_{\mathbf{s}}$ . Let also  $\delta$  be the null root of  $\widehat{\mathfrak{sl}}_{\mathbf{s}}$  satisfying  $\langle \delta | \delta \rangle = \langle \delta | \alpha_i \rangle = 0$ ,  $i \in I$ . Set  $\alpha_0 = \delta + \varepsilon_N - \varepsilon_1$ . Then,  $\hat{\Delta}_{\mathbf{s}} = \{\alpha_i | i \in \hat{I}\}$  is the set of simple roots of  $\widehat{\mathfrak{sl}}_{\mathbf{s}}$ . Note that  $\langle \alpha_i | \alpha_j \rangle = A_{i,j}^{\mathbf{s}}$ ,  $i, j \in \hat{I}$ , and the parity of the simple root  $\alpha_i$  is given by  $|\alpha_i| =: |i| = (1 - s_i s_{i+1})/2$ .

## 2.2. Quantum affine superalgebra $U_q \widehat{\mathfrak{sl}}_{m|n}$

Fix  $q \in \mathbb{C}^\times$  not a root of unity and let  $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ ,  $k \in \mathbb{Z}$ . We also use the notation  $[X, Y]_a = XY - (-1)^{|X||Y|} a YX$ . For simplicity, we write  $[X, Y]_1 = [X, Y]$ . The bracket  $[X, Y]_a$  satisfies the following Jacobi identity

$$[[X, Y]_a, Z]_b = [X, [Y, Z]_c]_{abc^{-1}} + (-1)^{|Y||Z|} c [[X, Z]_{bc^{-1}}, Y]_{ac^{-1}}. \quad (2.2)$$

Let  $\mathbf{s}$  be a parity sequence.

In the *Drinfeld-Jimbo realization*, the quantum affine superalgebra  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is generated by *Chevalley generators*  $e_i, f_i, t_i$ ,  $i \in \hat{I}$ . The parity of generators is given by  $|e_i| = |f_i| = |i| = (1 - s_i s_{i+1})/2$ , and  $|t_i| = 0$ .

The defining relations are as follows.

$$\begin{aligned} t_i t_j &= t_j t_i, & t_i e_j t_i^{-1} &= q^{A_{i,j}^{\mathbf{s}}} e_j, & t_i f_j t_i^{-1} &= q^{-A_{i,j}^{\mathbf{s}}} f_j, \\ [e_i, f_j] &= \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \\ [e_i, e_j] &= [f_i, f_j] = 0 & (A_{i,j}^{\mathbf{s}} = 0), \\ [e_i, [e_i, e_{i \pm 1}]] &= [f_i, [f_i, f_{i \pm 1}]] = 0 & (A_{i,i}^{\mathbf{s}} \neq 0), \\ [e_i, [e_{i+1}, [e_i, e_{i-1}]]] &= [f_i, [f_{i+1}, [f_i, f_{i-1}]]] = 0 & (mn \neq 2, A_{i,i}^{\mathbf{s}} = 0), \\ [e_{i+1}, [e_{i-1}, [e_{i+1}, [e_{i-1}, e_i]]]] &= [e_{i-1}, [e_{i+1}, [e_{i-1}, [e_{i+1}, e_i]]]] & (mn = 2, A_{i,i}^{\mathbf{s}} \neq 0), \\ [f_{i+1}, [f_{i-1}, [f_{i+1}, [f_{i-1}, f_i]]]] &= [f_{i-1}, [f_{i+1}, [f_{i-1}, [f_{i+1}, f_i]]]] & (mn = 2, A_{i,i}^{\mathbf{s}} \neq 0), \end{aligned}$$

where  $\llbracket X, Y \rrbracket = [X, Y]_{q^{-\langle \beta | \gamma \rangle}}$  if  $X, Y$  have weights  $\beta, \gamma \in Q_{\mathbf{s}}$ , i.e., if  $t_i X t_i^{-1} = q^{\langle \alpha_i | \beta \rangle}$  and  $t_i Y t_i^{-1} = q^{\langle \alpha_i | \gamma \rangle}$  for  $i \in I$ .

The element  $t_0 t_1 \dots t_{N-1}$  is central.

The subalgebra of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  generated by  $e_i, f_i, t_i$ ,  $i \in I$ , is isomorphic to the quantum superalgebra  $U_q \mathfrak{sl}_{\mathbf{s}}$ .

The superalgebra  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  in the Drinfeld-Jimbo realization has a  $\mathbb{Z}^N$ -grading given by

$$\deg^h(x) = \left( \deg_0^h(x), \deg_1^h(x), \dots, \deg_{N-1}^h(x) \right), \quad (2.3)$$

where

$$\deg_i^h(e_j) = \delta_{i,j}, \quad \deg_i^h(f_j) = -\delta_{i,j}, \quad \deg_i^h(t_j) = 0 \quad (i, j \in \hat{I}).$$

In the *new Drinfeld realization*, the quantum affine superalgebra  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is generated by *current generators*<sup>1</sup>  $x_{i,r}^{\pm}, h_{i,r}, k_i^{\pm 1}, c^{\pm 1}$ ,  $i \in I$ ,  $r \in \mathbb{Z}'$ . Here and below, we use the following convention:  $r \in \mathbb{Z}'$  means  $r \in \mathbb{Z}$  if  $r$  is an index of a non-Cartan current generator  $x_{i,r}^{\pm}$ , and  $r \in \mathbb{Z}'$  means  $r \in \mathbb{Z} \setminus \{0\}$  if  $r$  is an index of a Cartan current generator  $h_{i,r}$ .

The parity of generators is given by  $|x_{i,r}^{\pm}| = |i| = (1 - s_i s_{i+1})/2$ , and all remaining generators have parity 0.

The defining relations are as follows.

$$\begin{aligned} c \text{ is central, } \quad k_i k_j &= k_j k_i, \quad k_i x_j^{\pm}(z) k_i^{-1} = q^{\pm A_{i,j}^{\mathbf{s}}} x_j^{\pm}(z), \\ [h_{i,r}, h_{j,s}] &= \delta_{r+s,0} \frac{[r A_{i,j}^{\mathbf{s}}]}{r} \frac{c^r - c^{-r}}{q - q^{-1}}, \\ [h_{i,r}, x_j^{\pm}(z)] &= \pm \frac{[r A_{i,j}^{\mathbf{s}}]}{r} c^{-(r \pm |r|)/2} z^r x_j^{\pm}(z), \\ [x_i^+(z), x_j^-(w)] &= \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta \left( c \frac{w}{z} \right) k_i^+(w) - \delta \left( c \frac{z}{w} \right) k_i^-(z) \right), \\ (z - q^{\pm A_{i,j}^{\mathbf{s}}} w) x_i^{\pm}(z) x_j^{\pm}(w) &+ (-1)^{|i||j|} (w - q^{\pm A_{i,j}^{\mathbf{s}}} z) x_j^{\pm}(w) x_i^{\pm}(z) = 0 \quad (A_{i,j}^{\mathbf{s}} \neq 0), \\ [x_i^{\pm}(z), x_j^{\pm}(w)] &= 0 \quad (A_{i,j}^{\mathbf{s}} = 0), \\ \text{Sym}_{z_1, z_2} \llbracket x_i^{\pm}(z_1), \llbracket x_i^{\pm}(z_2), x_{i \pm 1}^{\pm}(w) \rrbracket \rrbracket &= 0 \quad (A_{i,i}^{\mathbf{s}} \neq 0, \ i \pm 1 \in I), \\ \text{Sym}_{z_1, z_2} \llbracket x_i^{\pm}(z_1), \llbracket x_{i+1}^{\pm}(y), \llbracket x_i^{\pm}(z_2), x_{i-1}^{\pm}(w) \rrbracket \rrbracket \rrbracket &= 0 \quad (A_{i,i}^{\mathbf{s}} = 0, \ i \pm 1 \in I), \end{aligned}$$

where  $x_i^{\pm}(z) = \sum_{k \in \mathbb{Z}} x_{i,k}^{\pm} z^{-k}$ ,  $k_i^{\pm}(z) = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{r>0} h_{i,\pm r} z^{\mp r} \right)$ .

The superalgebra  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  in the new Drinfeld realization has a  $\mathbb{Z}^N$ -grading given by

$$\deg^v(x) = \left( \deg_1^v(x), \dots, \deg_{N-1}^v(x); \deg_{\delta}(x) \right), \quad (2.4)$$

where

<sup>1</sup> Our generators  $x_{i,r}^{\pm}, h_{i,r}, c^{\pm 1}$  correspond to  $x_{i,r}^{\pm}, K_{\delta}^{-r/2} h_{i,r}, K_{\delta}^{\pm 1}$  in [12]. In particular,  $k_i^+(z), k_i^-(z)$  correspond to  $\psi_i(K_{\delta}^{-1/2} z^{-1}), \phi_i(K_{\delta}^{-1/2} z)$  in [12].

$$\begin{aligned} \deg_i^v(x_{j,r}^\pm) &= \pm\delta_{i,j}, & \deg_i^v(k_j) &= \deg_i^v(h_{j,r}) = \deg_i^v(c) = 0 & (i, j \in I, r \in \mathbb{Z}'), \\ \deg_\delta(x_{i,r}^\pm) &= \deg_\delta(h_{i,r}) = r, & \deg_\delta(k_i) &= \deg_\delta(c) = 0 & (i \in I, r \in \mathbb{Z}'). \end{aligned}$$

The isomorphism between Drinfeld-Jimbo and new Drinfeld realizations is described in Proposition 3.3.

For  $J \subset I$ , we call the subalgebra of  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$  generated by  $a_{j,r}, k_j^{\pm 1}, c^{\pm 1}$ ,  $j \in J$ ,  $r \in \mathbb{Z}'$ , where  $a = x^+, x^-, h$ , the *diagram subalgebra associated with  $J$*  and denote it by  $U_q^J\widehat{\mathfrak{sl}}_{\mathbf{s}}$ . Any diagram subalgebra is isomorphic to a tensor product of  $U_q\widehat{\mathfrak{sl}}_{k|l}$  algebras with central elements  $c$  in each factor set equal to each other.

### 3. Affine braid group

It is well known that the role of the Weyl group for simple Lie algebras is played by an appropriate braid group in the quantum setting, see [7]. In this section we recall the action of extended affine braid group of type  $A$  on  $U_q\widehat{\mathfrak{sl}}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ . We follow [12].

In this section, we always assume  $N \geq 4$ .

#### 3.1. Extended affine braid group of type $A$

We recall the extended affine braid group of type  $A$ .

Let  $\mathcal{B}_N$  be the group generated by elements  $\tau, T_i, i \in \hat{I}$ , with defining relations

$$T_i T_j = T_j T_i \quad (j \neq i, i \pm 1), \quad (3.1)$$

$$T_j T_i T_j = T_i T_j T_i \quad (j = i \pm 1), \quad (3.2)$$

$$\tau T_{i-1} \tau^{-1} = T_i \quad (i \in \hat{I}). \quad (3.3)$$

The group  $\mathcal{B}_N$  is called the *extended affine braid group* of type  $A$ .

Alternatively,  $\mathcal{B}_N$  can be described as the group generated by elements  $\mathcal{X}_i, T_i, i \in I$ , with defining relations

$$T_i T_j = T_j T_i \quad (j \neq i, i \pm 1), \quad (3.4)$$

$$T_j T_i T_j = T_i T_j T_i \quad (j = i \pm 1), \quad (3.5)$$

$$\mathcal{X}_i \mathcal{X}_j = \mathcal{X}_j \mathcal{X}_i \quad (i, j \in I), \quad (3.6)$$

$$T_i \mathcal{X}_j = \mathcal{X}_j T_i \quad (i \neq j), \quad (3.7)$$

$$T_1^{-1} \mathcal{X}_1 T_1^{-1} = \mathcal{X}_2 \mathcal{X}_1^{-1}, \quad (3.8)$$

$$T_{N-1}^{-1} \mathcal{X}_{N-1} T_{N-1}^{-1} = \mathcal{X}_{N-2} \mathcal{X}_{N-1}^{-1}, \quad (3.9)$$

$$T_i^{-1} \mathcal{X}_i T_i^{-1} = \mathcal{X}_{i-1} \mathcal{X}_{i+1} \mathcal{X}_i^{-1} \quad (2 \leq i \leq N-2). \quad (3.10)$$

Note that  $\mathcal{B}_N$  is actually generated by  $T_i$ ,  $i \in I$ , and  $\mathcal{X}_1$ .

An isomorphism  $\gamma$  between the two realizations is given by

$$\gamma : \mathcal{X}_1 \mapsto \tau T_{N-1} \cdots T_1, \quad T_i \mapsto T_i \quad (i \in I). \quad (3.11)$$

We have a surjective group homomorphism

$$\pi : \mathcal{B}_N \rightarrow \mathfrak{S}_N, \quad \tau \mapsto \tau, \quad T_i \mapsto \sigma_i \quad (i \in \hat{I}), \quad (3.12)$$

where we denoted  $\sigma_i = (i, i+1)$ ,  $i \in I$ ,  $\sigma_0 = (1, N)$ , and, by an abuse of notation,  $\tau = (1, 2, \dots, N)$ .

### 3.2. Action of $\mathcal{B}_N$ on Drinfeld-Jimbo realization of $U_q \widehat{\mathfrak{sl}}_\bullet$

The symmetric group  $\mathfrak{S}_N$  acts naturally on  $\mathcal{S}_{m|n}$  by permuting indices,  $\sigma \mathbf{s} := (s_{\sigma^{-1}(1)}, \dots, s_{\sigma^{-1}(N)})$  for all  $\sigma \in \mathfrak{S}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ .

The extended affine braid group also acts on  $\mathcal{S}_{m|n}$  by  $T\mathbf{s} = \pi(T)\mathbf{s}$ , for  $T \in \mathcal{B}_N$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , see (3.12).

The next proposition describes a family of isomorphisms of quantum affine superalgebras.

**Proposition 3.1.** [12, Prop. 8.2.1] *We have the following.*

- (i) For  $i \in \hat{I}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , there exists an isomorphism of superalgebras  $T_{i,\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$  given on Chevalley generators by

$$\begin{aligned} T_{i,\mathbf{s}}(t_i) &= t_i^{-1}, & T_{i,\mathbf{s}}(t_{i\pm 1}) &= t_i t_{i\pm 1}, \\ T_{i,\mathbf{s}}(e_i) &= -s_i f_i t_i, & T_{i,\mathbf{s}}(f_i) &= -s_{i+1} t_i^{-1} e_i, \\ T_{i,\mathbf{s}}(e_{i-1}) &= s_{i+1} q^{-s_{i+1}} \llbracket e_{i-1}, e_i \rrbracket, & T_{i,\mathbf{s}}(e_{i+1}) &= s_i q^{-s_i} (-1)^{|e_i||e_{i+1}|} \llbracket e_{i+1}, e_i \rrbracket, \\ T_{i,\mathbf{s}}(f_{i-1}) &= -(-1)^{|f_i||f_{i-1}|} \llbracket f_{i-1}, f_i \rrbracket, & T_{i,\mathbf{s}}(f_{i+1}) &= -\llbracket f_{i+1}, f_i \rrbracket, \\ T_{i,\mathbf{s}}(e_j) &= e_j, & T_{i,\mathbf{s}}(f_j) &= f_j, & T_{i,\mathbf{s}}(t_j) &= t_j \quad (j \neq i, i \pm 1). \end{aligned}$$

The parities on the r.h.s. correspond to the generators of the target algebra  $U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$ .

- (ii) The left-inverse of  $T_{i,\mathbf{s}}$ ,  $(T_{i,\mathbf{s}})^{-1} : U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}} \rightarrow U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$ , is given by

$$\begin{aligned} (T_{i,\mathbf{s}})^{-1}(t_i) &= t_i^{-1}, & (T_{i,\mathbf{s}})^{-1}(t_{i\pm 1}) &= t_i t_{i\pm 1}, \\ (T_{i,\mathbf{s}})^{-1}(e_i) &= -s_{i+1} t_i^{-1} f_i, & (T_{i,\mathbf{s}})^{-1}(f_i) &= -s_i e_i t_i, \\ (T_{i,\mathbf{s}})^{-1}(e_{i-1}) &= s_i q^{-s_i} (-1)^{|e_i||e_{i-1}|} \llbracket e_i, e_{i-1} \rrbracket, & (T_{i,\mathbf{s}})^{-1}(e_{i+1}) &= s_{i+1} q^{-s_{i+1}} \llbracket e_i, e_{i+1} \rrbracket, \\ (T_{i,\mathbf{s}})^{-1}(f_{i+1}) &= -(-1)^{|f_i||f_{i+1}|} \llbracket f_i, f_{i+1} \rrbracket, & (T_{i,\mathbf{s}})^{-1}(f_{i-1}) &= -\llbracket f_i, f_{i-1} \rrbracket, \\ (T_{i,\mathbf{s}})^{-1}(e_j) &= e_j, & (T_{i,\mathbf{s}})^{-1}(f_j) &= f_j, & (T_{i,\mathbf{s}})^{-1}(t_j) &= t_j \quad (j \neq i, i \pm 1). \end{aligned}$$

The parities on the r.h.s. correspond to the generators of the target algebra  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ .

- (iii) For  $\mathbf{s} \in \mathcal{S}_{m|n}$ , there exist an isomorphism of superalgebras  $\tau_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow U_q\widehat{\mathfrak{sl}}_{\tau_{\mathbf{s}}}$  given on Chevalley generators by

$$\tau_{\mathbf{s}}(x_i) = x_{i+1} \quad (x = e, f, t). \quad \square$$

We note the following useful formula

$$(T_i T_{i\pm 1})_{\mathbf{s}}(x_i) = x_{i\pm 1} \quad (x = e, f, t). \quad (3.13)$$

The isomorphisms  $T_{i,\mathbf{s}}$  and  $\tau_{\mathbf{s}}$  change the grading in the Drinfeld-Jimbo realization as follows.

If  $\deg^h(x) = (d_0, d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_{N-1})$ , then

$$\begin{aligned} \deg^h(T_{i,\mathbf{s}}(x)) &= (d_0, d_1, \dots, d_{i-1}, d_{i-1} + d_{i+1} - d_i, d_{i+1}, \dots, d_{N-1}) \quad (i \in \hat{I}), \\ \deg^h(\tau_{\mathbf{s}}(x)) &= (d_{N-1}, d_0, d_1, \dots, d_{N-2}). \end{aligned} \quad (3.14)$$

The isomorphisms generate a groupoid if one considers the category whose objects are the superalgebras  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , and whose morphisms are  $\tau_{\mathbf{s}}$ ,  $T_{i,\mathbf{s}}$ ,  $i \in \hat{I}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , their inverses, and compositions.

In our situation, the groupoid structure is equivalent to the group action as follows.

Define the following automorphisms of  $U_q\widehat{\mathfrak{sl}}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$

$$\tau = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \tau_{\mathbf{s}}, \quad T_i = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} T_{i,\mathbf{s}} \quad (i \in \hat{I}). \quad (3.15)$$

Note that, by abuse of notation, we denote by  $\tau$  both the automorphism above and the element of  $\mathfrak{S}_N$ .

**Proposition 3.2.** [12, Prop. 8.2.2] *The automorphisms  $\tau$ ,  $T_i$ ,  $i \in \hat{I}$ , define an action of the extended affine braid group  $\mathcal{B}_N$  on  $U_q\widehat{\mathfrak{sl}}_{\bullet}$ , i.e., they satisfy the relations (3.1)-(3.3).  $\square$*

We adopt the following convention. For  $T \in \mathcal{B}_N$ , we denote  $T_{\mathbf{s}}$  the restriction of  $T$  to the  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$  summand in  $U_q\widehat{\mathfrak{sl}}_{\bullet}$ . Note that the image of  $T_{\mathbf{s}}$  is also a particular summand in  $U_q\widehat{\mathfrak{sl}}_{\bullet}$ , namely  $U_q\widehat{\mathfrak{sl}}_{T_{\mathbf{s}}}$ . For example,  $(\tau T_i T_j T_k)_{\mathbf{s}} = \tau_{\sigma_i \sigma_j \sigma_k \mathbf{s}} T_{i, \sigma_j \sigma_k \mathbf{s}} T_{j, \sigma_k \mathbf{s}} T_{k, \mathbf{s}}$  is mapping  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$  to  $U_q\widehat{\mathfrak{sl}}_{\tau \sigma_i \sigma_j \sigma_k \mathbf{s}}$ . We use a similar convention with other maps, see, for example, Theorem 5.1 below.

Note that the action of  $\mathcal{B}_N$  on  $\mathcal{S}_{m|n}$  is transitive. In particular, Proposition 3.1 implies that all superalgebras  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , are isomorphic.

### 3.3. Action of $\mathcal{B}_N$ on new Drinfeld realization of $U_q\widehat{\mathfrak{sl}}_{\bullet}$

We have an action of extended affine braid group  $\mathcal{B}_N$  on  $U_q\widehat{\mathfrak{sl}}_{\bullet}$  given in Chevalley generators, see Section 3.2. The group  $\mathcal{B}_N$  contains elements  $\mathcal{X}_i$ ,  $i \in I$ , see Section 3.1.



The elements  $\mathcal{X}_i$  preserve the parity,  $\mathcal{X}_i \mathbf{s} = \mathbf{s}$ , for all  $\mathbf{s} \in \mathcal{S}_{m|n}$ , and, therefore,  $(\mathcal{X}_i)_{\mathbf{s}}$  is an automorphism of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$ . These automorphisms are used to obtain an isomorphism between the two different realizations of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  (similar to the even case, see [1]).

**Proposition 3.3.** [12, Theorem 8.5.1] *There exists an isomorphism  $\iota_{\mathbf{s}}$  from the new Drinfeld to the Drinfeld-Jimbo realization of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  mapping:*

$$\begin{aligned} x_{i,r}^+ &\mapsto (-1)^{ir} \mathcal{X}_{i,\mathbf{s}}^{-r}(e_i), & x_{i,r}^- &\mapsto (-1)^{ir} \mathcal{X}_{i,\mathbf{s}}^r(f_i), & k_i &\mapsto t_i, \\ c &\mapsto t_0 t_1 \cdots t_{N-1} & (r \in \mathbb{Z}, i \in I). & \square \end{aligned} \quad (3.16)$$

The identifications  $\iota_{\mathbf{s}}$  allow us to study action of  $\mathcal{B}_N$  on the new Drinfeld realization. One can describe action of the  $\mathcal{X}_{i,\mathbf{s}}$  in current generators explicitly.

**Proposition 3.4.** *For  $i \in I$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , the action of  $\mathcal{X}_{i,\mathbf{s}}$  in current generators is given by*

$$\begin{aligned} \mathcal{X}_{i,\mathbf{s}}(x_{j,r}^{\pm}) &= (-1)^{i\delta_{ij}} x_{j,r \mp \delta_{ij}}^{\pm}, & \mathcal{X}_{i,\mathbf{s}}(k_j) &= c^{-\delta_{ij}} k_j, \\ \mathcal{X}_{i,\mathbf{s}}(h_{j,r}) &= h_{j,r}, & \mathcal{X}_{i,\mathbf{s}}(c) &= c \end{aligned} \quad (r \in \mathbb{Z}', j \in I).$$

**Proof.** The above equalities with  $i = j$  follow from (3.16). If  $i \neq j$ , by [12, Prop. 8.2.3], we have  $\mathcal{X}_{i,\mathbf{s}}(a_j) = a_j$ , ( $a = e, f, t$ ), and the proposition follows from the commutativity (3.6) of the operators  $\mathcal{X}_{i,\mathbf{s}}$ .  $\square$

For the action of  $T_{i,\mathbf{s}}$  in current generators we have some partial information.

**Lemma 3.5.** *For  $i \in I$ , we have*

$$T_{i,\mathbf{s}}(a_{j,r}) = a_{j,r} \quad (r \in \mathbb{Z}', j \in I, i \neq j, j \pm 1, a = x^+, x^-, h). \quad (3.17)$$

Moreover,

$$T_{i,\mathbf{s}}(x_{i+1,r}^+) = s_i q^{-s_i} (-1)^{|i||i+1|} \llbracket x_{i+1,r}^+, x_{i,0}^+ \rrbracket \quad (r \in \mathbb{Z}), \quad (3.18)$$

$$T_{i,\mathbf{s}}(x_{i-1,r}^+) = s_{i+1} q^{-s_{i+1}} \llbracket x_{i-1,r}^+, x_{i,0}^+ \rrbracket \quad (r \in \mathbb{Z}), \quad (3.19)$$

$$T_{i,\mathbf{s}}(x_{i+1,r}^-) = -\llbracket x_{i+1,r}^-, x_{i,0}^- \rrbracket \quad (r \in \mathbb{Z}), \quad (3.20)$$

$$T_{i,\mathbf{s}}(x_{i-1,r}^-) = -(-1)^{|i||i-1|} \llbracket x_{i-1,r}^-, x_{i,0}^- \rrbracket \quad (r \in \mathbb{Z}). \quad (3.21)$$

The parities on the r.h.s. correspond to the generators of target algebra  $U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$ .

We also have,  $T_{i,\mathbf{s}} U_q^{\{i\}} \widehat{\mathfrak{sl}}_{\mathbf{s}} \subset U_q^{\{i\}} \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$  if  $i \neq 1, N-1$ .

Finally,  $T_{1,\mathbf{s}} U_q^{\{1\}} \widehat{\mathfrak{sl}}_{\mathbf{s}} \subset U_q^{\{1,2\}} \widehat{\mathfrak{sl}}_{\sigma_1 \mathbf{s}}$  and  $T_{N-1,\mathbf{s}} U_q^{\{N-1\}} \widehat{\mathfrak{sl}}_{\mathbf{s}} \subset U_q^{\{N-1, N-2\}} \widehat{\mathfrak{sl}}_{\sigma_{N-1} \mathbf{s}}$ .

**Proof.** Equations (3.17)-(3.21) follow from relation (3.7) and Proposition 3.1.

To prove the last part, we note that if  $i \neq 1$  then  $x_{i,0}^\pm$  and  $h_{i-1,\pm 1}$  generate a subalgebra containing  $(U_q^{\{i\}} \widehat{\mathfrak{sl}}_{\mathbf{s}})^\pm$ . Here and below we denote by suffix  $+$  (resp.  $-$ ) the subalgebras generated by non-negative (resp. non-positive) modes of the generating currents in Drinfeld realization. Note that such subalgebras are  $\mathbb{Z}_{\geq 0}$ -graded with finite-dimensional graded components. Therefore,  $T_{i,\mathbf{s}}(U_q^{\{i\}} \widehat{\mathfrak{sl}}_{\mathbf{s}})^\pm \subset (U_q^{\{i,i-1\}} \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}})^\pm$ . Similarly, if  $i \neq N-1$ , we have  $T_{i,\mathbf{s}}(U_q^{\{i\}} \widehat{\mathfrak{sl}}_{\mathbf{s}})^\pm \subset (U_q^{\{i,i+1\}} \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}})^\pm$ . Now, from the PBW theorem [11, Theorem 5.7], we have the intersection  $(U_q^{\{i,i-1\}})^\pm \cap (U_q^{\{i,i+1\}})^\pm = (U_q^{\{i\}})^\pm$ . The lemma follows.  $\square$

We can also write the inverse of the isomorphism  $\iota_{\mathbf{s}}$ .

**Lemma 3.6.** *The isomorphism  $\iota_{\mathbf{s}}^{-1}$  maps*

$$e_i \mapsto x_{i,0}^+, \quad f_i \mapsto x_{i,0}^-, \quad t_i \mapsto k_i \quad (i \in I), \quad (3.22)$$

$$t_0 \mapsto c(k_1 k_2 \cdots k_{m+n-1})^{-1}, \quad (3.23)$$

$$e_0 \mapsto (\mathcal{X}_1 T_{N-1} \cdots T_2 T_1^{-1})_{\tau \mathbf{s}}(x_{1,0}^+), \quad (3.24)$$

$$f_0 \mapsto (\mathcal{X}_1 T_{N-1} \cdots T_2 T_1^{-1})_{\tau \mathbf{s}}(x_{1,0}^-). \quad (3.25)$$

**Proof.** Equations (3.22) and (3.23) are clear.

We check (3.24), the check for equation (3.25) is analogous. By equation (3.11) we have

$$e_0 = (\tau^{-1})_{\tau \mathbf{s}}(e_1) = (T_{N-1} \cdots T_2 T_1 \mathcal{X}_1^{-1})_{\tau \mathbf{s}}(x_{1,0}^+),$$

using equations (3.7) and (3.8), and noting that  $\mathcal{X}_2$  commutes with  $T_1$  and acts trivially on  $x_{1,0}^+$ , we have

$$\begin{aligned} (T_{N-1} \cdots T_2 T_1 \mathcal{X}_1^{-1})_{\tau \mathbf{s}}(x_{1,0}^+) &= (T_{N-1} \cdots T_2 \mathcal{X}_1 \mathcal{X}_2^{-1} T_1^{-1})_{\tau \mathbf{s}}(x_{1,0}^+) \\ &= (\mathcal{X}_1 T_{N-1} \cdots T_2 T_1^{-1})_{\tau \mathbf{s}}(x_{1,0}^+). \quad \square \end{aligned}$$

Note that in Lemma 3.6 we apply  $T_i$  only to Chevalley generators, therefore the formulas are explicit.

The correspondence between the  $\mathbb{Z}^N$ -grading in the two realizations of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is as follows.

If  $x \in U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is given in the new Drinfeld realization and  $\deg^v(x) = (d_1^v, \dots, d_{N-1}^v; d_\delta)$ , then the grading in the Drinfeld-Jimbo realization is

$$\deg^h(\iota_{\mathbf{s}}(x)) = (d_\delta, d_1^v + d_\delta, \dots, d_{N-1}^v + d_\delta). \quad (3.26)$$

Similarly, if  $x \in U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is given in the Drinfeld-Jimbo realization and  $\deg^h(x) = (d_0^h, d_1^h, \dots, d_{N-1}^h)$ , then the grading in the new Drinfeld realization is

$$\deg^v(\iota_{\mathbf{s}}^{-1}(x)) = (d_1^h - d_0^h, \dots, d_{N-1}^h - d_0^h; d_0^h). \quad (3.27)$$

### 3.4. The anti-automorphisms $\varphi$ and $\eta$

We have two anti-automorphisms of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  which will be used in Sections 4 and 5.

**Lemma 3.7.** *There exists a superalgebra anti-automorphism  $\varphi : U_q \widehat{\mathfrak{sl}}_{\bullet} \rightarrow U_q \widehat{\mathfrak{sl}}_{\bullet}$ , where  $\varphi = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \varphi_{\mathbf{s}}$ , and for  $\mathbf{s} \in \mathcal{S}_{m|n}$ , the anti-automorphism  $\varphi_{\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is given on Chevalley generators by*

$$\varphi_{\mathbf{s}}(e_i) = e_i, \quad \varphi_{\mathbf{s}}(f_i) = f_i, \quad \varphi_{\mathbf{s}}(t_i) = t_i^{-1} \quad (i \in \hat{I}).$$

Moreover,

$$(\varphi T_i \varphi)_{\mathbf{s}} = (T_{i, \sigma_i \mathbf{s}})^{-1} \quad (i \in \hat{I}).$$

**Proof.** This is checked by a straightforward computation.  $\square$

Note that  $\varphi_{\mathbf{s}}$  preserves the grading (2.3).

**Lemma 3.8.** *There exists a superalgebra anti-automorphism  $\eta : U_q \widehat{\mathfrak{sl}}_{\bullet} \rightarrow U_q \widehat{\mathfrak{sl}}_{\bullet}$ , where  $\eta = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \eta_{\mathbf{s}}$ , and for  $\mathbf{s} \in \mathcal{S}_{m|n}$ , the anti-automorphism  $\eta_{\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is given on current generators by*

$$\eta_{\mathbf{s}}(c) = c, \quad \eta_{\mathbf{s}}(k_i^{\pm}(z)) = k_i^{\mp}(cz^{-1}), \quad \eta_{\mathbf{s}}(x_i^{\pm}(z)) = x_i^{\pm}(z^{-1}) \quad (i \in I).$$

Moreover,

$$(\eta T_i \eta)_{\mathbf{s}} = (T_{i, \sigma_i \mathbf{s}})^{-1} \quad (i \in I).$$

**Proof.** The existence of this anti-automorphism is checked directly.

For the last equality, note that  $\eta_{\mathbf{s}}$  coincides with  $\varphi_{\mathbf{s}}$  on the subalgebra generated by  $x_{i,0}^{\pm}, k_i, c, i \in I$ . Also, by the isomorphism between the new Drinfeld and Drinfeld–Jimbo realizations of  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$ , it suffices only to check the identity on  $x_{1,\mp 1}^{\pm}$ .

We verify  $(\eta T_i \eta)_{\mathbf{s}} = (T_{i, \sigma_i \mathbf{s}})^{-1}$  on  $x_{1,1}^{-}$  for  $i = 1, 2$ . The remaining values of  $i$  are trivial. The check for  $x_{1,-1}^{+}$  is analogous.

Using the relation (3.8) we have

$$\begin{aligned} (\eta T_1 \eta)_{\mathbf{s}}(x_{1,1}^{-}) &= -(\eta T_1 \mathcal{X}_1^{-1})_{\mathbf{s}}(x_{1,0}^{-}) = -(\eta \mathcal{X}_1 \mathcal{X}_2^{-1} T_1^{-1})_{\mathbf{s}}(x_{1,0}^{-}) = \eta_{\sigma_1 \mathbf{s}}(-s_1 c^{-1} x_{1,-1}^{+} k_1) \\ &= -s_1 c^{-1} k_1^{-1} x_{1,1}^{+}, \\ (T_{1, \sigma_1 \mathbf{s}})^{-1}(x_{1,1}^{-}) &= -(T_1^{-1} \mathcal{X}_1)_{\mathbf{s}}(x_{1,0}^{-}) = -(\mathcal{X}_2 \mathcal{X}_1^{-1} T_1)_{\mathbf{s}}(x_{1,0}^{-}) = -s_1 c^{-1} k_1^{-1} x_{1,1}^{+}. \end{aligned}$$

And using the relation (3.7) we have

$$\begin{aligned}(\eta T_2 \eta)_s(x_{1,1}^-) &= -(\eta T_2 \mathcal{X}_1^{-1})_s(x_{1,0}^-) = (\eta \mathcal{X}_1^{-1})_{\sigma_2 s}((-1)^{|1||2|} \llbracket x_{1,0}^-, x_{2,0}^- \rrbracket) = -\llbracket x_{2,0}^-, x_{1,1}^- \rrbracket, \\(T_{2, \sigma_2 s})^{-1}(x_{1,1}^-) &= -(T_2^{-1} \mathcal{X}_1)_s(x_{1,0}^-) = \mathcal{X}_{1, \sigma_2 s}(\llbracket x_{2,0}^-, x_{1,0}^- \rrbracket) = -\llbracket x_{2,0}^-, x_{1,1}^- \rrbracket.\end{aligned}$$

This completes the proof.  $\square$

Note that, if  $x \in U_q \widehat{\mathfrak{sl}}_s$  is given in the new Drinfeld realization and  $\deg^v(x) = (d_1, \dots, d_{N-1}; d_\delta)$ , then

$$\deg^v(\eta_s(x)) = (d_1, \dots, d_{N-1}; -d_\delta). \quad (3.28)$$

Both anti-automorphisms  $\varphi_s$  and  $\eta_s$  are anti-involutions:  $\varphi_s^2 = \eta_s^2 = 1$ .

#### 4. Quantum toroidal superalgebra $\mathcal{E}_s$

The quantum toroidal algebra associated with  $\mathfrak{gl}_{m|n}$  and standard parity was introduced in [2]. In this section, we introduce the quantum toroidal algebra  $\mathcal{E}_s$  associated with  $\mathfrak{gl}_{m|n}$  for any choice of parity  $s$ . We give a few properties of these algebras.

##### 4.1. Definition of $\mathcal{E}_s$

Fix  $d, q \in \mathbb{C}^\times$  and define

$$q_1 = dq^{-1}, \quad q_2 = q^2, \quad q_3 = d^{-1}q^{-1}.$$

Note that  $q_1 q_2 q_3 = 1$ . In this paper we always assume that  $q_1, q_2$  are generic, meaning that  $q_1^{n_1} q_2^{n_2} q_3^{n_3} = 1$ ,  $n_1, n_2, n_3 \in \mathbb{Z}$ , iff  $n_1 = n_2 = n_3$ . Fix also  $d^{1/2}, q^{1/2} \in \mathbb{C}^\times$  such that  $(d^{1/2})^2 = d$ ,  $(q^{1/2})^2 = q$ .

Recall the affine Cartan matrix  $\hat{A}^s = (A_{i,j}^s)_{i,j \in \hat{I}}$ , see (2.1).

We also define the matrix  $M^s = (M_{i,j}^s)_{i,j \in \hat{I}}$  by  $M_{i+1,i}^s = -M_{i,i+1}^s = s_{i+1}$ , and  $M_{i,j}^s = 0$ ,  $i \neq j \pm 1$ .

**Definition 4.1.** The quantum toroidal algebra associated with  $\mathfrak{gl}_{m|n}$  and a parity sequence  $s$  is the unital associative superalgebra  $\mathcal{E}_s = \mathcal{E}_s(q_1, q_2, q_3)$  generated by  $E_{i,r}, F_{i,r}, H_{i,r}$ , and invertible elements  $K_i, C$ , where  $i \in \hat{I}$ ,  $r \in \mathbb{Z}'$ , subject to the defining relations (4.1)–(4.15) below. The parity of the generators is given by  $|E_{i,r}| = |F_{i,r}| = |i| = (1 - s_i s_{i+1})/2$ , and all remaining generators have parity 0.

We use generating series

$$E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k},$$

$$K_i^\pm(z) = K_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{r>0} H_{i,\pm r} z^{\mp r}\right).$$

Let also  $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$  be the formal delta function.

Then the defining relations are as follows.

### $C, K$ relations

$$\begin{aligned} C \text{ is central, } \quad K_i K_j &= K_j K_i, \quad K_i E_j(z) K_i^{-1} = q^{A_{i,j}^s} E_j(z), \\ K_i F_j(z) K_i^{-1} &= q^{-A_{i,j}^s} F_j(z). \end{aligned} \quad (4.1)$$

### $K$ - $K$ , $K$ - $E$ and $K$ - $F$ relations

$$K_i^\pm(z) K_j^\pm(w) = K_j^\pm(w) K_i^\pm(z), \quad (4.2)$$

$$\frac{d^{M_{i,j}^s} C^{-1} z - q^{A_{i,j}^s} w}{d^{M_{i,j}^s} C z - q^{A_{i,j}^s} w} K_i^-(z) K_j^+(w) = \frac{d^{M_{i,j}^s} q^{A_{i,j}^s} C^{-1} z - w}{d^{M_{i,j}^s} q^{A_{i,j}^s} C z - w} K_j^+(w) K_i^-(z), \quad (4.3)$$

$$(d^{M_{i,j}^s} z - q^{A_{i,j}^s} w) K_i^\pm(C^{-(1 \pm 1)/2} z) E_j(w) = (d^{M_{i,j}^s} q^{A_{i,j}^s} z - w) E_j(w) K_i^\pm(C^{-(1 \pm 1)/2} z), \quad (4.4)$$

$$(d^{M_{i,j}^s} z - q^{-A_{i,j}^s} w) K_i^\pm(C^{-(1 \mp 1)/2} z) F_j(w) = (d^{M_{i,j}^s} q^{-A_{i,j}^s} z - w) F_j(w) K_i^\pm(C^{-(1 \mp 1)/2} z). \quad (4.5)$$

### $E$ - $F$ relations

$$[E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta\left(C \frac{w}{z}\right) K_i^+(w) - \delta\left(C \frac{z}{w}\right) K_i^-(z) \right). \quad (4.6)$$

### $E$ - $E$ and $F$ - $F$ relations

$$[E_i(z), E_j(w)] = 0, \quad [F_i(z), F_j(w)] = 0 \quad (A_{i,j}^s = 0), \quad (4.7)$$

$$(d^{M_{i,j}^s} z - q^{A_{i,j}^s} w) E_i(z) E_j(w) = (-1)^{|i||j|} (d^{M_{i,j}^s} q^{A_{i,j}^s} z - w) E_j(w) E_i(z) \quad (A_{i,j}^s \neq 0), \quad (4.8)$$

$$(d^{M_{i,j}^s} z - q^{-A_{i,j}^s} w) F_i(z) F_j(w) = (-1)^{|i||j|} (d^{M_{i,j}^s} q^{-A_{i,j}^s} z - w) F_j(w) F_i(z) \quad (A_{i,j}^s \neq 0). \quad (4.9)$$

### Serre relations

$$\text{Sym}_{z_1, z_2} \llbracket E_i(z_1), \llbracket E_i(z_2), E_{i \pm 1}(w) \rrbracket \rrbracket = 0 \quad (A_{i,i}^s \neq 0), \quad (4.10)$$

$$\text{Sym}_{z_1, z_2} \llbracket F_i(z_1), \llbracket F_i(z_2), F_{i \pm 1}(w) \rrbracket \rrbracket = 0 \quad (A_{i,i}^s \neq 0). \quad (4.11)$$

If  $mn \neq 2$ ,

$$\text{Sym}_{z_1, z_2} \llbracket E_i(z_1), \llbracket E_{i+1}(y), \llbracket E_i(z_2), E_{i-1}(w) \rrbracket \rrbracket \rrbracket = 0 \quad (A_{i,i}^s = 0), \quad (4.12)$$

$$\text{Sym}_{z_1, z_2} \llbracket F_i(z_1), \llbracket F_{i+1}(y), \llbracket F_i(z_2), F_{i-1}(w) \rrbracket \rrbracket \rrbracket = 0 \quad (A_{i,i}^s = 0). \quad (4.13)$$

If  $mn = 2$ ,

$$\text{Sym}_{z_1, z_2} \text{Sym}_{w_1, w_2} \llbracket E_{i-1}(z_1), \llbracket E_{i+1}(w_1), \llbracket E_{i-1}(z_2), \llbracket E_{i+1}(w_2), E_i(y) \rrbracket \rrbracket \rrbracket \rrbracket = \quad (4.14)$$

$$= \text{Sym}_{z_1, z_2} \text{Sym}_{w_1, w_2} \llbracket E_{i+1}(w_1), \llbracket E_{i-1}(z_1), \llbracket E_{i+1}(w_2), \llbracket E_{i-1}(z_2), E_i(y) \rrbracket \rrbracket \rrbracket \rrbracket \quad (A_{i,i}^s \neq 0),$$

$$\text{Sym}_{z_1, z_2} \text{Sym}_{w_1, w_2} \llbracket F_{i-1}(z_1), \llbracket F_{i+1}(w_1), \llbracket F_{i-1}(z_2), \llbracket F_{i+1}(w_2), F_i(y) \rrbracket \rrbracket \rrbracket \rrbracket = \quad (4.15)$$

$$= \text{Sym}_{z_1, z_2} \text{Sym}_{w_1, w_2} \llbracket F_{i+1}(w_1), \llbracket F_{i-1}(z_1), \llbracket F_{i+1}(w_2), \llbracket F_{i-1}(z_2), F_i(y) \rrbracket \rrbracket \rrbracket \rrbracket \quad (A_{i,i}^s \neq 0).$$

The relations (4.2)–(4.5) are equivalent to

$$[H_{i,r}, E_j(z)] = \frac{[rA_{i,j}^s]}{r} d^{-rM_{i,j}^s} C^{-(r+|r|)/2} z^r E_j(z), \quad (4.16)$$

$$[H_{i,r}, F_j(z)] = -\frac{[rA_{i,j}^s]}{r} d^{-rM_{i,j}^s} C^{-(r-|r|)/2} z^r F_j(z), \quad (4.17)$$

$$[H_{i,r}, H_{j,s}] = \delta_{r+s,0} \cdot \frac{[rA_{i,j}^s]}{r} d^{-rM_{i,j}^s} \frac{C^r - C^{-r}}{q - q^{-1}}, \quad (4.18)$$

for all  $r \in \mathbb{Z}'$ ,  $i, j \in \hat{I}$ .

The relations (4.10) and (4.11) are also satisfied if  $A_{i,i}^s = 0$ , due to the quadratic relations (4.7).

The element  $K := K_0 K_1 \cdots K_{N-1}$  is central.

For any  $J \subset \hat{I}$ , let  $\mathcal{E}_s^J \subset \mathcal{E}_s$  be the subalgebra generated by  $E_i(z), F_i(z), K_i^\pm(z), C, i \in J$ . We call  $\mathcal{E}_s^J$  the *diagram subalgebra associated with  $J$*  of  $\mathcal{E}_s$ .

#### 4.2. Some properties of $\mathcal{E}_s$

For each  $i \in \hat{I}$ , the superalgebra  $\mathcal{E}_s$  has a  $\mathbb{Z}$ -grading given by

$$\deg_i(E_{j,r}) = \delta_{i,j}, \quad \deg_i(F_{j,r}) = -\delta_{i,j}, \quad \deg_i(H_{j,r}) = \deg_i(K_j) = \deg_i(C) = 0 \\ (j \in \hat{I}, r \in \mathbb{Z}').$$

There is also the *homogeneous  $\mathbb{Z}$ -grading* given by

$$\deg_\delta(E_{j,r}) = \deg_\delta(F_{j,r}) = r, \quad \deg_\delta(H_{j,r}) = r, \quad \deg_\delta(K_j) = \deg_\delta(C) = 0 \\ (j \in \hat{I}, r \in \mathbb{Z}').$$

Thus, the superalgebra  $\mathcal{E}_{\mathbf{s}}$  has a  $\mathbb{Z}^{N+1}$ -grading given on a homogeneous element  $X \in \mathcal{E}_{\mathbf{s}}$  by

$$\deg(X) = (\deg_0(X), \deg_1(X), \dots, \deg_{N-1}(X); \deg_{\delta}(X)). \quad (4.19)$$

The superalgebra  $\mathcal{E}_{\mathbf{s}}$  has a graded topological Hopf superalgebra structure given on generators by

$$\begin{aligned} \Delta E_i(z) &= E_i(z) \otimes 1 + K_i^-(z) \otimes E_i(C_1 z), \\ \Delta F_i(z) &= F_i(C_2 z) \otimes K_i^+(z) + 1 \otimes F_i(z), \\ \Delta K_i^+(z) &= K_i^+(C_2 z) \otimes K_i^+(z), \\ \Delta K_i^-(z) &= K_i^-(z) \otimes K_i^-(C_1 z), \\ \Delta C &= C \otimes C, \\ \varepsilon(E_i(z)) &= \varepsilon(F_i(z)) = 0, \quad \varepsilon(K_i^{\pm}(z)) = \varepsilon(C) = 1, \\ S(E_i(z)) &= -(K_i^-(C^{-1}z))^{-1} E_i(C^{-1}z), \\ S(F_i(z)) &= -F_i(C^{-1}z) (K_i^+(C^{-1}z))^{-1}, \\ S(K_i^{\pm}(z)) &= (K_i^{\pm}(C^{-1}z))^{-1}, \quad S(C) = C^{-1}, \end{aligned}$$

where  $C_1 = C \otimes 1$ ,  $C_2 = 1 \otimes C$ . In particular, we have  $S(xy) = (-1)^{|x||y|} S(y)S(x)$ . Note that we always use the graded tensor product multiplication defined for homogeneous elements  $x_1, x_2, y_1, y_2 \in \mathcal{E}_{\mathbf{s}}$  by  $(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{|y_1||x_2|} x_1 x_2 \otimes y_1 y_2$  and extended to the whole algebra by linearity.

#### 4.3. Horizontal and vertical subalgebras

Let  $\mathbf{s}$  be a parity sequence. For  $i \in I$ , define  $\mu_{\mathbf{s}}(i) = -\sum_{j=1}^i s_j$ . Define the *vertical homomorphism* of superalgebras  $v_{\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow \mathcal{E}_{\mathbf{s}}$  by

$$\begin{aligned} v_{\mathbf{s}}(x_i^+(z)) &= E_i(d^{\mu_{\mathbf{s}}(i)} z), \quad v_{\mathbf{s}}(x_i^-(z)) = F_i(d^{\mu_{\mathbf{s}}(i)} z), \quad v_{\mathbf{s}}(k_i^{\pm}(z)) = K_i^{\pm}(d^{\mu_{\mathbf{s}}(i)} z), \\ v_{\mathbf{s}}(c) &= C \quad (i \in I). \end{aligned}$$

Note that if  $x \in U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and  $\deg^v(x) = (d_1, d_2, \dots, d_{N-1}; d_{\delta})$ , then

$$\deg(v_{\mathbf{s}}(x)) = (0, d_1, d_2, \dots, d_{N-1}; d_{\delta}). \quad (4.20)$$

**Proposition 4.2.** *The vertical homomorphism  $v_{\mathbf{s}}$  is injective for generic values of parameters.*

**Proof.** This proposition was proved in [2] in the case  $\mathbf{s}$  is the standard parity sequence using the existence of the evaluation map. An evaluation map for any choice of parity  $\mathbf{s}$  under the resonance condition  $q_3^{m-n} = C^2$  is given in Appendix A. It provides a left inverse for  $v_{\mathbf{s}}$  and therefore,  $v_{\mathbf{s}}$  is injective under the resonance condition. The property of being injective is open. Thus  $v_{\mathbf{s}}$  is injective for generic parameters.  $\square$

The image of the vertical homomorphism coincides with  $\mathcal{E}_{\mathbf{s}}^I$ . We denote this subalgebra  $U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and call it the *vertical quantum affine*  $\mathfrak{sl}_{\mathbf{s}}$ .

The vertical subalgebra  $U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is a Hopf subalgebra of  $\mathcal{E}_{\mathbf{s}}$ .

**Corollary 4.3.** *Let  $J \subset \hat{I}$ ,  $J \neq \hat{I}$ . Then for generic values of parameters the diagram subalgebra  $\mathcal{E}_{\mathbf{s}}^J$  is isomorphic to tensor product of quantum affine superalgebras  $U_q \widehat{\mathfrak{sl}}_{k|l}$  with central elements  $c$  in all factors set equal to each other.*  $\square$

We denote by  $U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  the subalgebra of  $\mathcal{E}_{\mathbf{s}}$  generated by  $E_{i,0}, F_{i,0}, K_i, i \in \hat{I}$ , and we call it the *horizontal quantum affine*  $\mathfrak{sl}_{\mathbf{s}}$ .

We have a *horizontal homomorphism* of superalgebras  $h_{\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow \mathcal{E}_{\mathbf{s}}$  given by

$$e_i \mapsto E_{i,0}, \quad f_i \mapsto F_{i,0}, \quad t_i \mapsto K_i \quad (i \in \hat{I}),$$

with image  $U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$ .

We will later prove (for  $N > 3$ ) that for generic values of parameters the horizontal homomorphism  $h_{\mathbf{s}}$  is injective, see Corollary 5.10. Note that it is not a Hopf algebra map.

Note that, if  $x \in U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and  $\deg^h(x) = (d_0, d_1, d_2, \dots, d_{N-1})$ , then

$$\deg(h_{\mathbf{s}}(x)) = (d_0, d_1, d_2, \dots, d_{N-1}; 0). \quad (4.21)$$

**Lemma 4.4.** *The quantum toroidal algebra  $\mathcal{E}_{\mathbf{s}}$  is generated by the vertical and horizontal subalgebras  $U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and  $U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$ .*

**Proof.** The only generators which are not generators of either the vertical or horizontal subalgebras are  $E_{0,r}, F_{0,r}, r \in \mathbb{Z}^{\times}$ . These generators are obtained as commutators of  $E_{0,0}, F_{0,0} \in U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and  $H_{1,\pm 1} \in U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$ , see (4.16), (4.17).  $\square$

We will often use a shortcut notation

$$\begin{aligned} X_0^+(z) &:= E_0(z), & X_0^-(z) &:= F_0(z), & \tilde{K}_0^{\pm}(z) &:= K_0^{\pm}(z), \\ X_i^{\pm}(z) &:= v_{\mathbf{s}}(x_i^{\pm}(z)), & \tilde{K}_i^{\pm}(z) &:= v_{\mathbf{s}}(k_i^{\pm}(z)) & & (i \in I). \end{aligned} \quad (4.22)$$

#### 4.4. Morphisms

We list some symmetries of the superalgebras  $\mathcal{E}_{\mathbf{s}}$ .



Set

$$\mathcal{E}_\bullet = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \mathcal{E}_\mathbf{s}.$$

We have a number of maps which depend on a parity sequence  $\mathbf{s}$ . We always consider the direct sums of such maps. For example,  $h = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} h_\mathbf{s}$  and  $v = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} v_\mathbf{s}$  are maps from  $U_q \widehat{\mathfrak{sl}}_\bullet$  to  $\mathcal{E}_\bullet$ .

Given  $\mathbf{s} \in \mathcal{S}_{m|n}$ , let  $\mathbf{s}' = (s_{m-1}, s_{m-2}, \dots, s_{-n})$ . The *diagram isomorphism*  $\omega_\mathbf{s} : \mathcal{E}_\mathbf{s}(q_1, q_2, q_3) \rightarrow \mathcal{E}_{\mathbf{s}'}(q_3, q_2, q_1)$  is defined by

$$\omega_\mathbf{s}(C) = C, \quad \omega_\mathbf{s}(A_i(z)) = A_{m-i}(z) \quad (i \in \hat{I}, A = K^\pm, E, F).$$

Note that it changes  $d$  to  $d^{-1}$ .

Given  $\mathbf{s} \in \mathcal{S}_{m|n}$ , let  $-\mathbf{s} = (-s_1, -s_2, \dots, -s_N) \in \mathcal{S}_{n|m}$ . The *change of parity isomorphism*  $\nu_\mathbf{s} : \mathcal{E}_\mathbf{s}(q_1, q_2, q_3) \rightarrow \mathcal{E}_{-\mathbf{s}}(q_3^{-1}, q_2^{-1}, q_1^{-1})$  is defined by

$$\begin{aligned} \nu_\mathbf{s}(E_i(z)) &= E_{-i}(z), \quad \nu_\mathbf{s}(F_i(z)) = F_{-i}(z), \quad \nu_\mathbf{s}(K_i^\pm(z)) = -K_{-i}^\pm(z), \\ \nu_\mathbf{s}(C) &= C \quad (i \in \hat{I}). \end{aligned}$$

Note that it changes  $q$  to  $q^{-1}$ .

For  $u \in \mathbb{C}^\times$ , the *shift of spectral parameter*  $\gamma_{u,\mathbf{s}} : \mathcal{E}_\mathbf{s} \rightarrow \mathcal{E}_\mathbf{s}$  is an isomorphism defined by

$$\gamma_{u,\mathbf{s}}(C) = C, \quad \gamma_{u,\mathbf{s}}(A_i(z)) = A_i(uz) \quad (i \in \hat{I}, A = K^\pm, E, F).$$

For  $\mathbf{s} \in \mathcal{S}_{m|n}$ , there exists an isomorphism  $\widehat{\tau}_\mathbf{s} : \mathcal{E}_\mathbf{s} \rightarrow \mathcal{E}_{\tau\mathbf{s}}$  given by

$$\widehat{\tau}_\mathbf{s}(C) = C, \quad \widehat{\tau}_\mathbf{s}(A_i(z)) = A_{i+1}(-d^{-s_N} z) \quad (i \in \hat{I}, A = K^\pm, E, F). \quad (4.23)$$

In the notation (4.22), the map  $\widehat{\tau}_\mathbf{s}$  takes the form

$$\widehat{\tau}_\mathbf{s}(C) = C, \quad \widehat{\tau}_\mathbf{s}(A_i(z)) = A_{i+1}(-d^{(n-m)\delta_{i,N-1}} z) \quad (i \in \hat{I}, A = \tilde{K}^\pm, X^+, X^-).$$

**Proposition 4.5.** *The isomorphisms  $\widehat{\tau}_\mathbf{s}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , satisfy*

$$(\widehat{\tau} h)_\mathbf{s} = (h \tau)_\mathbf{s}, \quad (4.24)$$

$$(\widehat{\tau} v)_\mathbf{s}(a_i(z)) = v_{\tau\mathbf{s}}(a_{i+1}(-z)) \quad (i \in I \setminus \{N-1\}, a = k^\pm, x^\pm), \quad (4.25)$$

$$(\widehat{\tau} v T_i)_\mathbf{s}(a_j(z)) = (v T_{i+1})_{\tau\mathbf{s}}(a_{j+1}(-z)) \quad (i, j \in I \setminus \{N-1\}, a = k^\pm, x^\pm). \quad (4.26)$$

The maps  $\widehat{\tau}_\mathbf{s}$  preserve the homogeneous grading and  $\deg_i(\widehat{\tau}_\mathbf{s}(X)) = \deg_{i-1}(X)$ ,  $i \in \hat{I}$ .

**Proof.** A straightforward computation shows that  $\widehat{\tau}_{\mathbf{s}}$  preserve the homogeneous grading, satisfy equality (4.24), and  $\deg_i(\widehat{\tau}_{\mathbf{s}}(X)) = \deg_{i-1}(X)$ ,  $i \in \hat{I}$ , if  $X \in \mathcal{E}_{\mathbf{s}}$  is homogeneous.

We check  $(\widehat{\tau} v)_{\mathbf{s}}(x_i^+(z)) = v_{\tau\mathbf{s}}(x_{i+1}^+(-z))$  for  $1 \leq i \leq N-2$ .

By definition, we have

$$\begin{aligned}(\widehat{\tau} v)_{\mathbf{s}}(x_i^+(z)) &= \widehat{\tau}_{\mathbf{s}}(E_i(d^{\mu_{\mathbf{s}}(i)} z)) = E_{i+1}(-d^{\mu_{\mathbf{s}}(i)-s_N} z), \\ v_{\tau\mathbf{s}}(x_{i+1}^+(-z)) &= E_{i+1}(-d^{\mu_{\tau\mathbf{s}}(i+1)} z).\end{aligned}$$

But  $\tau\mathbf{s} = (s_N, s_1, \dots, s_{N-1})$ , thus

$$\mu_{\tau\mathbf{s}}(i+1) = -s_N - s_1 - \dots - s_i = \mu_{\mathbf{s}}(i) - s_N.$$

The proofs for  $x_i^-(z)$  and  $k_i^{\pm}(z)$  are analogous.

Equation (4.26) for  $i \neq j$  follows from Lemma 3.5 and equation (4.25).

To show (4.26) with  $i = j$ , set  $l = i - 1$  if  $i \neq 1$ , and  $l = 2$  if  $i = 1$ . In particular,  $A_{l,i}^{\mathbf{s}} \neq 0$ . Then, since  $l \neq i$ , we have

$$(\widehat{\tau} v T_i)_{\mathbf{s}}(h_{l,\pm 1}) = -(v T_{i+1})_{\tau\mathbf{s}}(h_{l+1,\pm 1}).$$

Also, by a direct computation, we have

$$(\widehat{\tau} v T_i)_{\mathbf{s}}(x_{i,0}^{\pm}) = (v T_{i+1})_{\tau\mathbf{s}}(x_{i+1,0}^{\pm}).$$

Therefore, the constant terms of left hand side and right hand side of (4.26) coincide. The equality of other terms follows from the commutator

$$[h_{l,\pm 1}, x_i^{\pm}(z)] = [A_{l,i}^{\mathbf{s}}] c^{-(1\pm 1)/2} z^{\pm 1} x_i^{\pm}(z)$$

and a similar one for  $x_i^-(z)$ .  $\square$

The homomorphisms  $v_{\mathbf{s}}, h_{\mathbf{s}}$  and  $\widehat{\tau}_{\mathbf{s}}$  previously defined correspond to the algebra  $\mathcal{E}_{\mathbf{s}}(q_1, q_2, q_3)$ . Let  $v'_{\mathbf{s}}, h'_{\mathbf{s}}$  and  $\widehat{\tau}'_{\mathbf{s}}$  be the analogous homomorphisms corresponding to the algebra  $\mathcal{E}_{\mathbf{s}}(q_3, q_2, q_1)$ , i.e., the parameter  $d$  is switched to  $d^{-1}$ .

The map  $\eta_{\mathbf{s}}$  defined in Lemma 3.8 has the following toroidal counterpart.

For  $\mathbf{s} \in \mathcal{S}_{m|n}$ , there exists an anti-isomorphism  $\hat{\eta}_{\mathbf{s}} : \mathcal{E}_{\mathbf{s}}(q_1, q_2, q_3) \rightarrow \mathcal{E}_{\mathbf{s}}(q_3, q_2, q_1)$  given by

$$\begin{aligned}\hat{\eta}_{\mathbf{s}}(C) &= C, & \hat{\eta}_{\mathbf{s}}(K_i^{\pm}(z)) &= K_i^{\mp}(Cz^{-1}), & \hat{\eta}_{\mathbf{s}}(E_i(z)) &= E_i(z^{-1}), \\ \hat{\eta}_{\mathbf{s}}(F_i(z)) &= F_i(z^{-1}) & (i \in \hat{I}).\end{aligned}$$

The anti-isomorphism  $\hat{\eta}_{\mathbf{s}}$  preserves  $\deg_i$ ,  $i \in \hat{I}$ , and  $\deg_{\delta}(\hat{\eta}_{\mathbf{s}}(X)) = -\deg_{\delta}(X)$  if  $X \in \mathcal{E}_{\mathbf{s}}$  is homogeneous.

**Lemma 4.6.** *The anti-isomorphisms  $\hat{\eta}_{\mathbf{s}}, \mathbf{s} \in \mathcal{S}_{m|n}$ , satisfy*

$$(\hat{\eta} v)_{\mathbf{s}} = (v' \eta)_{\mathbf{s}}, \quad (\hat{\eta} h)_{\mathbf{s}} = (h' \varphi)_{\mathbf{s}}, \quad (\hat{\eta} \hat{\tau})_{\mathbf{s}} = (\hat{\tau}' \hat{\eta})_{\mathbf{s}}. \quad (4.27)$$

**Proof.** The equality  $(\hat{\eta} h)_{\mathbf{s}} = (h' \varphi)_{\mathbf{s}}$  is clear.

We check, for example,  $(\hat{\eta} v)_{\mathbf{s}} = (v' \eta)_{\mathbf{s}}$  on  $x_i^+(z)$  and  $(\hat{\eta} \hat{\tau})_{\mathbf{s}} = (\hat{\tau}' \hat{\eta})_{\mathbf{s}}$  on  $E_{i,r}$ . The other cases are analogous. Note that  $\hat{\eta}_{\mathbf{s}}$  interchanges the parameters  $q_1$  and  $q_3$ , i.e.,  $d$  and  $d^{-1}$  are interchanged. Thus,

$$\begin{aligned} (\hat{\eta} v)_{\mathbf{s}}(x_{i,r}^+) &= \hat{\eta}_{\mathbf{s}}(d^{-r\mu_{\mathbf{s}}(i)} E_{i,r}) = d^{-r\mu_{\mathbf{s}}(i)} E_{i,-r} = v'_{\mathbf{s}}(x_{i,-r}^+) = (v' \eta)_{\mathbf{s}}(x_{i,r}^+), \\ (\hat{\eta} \hat{\tau})_{\mathbf{s}}(E_{i,r}) &= \hat{\eta}_{\tau\mathbf{s}}(-d^{r\mathbf{s}N} E_{i+1,r}) = -d^{r\mathbf{s}N} E_{i+1,-r} = \hat{\tau}'_{\mathbf{s}}(E_{i,-r}) = (\hat{\tau}' \hat{\eta})_{\mathbf{s}}(E_{i,r}). \quad \square \end{aligned}$$

The maps  $\mathcal{X}_{i,\mathbf{s}}$  defined in Proposition 3.4 also have toroidal analogs as follows.

There exist automorphisms of superalgebras  $\hat{\mathcal{X}}_{i,\mathbf{s}} : \mathcal{E}_{\mathbf{s}} \rightarrow \mathcal{E}_{\mathbf{s}}, i \in \hat{I}, \mathbf{s} \in \mathcal{S}_{m|n}$ , given by

$$\begin{aligned} \hat{\mathcal{X}}_{i,\mathbf{s}}(C) &= C, \quad \hat{\mathcal{X}}_{i,\mathbf{s}}(X_j^{\pm}(z)) = ((-1)^i z^{\mp 1})^{\delta_{ij}} X_j^{\pm}(z), \\ \hat{\mathcal{X}}_{i,\mathbf{s}}(K_j^{\pm}(z)) &= C^{\mp \delta_{ij}} K_j^{\pm}(z) \quad (j \in \hat{I}). \end{aligned} \quad (4.28)$$

The automorphism  $\hat{\mathcal{X}}_{i,\mathbf{s}}$  preserves  $\deg_j, j \in \hat{I}$ , and  $\deg_{\delta}(\hat{\mathcal{X}}_{i,\mathbf{s}}(X)) = \deg_{\delta}(X) - \deg_i(X)$  if  $X \in \mathcal{E}_{\mathbf{s}}$  is homogeneous. Let also  $\hat{\mathcal{X}}'_{i,\mathbf{s}}$  be the analogous automorphism corresponding to the algebra  $\mathcal{E}_{\mathbf{s}}(q_3, q_2, q_1)$ .

Let  $\zeta_{\mathbf{s}} : \mathcal{E}_{\mathbf{s}} \rightarrow \mathcal{E}_{\mathbf{s}}, \mathbf{s} \in \mathcal{S}_{m|n}$ , be the rescaling automorphism given by

$$\begin{aligned} \zeta_{\mathbf{s}}(C) &= C, \quad \zeta_{\mathbf{s}}(X_i^{\pm}(z)) = ((-1)^N d^{n-m})^{\pm \delta_{i,0}} X_i^{\pm}(z), \\ \zeta_{\mathbf{s}}(K_i^{\pm}(z)) &= K_i^{\pm}(z) \quad (i \in \hat{I}). \end{aligned}$$

**Proposition 4.7.** *The automorphisms  $\hat{\mathcal{X}}_{i,\mathbf{s}}, \zeta_{\mathbf{s}}$  satisfy*

$$(\hat{\eta}' \hat{\mathcal{X}}'_i \hat{\eta})_{\mathbf{s}} = \hat{\mathcal{X}}_{i,\mathbf{s}}^{-1}, \quad (\hat{\mathcal{X}}_j v)_{\mathbf{s}} = (v \mathcal{X}_j)_{\mathbf{s}} \quad (i \in \hat{I}, j \in I), \quad (4.29)$$

$$(\hat{\tau} \hat{\mathcal{X}}_{j-1})_{\mathbf{s}} = (\hat{\mathcal{X}}_j \hat{\tau})_{\mathbf{s}}, \quad (\hat{\tau} \hat{\mathcal{X}}_{N-1})_{\mathbf{s}} = (\zeta \hat{\mathcal{X}}_0 \hat{\tau})_{\mathbf{s}} \quad (j \in I). \quad (4.30)$$

**Proof.** Identities (4.29) and the first equality of (4.30) are clear.

We check  $(\hat{\tau} \hat{\mathcal{X}}_{N-1})_{\mathbf{s}} = (\zeta \hat{\mathcal{X}}_0 \hat{\tau})_{\mathbf{s}}$  applied to  $X_{N-1,r}^{\pm}$ :

$$\begin{aligned} (\hat{\tau} \hat{\mathcal{X}}_{N-1})_{\mathbf{s}}(X_{N-1,r}^{\pm}) &= \hat{\tau}_{\mathbf{s}}((-1)^{N-1} X_{N-1,r\mp 1}^{\pm}) = (-1)^{N+r} d^{(r\pm 1)(m-n)} X_{0,r\mp 1}^{\pm}, \\ (\zeta \hat{\mathcal{X}}_0 \hat{\tau})_{\mathbf{s}}(X_{N-1,r}^{\pm}) &= (\zeta \hat{\mathcal{X}}_0)_{\mathbf{s}}((-1)^r d^{r(m-n)} X_{0,r}^{\pm}) = \zeta_{\mathbf{s}}((-1)^r d^{r(m-n)} X_{0,r\mp 1}^{\pm}) \\ &= (-1)^{N+r} d^{(r\pm 1)(m-n)} X_{0,r\mp 1}^{\pm}. \end{aligned}$$

The check on the remaining generators is similar.  $\square$

## 5. Toroidal braid group

We construct an action of the toroidal braid group  $\widehat{\mathcal{B}}_N$  associated with  $\mathfrak{sl}_{m|n}$  on  $\mathcal{E}_\bullet = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \mathcal{E}_\mathbf{s}$ . As a consequence, we show that the algebras  $\mathcal{E}_\mathbf{s}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , are all isomorphic. It also gives us the *Miki automorphism* of  $\mathcal{E}_\mathbf{s}$ , which interchanges the horizontal and vertical subalgebras.

In this section, we assume  $N \geq 4$ .

### 5.1. Action of $\mathcal{B}_N$ on $\mathcal{E}_\bullet$

We start with extending the action of affine braid group  $\mathcal{B}_N$  from the vertical subalgebra  $U_q^{ver} \widehat{\mathfrak{sl}}_\bullet$  given in Proposition 3.1 to the toroidal algebra  $\mathcal{E}_\bullet$ .

Note that the map  $\widehat{\tau}_\mathbf{s}$  was already defined in (4.23). We also recall that the prime indicates the action of the operator with  $q_3$  and  $q_1$  switched, see Section 4.4. The following theorem is a supersymmetric analog of [8, Proposition 1].

**Theorem 5.1.** *Let  $N > 3$ . For  $i \in \hat{I}$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , there exists an isomorphism of superalgebras  $\widehat{T}_{i,\mathbf{s}} : \mathcal{E}_\mathbf{s} \rightarrow \mathcal{E}_{\sigma_i \mathbf{s}}$  satisfying*

$$(\widehat{T}_i v)_\mathbf{s} = (v T_i)_\mathbf{s} \quad (i \in I), \quad (5.1)$$

$$(\widehat{T}_i h)_\mathbf{s} = (h T_i)_\mathbf{s} \quad (i \in \hat{I}), \quad (5.2)$$

$$(\widehat{\tau} \widehat{T}_i)_\mathbf{s} = (\widehat{T}_{i+1} \widehat{\tau})_\mathbf{s} \quad (i \in \hat{I}), \quad (5.3)$$

$$(\widehat{\eta}' \widehat{T}_i' \widehat{\eta})_\mathbf{s} = (\widehat{T}_i^{-1})_\mathbf{s} \quad (i \in \hat{I}). \quad (5.4)$$

Moreover, the isomorphisms  $\widehat{T}_{i,\mathbf{s}}$  satisfy the Coxeter relations

$$(\widehat{T}_{i+1} \widehat{T}_i \widehat{T}_{i+1})_\mathbf{s} = (\widehat{T}_i \widehat{T}_{i+1} \widehat{T}_i)_\mathbf{s} \quad (i \in \hat{I}), \quad (5.5)$$

$$(\widehat{T}_i \widehat{T}_j)_\mathbf{s} = (\widehat{T}_j \widehat{T}_i)_\mathbf{s} \quad (i \neq j \pm 1). \quad (5.6)$$

Finally,  $\widehat{T}_{i,\mathbf{s}}$  are graded with respect to homogeneous grading.

**Proof.** Throughout this proof we write similar formulas for  $X_i^\pm(z)$  and  $\tilde{K}_i^\pm(z)$ . To avoid repeating the same formula four times, we use the letters  $A_i(z)$  to denote  $X_i^\pm(z)$  or  $\tilde{K}_i^\pm(z)$ , and  $a_i(z)$  to denote  $x_i^\pm(z)$  or  $k_i^\pm(z)$ . Note that  $A_i(z)$  and  $a_i(z)$  in the same formula are all of the same kind - e.g. all  $X_i^+(z)$  and  $x_i^+(z)$ .

Define the map  $\widehat{T}_{1,\mathbf{s}}$  on generators of  $\mathcal{E}_\mathbf{s}$  by

$$\widehat{T}_{1,\mathbf{s}}(A_0(z)) = (\widehat{\tau}^{-1} v T_2)_{\tau \mathbf{s}}(a_1(-z)), \quad \widehat{T}_{1,\mathbf{s}}(A_i(z)) = (v T_1)_\mathbf{s}(a_i(z)) \quad (i \in I).$$

Note that  $\widehat{T}_{1,\mathbf{s}}(A_i(z)) = A_i(z)$  if  $i = 3, \dots, N-1$ . Moreover, the action of  $\widehat{T}_{1,\mathbf{s}}$  on  $A_0(z), A_2(z), A_{1,0}$  is explicit by Lemma 3.5. The map  $\widehat{T}_{1,\mathbf{s}}$  respects homogeneous grading because  $T_{1,\mathbf{s}}$  and  $T_{2,\mathbf{s}}$  do.

We claim that this extends to an isomorphism of superalgebras. In fact all relations which do not involve node 1 (that is the ones which do not contain  $E_1(z), F_1(z), K_1^\pm(z)$ ) can be checked by a direct computation. To check the relations which do involve node 1 and to reduce the calculations in other cases we can use the following arguments.

The relations not involving node 0 are satisfied since we can compute in the vertical algebra and  $T_{1,s}$  is a homomorphism.

To check the relations involving node 0, note that by Proposition 4.5 we have

$$\begin{aligned}\widehat{\tau}_s(A_{N-1}(z)) &= A_0(-d^{m-n}z), & \widehat{\tau}_s(A_i(z)) &= A_{i+1}(-z) & (i \neq N-1), \\ \widehat{T}_{1,s}(A_i(z)) &= (\widehat{\tau}^{1-k}vT_k)_{\tau^{k-1}s}(a_{k+i-1}((-1)^{1-k}z)) & (k, k+i-1 \in I), \\ \widehat{T}_{1,s}(A_{N-1}(d^{m-n}z)) &= (\widehat{\tau}^{-2}vT_3)_{\tau^2s}(a_1(z)) = (\widehat{\tau}^{-3}vT_4)_{\tau^3s}(a_2(-z)) & (N > 4), \\ \widehat{T}_{1,s}(A_{N-2}(d^{m-n}z)) &= (\widehat{\tau}^{-3}vT_4)_{\tau^3s}(a_2(-z)) & (N > 4).\end{aligned}$$

Let us first assume that  $N > 4$ .

We prove the relations involving the nodes 0, 1, and 2 by moving these nodes to 1, 2 and 3 using  $\tau$  and  $\widehat{\tau}$ . Namely, by (3.3) and (4.25), we have

$$\widehat{T}_{1,s}(A_i(z)) = (\widehat{\tau}^{-1}vT_2)_{\tau s}(a_{i+1}(-z)) \quad (i = 0, 1, 2).$$

Observe that the defining relations between generators of  $\mathcal{E}_s$  involving nodes 0, 1, 2 are the same as the defining relations in  $U_q\widehat{\mathfrak{sl}}_{\tau s}$  involving nodes 1, 2, 3. Since  $(\widehat{\tau}^{-1}vT_2)_{\tau s}$  is a homomorphism, it maps the relations of  $U_q\widehat{\mathfrak{sl}}_{\tau s}$  involving the nodes 1, 2, 3 to zero. Therefore  $\widehat{T}_{1,s}$  maps defining relations of  $\mathcal{E}_s$  involving nodes 0, 1, 2 to zero.

The relations involving the nodes  $N-1, 0, 1$  are treated similarly: we go to nodes 1, 2, 3 again by using

$$\begin{aligned}\widehat{T}_{1,s}(A_i(z)) &= (\widehat{\tau}^{-2}vT_3)_{\tau^2s}(a_{i+2}(z)) & (i = 0, 1), \\ \widehat{T}_{1,s}(A_{N-1}(d^{m-n}z)) &= (\widehat{\tau}^{-2}vT_3)_{\tau^2s}(a_1(z)).\end{aligned}$$

Note that the defining relations between generators of  $\mathcal{E}_s$  involving nodes  $N-1, 0, 1$  are the same as the defining relations in  $U_q\widehat{\mathfrak{sl}}_{\tau^2s}$  involving nodes 1, 2, 3 if the shift of spectral parameter in the generating series related to node  $N-1$  is taken into account. Therefore as before,  $\widehat{T}_{1,s}$  maps defining relations of  $\mathcal{E}_s$  involving nodes  $N-1, 0, 1$  to zero.

For the relations involving the nodes  $N-2, N-1, 0$  we proceed in the same way by using

$$\begin{aligned}\widehat{T}_{1,s}(A_{N-i}(d^{m-n}z)) &= (\widehat{\tau}^{-3}vT_4)_{\tau^3s}(a_{3-i}(-z)) & (i = 1, 2), \\ \widehat{T}_{1,s}(A_0(z)) &= (\widehat{\tau}^{-3}vT_4)_{\tau^3s}(a_3(-z)),\end{aligned}$$

and reducing to nodes 1, 2, 3 once again. We omit further details. Thus for  $N > 4$  all relations follow without extra computations.

If  $N = 4$ , the previous argument applies for the relations involving the nodes  $0, 1, 2$ , or  $-1, 0, 1$ , or  $0, 1$  or the nodes involving  $2, 0$ . The additional relations (4.12) and (4.13) for  $i = 3$  in the case  $A_{3,3}^s = 0$  are checked directly. We check (4.12) with  $i = 3$  as an example.

First, by identity (2.2) and relation (4.12) we have

$$\begin{aligned} 0 &= \text{Sym}_{z_1, z_2} [\llbracket \llbracket E_3(z_1), \llbracket E_0(w_1), \llbracket E_3(z_2), E_2(w_2) \rrbracket \rrbracket \rrbracket, E_{1,0}], E_{1,0}] \\ &= (q + q^{-1}) \text{Sym}_{z_1, z_2} [\llbracket E_3(z_1), \llbracket E_0(w_1), E_{1,0} \rrbracket \rrbracket, \llbracket E_3(z_2), \llbracket E_2(w_2), E_{1,0} \rrbracket \rrbracket]. \end{aligned}$$

Using Lemma 3.5 we can compute the action of  $\widehat{T}_{1,s}$  on (4.12) explicitly. Note that  $A_{3,3}^s = 0$ ,  $N = 4$ ,  $m \neq n$  imply  $mn = 3$  and  $|1| = 0$ . We have

$$\begin{aligned} &\widehat{T}_{1,s}(\text{Sym}_{z_1, z_2} [\llbracket E_3(z_1), \llbracket E_0(w_1), \llbracket E_3(z_2), E_2(w_2) \rrbracket \rrbracket \rrbracket]) \\ &= \widehat{T}_{1,s}(\text{Sym}_{z_1, z_2} [\llbracket E_3(z_1), E_0(w_1) \rrbracket, \llbracket E_3(z_2), E_2(w_2) \rrbracket \rrbracket]) \\ &= q^{-A_{1,1}^s} \text{Sym}_{z_1, z_2} [\llbracket E_3(z_1), \llbracket E_0(w_1), E_{1,0} \rrbracket \rrbracket, \llbracket E_3(z_2), \llbracket E_2(w_2), E_{1,0} \rrbracket \rrbracket] = 0. \end{aligned}$$

This shows that  $\widehat{T}_{1,s}$  is a homomorphism.

For  $i \in \hat{I}$  define

$$\widehat{T}_{i,s} = (\widehat{\tau}^{i-1} \widehat{T}_1 \widehat{\tau}^{1-i})_s. \quad (5.7)$$

Since  $\widehat{T}_{1,s}$  and  $\widehat{\tau}_s$  are homomorphisms for all  $s \in \mathcal{S}_{m|n}$ , the maps  $\widehat{T}_{i,s}$  are well defined homomorphisms for all  $i \in \hat{I}$  and  $s \in \mathcal{S}_{m|n}$ .

Note that

$$\widehat{T}_{i,s}(A_j(z)) = A_j(z) \quad (j \neq i, i \pm 1). \quad (5.8)$$

Also,  $\widehat{T}_{i,s}(A_j(z))$  is explicit if  $j = i \pm 1$  and so is  $\widehat{T}_{i,s}(A_{i,0})$ .

Now, we show that the homomorphisms  $\widehat{T}_{i,s}$  satisfy equations (5.1) using induction on  $i$ . For  $i = 1$  the statement follows from definition of  $\widehat{T}_1$ . Suppose (5.1) is true for  $i = j \leq N - 2$ . Let us prove it for  $i = j + 1$ .

If  $l \in I$ ,  $l \neq 1$ , we have

$$\begin{aligned} (\widehat{T}_{j+1} v)_s(a_l(z)) &= (\widehat{\tau} \widehat{T}_j \widehat{\tau}^{-1} v)_s(a_l(z)) = (\widehat{\tau} \widehat{T}_j v)_{\tau^{-1}s}(a_{l-1}(-z)) \\ &= (\widehat{\tau} v T_j)_{\tau^{-1}s}(a_{l-1}(-z)) = (v T_{j+1})_s(a_l(z)). \end{aligned}$$

Here the second equality is (4.25), the third equality is the induction hypothesis, and the last equality is (4.26).

If  $l = 1$  then

$$(\widehat{T}_{j+1} v)_s(a_1(z)) = (\widehat{\tau} \widehat{T}_j \widehat{\tau}^{-1})_s(A_1(z)) = (\widehat{\tau} \widehat{T}_j)_{\tau^{-1}s}(A_0(-z)) = (v T_{j+1})_s(a_1(z)).$$

Here the last equation is a definition if  $j = 1$  and a trivial statement if  $j > 1$  since in that case  $\widehat{T}_{j,s}(A_0(z)) = A_0(z)$  and  $T_{j+1,s}(a_1(z)) = a_1(z)$ .

Thus,  $\widehat{T}_{i,s}$  satisfy equation (5.1).

Next we show (5.4) for  $i = 1$ . By (5.1), Lemmas 3.8 and 4.6, we have

$$\begin{aligned} (\widehat{T}_1 \widehat{\eta}' \widehat{T}_1' \widehat{\eta})_s(A_i(z)) &= (v T_1 \eta T_1 \eta)_s(a_i(z)) = v_s(a_i(z)) = A_i(z) & (i \in I), \\ (\widehat{T}_1 \widehat{\eta}' \widehat{T}_1' \widehat{\eta})_s(A_0(z)) &= (\widehat{\tau}^{-1} v T_2 \eta T_2 \eta)_{\tau s}(a_1(-z)) = (\widehat{\tau}^{-1} v)_{\tau s}(a_1(-z)) = A_0(z). \end{aligned}$$

Thus, equation (5.4) holds for  $i = 1$ . In particular, it implies that  $\widehat{T}_{1,s}$  has an inverse and therefore is an isomorphism.

By Lemma 4.6,  $\widehat{\eta}_s$  commutes with  $\widehat{\tau}_s$ . It implies (5.4) for all  $i \in \widehat{I}$ . In particular,  $\widehat{T}_{i,s}$  is isomorphism for all  $i \in \widehat{I}$ .

By (4.24), the isomorphisms  $\widehat{T}_{i,s}$  satisfy equation (5.2).

Finally, we show Coxeter relations (5.5) and (5.6). By equation (5.3), it is sufficient to show these relations when  $i = 1$  and  $j \neq 0$ . By Proposition 3.2, Coxeter relations are satisfied by the homomorphisms  $T_{i,s}$ . Then by (5.1), relations (5.5) and (5.6) are satisfied on the image of  $v_s$ . By (5.2), these relations are also satisfied on the image of  $h_s$ . Since horizontal and vertical subalgebras generate the whole algebra  $\mathcal{E}_s$ , we obtain the proof of (5.5) and (5.6)  $\square$

**Corollary 5.2.** *The superalgebras  $\mathcal{E}_s$  are isomorphic for all  $s \in \mathcal{S}_{m|n}$ .*

**Proof.** The corollary follows from Theorem 5.1.  $\square$

**Remark 5.3.** The corollary above treats the case  $N \geq 4$ . For  $N = 3$ , the isomorphisms between all three algebras  $\mathcal{E}_s$  are given by the map  $\widehat{\tau}$ .

**Corollary 5.4.** *Let  $N > 3$ . The automorphisms  $\widehat{\tau}, \widehat{T}_i, i \in \widehat{I}$ , define an action of the extended affine braid group  $\mathcal{B}_N$  on  $\mathcal{E}_\bullet$ , i.e., they satisfy the relations (3.1)-(3.3).*

## 5.2. Toroidal braid group

We recall the definition of the toroidal braid group of  $\mathfrak{sl}_N$ , cf. [8].

**Definition 5.5.** The toroidal braid group  $\widehat{\mathcal{B}}_N$  of  $\mathfrak{sl}_N$  is the group generated by elements  $\widehat{\tau}, \widehat{T}_i, \widehat{\mathcal{Y}}_j, i \in I, j \in \widehat{I}$ , satisfying the relations

$$\widehat{T}_i \widehat{T}_j = \widehat{T}_j \widehat{T}_i \quad (j \neq i, i \pm 1), \quad (5.9)$$

$$\widehat{T}_j \widehat{T}_i \widehat{T}_j = \widehat{T}_i \widehat{T}_j \widehat{T}_i \quad (j = i \pm 1), \quad (5.10)$$

$$\widehat{\mathcal{Y}}_i \widehat{\mathcal{Y}}_j = \widehat{\mathcal{Y}}_j \widehat{\mathcal{Y}}_i, \quad (5.11)$$

$$\widehat{T}_i \widehat{\mathcal{Y}}_j = \widehat{\mathcal{Y}}_j \widehat{T}_i \quad (j \neq i, i+1), \quad (5.12)$$

$$\widehat{T}_i^{-1} \widehat{\mathcal{Y}}_i \widehat{T}_i^{-1} = \widehat{\mathcal{Y}}_{i+1} \quad (i \in I), \quad (5.13)$$

$$\widehat{\tau} \widehat{T}_i \widehat{\tau}^{-1} = \widehat{T}_{i+1}, \quad (1 \leq i \leq N-2), \quad (5.14)$$

$$\widehat{\tau}^2 \widehat{T}_{N-1} \widehat{\tau}^{-2} = \widehat{T}_1, \quad (5.15)$$

$$\widehat{\tau} \widehat{\mathcal{Y}}_i \widehat{\tau}^{-1} = \widehat{\mathcal{Y}}_{i+1} \quad (i \in I). \quad (5.16)$$

We remark that the quotient of the toroidal braid group  $\widehat{\mathcal{B}}_N$  by the relation  $\widehat{\tau} \widehat{\mathcal{Y}}_0 \widehat{\tau}^{-1} = \widehat{\mathcal{Y}}_1$  is isomorphic to double affine Hecke group with central element set to 1, see Definition 4.1 in [3].

The toroidal braid group has the following Fourier transform given by qKZ elements.

**Lemma 5.6.** [3][8, (2.38)] *There exists an automorphism  $\Phi$  of  $\widehat{\mathcal{B}}_N$  given by*

$$\Phi(\widehat{T}_i) = \widehat{T}_i, \quad \Phi(\widehat{\mathcal{Y}}_j) = \widehat{T}_{j-1}^{-1} \cdots \widehat{T}_1^{-1} \widehat{\tau} \widehat{T}_{N-1} \cdots \widehat{T}_j, \quad \Phi(\widehat{\tau}) = \widehat{\mathcal{Y}}_1^{-1} \widehat{T}_1 \cdots \widehat{T}_{N-1}. \quad \square$$

Note that the subgroup  $G \subset \widehat{\mathcal{B}}_N$  generated by  $\widehat{T}_1$  and  $\widehat{\tau}$ , and the subgroup  $H \subset \widehat{\mathcal{B}}_N$  generated by  $\widehat{T}_i$ ,  $i \in I$ , and  $\widehat{\mathcal{Y}}_1$  are both isomorphic to the extended affine braid group  $\mathcal{B}_N$ . The isomorphism  $\gamma$  between these two presentations of  $\mathcal{B}_N$  is described in (3.11).

Let  $i_G$  be the inclusion  $i_G : G \cong \mathcal{B}_N \rightarrow \widehat{\mathcal{B}}_N$  given by

$$i_G(T_1) = \widehat{T}_1, \quad i_G(\tau) = \widehat{\tau},$$

and  $i_H$  be the inclusion  $i_H : H \cong \mathcal{B}_N \rightarrow \widehat{\mathcal{B}}_N$  given by

$$i_H(\mathcal{X}_1) = \widehat{\mathcal{Y}}_1, \quad i_H(T_i) = \widehat{T}_i \quad (i \in I).$$

The following lemma is easily checked on the generators.

**Lemma 5.7.** *The homomorphisms  $\gamma$ ,  $i_G$ ,  $i_H$  and  $\Phi$  satisfy the following commutative diagram*

$$\begin{array}{ccc} \widehat{\mathcal{B}}_N & \xrightarrow{\sim \Phi} & \widehat{\mathcal{B}}_N \\ i_H \uparrow & & \uparrow i_G \\ \mathcal{B}_N & \xrightarrow{\sim \gamma} & \mathcal{B}_N \end{array} \quad \square$$

Recall automorphisms  $\zeta_s, \widehat{\mathcal{X}}_{i,s}$  of  $\mathcal{E}_s$  described in Proposition 4.7. Define the following automorphisms of  $\mathcal{E}_s$

$$\widehat{\mathcal{Y}}_{0,s} = (\zeta \widehat{\mathcal{X}}_0 \widehat{\mathcal{X}}_{N-1}^{-1})_s, \quad \widehat{\mathcal{Y}}_{i,s} = (\widehat{\mathcal{X}}_i \widehat{\mathcal{X}}_{i-1}^{-1})_s \quad (i \in I). \quad (5.17)$$

The following is the super-analog of [8, Corollary 1].



**Proposition 5.8.** *The automorphisms  $\hat{\tau}, \hat{T}_i, \hat{\mathcal{Y}}_j$ ,  $i \in I$ ,  $j \in \hat{I}$ , define an action of the toroidal braid group  $\hat{\mathcal{B}}_N$  on  $\mathcal{E}_\bullet$ .*

**Proof.** The relations for  $\hat{T}_i$  and  $\hat{\tau}$  follow from Theorem 5.1. Relation (5.16) between  $\hat{\mathcal{Y}}_i$  and  $\hat{\tau}$  follows from (4.30). Relations (5.12) are clear due to (5.8). Equation (5.13) between  $\hat{\mathcal{Y}}_i$  and  $\hat{T}_j$  on vertical subalgebra follows from (3.10) and (3.8) (note that  $(\hat{\mathcal{X}}_0 v)_s = v_s$ ). To check the relations on  $A_0(z)$  we write  $A_0(z) = \hat{\tau}_s^{-1}(A_1(-z))$  and use the already established relations with  $\hat{\tau}$ .  $\square$

### 5.3. Miki automorphism

Now we are ready to prove the existence of the Miki automorphism of  $\mathcal{E}_s$  which switches horizontal and vertical subalgebras.

Recall the isomorphism  $\iota_s$  identifying the new Drinfeld and Drinfeld-Jimbo realizations of  $U_q \widehat{\mathfrak{sl}}_s$ , see Proposition 3.3.

The following is the super-analog of [8, Theorem 1].

**Theorem 5.9.** *Let  $N > 3$ . There exists a superalgebra automorphism  $\psi_s$  of  $\mathcal{E}_s$  satisfying*

$$(\psi v)_s = (h \iota)_s, \quad (\psi h \iota)_s = (v \eta \iota^{-1} \varphi \iota)_s, \quad \psi_s^{-1} = (\hat{\eta}' \psi' \hat{\eta})_s, \quad (5.18)$$

where the first two equalities are equalities of maps from the new Drinfeld realization of  $U_q \widehat{\mathfrak{sl}}_s$  to  $\mathcal{E}_s$ .

**Proof.** We often write equation of maps from  $U_q \widehat{\mathfrak{sl}}_s$  to  $\mathcal{E}_s$ , similar to (5.18). We understand that new Drinfeld realization is identified with the Drinfeld-Jimbo realization via map  $\iota$  and do not distinguish between them. In particular, we skip  $\iota$  from our formulas.

Recall notation (4.22). For  $i \in I$ , let  $\mathcal{Z}_{i,s} = (\hat{\mathcal{Y}}_1 \cdots \hat{\mathcal{Y}}_i)_s$ .

Using equations (3.11), (4.24) and (5.2) we get

$$\Phi(\hat{\mathcal{Y}}_{1,s})h_s = h_s(\tau T_{N-1} \cdots T_1)_s = (h \mathcal{X}_1)_s.$$

And, by relations (3.8)-(3.10), we have

$$\Phi(\hat{\mathcal{Y}}_{i,s})h_s = (h \mathcal{X}_i \mathcal{X}_{i-1}^{-1})_s \quad (i \in I \setminus \{1\}).$$

Thus,

$$\Phi(\mathcal{Z}_{i,s})h_s = h_s \mathcal{X}_{i,s} \quad (i \in I). \quad (5.19)$$

Define  $\psi_s$  on the  $\mathcal{E}_s$  generators by

$$\begin{aligned} \psi_s(X_{i,r}^\pm) &= (-1)^{ir} \Phi(\mathcal{Z}_i^{\mp r})_s(X_{i,0}^\pm), & \psi_s(K_i) &= K_i & (i \in I, r \in \mathbb{Z}), \\ \psi_s(X_{0,r}^\pm) &= (-1)^r \Phi(\hat{\tau}^{-1} \mathcal{Z}_1^{\mp r})_{\tau s}(X_{1,0}^\pm), & \psi_s(K_0) &= \Phi(\hat{\tau}^{-1})_{\tau s}(K_1) & (r \in \mathbb{Z}). \end{aligned}$$

For  $i \in I$ ,  $r \in \mathbb{Z}'$ , by equation (5.19), we have

$$\begin{aligned} (\psi v)_s(x_{i,r}^\pm) &= (-1)^{ir} \Phi(\mathcal{Z}_i^{\mp r})_s(X_{i,0}^\pm) = (-1)^{ir} (\Phi(\mathcal{Z}_i^{\mp r})h)_s(x_{i,0}^\pm) = (-1)^{ir} h_s \mathcal{X}_{i,s}^{\mp r}(x_{i,0}^\pm) \\ &= h_s(x_{i,r}^\pm). \end{aligned}$$

This implies  $\psi_s v_s = h_s$ . Thus,  $\psi_s$  extends to a homomorphism  $U_q^{ver} \widehat{\mathfrak{sl}}_s \rightarrow \mathcal{E}_s$ .

We now check that  $\psi_s$  satisfy the relations involving the node 0. This is done in a similar way as in Theorem 5.1. For  $i \in I$ , using (3.13) and the relations in the group, we obtain

$$\begin{aligned} \Phi(\hat{\tau})_s(X_{i,0}^\pm) &= (\hat{\mathcal{Y}}_1^{-1} \hat{T}_1 \cdots \hat{T}_{N-1})_s(X_{i,0}^\pm) = X_{i+1,0}^\pm \quad (i \neq N-1), \\ \Phi(\hat{\tau}^2)_s(X_{N-1,0}^\pm) &= (\mathcal{Z}_2^{-1} \hat{T}_2 \hat{T}_1 \hat{T}_3 \hat{T}_2 \cdots \hat{T}_{N-1} \hat{T}_{N-2})_s(X_{N-1,0}^\pm) = X_{1,0}^\pm. \end{aligned}$$

Thus, the relations involving the nodes 0, 1 and 2 follow from the relations involving the nodes 1, 2 and 3 in  $U_q^{ver} \widehat{\mathfrak{sl}}_{\tau s}$  using the equation

$$\psi_s(X_{i,r}^\pm) = (-1)^r \Phi(\hat{\tau}^{-1})_{\tau s} \psi_{\tau s}(X_{i+1,r}^\pm) \quad (i \in \hat{I}). \quad (5.20)$$

For the relations involving the nodes 0, 1 and  $N-1$  we use

$$\psi_s(X_{i,r}^\pm) = (-1)^r (d^{m-n})^{\pm r \delta_{i,N-1}} \Phi(\hat{\tau}^{-2})_{\tau^2 s} \psi_{\tau^2 s}(X_{i+2,r}^\pm) \quad (i = 0, 1, N-1).$$

And for the relations involving the nodes 0,  $N-1$  and  $N-2$  we use

$$\psi_s(X_{i,r}^\pm) = (-1)^r (d^{m-n})^{\pm r(1-\delta_{i,0})} \Phi(\hat{\tau}^{-3})_{\tau^3 s} \psi_{\tau^3 s}(X_{i+3,r}^\pm) \quad (i = 0, N-1, N-2).$$

We check the equation  $(\psi h)_s = (v \eta \varphi)_s$  on the Chevalley generators  $e_i$ ,  $i \in \hat{I}$ . The proof for  $f_i, t_i$ ,  $i \in \hat{I}$ , is analogous. By (3.11) and (3.24), we have

$$\begin{aligned} (\psi h)_s(e_i) &= \psi_s(X_{i,0}^+) = X_{i,0}^+ = v_s(x_{i,0}^+) = (v \eta \varphi)_s(e_i) \quad (i \in I), \\ (\psi h)_s(e_0) &= \psi_s(X_{0,0}^+) = \Phi(\hat{\tau}^{-1})_{\tau s}(X_{1,0}^+) = (\hat{T}_{N-1}^{-1} \cdots \hat{T}_1^{-1} \hat{\mathcal{X}}_1 \hat{\mathcal{X}}_2^{-1})_{\tau s}(X_{1,0}^+) \\ &= (\hat{T}_{N-1}^{-1} \cdots \hat{T}_1^{-1} \hat{\mathcal{X}}_1)_{\tau s}(X_{1,0}^+), \\ (v \eta \varphi)_s(e_0) &= v_s \eta_s(T_{N-1} \cdots T_1 \mathcal{X}_1^{-1})_s(x_{1,0}^+) = v_s(T_{N-1}^{-1} \cdots T_1^{-1} \mathcal{X}_1)_{\tau s}(x_{1,0}^+) \\ &= (\hat{T}_{N-1}^{-1} \cdots \hat{T}_1^{-1} \hat{\mathcal{X}}_1)_{\tau s}(X_{1,0}^+). \end{aligned}$$

Finally, we show  $\psi_s^{-1} = (\hat{\eta}' \psi' \hat{\eta})_s$ .

By Lemma 4.6 and the identities  $(\psi v)_s = h_s$ ,  $(\psi h)_s = (v \eta \varphi)_s$ , we have

$$\begin{aligned} (\psi \hat{\eta}' \psi' (\hat{\eta} v))_s &= (\psi \hat{\eta}' (\psi' v') \eta)_s = (\psi (\hat{\eta}' h') \eta)_s = ((\psi h) \varphi \eta)_s = (v \eta \varphi \varphi \eta)_s = v_s, \\ (\psi \hat{\eta}' \psi' (\hat{\eta} h))_s &= (\psi \hat{\eta}' (\psi' h') \varphi)_s = (\psi (\hat{\eta}' v') \eta)_s = (\psi v)_s = h_s. \end{aligned}$$

Thus,  $(\psi \hat{\eta}' \psi' \hat{\eta})_{\mathbf{s}} = 1_{\mathbf{s}}$  on both  $U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  and  $U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$ , but they generate  $\mathcal{E}_{\mathbf{s}}$ . Therefore  $(\psi \hat{\eta}' \psi' \hat{\eta})_{\mathbf{s}} = 1_{\mathbf{s}}$  on  $\mathcal{E}_{\mathbf{s}}$ .

This completes the proof.  $\square$

Since the Miki automorphism  $\psi_{\mathbf{s}}$  sends the vertical subalgebra  $U_q^{ver} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  to the horizontal  $U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  subalgebra, from Proposition 4.2 we obtain the following corollary.

**Corollary 5.10.** *Let  $N > 3$ . For generic values of parameters, the horizontal map  $h_{\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow \mathcal{E}_{\mathbf{s}}$  is injective. In particular,  $U_q^{hor} \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is isomorphic to  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$ .  $\square$*

The key property used in the proof of Theorem 5.9 was the compatibility of  $\psi$  with the braid group action which we describe in the next proposition.

**Proposition 5.11.** *The automorphism  $\psi_{\mathbf{s}}$  satisfies*

$$(\psi B)_{\mathbf{s}} = (\Phi(B)\psi)_{\mathbf{s}} \quad (B \in \widehat{\mathcal{B}}_N). \quad (5.21)$$

**Proof.** If  $B = \widehat{T}_i$ ,  $i \in I$ , we use (5.1), (5.2) and the first two equalities of (5.18) to show (5.21) is satisfied on the horizontal and vertical subalgebras. We have

$$\begin{aligned} (\psi \widehat{T}_i v)_{\mathbf{s}} &= (\psi v T_i)_{\mathbf{s}} = (h T_i)_{\mathbf{s}} = (\widehat{T}_i h)_{\mathbf{s}} = (\widehat{T}_i \psi v)_{\mathbf{s}} = (\Phi(\widehat{T}_i) \psi v)_{\mathbf{s}}, \\ (\psi \widehat{T}_i h)_{\mathbf{s}} &= (\psi h T_i)_{\mathbf{s}} = (v \eta \varphi T_i)_{\mathbf{s}} = (\widehat{T}_i v \eta \varphi)_{\mathbf{s}} = (\widehat{T}_i \psi h)_{\mathbf{s}} = (\Phi(\widehat{T}_i) \psi h)_{\mathbf{s}}. \end{aligned}$$

Since the horizontal and vertical subalgebras generate  $\mathcal{E}_{\mathbf{s}}$ , we have  $(\psi \widehat{T}_i)_{\mathbf{s}} = (\Phi(\widehat{T}_i)\psi)_{\mathbf{s}}$  on  $\mathcal{E}_{\mathbf{s}}$ .

In the case  $B = \widehat{\tau}$ , equation (5.21) is precisely the equation (5.20).

Since  $\widehat{\mathcal{B}}_N$  is generated by  $\widehat{T}_i$ ,  $\widehat{\mathcal{Y}}_1$  and  $\widehat{\tau}$ , it remains to check (5.21) for  $B = \widehat{\mathcal{Y}}_1$ . From the previous cases we have

$$(\psi \widehat{\tau})_{\mathbf{s}} = (\Phi(\widehat{\tau})\psi)_{\mathbf{s}} = (\widehat{\mathcal{Y}}_1^{-1} \widehat{T}_1 \cdots \widehat{T}_{N-1} \psi)_{\mathbf{s}}.$$

Thus,

$$(\widehat{\tau}^{-1} \psi^{-1})_{\mathbf{s}} = (\psi^{-1} \widehat{T}_{N-1}^{-1} \cdots \widehat{T}_1^{-1} \widehat{\mathcal{Y}}_1)_{\mathbf{s}} = (\widehat{T}_{N-1}^{-1} \cdots \widehat{T}_1^{-1} \psi^{-1} \widehat{\mathcal{Y}}_1)_{\mathbf{s}},$$

or equivalently

$$(\psi^{-1} \widehat{\mathcal{Y}}_1^{-1})_{\mathbf{s}} = (\widehat{\tau} \widehat{T}_{N-1}^{-1} \cdots \widehat{T}_1^{-1} \psi^{-1})_{\mathbf{s}}.$$

By the first equality of (4.29) and the last equality of (5.18), we have

$$(\psi \widehat{\mathcal{Y}}_1)_{\mathbf{s}} = (\hat{\eta}'(\psi')^{-1}(\widehat{\mathcal{Y}}_1')^{-1}\hat{\eta})_{\mathbf{s}},$$

and by equation (5.4) and the last equality of (4.27), we have

$$\begin{aligned}(\hat{\eta}'(\psi')^{-1}(\hat{\mathcal{Y}}_1')^{-1}\hat{\eta})_{\mathbf{s}} &= (\hat{\eta}'\hat{\tau}'(\hat{T}_{N-1}')^{-1}\cdots(\hat{T}_1')^{-1}(\psi')^{-1}\hat{\eta})_{\mathbf{s}} \\ &= (\hat{\tau}\hat{T}_{N-1}\cdots\hat{T}_1\psi)_{\mathbf{s}} = (\Phi(\hat{\mathcal{Y}}_1)\psi)_{\mathbf{s}}. \quad \square\end{aligned}$$

Finally, we describe how Miki automorphism changes the grading, cf. [8, Theorem 1].

**Proposition 5.12.** *If  $X \in \mathcal{E}_{\mathbf{s}}$  is homogeneous, then*

$$\deg_{\delta}(\psi_{\mathbf{s}}(X)) = -\deg_0(X), \quad \deg_i(\psi_{\mathbf{s}}(X)) = \deg_{\delta}(X) + \deg_i(X) - \deg_0(X) \quad (i \in \hat{I}). \quad (5.22)$$

**Proof.** Recall (4.20) and (4.21).

If  $X \in U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ , then (5.22) reduces to (3.26). Similarly if  $X \in U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ , then (5.22) reduces to (3.27) thanks to (3.28).

The proposition follows since vertical and horizontal subalgebras generate  $\mathcal{E}_{\mathbf{s}}$ .  $\square$

## Appendix A

In this appendix, we define another action of the braid group of finite type  $A$  on a suitable completion  $\widetilde{U}_q\widehat{\mathfrak{sl}}_{\bullet}$  of  $U_q\widehat{\mathfrak{sl}}_{\bullet}$ . We write an evaluation homomorphism from  $\mathcal{E}_{\mathbf{s}}$  to  $\widetilde{U}_q\widehat{\mathfrak{gl}}_{m|n}$  with parity  $\mathbf{s}$  in terms of the braid group action.

### A.1. The superalgebra $U_q\widehat{\mathfrak{gl}}_{m|n}$

Let  $\mathbf{s}$  be a parity sequence. We denote the superalgebra  $U_q\widehat{\mathfrak{gl}}_{m|n}$  with parity  $\mathbf{s}$  by  $U_q\widehat{\mathfrak{gl}}_{\mathbf{s}}$ . Here we use a presentation of  $U_q\widehat{\mathfrak{gl}}_{m|n}$  similar to the presentation of  $U_q\widehat{\mathfrak{gl}}_m$  given in [4].

The algebra  $U_q\widehat{\mathfrak{gl}}_{\mathbf{s}}$  is obtained by adding to  $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$  a Heisenberg current  $H(z)$  commuting with it and extending the root lattice generated by  $k_i$ . For our purposes it is convenient to write the generators and relations in the following way.

Define the matrix  $B^{\mathbf{s}} = (B_{i,j}^{\mathbf{s}})_{i,j \in \hat{I}}$ , where  $B_{i,j}^{\mathbf{s}} = s_i(\delta_{i,j} - \delta_{i,j+1})$ . Note that  $A_{i,j}^{\mathbf{s}} = B_{i,j}^{\mathbf{s}} - B_{i+1,j}^{\mathbf{s}}$ .

For  $i, j \in \hat{I}$ , define  $\delta_{i>j} = 1$  if  $i > j$  and  $\delta_{i>j} = 0$  if  $i \leq j$ . In this notation, we still consider the elements of  $\hat{I}$  modulo  $N$ , but we identify  $\hat{I}$  with the set  $\{1, 2, \dots, N-1, N\}$ . For example,  $\delta_{0>1} = \delta_{N>1} = 1$ .

The superalgebra  $U_q\widehat{\mathfrak{gl}}_{\mathbf{s}}$  is generated by current generators  $c, \phi_i^{\pm 1}, \phi_{i,r}, x_{j,r}^{\pm}, i \in \hat{I}, j \in I, r \in \mathbb{Z}'$ .

Let  $\phi_i^{\pm}(z) = \phi_i^{\pm 1} \exp(\pm(q - q^{-1}) \sum_{r>0} \phi_{i,\pm r} z^{\mp r})$ ,  $i \in \hat{I}$ .

For  $i \in I$ , define  $k_i^{\pm}(z) = \phi_i^{\pm}(q^{\mu_{\mathbf{s}}(i)} z) \phi_{i+1}^{\pm}(q^{\mu_{\mathbf{s}}(i)} z)^{-1}$ , where  $\mu_{\mathbf{s}}(i) = -\sum_{j=1}^i s_j$ , see Section 4.3.

The defining relations are as follows. First, the currents  $k_i^\pm(z)$ ,  $x_i^\pm(z)$ ,  $i \in I$  satisfy the relations of  $U_q \widehat{\mathfrak{sl}}_s$  given in the new Drinfeld realization as in Section 2.2.

The remaining defining relations are as follows.

For  $i, j \in \hat{I}$

$$\phi_i^\pm(z) \phi_j^\pm(w) = \phi_j^\pm(w) \phi_i^\pm(z),$$

$$\frac{c^{-1}z - w}{cz - w} \frac{c^{-1}z - q^{2s_i \delta_j > i} w}{cz - q^{2s_i \delta_j > i} w} \phi_i^+(z) \phi_j^-(w) = \frac{c^{-1}z - q^{2s_i} w}{cz - q^{2s_i} w} \frac{c^{-1}z - q^{-2s_i \delta_i > j} w}{cz - q^{-2s_i \delta_i > j} w} \phi_j^-(w) \phi_i^+(z).$$

For  $i \in \hat{I}$  and  $j \in I$

$$\begin{aligned} (z - q^{\mu_s(i) + \frac{1}{2}s_i + \frac{3}{2}B_{i,j}^s} w) \phi_i^\pm(c^{-(1 \pm 1)/2} z) x_j^\pm(w) \\ = q^{B_{i,j}^s} (z - q^{\mu_s(i) + \frac{1}{2}s_i - \frac{1}{2}B_{i,j}^s} w) x_j^\pm(w) \phi_i^\pm(c^{-(1 \pm 1)/2} z), \\ (z - q^{\mu_s(i) + \frac{1}{2}s_i - \frac{1}{2}B_{i,j}^s} w) \phi_i^\pm(c^{-(1 \mp 1)/2} z) x_j^\mp(w) \\ = q^{-B_{i,j}^s} (z - q^{\mu_s(i) + \frac{1}{2}s_i + \frac{3}{2}B_{i,j}^s} w) x_j^\mp(w) \phi_i^\pm(c^{-(1 \mp 1)/2} z). \end{aligned}$$

It is known that the subalgebra generated by the coefficients of  $k_i^\pm(z)$ ,  $x_i^\pm(z)$ ,  $i \in I$ , is isomorphic to  $U_q \widehat{\mathfrak{sl}}_s$ .

The Heisenberg current commuting with  $U_q \widehat{\mathfrak{sl}}_s$  is given by

$$H^\pm(z) = H^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{r>0} H_{\pm r} z^{\mp r}\right) = \prod_{i=1}^N \left(\phi_i^\pm(z q^{-2\mu_s(i) + s_i})\right)^{s_i}.$$

Note that  $H = \prod_i \phi_i^{s_i}$  is central.

It is useful to write the relations between the Cartan currents  $k_i^\pm(z)$  and  $\phi_j^\pm(w)$ :

$$\begin{aligned} \frac{c^{-1} q^{-\mu_s(i) - \frac{1}{2}s_i + \frac{1}{2}B_{i,j}^s} z - w}{c q^{-\mu_s(i) - \frac{1}{2}s_i + \frac{1}{2}B_{i,j}^s} z - w} \phi_i^+(z) k_j^-(w) &= \frac{c^{-1} q^{-\mu_s(i) - \frac{1}{2}s_i - \frac{3}{2}B_{i,j}^s} z - w}{c q^{-\mu_s(i) - \frac{1}{2}s_i - \frac{3}{2}B_{i,j}^s} z - w} k_j^-(w) \phi_i^+(z), \\ \frac{c^{-1} q^{-\mu_s(i) - \frac{1}{2}s_i - \frac{3}{2}B_{i,j}^s} z - w}{c q^{-\mu_s(i) - \frac{1}{2}s_i - \frac{3}{2}B_{i,j}^s} z - w} \phi_i^-(z) k_j^+(w) &= \frac{c^{-1} q^{-\mu_s(i) - \frac{1}{2}s_i + \frac{1}{2}B_{i,j}^s} z - w}{c q^{-\mu_s(i) - \frac{1}{2}s_i + \frac{1}{2}B_{i,j}^s} z - w} k_j^+(w) \phi_i^-(z). \end{aligned}$$

There is the *homogeneous  $\mathbb{Z}$ -grading* of  $U_q \widehat{\mathfrak{gl}}_s$  given by

$$\deg_\delta(x_{j,r}^\pm) = r, \quad \deg_\delta(\phi_{j,r}) = r, \quad \deg_\delta(\phi_j^{\pm 1}) = \deg_\delta(c) = 0 \quad (j \in I, r \in \mathbb{Z}').$$

Moreover,  $\deg^v(X) = (\deg_1^v(X), \dots, \deg_{N-1}^v(X); \deg_\delta(X))$ , where  $X \in U_q \widehat{\mathfrak{gl}}_s$  and for  $i \in I$ ,

$$\deg_i^v(x_{j,r}^\pm) = \pm \delta_{i,j}, \quad \deg_i^v(\phi_{j,r}) = \deg_i^v(\phi_j^{\pm 1}) = \deg_i^v(c) = 0 \quad (j \in I, r \in \mathbb{Z}'),$$

defines a  $\mathbb{Z}^{m+n}$ -grading of superalgebra  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ .

We define  $\widetilde{U}_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$  to be the completion of  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$  with respect to the homogeneous grading in the positive direction. Thus the elements of  $\widetilde{U}_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$  are series of the form  $\sum_{j=-M}^{\infty} g_j$ , with  $g_j \in U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ ,  $\deg_{\delta} g_j = j$ . The algebra  $\widetilde{U}_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$  is defined in the same way.

**Lemma A.1.** *We have an embedding*

$$U_q \widehat{\mathfrak{gl}}_{\mathbf{s}} \rightarrow \widetilde{U}_q \widehat{\mathfrak{gl}}_{\mathbf{s}}. \quad \square$$

A  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ -module  $V$  is *admissible* if for any  $v \in V$  there exist  $M = M_v > 0$  such that  $xv = 0$  for all  $x \in U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$  with  $\deg_{\delta} x > M$ . Any admissible  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ -module is also an  $\widetilde{U}_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ -module.

A vector  $v$  in a  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ -module is called *highest weight vector* if it is highest weight vector with respect to  $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \subset U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$  and is annihilated by the positive modes of the current  $H(z)$ :

$$e_i v = 0, \quad \phi_i v = q^{\lambda_i} v \quad (i \in \hat{I}), \quad H_r v = 0, \quad (r > 0).$$

A  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ -module  $V$  is called *highest weight module* if  $V$  is generated by the highest weight vector  $v$ ,  $V = U_q \widehat{\mathfrak{gl}}_{\mathbf{s}} v$ .

Highest weight  $U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ -modules are admissible.

As before, define  $U_q \widehat{\mathfrak{gl}}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} U_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ ,  $\widetilde{U}_q \widehat{\mathfrak{gl}}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \widetilde{U}_q \widehat{\mathfrak{gl}}_{\mathbf{s}}$ , and  $\widetilde{U}_q \widehat{\mathfrak{sl}}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \widetilde{U}_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$ .

Let  $\mathcal{B}_N^{fin}$  be the subgroup of  $\mathcal{B}_N$  generated by  $T_i$ ,  $i \in I$ , see Section 3.1. We call  $\mathcal{B}_N^{fin}$  the *braid group associated with  $\mathfrak{sl}_N$* .

The “fused currents” technique can be used to define an action of the group  $\mathcal{B}_N^{fin}$  on  $\widetilde{U}_q \widehat{\mathfrak{sl}}_{\bullet}$  as follows. See [6] for a more detailed discussion of fused currents.

**Proposition A.2.** *For  $i \in I$ ,  $\mathbf{s} \in \mathcal{S}_{m|n}$ , there exists an isomorphism  $T_{i,\mathbf{s}} : \widetilde{U}_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \rightarrow \widetilde{U}_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$  given by*

$$T_{i,\mathbf{s}}(x_{i+1}^+(z)) = s_{i+1} \lim_{z' \rightarrow z} \left(1 - \frac{z}{z'}\right) x_i^+(q^{-s_i} z') x_{i+1}^+(z), \quad (\text{A.1})$$

$$T_{i,\mathbf{s}}(x_{i-1}^+(z)) = s_{i+1} \lim_{z' \rightarrow z} \left(1 - \frac{z}{z'}\right) x_i^+(q^{-s_{i+1}} z') x_{i-1}^+(z), \quad (\text{A.2})$$

$$T_{i,\mathbf{s}}(x_{i+1}^-(z)) = s_i \lim_{z' \rightarrow z} \left(1 - \frac{z}{z'}\right) x_{i+1}^-(z') x_i^-(q^{-s_i} z), \quad (\text{A.3})$$

$$T_{i,\mathbf{s}}(x_{i-1}^-(z)) = s_i \lim_{z' \rightarrow z} \left(1 - \frac{z}{z'}\right) x_{i-1}^-(z') x_i^-(q^{-s_{i+1}} z), \quad (\text{A.4})$$

$$T_{i,\mathbf{s}}(k_{i+1}^{\pm}(z)) = k_i^{\pm}(q^{-s_i} z) k_{i+1}^{\pm}(z), \quad (\text{A.5})$$

$$T_{i,\mathbf{s}}(k_{i-1}^{\pm}(z)) = k_i^{\pm}(q^{-s_{i+1}} z) k_{i-1}^{\pm}(z), \quad (\text{A.6})$$

$$T_{i,\mathbf{s}}(x_i^+(z)) = s_i x_i^-(c^{-1} q^{-s_i - s_{i+1}} z) k_i^+(c^{-1} q^{-s_i - s_{i+1}} z)^{-1}, \quad (\text{A.7})$$

$$T_{i,\mathbf{s}}(x_i^-(z)) = s_{i+1}k_i^-(c^{-1}q^{-s_i-s_{i+1}}z)^{-1}x_i^+(c^{-1}q^{-s_i-s_{i+1}}z), \quad (\text{A.8})$$

$$T_{i,\mathbf{s}}(k_i^\pm(z)) = k_i^\pm(q^{-s_i-s_{i+1}}z)^{-1}, \quad (\text{A.9})$$

$$T_{i,\mathbf{s}}(A_j(z)) = A_j(z) \quad (j \neq i, i \pm 1 \text{ } A = x^\pm, k^\pm). \quad (\text{A.10})$$

Moreover, the isomorphisms  $T_{i,\mathbf{s}}$  satisfy

$$\begin{aligned} (\eta T_i \eta)_{\mathbf{s}} &= (T_{i,\sigma_i \mathbf{s}})^{-1}, \\ (T_i T_j)_{\mathbf{s}} &= (T_j T_i)_{\mathbf{s}} \quad (j \neq i, i \pm 1), \\ (T_j T_i T_j)_{\mathbf{s}} &= (T_i T_j T_i)_{\mathbf{s}} \quad (j = i \pm 1). \end{aligned}$$

**Proof.** The proposition is checked directly. In some cases it is useful to consider the following alternative formula for the fused currents. This formula for  $n = 0$ , i.e., in the purely even case, was obtained in [5], see also [9].

Set  $\text{Res}_z(\sum_{i \in \mathbb{Z}} a_i z^{-i}) = a_1$ . Then,

$$\begin{aligned} &\lim_{z' \rightarrow z} \left(1 - \frac{z}{z'}\right) x_i^+(q^{A_{i,j}} z') x_j^+(z) \\ &= \text{Res}_w \left( x_i^+(q^{A_{i,j}} z) x_j^+(w) - (-1)^{|i||j|} \frac{q^{2A_{i,j}} z - w}{q^{A_{i,j}}(z - w)} x_j^+(w) x_i^+(q^{A_{i,j}} z) \right), \\ &\lim_{z' \rightarrow z} \left(1 - \frac{z}{z'}\right) x_i^-(z') x_j^-(q^{A_{i,j}} z) \\ &= \text{Res}_w \left( x_i^-(z) x_j^-(q^{A_{i,j}} w) - (-1)^{|i||j|} \frac{q^{-2A_{i,j}} z - w}{q^{-A_{i,j}}(z - w)} x_j^-(q^{A_{i,j}} w) x_i^-(z) \right), \end{aligned}$$

where the rational functions must be expanded in the region  $|w| > |z|$ .  $\square$

Define

$$T_i = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} T_{i,\mathbf{s}} \quad (i \in \hat{I}).$$

**Corollary A.3.** The automorphisms  $T_i$ ,  $i \in I$ , define an action of  $\mathcal{B}_N^{fin}$  on  $\tilde{U}_q \hat{\mathfrak{gl}}_{\bullet}$ .

We are now able to construct an evaluation homomorphism from  $\mathcal{E}_{\mathbf{s}}$  to  $\tilde{U}_q \hat{\mathfrak{gl}}_{\mathbf{s}}$ . The evaluation map for the standard parity sequence was constructed in [2] without explicitly using the above braid action. The evaluation map for all choices of  $\mathbf{s}$  given in the following theorem coincides with the previous one if  $\mathbf{s}$  is the standard parity sequence. This construction is similar to the  $n = 0$  case, which was done in [9].

**Theorem A.4.** Fix  $u \in \mathbb{C}^\times$ . The following map is a surjective homomorphism of superalgebras  $\text{ev}_u^{\mathbf{s}} : \mathcal{E}_{\mathbf{s}} \rightarrow \tilde{U}_q \hat{\mathfrak{gl}}_{\mathbf{s}}$  with  $C^2 = q_3^{m-n}$ :

$$K \mapsto 1, \quad C \mapsto c,$$

$$E_i(z) \mapsto x_i^+(d^{-\mu_s(i)}z), \quad F_i(z) \mapsto x_i^-(d^{-\mu_s(i)}z), \quad K_i^\pm(z) \mapsto k_i^\pm(d^{-\mu_s(i)}z) \quad (i \in I),$$

$$E_0(z) \mapsto u^{-1} \phi_0^-(c^{-2}q^{2(n-m)}z) \times (T_{N-1} \dots T_1)_{\tau s}(x_1^+(zq^{s_N})) \times \phi_0^+(c^{-1}q^{2(n-m)}z)^{-1},$$

$$F_0(z) \mapsto u \phi_0^-(c^{-1}q^{2(n-m)}z)^{-1} \times (T_{N-1} \dots T_1)_{\tau s}(x_1^-(zq^{s_N})) \times \phi_0^+(c^{-2}q^{2(n-m)}z),$$

$$K_0^\pm(z) \mapsto (T_{N-1} \dots T_1)_{\tau s}(k_1^\pm(zq^{s_N})) \times \phi_0^\pm(c^{-2}q^{2(n-m)}z)\phi_0^\pm(q^{2(n-m)}z)^{-1}.$$

**Proof.** The proof of the theorem is the same as in [2] with the appropriate changes on the shifts in the formulae.  $\square$

Note that if  $X \in \mathcal{E}_s$  has grading  $\deg X = (d_0, d_1, \dots, d_{N-1}, d_\delta)$  then the grading of the image is given by  $\deg^v(ev_u^s(X)) = (d_1 - d_0, d_2 - d_0, \dots, d_{N-1} - d_0, d_\delta)$ .

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