

On the Stabilization Problem for Submodules of Specht Modules

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A finite basis problem in Specht modules is considered. Some criteria are proved for a submodule to be generated by a submodule of the Specht module with fewer columns. Also a positive solution is obtained for two-row diagrams.

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1. INTRODUCTION

Positive solutions of finite basis property problems are often based on some standard fact like the Hilbert basis theorem or the results from [3]. In fact, the results from [3] imply the Hilbert basis theorem and are probably the strongest known conditions of the finite basis property of purely combinatorial objects. In particular, the solutions of Specht's problem for T -ideals as well as for T -spaces, when the base field characteristic is zero, as they are given in [2] and [8] are developed according to the following scheme. First a system of reductions directed to localization, embedding in the algebra of generic matrices with the trace, and replacing substitution by insertion, is given. Then the Hilbert basis theorem is applied.

This scheme prompts the following problem. Let $X = \{x_i: i \in \mathbb{N}\}$ be a countable alphabet and let f_1, \dots, f_k be arbitrary formal symbols. We will call f_1, \dots, f_k form symbols and assign to each f_i its arity $n(i)$. Consider an infinite field K . We are interested in the case $\text{char } K = p > 0$. Consider

the commutative algebra A freely generated over K by the formal terms $f_i(x_{t_1}, \dots, x_{t_{n(i)}})$.

The group $GL_\infty(K)$ allows the following interpretation. Let V be the linear space over K with the basis X . Then $GL_\infty(K)$ is the set of all invertible operators on V . This group acts on A via its generators by the formula $\varphi f_s(x_{t_1}, \dots, x_{t_{n(s)}}) = \sum_{i_1=1}^\infty \cdots \sum_{i_{n(s)}=1}^\infty a_{i_1, t_1} \cdots a_{i_{n(s)}, t_{n(s)}} f_i(x_{i_1}, \dots, x_{i_{n(s)}})$, where $\varphi(x_t) = \sum_{i=1}^\infty a_{i, t} x_i$. Let us call an ideal I of A an S -ideal if it is closed under the above action of $GL_\infty(K)$.

We say that an element $g \in A$ follows from elements $g_1, \dots, g_n \in A$ if g belongs to the minimal S -ideal containing all g_1, \dots, g_n . This latter S -ideal is called the S -ideal generated by g_1, \dots, g_n . A sequence g_1, \dots, g_n, \dots of polynomials from A is called finitely generated if there exists some N such that g_i follows from g_1, \dots, g_N for $i > N$.

Let us call a polynomial $h \in A$ multilinear if every variable x_i either appears once in every monomial of h or does not enter to h at all. For example, the polynomial $f_1(x_1, x_2)f_2(x_3) + f_1(x_1, x_3)f_2(x_2)$ is multilinear, while $f_1(x_1, x_2)f_2(x_3) + f_1(x_2, x_3)f_2(x_4)$ is not.

Problem 1. Suppose $\text{char } K = p > 0$. Is every sequence of multilinear polynomials from A finitely based?

Note that when we deal with multilinear polynomials, we can replace $GL_\infty(K)$ with its subgroup that permutes elements of X . We also can expand the base field K . If we omit the word “multilinear” in Problem 1, then the answer will be negative, as it is shown in [7]. These problems are closely related to the classic Specht’s problem. In our case, elements of A play the role of identities in algebras. An example of such form identities is Garnir relations in Specht modules.

In this paper we specialize the variables x_i to finite dimensional vectors and form symbols to determinants. This special case seems to be very important. In addition, the standard technique of the representation theory of the symmetric and general linear groups is applicable.

Let us pass now to the strict definitions. We base our notation on that of [1] and [5]. Let $I(n, r)$ be the set of sequences of length r whose entries are integers from 1 to n . The symmetric group $G(r)$ of degree r acts on $I(n, r)$ on the right by the formula $i\pi = (i_{\pi_1}, \dots, i_{\pi_r})$. We write $i \sim j$ if $i = j\pi$ for some $\pi \in G(r)$. Denote by $\Lambda(n, r)$ the set of all sequences $(\lambda_1, \dots, \lambda_m)$ such that $\lambda_1, \dots, \lambda_m$ are nonnegative integers and $\lambda_1 + \dots + \lambda_m = r$. The subset of $\Lambda(n, r)$ consisting of λ such that $\lambda_1 \geq \dots \geq \lambda_m$ is denoted by $\Lambda^+(n, r)$. Elements of $\Lambda(n, r)$ are called weights and elements of $\Lambda^+(n, r)$ are called dominant weights. We say that $i \in I(n, r)$ belongs to the weight $\lambda \in \Lambda(n, r)$ if exactly λ_t entries in i are equal to t , for each $t = 1, \dots, n$. So $G(r)$ -orbits of $I(n, r)$ correspond to sets of elements of $I(n, r)$ belonging to the same weight. For $\lambda \in \Lambda^+(n, r)$, we put $[\lambda] = \{(i, j): i \in \mathbb{N}, 1 \leq j \leq \lambda_i\}$.

This set is called the diagram of λ . By λ' we denote any conjugate partition of λ . In fact, all such partitions differ by the number of zeros at the end. In every case, when this matters, we will specify to what set $\Lambda^+(N, r)$ the partition λ' belongs.

Let us make the following stipulations. We put $I(\infty, r) = \bigcup_{k \in \mathbb{N}} I(k, r)$. For two sequences of integers $i = (i_1, \dots, i_{r_1})$, $j = (j_1, \dots, j_{r_2})$, and two integers s and t , we define $si + tj = (si_1 + tj_1, \dots, si_{r_1} + tj_{r_1}, tj_{r_1+1}, \dots, tj_{r_2})$ if $r_1 < r_2$, $si + tj = (si_1 + tj_1, \dots, si_{r_2} + tj_{r_2}, si_{r_2+1}, \dots, si_{r_1})$ if $r_2 < r_1$, and $si + tj = (si_1 + tj_1, \dots, si_{r_1} + tj_{r_1})$ if $r_1 = r_2$. We also put $ij = (i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2})$. By $(i_1^{s_1}, \dots, i_k^{s_k})$ we denote the sequence obtained by the following rule: we write s_1 numbers i_1 , then s_2 numbers i_2 , and so on, ending with s_k numbers i_k . For $i \in I(\infty, r)$, we put $\{i\} = \{i_1, \dots, i_r\}$. For any set A , we denote by $G(A)$ the set of all bijections of A . Thus $G(r) = G(\{1, \dots, r\})$. For any $\pi \in G(r)$, we denote by $s(\pi)$ the sign of π .

Let $X_n = \{x_i^s : i \in \mathbb{N}, s = 1, \dots, n\}$ be a set of commutative variables, let K be a field, and let $F_n = K[X_n]$ be the free commutative algebra with 1. For $i \in I(\infty, r)$ and $j \in I(n, r)$, we put $x_i^j = x_{i_1}^{j_1} \dots x_{i_r}^{j_r}$. If j belongs to the weight $\lambda \in \Lambda^+(n, r)$, then the λ -tabloid in the notation from [4] that has the numbers $\{i_t : j_t = s\}$ in row s corresponds to x_i^j . Then polytabloids correspond to linear combinations of such monomials.

For any $r \in \mathbb{N}$, we put $u(r) = (1, \dots, r)$. This element belongs to $I(r, r)$. We will keep this notation for $u(r)$ throughout this text. Let F_n^r be the K -linear space spanned by all monomials of the form x_i^j , where $i \sim u(r)$ and $j \in I(n, r)$. The group $G(r)$ acts on F_n^r on the left as follows: $\pi x_i^j = x_{(\pi i_1, \dots, \pi i_r)}^j$.

Let us define $[x_{t_1}, \dots, x_{t_k}] = \sum_{\sigma \in G(k)} s(\sigma) x_{t_1}^{\sigma(1)} \dots x_{t_k}^{\sigma(k)}$, for $1 \leq k \leq n$. In fact, this is the minor of the infinite matrix $\{x_t^s\}_{1 \leq s \leq n, t \in \mathbb{N}}$ that corresponds to the rows $1, \dots, k$ and the columns t_1, \dots, t_k with regard to the sign.

Let $\lambda \in \Lambda^+(n, r)$. By M^λ we denote the K -linear space spanned by all monomials of the form x_i^j , where $i \sim u(r)$ and j belongs to the weight λ . Let $\mu = (\mu_1, \dots, \mu_m)$, where $m = \lambda_1$, be the conjugate partition of λ . By S^λ we denote the K -linear space spanned by polynomials of the form $s(T) = s(T, 1) \dots s(T, m)$, where T is a bijection from $[\lambda]$ to $\{1, \dots, r\}$ and $s(T, t) = [x_{T(1,t)}, \dots, x_{T(\mu_t,t)}]$. Then we have $S^\lambda \subset M^\lambda \subset F_n^r$. These inclusions make M^λ and S^λ into $KG(r)$ -modules.

For a $KG(r)$ -submodule $U \subset F_n^r$, we denote by $U\uparrow$, or $U\uparrow_n$ if there can be confusion about n , the $KG(r+n)$ -submodule of F_{n+n}^{r+n} generated by $U[x_{r+1}, \dots, x_{r+n}]$. Note that $U \subset M^\lambda$ or $U \subset S^\lambda$ implies $U\uparrow \subset M^{\lambda+(1^n)}$ or $U\uparrow \subset S^{\lambda+(1^n)}$, respectively.

For a $KG(r+n)$ -submodule $U \subset F_{n+n}^{r+n}$, we denote by $U\downarrow$ ($U\downarrow_n$) the set of all $f \in F_n^r$ such that $f[x_{r+1}, \dots, x_{r+n}] \in U$. It is evident that $U\downarrow$

is a $KG(r)$ -module. If $U \subset M^{\lambda+(1^n)}$ or $U \subset S^{\lambda+(1^n)}$, then $U \downarrow \subset M^\lambda$ or $U \downarrow \subset S^\lambda$, respectively. For the proof of the latter assertion, one can apply [4, Corollary 17.18].

We denote by $U \uparrow^t$ or $U \downarrow^t$ the result of the t -fold application of \uparrow or \downarrow , respectively, to U , of course if such operations are applicable. Notice that $U \downarrow^t \uparrow^t \subset U \subset U \uparrow^t \downarrow^t$. In this paper, we study the actions of \uparrow and \downarrow on submodules and composition series of S^λ . Problem 1 in our special case turns to

Problem 2. Let $V_i \subset S^{\lambda+(q_i^n)}$, $i \in \mathbb{N}$, where $0 \leq q_1 < q_2 < \dots$, be a sequence of submodules such that $V_i \uparrow^{q_{i+1}-q_i} \subset V_{i+1}$ for every $i \in \mathbb{N}$. Does there exist any $N \in \mathbb{N}$ such that, for every $i > N$, we have $V_N \uparrow^{q_i-q_N} = V_i$?

2. MULTILINEAR MULTIPLICATION

We call an element $i \in I(\infty, r)$ multilinear if the numbers i_1, \dots, i_r are distinct. By $R(\infty, r)$ we denote the set of all multilinear elements of $I(\infty, r)$. A monomial x_i^j , where $i \in R(\infty, r)$ and $j \in I(n, r)$, is also called multilinear. We put $\text{vr}(x_i^j) = \{i\}$. By multilinear polynomials we mean polynomials of F_n that have the form $f = \sum_i \alpha_i u_i$, where $\alpha_i \in K$, u_i are multilinear monomials, and $\text{vr}(u_i) = \{i\}$ for all i and some fixed i of $R(\infty, r)$. If $f \neq 0$, then we put $\text{vr}(f) = \{i\}$. Polynomials of the form $\alpha \cdot 1_{F_n}$, for $\alpha \in K$, are also said to be multilinear and we assume $\text{vr}(\alpha \cdot 1_{F_n}) = \emptyset$. We denote by P_n the set of all multilinear polynomials of the algebra F_n .

Let us fix the following notation $G(\infty) = G(\mathbb{N})$. We will introduce the action of $G(\infty)$ on P_n as follows. If $\pi \in G(\infty)$ and x_i^j , where $i \in R(\infty, r)$, is a multilinear monomial of F_n , then we define $\pi x_i^j = x_{(\pi i_1, \dots, \pi i_r)}^j$. We also assume that $G(\infty)$ acts on $K \cdot 1_{F_n}$ identically. The action of $G(\infty)$ on the whole of P_n is defined by linearity.

Let us introduce the operations $*$. Let $i \in R(\infty, r)$, $i' \in R(\infty, r')$, $j \in I(n, r)$, and $j' \in I(n, r')$. If there exist numbers s and t such that $i_s = i'_t$ and $j_s \neq j'_t$, then we put $x_i^j * x_{i'}^{j'} = 0$. Suppose now that $i_s = i'_t$ implies $j_s = j'_t$. Then let $x_i^j * x_{i'}^{j'}$ be equal to the result of the substitutions $x_t^s \rightarrow 1$ for $s = 1, \dots, n$ and $t \in \text{vr}(x_i^j) \cap \text{vr}(x_{i'}^{j'})$ in the monomial $x_i^j x_{i'}^{j'}$. For two polynomials $f, g \in P_n$, we define $f * g$ by linearity, additionally assuming that, for $\text{vr}(f) = \emptyset$ or $\text{vr}(g) = \emptyset$, we have $f * g = fg$. We always have $f * g \in P_n$.

If we put $\langle f, g \rangle = f * g$ for $f, g \in M^\lambda$, then we have the form \langle, \rangle from [1].

The operation $*$ also admits the following interpretation. Let P'_n be the linear span of polynomials uv for $u, v \in P_n$. Consider the system of reductions acting on elements of P'_n defined by the rule $x_t^{s'} x_t^{s''} \rightarrow 0$, where $s' \neq s''$,

and $x_i^s x_i^s \rightarrow 1$. It is easy to see that every element of P'_n has a unique normal form. If $f, g \in P_n$, then $f * g$ is the normal form of fg with respect to the described system of reductions.

This remark immediately yields the following properties:

- (1) The operation $*$ is commutative.
- (2) If $f, g, h \in P_n$ and $\text{vr}(f) \cap \text{vr}(g) \cap \text{vr}(h) = \emptyset$, then $f * (g * h) = (f * g) * h$.
- (3) If $f, g \in P_n$ and $\text{vr}(f) \cap \text{vr}(g) = \emptyset$, then $f * g = fg$.
- (4) If $f, g \in P_n$ and $\sigma \in G(\infty)$, then $\sigma(f * g) = \sigma f * \sigma g$.

THEOREM 1. *Let U be a $G(r)$ -submodule of F_n^r . Then $U \uparrow * [x_{r+1}, \dots, x_{r+n}] \subset U$.*

Proof. Let $A = \{a_1, \dots, a_n\}$, where $a_1 < \dots < a_n$, be a subset of $\{1, \dots, r+n\}$. Suppose that we have $i \in R(\infty, r)$ such that $i_1 < \dots < i_r$ and $\{i_1, \dots, i_r\} \sqcup A = \{1, \dots, r+n\}$, and $j \in I(n, r)$. The element i is uniquely defined by A . Let us calculate the value of the expression $x = x_i^j [x_{a_1}, \dots, x_{a_n}] * [x_{r+1}, \dots, x_{r+n}]$. By properties (1)–(3), we have $x = x_i^j * [x_{a_1}, \dots, x_{a_n}] * [x_{r+1}, \dots, x_{r+n}]$.

Let $c = n - |A \cap \{r+1, \dots, r+n\}|$, $A \cap \{r+1, \dots, r+n\} = \{b_1, \dots, b_{n-c}\}$, $\{r+1, \dots, r+n\} \setminus A = \{i'_1, \dots, i'_c\}$, and $A \setminus \{r+1, \dots, r+n\} = \{a'_1, \dots, a'_c\}$, where $b_1 < \dots < b_{n-c}$, $i'_1 < \dots < i'_c$, and $a'_1 < \dots < a'_c$. We can write now

$$[x_{r+1}, \dots, x_{r+n}] = \alpha_1 \sum_{\sigma \in G(n)} s(\sigma) x_{b_1}^{\sigma(1)} \dots x_{b_{n-c}}^{\sigma(n-c)} x_{i'_1}^{\sigma(n-c+1)} \dots x_{i'_c}^{\sigma(n)},$$

$$[x_{a_1}, \dots, x_{a_n}] = \alpha_2 \sum_{\pi \in G(n)} s(\pi) x_{b_1}^{\pi(1)} \dots x_{b_{n-c}}^{\pi(n-c)} x_{a'_1}^{\pi(n-c+1)} \dots x_{a'_c}^{\pi(n)},$$

where $\alpha_1, \alpha_2 = \pm 1$.

Notice that all numbers $c, i'_t, a'_t, b_t, \alpha_1, \alpha_2$ are uniquely defined by A and therefore are independent of choice of j .

For each $t = 1, \dots, c$, there exists some $s = 1, \dots, r$ such that $i_s = i'_t$. Then we put $j'_t = j_s$. We have $x_i^j = u x_{i'_t}^{j'_t} = u * x_{i'_t}^{j'_t}$, where u is either a monomial or $u = 1_{F_n}$. Let $J = \{j'_1, \dots, j'_c\}$ and $\bar{J} = \{1, \dots, n\} \setminus J$. If any numbers j'_t coincide (this corresponds to the case $|J| < c$), then $y = x_{i'_t}^{j'_t} * [x_{r+1}, \dots, x_{r+n}] = 0$ and therefore $x = 0$.

Suppose now that all numbers j'_t are distinct ($|J| = c$). We have $y = \alpha_1 \sum_{\sigma \in G'} s(\sigma) x_{b_1}^{\sigma(1)} \dots x_{b_{n-c}}^{\sigma(n-c)}$, where G' is the subset consisting of $\sigma \in G(n)$ such that $\sigma(n-c+1) = j'_1, \dots, \sigma(n) = j'_c$. It is evident that $\sigma \in G'$ implies $\sigma\{1, \dots, n-c\} = \bar{J}$.

Let G_0 denote the set of bijections from $\{1, \dots, n-c\}$ to \bar{J} and let G_1 denote the set of bijections from $\{n-c+1, \dots, n\}$ to J . For $\tau_0 \in G_0$ and

$\tau_1 \in G_1$, we denote by $\tau_0 + \tau_1$ the element of $G(n)$ whose restriction to $\{1, \dots, n-c\}$ is τ_0 and to $\{n-c+1, \dots, n\}$ is τ_1 . We define $\omega_1 \in G_1$ by the formula $\omega_1(n-c+1) = j'_1, \dots, \omega_1(n) = j'_c$. Then $G' = G_0 + \omega_1$. We have

$$[x_{a_1}, \dots, x_{a_n}] * y = \alpha_1 \alpha_2 \sum_{\tau_0 \in G_0} \sum_{\tau_1 \in G_1} s(\tau_0 + \tau_1) s(\tau_0 + \omega_1) x_{a'_1}^{\tau_1(n-c+1)} \dots x_{a'_c}^{\tau_1(n)}.$$

To prove this, one must put $\sigma = \tau_0 + \omega_1$ and $\pi = \tau_0 + \tau_1$ in the suitable formulas.

It is evident that $s(\tau_0 + \tau_1) s(\tau_0 + \omega_1)$ is independent of τ_0 . Let us fix some ω_0 of G_0 . Then we have

$$[x_{a_1}, \dots, x_{a_n}] * y = \alpha_1 \alpha_2 (n-c)! \times \sum_{\tau_1 \in G_1} s(\omega_0 + \tau_1) s(\omega_0 + \omega_1) x_{a'_1}^{\tau_1(n-c+1)} \dots x_{a'_c}^{\tau_1(n)}.$$

Let us see with what sign the monomial $x_{a'_1}^{j'_1} \dots x_{a'_c}^{j'_c}$ occurs in the last sum. This monomial emerges when $\tau_1 = \omega_1$ and its coefficient is equal to $(s(\omega_0 + \omega_1))^2 = 1$. Hence

$$[x_{a_1}, \dots, x_{a_n}] * y = \alpha_1 \alpha_2 (n-c)! \sum_{\rho \in G(c)} s(\rho) x_{a'_{\rho(1)}}^{j'_1} \dots x_{a'_{\rho(c)}}^{j'_c}. \quad (1)$$

For $\rho \in G(c)$, we denote by π_ρ the permutation of $G(r+n)$ such that $\pi_\rho(i'_1) = a'_{\rho(1)}, \dots, \pi_\rho(i'_c) = a'_{\rho(c)}, \pi_\rho(a'_1) = i'_1, \dots, \pi_\rho(a'_c) = i'_c$, and π_ρ acts on the rest of numbers identically.

We have $x_{a'_{\rho(1)}}^{j'_1} \dots x_{a'_{\rho(c)}}^{j'_c} = \pi_\rho x_{i'_1}^{j'_1} \dots x_{i'_c}^{j'_c}$, and by formula (1), the following equation holds:

$$x = \alpha_1 \alpha_2 (n-c)! \sum_{\rho \in G(c)} s(\rho) \pi_\rho x_i^j. \quad (2)$$

Formula (1) is true also for $|J| < c$. In this case, both sides are equal to zero. Therefore, formula (2) is also true in this case. In addition, the definition of π_ρ depends solely on A and is independent of j .

For any fixed n -element subset $A = \{a_1, \dots, a_n\}$ of $\{1, \dots, r+n\}$, we chose some permutation $\sigma_A \in G(r+n)$ such that $\sigma_A\{r+1, \dots, r+n\} = A$. Then $U \uparrow$ is the sum of the subspaces $\sigma_A(U[x_{r+1}, \dots, x_{r+n}]) = (\sigma_A U)[x_{a_1}, \dots, x_{a_n}]$, where A runs over all n -element subsets of $\{1, \dots, r+n\}$. Formula (2) shows that $(\sigma_A U)[x_{a_1}, \dots, x_{a_n}] * [x_{r+1}, \dots, x_{r+n}] \subset U$. Thus the theorem follows. ■

Note that $U[x_{r+1}, \dots, x_{r+n}] * [x_{r+1}, \dots, x_{r+n}] = n!U$. This yields

COROLLARY 1. *Suppose that $n < p$. Let U be a $G(r)$ -submodule of F_n^r . Then $U \uparrow * [x_{r+1}, \dots, x_{r+n}] = U$ and $U \uparrow \downarrow = U$. If $U_1 \subsetneq U_2 \subset F_n^r$ are two $G(r)$ -submodules, then $U_1 \uparrow \subsetneq U_2 \uparrow$.*

Proof. Let $f[x_{r+1}, \dots, x_{r+n}] \in U \uparrow$. Multiplying both sides by $[x_{r+1}, \dots, x_{r+n}]$ via $*$, we have $n!f \in U$. Thus $f \in U$. If $U_1 \uparrow = U_2 \uparrow$, then again multiplying both sides by $[x_{r+1}, \dots, x_{r+n}]$ via $*$, we get $U_1 = U_2$, which is not true. ■

Let us notice another application of Theorem 1. For any μ , we put $P^\mu = \{f \in S^\mu: \text{for any } g \in S^\mu, \text{ the equality } f * g = 0 \text{ holds}\}$. If μ is p -regular, then P^μ is a unique maximal submodule in S^μ . If μ is p -singular, then $P^\mu = S^\mu$. We put $D^\mu = S^\mu / P^\mu$ and $D_{\mu'} = D^\mu \otimes S^{(1^r)}$, where μ' is conjugate of μ . We have

COROLLARY 2. *Let $\lambda \in \Lambda^+(n, r)$. Then $P^\lambda \uparrow \subset P^{\lambda+(1^n)}$. If $\lambda + (1^n)$ is p -regular, then $P^{\lambda+(1^n)} \downarrow = P^\lambda$.*

Proof. Let $g \in S^{\lambda+(1^n)}$ and let $A = \{a_1, \dots, a_n\}$ be an n -element subset of $\{1, \dots, r+n\}$. By properties (1)–(4), we have

$$((\sigma_A P^\lambda)[x_{a_1}, \dots, x_{a_n}]) * g = \sigma_A(P^\lambda * [x_{r+1}, \dots, x_{r+n}] * \sigma_A^{-1}g), \quad (3)$$

where, as above, σ_A is an element of $G(r+n)$ such that $\sigma_A\{r+1, \dots, r+n\} = A$. Since $S^\lambda \uparrow = S^{\lambda+(1^n)}$, by Theorem 1 we have $[x_{r+1}, \dots, x_{r+n}] * \sigma_A^{-1}g \in S^\lambda$. Therefore, expression (3) is equal to zero. Hence $P^\lambda \uparrow * g = 0$.

Now suppose that $\lambda + (1^n)$ is p -regular and let $f[x_{r+1}, \dots, x_{r+n}] \in P^{\lambda+(1^n)}$. If $f \notin P^\lambda$, then $KG(r)f = S^\lambda$ and $P^{\lambda+(1^n)} \supset KG(r+n)f[x_{r+1}, \dots, x_{r+n}] = S^{\lambda+(1^n)}$, which is not true. ■

Notice finally that if $A = \{a_1, \dots, a_n\}$ is an n -element subset of $\{1, \dots, r+n\}$ and $f \in S^{\lambda+(1^n)}$, then $\sum_{\sigma \in G(A)} s(\sigma) \sigma f = (f * [x_{a_1}, \dots, x_{a_n}]) \times [x_{a_1}, \dots, x_{a_n}]$.

The interesting question about when $U_1 \subsetneq U_2$ implies $U_1 \uparrow \subsetneq U_2 \uparrow$ without the restriction $n < p$, will be studied in Section 4 with the help of the representation theory of the general linear group. Our proofs there are based on one construction from [5]. In Section 4, we will also reprove some results of this section. However, here we are interested in properties of the operation $*$ and we want to derive our results directly not applying Weyl modules.

3. LIFTING HOMOMORPHISMS AND ASYMPTOTICS

In this section, we collect two results for the case $n < p$.

Let $\mu \in \Lambda^+(n, r)$ and let $\nu = (\nu_1, \dots, \nu_{q+1})$, where $q = \mu_1$, be the conjugate partition of $\mu + (1^n)$. Then $\nu_1 = n$. Let us fix the following element of S^μ : $\tilde{x} = [x_{i_1^2}, \dots, x_{i_{\nu_2}^2}] \dots [x_{i_1^{q+1}}, \dots, x_{i_{\nu_{q+1}}^{q+1}}]$, where $i^s \in I(\infty, \nu_s)$, for $s = 2, \dots, q+1$, and $i^2 \dots i^{q+1} \sim u(r)$. Let also $i^1 = (r+1, \dots, r+n)$ and $x = [x_{r+1}, \dots, x_{r+n}]\tilde{x}$. We have $x \in S^{\mu+(1^n)}$. For $k = 1, \dots, q$ and nonempty sets $C \subset \{i^k\}$ and $D \subset \{i^{k+1}\}$ such that $|C| + |D| = \nu_k + 1$, we define an element $\tau_{C,D}^k$ of $KG(r+n)$ that has the form $\sum_{\sigma}' s(\sigma)\sigma$, where the sum is taken over representatives of the left cosets of $G(C) \times G(D)$ in $G(C \cup D)$. This is the so-called Garnir element. There is a well known

PROPOSITION 1. *The annihilator of x in $KG(r+n)$ is generated as a left ideal by the following elements:*

- (1) $e + (i_s^k, i_t^k)$, where $k = 1, \dots, q+1$ and $1 \leq s < t \leq \nu_k$;
- (2) $\tau_{C,D}^k$, where $k = 1, \dots, q$, $C \subset \{i^k\}$ and $D \subset \{i^{k+1}\}$ are nonempty, and $|C| + |D| = \nu_k + 1$.

Let us take one more partition $\lambda \in \Lambda^+(n, r)$. Suppose that we have two submodules V_1 and V_2 of S^λ such that $V_2 \subset V_1$ and an epimorphism $\varphi: S^\mu \rightarrow V_1/V_2$.

THEOREM 2. *There exists an epimorphism $\varphi': S^{\mu+(1^n)} \rightarrow V_1\uparrow/V_2\uparrow$ such that, for any $f \in S^\mu$, we have $\varphi'(f[x_{r+1}, \dots, x_{r+n}]) = \varphi(f)[x_{r+1}, \dots, x_{r+n}]$.*

Proof. The modules S^μ and $S^{\mu+(1^n)}$ are cyclic and \tilde{x} and x , respectively, are their generators. Let us define φ' as follows: $\varphi'(\tau x) = \tau([x_{r+1}, \dots, x_{r+n}]\varphi(\tilde{x})) + V_2\uparrow$, where $\tau \in KG(r+n)$. To check that we get in this way a well-defined homomorphism, it suffices to show that the annihilator of x annihilates $y = [x_{r+1}, \dots, x_{r+n}]\varphi(\tilde{x})$ modulo $V_2\uparrow$. All elements of form (1) and (2) for $k \geq 2$ annihilate y modulo $V_2\uparrow$, since their actions can be carried under the sign φ . Also elements of form 1 for $k = 1$ annihilate y already in S^λ . It remains to check that elements of form (2) for $k = 1$ annihilate y modulo $V_2\uparrow$.

We will rearrange $\varphi(\tilde{x})$ as follows. Let $\tau_0 = \sum_{\sigma \in G(\{i^2\})} s(\sigma)\sigma$. We have $\varphi(\tilde{x}) = (1/\nu_2!)\varphi(\tau_0\tilde{x}) = (1/\nu_2!)\tau_0\varphi(\tilde{x}) \pmod{V_2}$. The element $(1/\nu_2!)\tau_0\varphi(\tilde{x})$ is a linear combination of expressions of the form $g \cdot \sum_{\rho \in G(\nu_2)} s(\rho)x_{i_1^2}^{j_{\rho(1)}} \dots x_{i_{\nu_2}^2}^{j_{\rho(\nu_2)}}$, where $j = (j_1, \dots, j_{\nu_2})$ is a multilinear element of $I(n, \nu_2)$. The latter sum is in some sense analogous to $[x_{i_1^2}, \dots, x_{i_{\nu_2}^2}]$. It is clear that elements (2) for $k = 1$ annihilate $(1/\nu_2!)\tau_0\varphi(\tilde{x})[x_{r+1}, \dots, x_{r+n}]$ and thus they annihilate y modulo $V_2\uparrow$. ■

This theorem implies $(\text{Ker } \varphi)\uparrow \subset \text{Ker } \varphi'$. If $V_2 \not\subset V_1$, then by Corollary 1 we have $\text{Ker } \varphi' \subset P^{\mu+(1^n)}$. This shows that there exists a unique maximal module V such that $V_2\uparrow \subset V \subset V_1\uparrow$ and, in addition, $V = \varphi'(P^{\mu+(1^n)})$ and $V_1\uparrow/V \cong D^{\mu+(1^n)}$.

Note also that we can get rid of the condition $n < p$, for example, in the case $V_2 = 0$ when $\text{char } K \neq 2$ or μ is 2-regular. In this case $\varphi = \sum_{i=1}^k \alpha_i \varphi_i$, where $\alpha_i \in K$ and φ_i are inverse semistandard homomorphisms if one uses the terminology from [4]. Let c_n be the column consisting of integers $n, \dots, 1$ going from the top to the bottom. Let φ'_i be the homomorphism whose tableau is obtained from the tableau of φ_i by adding c_n as the first column. Then we can put $\varphi' = \sum_{i=1}^k \alpha_i \varphi'_i$.

We will now prove one result that will be used to obtain an estimation of the length of Specht modules. Consider the following situation. Let S be a K -algebra and let e_1, \dots, e_k be some mutually commutative idempotents of S . We denote by $\text{mod } S$ the category of left S -modules and by $\text{mod } K$ the category of linear spaces over K . Consider the functor $f: \text{mod } S \rightarrow \text{mod } K$ defined by the rule: if $V \in \text{mod } S$, then $f(V) = e_1 V + \dots + e_k V$; if $\varphi: V \rightarrow U$ is a morphism in $\text{mod } S$, then $f(\varphi)$ is the restriction of φ to $f(V)$. The functor f is some analog of the functor from [1, Section 6.2].

LEMMA 1. Let $U, V \in \text{mod } S$ and $U \subset V$. Then $f(V) \cap U = f(U)$.

Proof. The inclusion $f(U) \subset f(V) \cap U$ is evident. Prove the inverse one. Let $u = e_1 v_1 + \dots + e_k v_k \in U$, where $v_1, \dots, v_k \in V$. Denote by $I'(k, s)$ the subset of $I(k, s)$ consisting of i such that $i_1 > \dots > i_s$. We will prove by induction over r the representation

$$u = \left(\sum_{s=r}^k \sum_{i \in I'(k, s)} e_{i_1} \dots e_{i_s} v_{i, r} \right) + u_r, \quad (4)$$

where $v_{i, r} \in V$ and $u_r \in f(U)$. For $r = 1$, this is true. Suppose that (4) is true for some $r < k$ and prove it for $r + 1$.

Take some $j \in I'(k, r)$ and put $s_j = e_{j_1} \dots e_{j_r}$. Let us calculate $s_j u$. We will carry out multiplication termwise. Let $i \in I'(k, s)$, where $s \geq r$. If $i = j$, then $s_j e_{i_1} \dots e_{i_s} v_{i, s} = e_{j_1} \dots e_{j_r} v_{j, r}$. If $i \neq j$, then $s_j e_{i_1} \dots e_{i_s} v_{i, s} = e_{h_1} \dots e_{h_t} v_{i, r}$, where $\{h_1, \dots, h_t\} = \{i\} \cup \{j\}$ and $h_1 > \dots > h_t$. In the latter case, we have $t > r$. Thus $s_j u$ is the sum of $e_{j_1} \dots e_{j_r} v_{j, r}$, elements of the form $e_{h_1} \dots e_{h_t} v_{i, s}$, where $t > s \geq r$ and $h = (h_1, \dots, h_t) \in I'(k, t)$, and an element from $f(U)$.

We put $y = \sum_{j \in I'(k, r)} s_j u$. Then $y \in f(U)$. It follows from our calculation of $s_j u$ that $u - y$ has form (4), where r is replaced by $r + 1$. Hence we have proved (4).

Putting $r = k$, we get $u = e_1 \dots e_k v_{u(k), k} + u_k$, where $v_{u(k), k} \in V$ and $u_k \in f(U)$. Therefore, $e_1 \dots e_k v_{u(k), k} = u - u_k \in U$ and $u = e_1 \dots e_k \times (e_1 \dots e_k v_{u(k), k}) + u_k \in f(U)$. ■

COROLLARY 3. Suppose that $0 \rightarrow V_1 \xrightarrow{\psi} V_2 \xrightarrow{\varphi} V_3 \rightarrow 0$ is a short exact sequence in $\text{mod } S$. Then f takes it to an exact sequence. Hence for $U, V \in \text{mod } S$ such that $U \subset V$, we have $f(U/V) \cong f(U)/f(V)$.

Proof. It suffices to check exactness only in the middle term. We have $\psi(V_1) = \text{Ker } \varphi$. By Lemma 1, $\text{Ker } f(\varphi) = f(V_1) \cap \text{Ker } \varphi = f(\text{Ker } \varphi) = f(\psi(V_1)) = \psi(f(V_1))$. ■

If V is finite dimensional, then $\dim_K f(V) = \dim_K f(U/V) + \dim_K f(U)$. Notice also that the lattice of linear subspaces of V generated by e_1V, \dots, e_kV is distributive. In addition, $e_{i_1}V \cap \dots \cap e_{i_r}V = e_{i_1} \dots e_{i_r}V$. These facts are well known in the theory of idempotents.

Suppose again that V is finite dimensional and that, for any $i \in I'(k, s)$, we have $\dim_K e_{i_1} \dots e_{i_s}V = d_{k-s}$, where d_{k-s} is an integer that depends only on s . Assume also $\dim_K V = d_k$. Then $\dim_K(V/f(V)) = \sum_{i=0}^k (-1)^{k-i} d_i C_k^i$.

We will apply now the obtained results to Specht modules. Let $\lambda \in \Lambda^+(n, r)$ and $k \in \mathbb{N}$. We put $S = KG(r + kn)$, $V = S^{\lambda + (k^n)}$, and $e_t = \frac{1}{n!} \sum \{s(\sigma)\sigma : \sigma \in G(\{r + (t-1)n + 1, \dots, r + tn\})\}$, for $t = 1, \dots, k$. Since $n < p$, we have $d_i = \dim_K S^{\lambda + (i^n)}$.

We will estimate the sum $x_k(a) = \sum_{i=0}^k (-1)^{k-i} d_i a^i C_k^i$, where a is a nonnegative real parameter. To do this, let us see what the polynomial $y_k(a) = \sum_{i=0}^k (-1)^{k-i} ((n!) / (i!)) a^i C_k^i$ is equal to. Our aim is to rearrange this sum to a sum with nonnegative terms.

Suppose that t_1, \dots, t_n are some commutative variables. Then $y_k(a)$ is the constant term of the following rational expression in t_1, \dots, t_n :

$$f_k = \left(a \frac{(t_1 + \dots + t_n)^n}{t_1 \dots t_n} - 1 \right)^k.$$

We have $a((t_1 + \dots + t_n)^n / t_1 \dots t_n) - 1 = a(g_n / t_1 \dots t_n) + (n!a - 1)$, where $g = (t_1 + \dots + t_n)^n - n!t_1 \dots t_n$ is a polynomial with positive coefficients. Then $f_k = \sum_{i=0}^k (g^i / t_1^i \dots t_n^i) a^i (n!a - 1)^{k-i} C_k^i$. Let G_i be the coefficient of $t_1^i \dots t_n^i$ in g^i . Then $y_k(a) = \sum_{i=0}^k G_i a^i (n!a - 1)^{k-i} C_k^i$. Since all the coefficients of g are positive, G_i is not greater than the value of g^i at $t_1 = 1, \dots, t_n = 1$, i.e., than $(n^n - n!)^i$. We obtain the following estimations: $0 \leq y_k(a) \leq (n^n a - 1)^k$ for $\frac{1}{n!} \leq a$ and $|y_k(a)| \leq ((n^n - 2n!)a + 1)^k$ for $0 \leq a \leq \frac{1}{n!}$.

Consider a more general situation. Let $B(i)$, $i = 1, \dots, k$, be some real sequence. Let us put $F_B(a) = \sum_{i=0}^k a^i B(i)$. It is easy to see that, for $C(i) = iB(i)$, we get $F_C(a) = aF'_B(a)$. We need to solve the reverse problem. Let $m \in \mathbb{N}$ and let $D(i) = \frac{B(i)}{i+m}$. Let us calculate F_D . We have $F_B(a) = aF'_D(a) + mF_D(a)$.

Our differential equation is equivalent to the following ones: $(a^m F_D \times (a))' = a^{m-1} F_B(a)$ and $a^m F_D(a) = \int_0^a t^{m-1} F_B(t) dt$. The integration constant is absent as both sides turn to zero at $a = 0$.

LEMMA 2. Suppose that for any $k \in \mathbb{N}$, we have a real sequence $B_1(1), \dots, B_k(k)$ and let the following estimations hold,

$$\begin{aligned} |F_{B_k}(a)| &\leq C((n^n - 2n!)a + 1)^k, & \text{for } 0 \leq a \leq \frac{1}{n!}, \\ |F_{B_k}(a)| &\leq C(n^n a - 1)^k, & \text{for } \frac{1}{n!} \leq a, \end{aligned} \quad (5)$$

where C is a positive constant. Let $D_k(i) = B_k(i)/(i + m)$ for $i = 1, \dots, k$ and $k \in \mathbb{N}$. Then similar estimations hold for F_{D_k} , where C is replaced by $\frac{C}{m}$.

Proof. For $0 \leq a \leq \frac{1}{n!}$, we have

$$|F_{D_k}(a)| \leq \frac{1}{a^m} \int_0^a t^{m-1} |F_{B_k}(t)| dt \leq \frac{C}{m} ((n^n - 2n!)a + 1)^k.$$

For $\frac{1}{n!} \leq a$, we have

$$\begin{aligned} |F_{D_k}(a)| &\leq \int_0^{1/n!} t^{m-1} |F_{B_k}(t)| dt + \int_{1/n!}^a t^{m-1} |F_{B_k}(t)| dt \\ &\leq \frac{1}{m(an!)^m} C \left(\frac{n^n}{n!} - 1 \right)^k + \left(\frac{1}{m} - \frac{1}{m(an!)^m} \right) C(n^n a - 1)^k \\ &\leq \frac{C}{m} (n^n a - 1)^k. \end{aligned}$$

■

By the hook formula, we get

$$d_i = L(i) \frac{(in)!}{(i!)^n},$$

where

$$L(i) = \prod_{1 \leq j < l \leq n} (\lambda_j - \lambda_l + l - j) \frac{\prod_{j=1}^r (in + j)}{\prod_{j=1}^n \prod_{l=1}^{\lambda_j + n - j} (i + l)}.$$

The function $L(i)$ can be represented as the sum of fractions $m_0/((i + m_1) \dots (i + m_s))$, where m_0, \dots, m_s are integers. We put $B_k(i) = (-1)^{k-i} \frac{(in)!}{(i!)^n} C_k^i$ and $D_k = (-1)^{k-i} d_i C_k^i$. Thus $D_k = LB_k$ and $x_k = F_{D_k}$. According to the estimations preceding Lemma 2, estimations (5) hold for F_{B_k} , with $C = 1$. Applying Lemma 2, if needed many times, and taking into account additivity, we obtain estimations (5) for F_{D_k} with $C = L(0)$. Hence we get

$$\text{LEMMA 3. } \dim_K S^{\lambda+(k^n)}/f(S^{\lambda+(1^k)}) \leq L(0)(n^n - 1)^k.$$

Proof. It suffices to notice that $\dim_K S^{\lambda+(k^n)}/f(S^{\lambda+(1^k)}) = x_k(1)$. ■

We will call an irreducible $KG(r + kn)$ -module D^μ , where $\mu \in \Lambda^+(n, r + kn)$, degenerate if $\mu_n = 0$. Clearly $f(D^\mu) = 0$. Let $S^{\lambda+(k^n)} = V_0 \supset V_1 \supset \cdots \supset V_{m-1} \supset V_m = 0$ be a composition series. Applying Lemma 1 and the remark on finite dimensional modules following it, we get $\dim_K f(S^{\lambda+(k^n)}) = \dim_K f(V_0/V_1) + \cdots + \dim_K f(V_{m-1}/V_m)$. In addition, $\dim_K S^{\lambda+(k^n)} = \dim_K V_0/V_1 + \cdots + \dim_K V_{m-1}/V_m$. Subtracting the former from the latter, we get

$$\dim_K S^{\lambda+(k^n)} / f(S^{\lambda+(k^n)}) = \sum_{i=0}^{m-1} \dim_K (V_i/V_{i+1}) - \dim_K f(V_i/V_{i+1}).$$

Each term of this sum is nonnegative. If V_i/V_{i+1} is degenerate, then term i of this sum is $\dim_K V_i/V_{i+1}$. By Lemma 3, we get

THEOREM 3. *The sum of dimensions of all degenerate factors of $S^{\lambda+(k^n)}$ does not exceed $\dim_K S^{\lambda+(k^n)} / f(S^{\lambda+(k^n)})$, which in turn does not exceed $L(0)(n^n - 1)^k$.*

Theorem 6 from [5] now gives

COROLLARY 4. *The length of $S^{\lambda+(k^n)}$ does not exceed $L(0)(n^n - 1)^k$.*

It is well known that $\dim_K S^{\lambda+(k^n)} \sim (n^n)^k$. Therefore, the estimation of Corollary 4 is not trivial.

4. ASCENT AND DESCENT OPERATORS IN WEYL MODULES

In this section, we will use the notation from [1] and [5, Sections 1 and 2]. We will make now some stipulations for the objects we are going to deal with.

Let us take arbitrary positive integers N and R such that $N \geq R$ and take an arbitrary partition $\delta \in \Lambda^+(N, R)$. We fix some δ' -tableau T' that is a bijection from $[\delta']$ to $\{1, \dots, R\}$. We suppose that the Specht module $S^{\delta'}$ is given by the definition from Section 1. Take the following cyclic generator $s(T') = s(T', 1) \dots s(T', \delta'_1)$ of $S^{\delta'}$, where $s(T', t) = [x_{T'(1, t)}, \dots, x_{T'(\delta, t)}]$. Modules $S_{T', K}$ and $\bar{S}_{T', K}$ are defined in [1, Sections 6.3 and 6.4]. Let us map $S^{\delta'}$ isomorphically onto $S_{T', K}$ by the map $\pi s(T') \rightarrow \pi\{C(T')\}[R(T')]$. Define an anti-isomorphism from $S_{T', K}$ to $\bar{S}_{T', K}$ by the formula $\pi\{C(T')\}[R(T')] \rightarrow \pi[R(T)]\{C(T)\}$, where T is the transposed tableau of T' , i.e., $T(i, j) = T'(j, i)$.

Consider the Weyl module $V_{\delta, K}$ with the basic tableau T (see [1, Section 5]). Let $\Omega = (1^R, 0^{N-R})$ be the weight of $u = u(R)$. The map $\pi[R(T)]\{C(T)\} \rightarrow e_u \pi[R(T)]\{C(T)\}$ gives an isomorphism between $\bar{S}_{T, K}$

and $\xi_\Omega V_{\delta,K}$. Let $\varphi^\delta: S^{\delta'} \rightarrow \xi_\Omega V_{\delta,K}$ be the composition of all above mentioned maps. This map is an isomorphism of linear spaces, and we have $\varphi^\delta(\pi x) = s(\pi) \xi_{u\pi,u} \varphi^\delta(x)$ for $\pi \in G(R)$ and $x \in S^{\delta'}$. By virtue of the isomorphism between $\xi_\Omega S_K(N, R) \xi_\Omega$ and $KG(R)$ given by $\xi_{u\pi,u} \rightarrow \pi$, we assume that any $\xi_\Omega S_K(N, R) \xi_\Omega$ -module is a $KG(R)$ -module.

For any $\xi_\Omega S_K(N, R) \xi_\Omega$ -submodule V of $\xi_\Omega V_{\delta,K}$, we denote by V^+ the maximal $S_K(N, R)$ -submodule W in $V_{\delta,K}$ such that $\xi_\Omega W = V$. It is clear that $S_K(N, R)V \subset V^+$. In general, this inclusion is strict.

In this section, we put $T^\delta(i, j) = R - \sum_{k=1}^i \delta_k + j$ for $(i, j) \in [\delta]$. We assume now that the basic tableau T for $V_{\delta,K}$ is equal to T^δ .

Now we fix some numbers m, n, r such that $n \geq r > m$ and a partition $\lambda \in \Lambda^+(n, r)$, where $\lambda_1 = m$. Since in this section we want to follow the notation from [1] and [5], we will interpret \uparrow and \downarrow as \uparrow_m and \downarrow_m , respectively. We put $S = S_K(n, r)$ and $S' = S_K(n-1, r-m)$.

Let us recall some definitions from [5]. We have $I^*(n, r) = \{i \in I(n, r): i_1, \dots, i_{r-m} \geq 2 \text{ and } i_{r-m+1} = \dots = i_r = 1\}$. For any $i \in I^*(n, r)$, we have $\bar{i} = (i_1 - 1, \dots, i_{r-m} - 1)$. Fix the idempotent η of S that is equal to the sum of all $\xi_{\bar{\alpha}}$, where $\alpha \in \Lambda(n, r)$ and $\alpha_1 = m$. Let $E = K \cdot e_1 \oplus \dots \oplus K \cdot e_n$ and $\bar{E} = K \cdot e_1 \oplus \dots \oplus K \cdot e_{n-1}$ be linear spaces over K . The map $\theta: E^{\otimes r} \rightarrow \bar{E}^{\otimes r-m}$ is defined by the formula

$$\theta(e_i) = \begin{cases} e_{\bar{i}}, & \text{if } i \in I^*(n, r), \\ 0, & \text{otherwise.} \end{cases}$$

Consider the subalgebra S_1 of $\eta S \eta$ that has a K -basic $\{\xi_{i,j}: i, j \in I^*(n, r)\}$. The map $\theta_1: S_1 \rightarrow S'$ defined by $\theta_1(\xi_{i,j}) = \xi_{\bar{i}, \bar{j}}$ is an algebra isomorphism. In addition, we have $\theta(\xi x) = \theta_1(\xi) \theta(x)$ for $\xi \in S_1$ and $x \in E^{\otimes r}$. Consider the restriction $\bar{\theta} = \theta|_{\eta V_{\lambda,K}}$. Then $\bar{\theta}$ is an isomorphism from $\eta V_{\lambda,K}$ to $V_{\bar{\lambda},K}$. For any $\mu \in \Lambda^+(n, r)$ such that $\mu_1 = m$, $\bar{\mu}$ denotes the partition (μ_2, \dots, μ_n) .

Let us introduce the operators \uparrow and \downarrow (more explicitly \uparrow_m and \downarrow_m) that act on submodules of Weyl modules. For a submodule V of $V_{\lambda,K}$, we put $V\downarrow = \bar{\theta}(\eta V)$. For a submodule U of $V_{\bar{\lambda},K}$, we put $U\uparrow = S\bar{\theta}^{-1}U$.

Let $\omega = (1^r, 0^{n-r})$ and let $\omega' = (1^{r-m}, 0^{n-r+m-1})$. We assume the natural inclusion $G(r-m) \subset G(r) = G$. Then we have $C(T^{\bar{\lambda}}) \subset C(T^\lambda)$. Let us choose a set A of representatives of the right cosets of $C(T^{\bar{\lambda}})$ in $C(T^\lambda)$. Then $\{C(T^\lambda)\} = \{C(T^{\bar{\lambda}})\}A$. When we write elements of modules of the form $E^{\otimes t}$, we usually omit the sign \otimes . Thus we have $e_i e_j = e_i \otimes e_j = e_{ij}$.

We have the following simple assertion establishing the relation between the operators $\uparrow, \downarrow, \uparrow$, and \downarrow .

LEMMA 4. *Let V be a submodule of $S^{\lambda'}$ and let U be a submodule of $S^{\lambda'-(1^m)}$. Then we have $V\downarrow = (\varphi^{\bar{\lambda}})^{-1} \xi_{\omega'}((\varphi^\lambda V)^+ \downarrow)$ and $U\uparrow = (\varphi^\lambda)^{-1} \xi_\omega(W\uparrow)$ for any W such that $(\varphi^{\bar{\lambda}} U)^+ \subset W \subset S' \varphi^{\bar{\lambda}} U$.*

Proof. Put $V' = (\varphi^\lambda)^{-1} \xi_{\omega'}((\varphi^\lambda V)^+ \downarrow)$ and $U' = (\varphi^\lambda)^{-1} \xi_\omega(W \uparrow)$. Let us fix the two sequences $v_1 = (1, \dots, r-m)$ and $v_2 = (2, \dots, r-m+1)$.

Take $f \in V \downarrow$. Then $f[x_{r-m+1}, \dots, x_r] \in V$. We get $\varphi^\lambda(f[x_{r-m+1}, \dots, x_r]) = \varphi^\lambda(f) \sum \{e_i; i \sim (r-m+1, \dots, r)\} \{A\} \in \varphi^\lambda V$. Let $x = (\xi_{v_2, v_1} \varphi^\lambda(f)) e_1^m \{A\}$. Then $x \in (\varphi^\lambda V)^+ \cap \eta V_{\lambda, K}$. Therefore $\varphi^\lambda(f) = \bar{\theta}(x) = \bar{\theta}(\eta x) \in \xi_{\omega'}((\varphi^\lambda V)^+ \downarrow)$. Hence $f \in V'$.

Conversely, take $f \in V'$. We put $g = \varphi^\lambda(f)$. Then $g \in \xi_{\omega'}((\varphi^\lambda V)^+ \downarrow)$. We have $g = \bar{\theta}(h\{C(T^\lambda)\})$ for some h , where $h\{C(T^\lambda)\} \in \xi_{\omega''}((\varphi^\lambda V)^+)$ and $\omega'' = (m, 1^{r-m}, 0^{n-r+m-1})$. The last formula is obtained from homogeneity of $(\varphi^\lambda V)^+$ and the fact that g belongs to the weight space $\xi_{\omega'} V_{\lambda, K}$. In addition, we ignore all homogeneous components that are taken to zero by $\bar{\theta}$. We can also assume that $h = \bar{h} e_1^m$. Then $h\{C(T^\lambda)\} = (\bar{h}\{C(T^\lambda)\}) e_1^m \{A\}$. Therefore $g = \xi_{v_1, v_2} \bar{h}\{C(T^\lambda)\}$. It follows that $(\xi_{v_2, v_1} g) e_1^m \{A\} = h\{C(T^\lambda)\} \in (\varphi^\lambda V)^+$. We have $\varphi^\lambda(f[x_{r-m+1}, \dots, x_r]) \in \xi_\omega S \cdot (\xi_{v_2, v_1} g) e_1^m \{A\} = \xi_\omega S \cdot h\{C(T^\lambda)\} \subset \varphi^\lambda V$. Then $f[x_{r-m+1}, \dots, x_r] \in V$ and $f \in V \downarrow$.

It follows from our formulas that $W \uparrow$ is the sum of elements of the form $\alpha(\xi_{v_2, v_1} f) e_1^m \{A\}$, where $f \in W$ and $\alpha \in S$. Then $\xi_\omega(W \uparrow)$ is the sum of elements of the same form except for $\alpha \in \xi_\omega S$. Hence we get that $\xi_\omega(W \uparrow)$ is also the sum of elements of the form $(\xi_{v, u'} g) \sum_{i \sim v'} e_i \{A\}$, where $u' = u(r-m)$, $vv' \sim u(r)$, and $g \in \varphi^\lambda U$. Thus the equality $U \uparrow = U'$ readily follows. ■

Let us fix the following notation: $I = I(n, r)$, $I^* = I^*(n, r)$, and $G = G(r)$.

LEMMA 5. *Let U be an S_1 -submodule of the module $\eta E^{\otimes r} \{C(T^\lambda)\}$. Then U is also an $\eta S \eta$ -submodule.*

Proof. The space $\eta E^{\otimes r}$ has a basis $\{e_i; i \in I^* G\}$. The expression $i \in I^* G$ means that exactly m integers in the sequence i_1, \dots, i_r are equal to 1. Let X_t be the set of numbers that stay in column t of T^λ and let \bar{X}_t be the set of numbers that stay in column t of $T^{\bar{\lambda}}$. Then we have $\bar{X}_t = X_t \cap \{1, \dots, r-m\}$, $\lambda_1 < t \Rightarrow X_t = \emptyset$, and $\lambda_2 < t \Rightarrow \bar{X}_t = \emptyset$.

We denote by I' the set of all elements i of $I(n, r-m)$ such that the numbers i_1, \dots, i_{r-m} are greater than 1. Let I'' be the subset of I' consisting of elements i such that, for any t and distinct a and b from \bar{X}_t , we have $i_a \neq i_b$. Then $\eta E^{\otimes r} \{C(T^\lambda)\}$ has the elements $e_i e_1^m \{C(T^\lambda)\}$, where $i \in I''$ and the sequences i are taken from different $C(T^\lambda)$ -orbits of I'' , as its basis. In addition, if $j = i\pi$ for $\pi \in C(T^{\bar{\lambda}})$, then $e_j e_1^m \{C(T^\lambda)\} = s(\pi) e_i e_1^m \{C(T^\lambda)\}$.

Let us take an element of $\eta S \eta$ of the form $\xi_{h, j}$, where $h, j \in I^* G$. We will prove that the action of $\xi_{h, j}$ on $\eta E^{\otimes r} \{C(T^\lambda)\}$ is the action on the same space of some element ξ of S_1 .

Let us take $i \in I''$ and calculate the value of $x = \xi_{h,j} e_i e_1^m \{C(T^\lambda)\}$. We can assume $j = j'(1^m)$. We have $j' \in I'$. Let $h = h'h''$, where h', h'' are sequences of lengths $r - m$ and m , respectively. We denote by c the number of 1s in h' . We have

$$x = (\xi_{h',j'} e_i) \sum_{k \sim h''} e_k \{C(T^\lambda)\}.$$

Let $\xi_{h',j'} e_i = \sum_{q \sim h'} \alpha_q e_q$, where $\alpha_q \in K$. We put $y_q = e_q \sum_{k \sim h''} e_k \{C(T^\lambda)\}$ for $q \sim h'$. Then we have $x = \sum_{q \sim h'} \alpha_q y_q$. Let us calculate y_q .

We denote by $\tilde{h} = (\tilde{h}_1, \dots, \tilde{h}_c)$ the sequence that remains of h'' after removing all 1s. Let z be the sequence obtained from h' by replacing the s th left 1 with \tilde{h}_s for every $s = 1, \dots, c$. As is easy to see, $z \in I'$.

We have now

$$y_q = (-1)^c (\xi_{z,h'} e_q) e_1^m \{C(T^\lambda)\}. \quad (6)$$

To prove formula (6), one must consider two cases. If, for any t , there exist distinct $a, b \in \bar{X}_t$ such that $q_a = q_b = 1$, then both sides of formula (6) are equal to zero. Suppose now that there does not exist such t . Let $a(1), \dots, a(c)$ be distinct numbers such that $q_{a(s)} = 1$ and $a(s) \in \bar{X}_{t(s)}$. By our assumption, all $t(1), \dots, t(c)$ are distinct. Thus all permutations $(a(s), r - m + t(s))$ belong to $C(T^\lambda)$ and $\{C(T^\lambda)\} = (-1)^c (a(1), r - m + t(1)) \dots (a(c), r - m + t(c)) \{C(T^\lambda)\}$. If k in the definition of y_q is such that $k_{t(s)} = 1$ for some $s = 1, \dots, c$, then $e_q e_k (a(s), r - m + t(s)) = 0$ and $e_q e_k \{C(T^\lambda)\} = 0$. If all $m - c$ 1s in k are at places with numbers from $\{1, \dots, m\} \setminus \{t(1), \dots, t(c)\}$, then $e_q e_k \{C(T^\lambda)\} = (-1)^c e_{q'} e_1^m \{C(T^\lambda)\}$, where $q'_{a(s)} = k_{t(s)}$ for $s = 1, \dots, c$ and $q'_t = q_t$ for other t . In addition, we have $(k_{t(1)}, \dots, k_{t(c)}) \sim \tilde{h}$. Thus formula (6) follows.

We have now $x = (-1)^c (\xi_{z,h'} \xi_{h',j'} e_i) e_1^m \{C(T^\lambda)\}$. Let us multiply two elements of the Schur algebra as follows: $\xi_{z,h'} \xi_{h',j'} = \sum_{i', i'' \in I'} \beta_{i', i''} \xi_{i', i''}$, where $\beta_{i', i''} \in K$. We define now the required element ξ by the formula $\xi = (-1)^c \sum_{i', i'' \in I'} \beta_{i', i''} \xi_{i'(1^m), i''(1^m)}$. We see that $\xi \in S_1$, since all $\xi_{i'(1^m), i''(1^m)}$ from the above sum belong to S_1 . We have $x = \xi e_i e_1^m \{C(T^\lambda)\}$. ■

THEOREM 4. (a) Let $\mu \in \Lambda^+(n, r)$, where $\mu_1 = m$, and let V_1, V_2 be two S -modules such that $V_2 \subset V_1 \subset V_{\lambda, K}$. Then the S -composition multiplicity of $F_{\mu, K}$ in V_1/V_2 is equal to the S' -composition multiplicity of $F_{\bar{\mu}, K}$ in $V_1 \downarrow / V_2 \downarrow$.

(b) Let $\nu \in \Lambda^+(n - 1, r - m)$ and let V_1, V_2 be two S' -modules such that $V_2 \subset V_1 \subset V_{\lambda, K}$. Then the S -composition multiplicity of $F_{(m)\nu, K}$ in $V_1 \uparrow / V_2 \uparrow$ is equal to the S' -composition multiplicity of $F_{\nu, K}$ in V_1/V_2 .

Proof. (a) immediately follows from [5] and our definitions.

(b) By [1, Lemma 6.6b], the S -composition multiplicity of $F_{(m)\nu, K}$ in $V_1 \uparrow / V_2 \uparrow$ is equal to the $\eta S \eta$ -composition multiplicity of $\eta F_{(m)\nu, K}$ in $\eta(V_1 \uparrow) / \eta(V_2 \uparrow) = \bar{\theta}^{-1} V_1 / \bar{\theta}^{-1} V_2$. For the last rearrangement, we used Lemma 5. By the results from [5, Section 2], the last composition multiplicity is equal to the S_1 -composition multiplicity of $\eta F_{(m)\nu, K}$ in $\bar{\theta}^{-1} V_1 / \bar{\theta}^{-1} V_2$. Applying the isomorphism $\bar{\theta}$, we obtain that this composition multiplicity is equal to the S' -composition multiplicity of $F_{\nu, K}$ in V_1 / V_2 . ■

If all composition factors of an S -module V are of the form $F_{\mu, K}$, where $\mu_1 < m$, then we call the module V degenerate (more precisely, m -degenerate).

COROLLARY 5. (1) For any submodule U of $V_{\lambda, K}$, we have $U \uparrow \downarrow = U$.

(2) For any submodule U of $V_{\lambda, K}$, we have $U \downarrow \uparrow \subset U$.

(3) If U and V are two submodules of $V_{\lambda, K}$ such that $V \subset U$ and U/V is degenerate, then $V \downarrow = U \downarrow$ and $U \downarrow \uparrow \subset V$.

(4) Let U be a submodule of $V_{\lambda, K}$ and let U' be the smallest submodule of U such that U/U' is degenerate. Then $U \downarrow \uparrow = U'$.

Proof. (1) We have $U \uparrow = S \bar{\theta}^{-1} U = S \eta \bar{\theta}^{-1} U$. It is evident that $\bar{\theta}^{-1} U$ is an S_1 -submodule of $\eta E^{\otimes r} \{C(T^\lambda)\}$. Therefore, by Lemma 5 we get that $\bar{\theta}^{-1} U$ is an $\eta S \eta$ -module and $\eta(U \uparrow) = \eta S \eta \bar{\theta}^{-1} U = \bar{\theta}^{-1} U$. Hence $U \uparrow \downarrow = \bar{\theta}(\eta(U \uparrow)) = U$.

(2) We have $U \downarrow \uparrow = S \bar{\theta}^{-1} \bar{\theta}(\eta U) = S \eta U \subset U$.

(3) We have $\eta V = \eta U$, since $\eta F_{\mu, K} = 0$ for any μ such that $\mu_1 < m$. Therefore $V \downarrow = U \downarrow$ and $U \downarrow \uparrow = V \downarrow \uparrow \subset V$.

(4) By (3), we have $U \downarrow = U' \downarrow$ and $U \downarrow \uparrow \subset U'$. Taking into account (1), we have $(U \downarrow \uparrow) \downarrow = U' \downarrow$. If we suppose that $U \downarrow \uparrow \subsetneq U'$, then by Theorem 4(a) the module $U'/U \downarrow \uparrow$ is nonzero and degenerate. This is a contradiction. ■

We can prove similar results about the operators \uparrow and \downarrow .

THEOREM 5. (a) Let $\mu \in \Lambda^+(n, r)$ be column p -regular, let $\mu_1 = m$, and let V_1, V_2 be two $KG(r)$ -modules such that $V_2 \subset V_1 \subset S^\lambda$. Then the $G(r)$ -composition multiplicity of $D^{\mu'}$ in V_1/V_2 is equal to the $G(r-m)$ -composition multiplicity of $D^{\mu'-(1^m)}$ in $V_1 \downarrow / V_2 \downarrow$.

(b) Let $\nu \in \Lambda^+(n-1, r-m)$ be a partition such that $(m)\nu$ is column p -regular and let V_1, V_2 be two $KG(r-m)$ -modules such that $V_2 \subset V_1 \subset S^{\bar{\lambda}}$. Then the $G(r-m)$ -composition multiplicity of $D^{\nu'}$ in V_1/V_2 is equal to the $G(r)$ -composition multiplicity of $D^{\nu'+(1^m)}$ in $V_1 \uparrow / V_2 \uparrow$.

Proof. (a)

the $G(r)$ -composition multiplicity of $D^{\mu'}$ in V_1/V_2
 = the $G(r)$ -composition multiplicity of D_{μ} in $\varphi^{\lambda}V_1/\varphi^{\lambda}V_2$
 = the S -composition multiplicity of $F_{\mu,K}$ in $(\varphi^{\lambda}V_1)^+ / (\varphi^{\lambda}V_2)^+$
 = the S' -composition multiplicity of $F_{\bar{\mu},K}$ in $(\varphi^{\lambda}V_1)^+ \downarrow / (\varphi^{\lambda}V_2)^+ \downarrow$
 (Theorem 4)
 = the $G(r-m)$ -composition multiplicity of $D_{\bar{\mu},K}$ in
 $\xi_{\omega'}((\varphi^{\lambda}V_1)^+ \downarrow) / \xi_{\omega'}((\varphi^{\lambda}V_2)^+ \downarrow)$
 = the $G(r-m)$ -composition multiplicity of $D^{\mu'-(1^m)}$ in $V_1 \downarrow / V_2 \downarrow$
 (Lemma 4).

(b)

the $G(r-m)$ -composition multiplicity of $D^{\nu'}$ in V_1/V_2
 = the $G(r-m)$ -composition multiplicity of D_{ν} in $\varphi^{\bar{\lambda}}V_1/\varphi^{\bar{\lambda}}V_2$
 = the S' -composition multiplicity of $F_{\nu,K}$ in $(\varphi^{\bar{\lambda}}V_1)^+ / (\varphi^{\bar{\lambda}}V_2)^+$
 = the S -composition multiplicity of $F_{(m)\nu,K}$ in $(\varphi^{\bar{\lambda}}V_1)^+ \uparrow / (\varphi^{\bar{\lambda}}V_2)^+ \uparrow$
 (Theorem 4)
 = the $G(r)$ -composition multiplicity of $D_{(m)\nu}$ in
 $\xi_{\omega}((\varphi^{\bar{\lambda}}V_1)^+ \uparrow) / \xi_{\omega}((\varphi^{\bar{\lambda}}V_2)^+ \uparrow)$
 = the $G(r)$ -composition multiplicity of $D^{\nu'+(1^m)}$ in $V_1 \uparrow / V_2 \uparrow$
 (Lemma 4).

By analogy with S -modules, we call a $KG(r)$ -module V degenerate (more precisely, m -degenerate) if all composition factors of V are of the form D^{μ} , where $\mu_m = 0$.

Now we can answer the question about what happens to a nonzero composition factor $V_1/V_2 \cong D^{\mu}$ when we pass to $V_1 \uparrow / V_2 \uparrow$. In fact, if $\mu + (1^m)$ is p -regular (this is always true if $m \leq p$), then the composition multiplicity of $D^{\mu+(1^m)}$ in $V_1 \uparrow / V_2 \uparrow$ is equal to 1 (in particular, $V_2 \uparrow \subsetneq V_1 \uparrow$) and all other composition factors of $V_1 \uparrow / V_2 \uparrow$ are degenerate. If $\mu + (1^m)$ is p -singular, then $V_1 \uparrow / V_2 \uparrow$ is degenerate.

We have the following analog of Corollary 5.

COROLLARY 6. *Suppose that $m \leq p$. Then*

- (1) *For any submodule U of $S^{\lambda'-(1^m)}$, we have $U\uparrow\downarrow = U$.*
- (2) *For any submodule U of $S^{\lambda'}$, we have $U\downarrow\uparrow \subset U$.*
- (3) *If U and V are two submodules of $S^{\lambda'}$ such that $V \subset U$ and U/V is degenerate, then $V\downarrow = U\downarrow$ and $U\uparrow\downarrow \subset V$.*
- (4) *Let U be a submodule of $S^{\lambda'}$ and let U' be the smallest submodule of U such that U/U' is degenerate. Then $U\uparrow\downarrow = U'$.*

Proof. As we noticed above, if $m \leq p$, then $\mu \in \Lambda^+(m, r-m)$ is p -regular implies $\mu + (1^m)$ is p -regular. We will use this fact in the succeeding proof to apply Theorem 5.

(1) By Theorem 5, the decomposition matrices of U and $U\uparrow\downarrow$ are the same. Taking into account the inclusion $U \subset U\uparrow\downarrow$, we get the needed result.

(2) is evident.

(3) Suppose that $V\downarrow \subsetneq U\downarrow$ and let D^μ , where μ is p -regular, be a composition factor of $U\downarrow/V\downarrow$. By Theorem 2(a), we get that $D^{\mu+(1^m)}$ is a composition factor of U/V . This contradicts the fact that U/V is degenerate. Then $V\downarrow = U\downarrow$ and $U\downarrow\uparrow = V\downarrow\uparrow \subset V$.

(4) By (3), we have $U\downarrow = U'\downarrow$ and $U\downarrow\uparrow \subset U'$. Taking into account (1), we have $(U\downarrow\uparrow)\downarrow = U'\downarrow$. If we suppose that $U\downarrow\uparrow \subsetneq U'$, then by Theorem 5(a), the module $U'/U\downarrow\uparrow$ is nonzero and degenerate. This is a contradiction. ■

It is easy to show that if $m > p$, then (2)–(4) may not be true. Indeed, let m be any integer greater than p and let $\lambda' = (k+1, 1^{m-1})$ for any positive integer k . We can see that the module $S^{\lambda'}$ is degenerate. But $S^{\lambda'}\downarrow = S^{(k)}$ is a nonzero module.

For Section 5, it is convenient to redefine the operators \uparrow and \downarrow as follows. Let $S'' = S_K(n-m, r-m)$ and let \tilde{E} be the subspace of \bar{E} spanned by e_1, \dots, e_{n-m} . We have the inclusion $\tilde{E}^{\otimes r-m} \subset \bar{E}^{\otimes r-m}$ and we can assume $V_{\tilde{\lambda}, K} = V_{\tilde{\lambda}, K} \cap \tilde{E}^{\otimes r-m}$, where $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_{n-m+1})$ is a partition from $\Lambda^+(n-m, r-m)$. For an S -submodule V of $V_{\tilde{\lambda}, K}$, we put $V\downarrow = \bar{\theta}(\eta V) \cap V_{\tilde{\lambda}, K}$, and for an S'' -submodule U of $V_{\tilde{\lambda}, K}$, we put $U\uparrow = S\bar{\theta}^{-1}(S'U)$.

Let us see what effect this redefinition has on Theorem 4 and Corollary 5. In the formulation of Theorem 4, we must have the following alterations. First we replace S' by S'' . In (a), we replace $\bar{\mu}$ by $\tilde{\mu} = (\mu_2, \mu_3, \dots, \mu_{n-m+1})$. In (b), we take $\nu \in \Lambda^+(n-m, r-m)$ and write $(m)\nu(0^{m-1})$ instead of $(m)\nu$. Then all results remain true by virtue of the fact that the categories $M_K(n-1, r-m)$ and $M_K(n-m, r-m)$ are isomorphic. See [1, Section 6.5] for exact formulations. Corollary 5 must be changed to: (1) $\eta(U\uparrow) = \bar{\theta}^{-1}S'U$

and $U\uparrow\downarrow = S'U \cap V_{\lambda, K} = U$; (2) $S'(U\downarrow) = \bar{\theta}(\eta U)$ and $U\downarrow\uparrow = S\eta U \subset U$; (3) and (4) the same.

5. STABILIZATION PROBLEM

We have the following assertion.

THEOREM 6. *For $n = 1, 2$, Problem 2 has a positive solution.*

For $n = 1$, all modules $S^{\lambda+(k^n)}$ are one-dimensional and there is nothing to prove. Assume now that $n = 2$. Let $\lambda = (g, c) \in \Lambda^+(2, r)$. Consider the following nonzero element of S^λ ,

$$\alpha[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \dots [x_{i_{2c-1}}, x_{i_{2c}}]x_{i_{2c+1}}^1 \dots x_{i_g}^1, \quad (7)$$

where $i \sim u(r)$ and $\alpha \in K$. We will associate now with element (7) its code. Let us reserve the following letters a_s, b_s, y , where $s = 1, \dots, c$. We put $j_s = \min\{i_{2s-1}, i_{2s}\}$ and $j'_s = \max\{i_{2s-1}, i_{2s}\}$. There exists a unique permutation $\sigma \in G(c)$ such that $j_{\sigma(1)} < \dots < j_{\sigma(c)}$. Define the code of element (7) to be the word in a_s, b_s, y of length r that has at place t from the left the letter a_s if $t = j_{\sigma(s)}$, the letter b_s if $t = j'_{\sigma(s)}$, and the letter y if $t = i_s$ for $s \geq 2c + 1$. The code is well defined, since all representations of an element in form (7) are the same up to the order of factors $[x_s, x_t]$ and x_k , rearrangements $[x_s, x_t] \rightarrow [x_t, x_s]$, and the sign.

For example, the code of $[x_2, x_4][x_3, x_1]x_5^1x_6^1$ is $a_1a_2b_1b_2yy$. Note that an element can be reconstructed by its code up to a nonzero coefficient.

Let us introduce an order on the letters a_s, b_s, y by the rule: $a_s > y > b_t$ for any s and t , $s < t \Rightarrow a_s < a_t$, and $s < t \Rightarrow b_s > b_t$. We compare words in a_s, b_s, y lexicographically. Notice that if u is a code, then for any $s = 1, \dots, c$, the letter a_s precedes b_s in u and $u = u_1a_su_2a_tu_3$ implies $s < t$.

A code u is called overlapping free if

- (1) u is not presentable in the form $u = u_1a_su_2yu_3b_su_4$.
- (2) u is not presentable in the form $u = u_1a_su_2a_tu_3b_su_4b_tu_5$.

If u is presentable in form (1), we say that the pair (a_s, b_s) and the letter y form an overlapping. If u is presentable in form (2), we say that the pairs (a_s, b_s) and (a_t, b_t) form an overlapping. We claim that any element of S^λ is presentable as a linear combination of elements of form (7) with overlapping free codes.

Suppose first that $u = u_1a_su_2yu_3b_su_4$ is the code of an element. Let us apply the Garnir relation acting on the variables x_l , where l is any number $|u_1a_s|$, $|u_1a_su_2y|$, or $|u_1a_su_2yu_3b_s|$. Then we represent the initial element as a linear combination of elements with the codes u_1yu' and $u_1a_su_2b_su_3yu_4$. Both these codes are strictly less than u .

Let now $u = u_1 a_s u_2 a_t u_3 b_s u_4 b_t u_5$ be the code of an element. In addition, we have $s < t$. Let us apply the Garnir relation acting on the variables x_l , where l is any number $|u_1 a_s u_2 a_t|$, $|u_1 a_s u_2 a_t u_3 b_s|$, or $|u_1 a_s u_2 a_t u_3 b_s u_4 b_t|$. Then we represent the initial element as a linear combination of elements with the codes $u_1 a_s u_2 b_s u'$ and $u_1 a_s u_2 a_t u_3 b_t u_4 b_s u_5$. We see again that both codes are strictly less than u .

Consider now some insertion operation. Suppose that we have $k = 1, \dots, r$. For any word u of length r in the letters a_s, b_s, y , we define the word $u^{(k)}$ as follows. Let $u = vw$, where v is a word of length k . Denote by w_0 the word obtained from w by the substitution: $a_s \rightarrow a_{s+1}$ and $b_s \rightarrow b_{s+1}$ for all $s = 1, \dots, c$ such that a_s occurs in w . We also denote by d the number of letters a_s in the word v . Then we put $u^{(k)} = va_{d+1}b_{d+1}w_0$.

Let σ_k be a permutation of $G(r+2)$ such that $\sigma_k(s) = s$ for $s \leq k$ and $\sigma_k(s) = s+2$ for $k+1 \leq s \leq r$. Assume that u is the code of element (7). Then $u^{(k)}$ is the code of

$$[x_{k+1}, x_{k+2}][x_{\sigma(i_1)}, x_{\sigma(i_2)}] \cdots [x_{\sigma(i_{2c-1})}, x_{\sigma(i_{2c})}] x_{\sigma(i_{2c+1})}^1 \cdots x_{\sigma(i_g)}^1.$$

If the code u is overlapping free, then the code $u^{(k)}$ is also overlapping free.

Let \bar{u} be the overlapping free code of some other element of form (7) and let $u > \bar{u}$. Assume $u = vw$ and $\bar{u} = \bar{v}\bar{w}$, where v and \bar{v} are words of length k , $u^{(k)} = va_{d+1}b_{d+1}w_0$, and $\bar{u}^{(k)} = \bar{v}a_{\bar{d}+1}b_{\bar{d}+1}\bar{w}_0$. There are two possible cases: (1) $v > \bar{v}$; (2) $v = \bar{v}$ and $w > \bar{w}$.

In case (1), we immediately obtain $u^{(k)} > \bar{u}^{(k)}$. Consider now case (2). We have $d = \bar{d}$. Let $w = w'xw''$ and $\bar{w} = w'\bar{x}\bar{w}''$, where x and \bar{x} are letters and $x > \bar{x}$. When we pass to the words w_0 and \bar{w}_0 , the part w' in both words w and \bar{w} is rearranged in the same way, since $v = \bar{v}$ and $d = \bar{d}$. There are the following five possible cases for the pair (x, \bar{x}) : (1) (a_s, a_t) for $s > t$; (2) (a_s, y) ; (3) (y, b_t) ; (4) (a_s, b_t) ; (5) b_s, b_t for $s < t$. When we pass to w_0 and \bar{w}_0 in the first four cases, the pair (x, \bar{x}) turns to (1) (a_{s+1}, a_{t+1}) ; (2) (a_{s+1}, y) ; (3) (y, b_t) or (y, b_{t+1}) ; (4) (a_{s+1}, b_t) or (a_{s+1}, b_{t+1}) . Thus in any case, the first component of the resulting pair is strictly greater than the second one.

Consider the fifth case. If a_s occurs in v , then (x, \bar{x}) turns to (b_s, b_t) or to (b_s, b_{t+1}) . Suppose now that a_s occurs in w' . If a_t occurs in w' , then (x, \bar{x}) turns to (b_{s+1}, b_{t+1}) . Suppose that a_t occurs in v . The letter b_s occurs neither in v nor in w' . Thus b_s is in the word \bar{w}'' . Then the pairs (a_s, b_s) and (a_t, b_t) form an overlapping in the code \bar{u} . Therefore, the last case is impossible. We see that in all possible cases, the first component is strictly greater than the second one. Therefore $u^{(k)} > \bar{u}^{(k)}$ in all cases.

Suppose that we have a representation of an element $f \in S^\lambda$ as a sum of elements of form (7) with overlapping free codes. Let us pick up the

summand with the greatest code. We associate this code with the given representation of f . According to the results from [3], the set of all overlapping free codes is finitely based with respect to the described insertion operation. Thus we obtain the assertion of the theorem. ■

Let us formulate the analog of Problem 2 for Weyl modules. We assume that the operators \uparrow and \downarrow are already redefined as in the end of Section 4.

Problem 3. Let $\lambda \in \Lambda^+(r, r)$, let $\lambda_1 \leq m$, and let $V_i \subset V_{(m^{q_i})\lambda, K}$, $i \in \mathbb{N}$, where $0 \leq q_1 < q_2 < \dots$, be a sequence of submodules such that $V_i \uparrow^{q_{i+1}-q_i} \subset V_{i+1}$ for every $i \in \mathbb{N}$. Does there exist any $N \in \mathbb{N}$ such that, for every $i > N$, we have $V_N \uparrow^{q_i-q_N} = V_i$?

Notice that if Problem 3 has a positive solution, so does Problem 2, where $n = m$ and λ in Problem 2 is replaced by λ' . The advantage of transition from Problem 2 to Problem 3 is that the operators \uparrow and \downarrow in the general case behave more regularly than \uparrow and \downarrow .

Let $\lambda \in \Lambda^+(r, r)$ and let $\lambda_1 \leq 2$. It suffices to consider only homogeneous elements of $V_{\lambda, K}$. Take any $\alpha \in \Lambda(r, r)$ such that $V_{\lambda, K}^\alpha \neq 0$. Then $\alpha_s \leq 2$ for $s = 1, \dots, r$. Denote by t the number of 2s in α . Let i_1, \dots, i_t be the places in α at which 2s stand. By ω we denote the result of replacing all 2s in α by 0s. Then $\lambda = (2')\mu$ for some μ . Every element of $V_{\lambda, K}^\alpha$ has the form $fe_{i_1}^2 \cdots e_{i_t}^2\{A\}$, where $f \in V_{\mu(0'), K}^\omega$ and A is a set of representatives of the right cosets of $C(T^\mu)$ in $C(T^\lambda)$. From this fact and Theorem 6, we obtain

THEOREM 7. For $m = 1, 2$, Problem 3 has a positive solution.

For $n, m \geq 3$, Problems 2 and 3 are still open.

Our main suspects for counterexamples to Problem 2 are sequences consisting of maximal submodules of Specht modules. In connection with this, we will prove one result that links Problem 2 with structure of injective hulls.

In the sequel, for any $KG(r)$ or $S_K(N, r)$ -module M , we denote by $d(M)$ the dual module of M .

LEMMA 6. Suppose that $\lambda \in \Lambda^+(n, r)$ is p -regular, that $\lambda'_1 = n$, and that V is a submodule of S^λ . For $n < p$, the following two conditions are equivalent:

$$(1) \quad V \downarrow \uparrow \not\subseteq V.$$

(2) There exist some module L and an embedding $\iota: d(S^\lambda/V) \rightarrow L$ such that $\text{Im } \iota$ is an essential submodule of L and $L/\text{Im } \iota \cong D^\mu$, where $\mu \in \Lambda^+(n-1, r)$.

For arbitrary n and p , (1) implies (2).

Proof. It is evident that (1) implies (2). Assume $n < p$ and (2). Take any $N \geq r$ and put $S = S_K(N, r)$, $G = G(r)$, $\omega = (1^r, 0^{N-r})$, and $e = \xi_\omega$. The map $\xi_{u(r)\pi, u} \rightarrow \pi$ is an isomorphism between rings eSe and KG and thus it allows us to make every KG -module into an eSe -module and vice versa. Let also $f: \text{mod } S \rightarrow \text{mod } eSe$ be the Schur functor (see [1, Section 6.2]).

It is known that $V_{\lambda', K}^\omega$ is isomorphic to the dual of $S^{\lambda'}$. Under the standard maps (see Section 4), V is taken to some submodule V' of $V_{\lambda', K}^\omega$ (V' is an eSe -module). In addition, $V \cong V' \otimes S^{(1^r)}$.

Put $U = SV'$. We have $e(V_{\lambda', K}/U) \cong (S^\lambda/V) \otimes S^{(1^r)}$. From (2), it follows that there exists a module M , which is isomorphic to $L \otimes S^{(1^r)}$, and an embedding $\varphi: d(S^\lambda/V) \otimes S^{(1^r)} \rightarrow M$ such that $\text{Im } \varphi$ is an essential submodule of M and $M/\text{Im } \varphi \cong D_{\mu'}$.

In [1], the following objects are introduced: $D_{\lambda', K}$, which is the dual of $V_{\lambda', K}$, ${}^{\lambda'}A_K(n, r)$, and the contravariant form $(,): V_{\lambda', K} \times D_{\lambda', K} \rightarrow K$. In addition, $D_{\lambda', K} \subset {}^{\lambda'}A_K(n, r)$. Let is put $W = \{w \in D_{\lambda', K}: (U, w) = 0\}$. We have

$$eW \cong d(S^\lambda/V) \otimes S^{(1^r)}.$$

Therefore, we can assume that M and φ are chosen so that $eW = \text{Im } \varphi$.

We put $h(M) = Se \otimes_{eSe} M$. Then $eh(M) \cong M$. Let H denote the subspace of $h(M)$ spanned by $\{s_1 e \otimes ew_1 - s_2 e \otimes ew_2: s_1, s_2 \in S, w_1, w_2 \in W, s_1 ew_1 = s_2 ew_2\}$. Clearly, H is a submodule of $h(M)$ and $eH = 0$.

Let W' denote the submodule of $h(M)/H$ generated by elements of the form $se \otimes ew + H$, where $s \in S$ and $w \in W$. Consider the map $\psi: W' \rightarrow W$ defined by $\psi(se \otimes ew + H) = sew$. This map is well defined and is an injection. Since W has no composition factors corresponding to column p -singular partitions, we have $SeW = W$ and $\psi(W') = W$. Thus ψ is an isomorphism.

Since $W \subset {}^{\lambda'}A_K(n, r)$ and the latter module is injective, there exists an extension $\bar{\psi}: h(M)/H \rightarrow {}^{\lambda'}A_K(n, r)$ of ψ . Applying the Schur functor f , we obtain the following commutative diagram:

$$\begin{array}{ccc} eW' & \subset & e(h(M)/H) \\ \downarrow f(\psi) & & \downarrow f(\bar{\psi}) \\ eW & \subset & e {}^{\lambda'}A_K(n, r) \end{array}.$$

The map $m \rightarrow e \otimes m + H$ takes M isomorphically to $e(h(M)/H)$. Therefore, there exists some homomorphism $\tau: M \rightarrow e {}^{\lambda'}A_K(n, r)$ that acts on eW identically. Since eW is an essential submodule of M , the map τ is an embedding.

The standard homomorphism takes $e {}^{\lambda'}A_K(n, r)$ to $M^{\lambda'}$ and its submodule $eD_{\lambda', K}$ to $S^{\lambda'}$. According to [4, Section 17], the module $e {}^{\lambda'}A_K(n, r)$ has

a filtration with factors isomorphic to S^ν , where $\nu \supseteq \lambda'$, such that $eD_{\lambda', K}$ is its minimal nonzero module.

The following two cases are possible: $\tau(M) \subset eD_{\lambda', K}$, or $D_{\mu'}$ is a bottom factor of some S^ν , where $\nu \supset \lambda'$. Let us show that the latter case is impossible. Suppose that $\varepsilon: D_{\mu'} \rightarrow S^\nu$ is an embedding. Since $S^\nu \subset M^\nu$ and the modules $D_{\mu'}$ and M^ν are self-dual, there exists an epimorphism $\pi: M^\nu \rightarrow D_{\mu'}$. Consider a cyclic generator of M^ν of the form $v = x_1^1 x_2^1 \dots x_{\nu_1}^1 u$. Since $\nu \supset \lambda'$, we get $\nu_1 \geq n$. Let $g = \sum_{\sigma \in G(n)} \sigma$. Then $gD_{\mu'} = 0$. We have $v = \frac{1}{n!}gv$. The last division is possible as $n < p$. We have $\pi(v) = \frac{1}{n!}g\pi(v) = 0$. Therefore $\pi = 0$. This is a contradiction.

We have $eW \subset \tau(M) \subset eD_{\lambda', K}$. Applying the form $(,)$ and multiplying by $S^{(1')}$, we obtain that there exists a submodule V_0 of V such that $V/V_0 \cong D^\mu$. Hence $V \downarrow \uparrow \subsetneq V$. ■

Consider the following example. Suppose that L is a $KG(r)$ -module and there exists $L_0 \subset L$ such that $L_0 \cong D^\mu$, where $\mu \in \Lambda^+(p-2, r)$, and $L/L_0 \cong S^{(1')}$. Take any $v \in L \setminus L_0$. Let $r = (p-1)k + l$, where $l < p-1$. Fix the following elements $e_i = \sum_{\sigma \in G(\{1+(i-1)(p-1), \dots, i(p-1)\})} s(\sigma)\sigma$, $i = 1, \dots, k$. We have $e_i^2 = (p-1)!e_i$ and $e_i D^\mu = 0$. Then $v_0 = e_1 \dots e_k v \in L \setminus L_0$. Suppose additionally that $k \geq 3$. Take any transposition $\alpha \in G(r)$. Then, for some $i = 1, \dots, k$, we have $[\alpha, e_i] = 0$. Thus we obtain $(1 + \alpha)v_0 = \frac{1}{(p-1)!}e_i(1 + \alpha)v_0 = 0$. Therefore, the module generated by v_0 is isomorphic to $S^{(1')}$. This shows that L is decomposable. It is well known that $S^{(1')} \cong D^{(k+1', k^{p-1-l})}$. Therefore, by Lemma 6, we have $P^{(k^l, k^{-1^{p-1-l}})} \uparrow = P^{(k+1', k^{p-1-l})}$.

REFERENCES

1. J. A. Green, "Polynomial representations of $GL_n(K)$," Lecture Notes in Mathematics, Vol. 830, Springer-Verlag, Berlin/Heidelberg/New York, 1980.
2. A. V. Grishin, On finite basis property for abstract T -spaces, *Fund. i Prikl. Mat.* **1** (1995), 669–700 (Russian).
3. G. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.* **2** (1952), 326–336.
4. G. D. James, "The Representation Theory of the Symmetric Group," Lecture Notes in Mathematics, Vol. 682, Springer-Verlag, Berlin, 1978.
5. G. D. James, On the decomposition matrices of the symmetric group, III, *J. Algebra* **71** (1981), 115–122.
6. A. R. Kemer, Varieties and Z_2 -graded algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.* **48** (1984), 1042–1059 (Russian).
7. V. V. Shchigolev, Finite basis property of S -ideal of finite dimensional forms, in "12th International Conference, Formal Power Series and Algebraic Combinatorics'00," 581–592. Moscow.
8. V. V. Shchigolev, Finite basis property of T -spaces over fields of characteristic zero, *Izv. Ross. Akad. Nauk. Ser. Mat.* **65** (2001), 191–224.