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Triads

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Abstract

Triads are four-termed complexes with end terms connected by a translation functor, similar to triangles in a triangular category. We use triads to give an adequate theory of *L-functors*, introduced in [W. Rump, The category of lattices over a lattice-finite ring, *Algebras and Representation Theory*, in press] to investigate the global structure of categories with almost split sequences. Roughly speaking, *L-functors* extend the Auslander–Reiten translate to morphisms and thereby make it functorial.
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Introduction

Soon after the discovery of almost split sequences in the early seventies, the significance of irreducible maps as a structural element of various categories of representations became apparent. To exemplify the categories we are thinking of in this article, let R be a complete regular local ring. An R -algebra Λ is said to be a *Cohen–Macaulay order* if ${}_R\Lambda$ is finitely generated and free. Accordingly, the category $\Lambda\text{-}\mathbf{CM}$ of (maximal) *Cohen–Macaulay modules* over Λ consists of the Λ -modules E such that ${}_RE$ is finitely generated and free. For example, Λ is a finite dimensional algebra over a field if $\dim R = 0$, and an order over a complete discrete valuation domain if $\dim R = 1$. In the first case, $\Lambda\text{-}\mathbf{CM} = \Lambda\text{-}\mathbf{mod}$, the category of finitely generated Λ -modules; in the second case, $\Lambda\text{-}\mathbf{CM}$ coincides with the

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category $\Lambda\text{-lat}$ of Λ -lattices. A Cohen–Macaulay order Λ is said to be *non-singular* if $\text{gld } \Lambda = \dim R$, and Λ is called an *isolated singularity* [2] if the localizations $\Lambda_{\mathfrak{p}}$ at prime ideals \mathfrak{p} of R are non-singular unless \mathfrak{p} is maximal. Auslander [2] proved that Λ is an isolated singularity if and only if $\Lambda\text{-CM}$ has almost split sequences, and the latter implies that the irreducible maps between indecomposables in $\Lambda\text{-CM}$ can be regarded as edges of a locally finite (valued) graph, the *Auslander–Reiten quiver* $\mathbb{A}(\Lambda\text{-CM})$ of Λ , which provides a combinatorial skeleton for $\Lambda\text{-CM}$.

For an almost split sequence $A' \twoheadrightarrow A \oplus B' \twoheadrightarrow B$ in $\Lambda\text{-CM}$, written as a commutative square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow f' & & \downarrow f \\ B' & \longrightarrow & B, \end{array} \quad (0)$$

the maps f, f' are irreducible. Moreover, f determines f' up to isomorphism. (This can be shown directly, or by use of [21, §3]. In particular, the isomorphism class of f' does not depend on the horizontal morphisms.) So one is tempted to ask whether some kind of process $f \mapsto f'$ can be iterated, hopefully in a functorial manner. Provided this works, one could build ladders across the Auslander–Reiten quiver to get an insight into its global structure.

A first progress in this direction was made by Igusa and Todorov [8] who obtained a characterization of finite Auslander–Reiten quivers $\mathbb{A}(\Lambda\text{-CM})$ for $\dim R = 0$ [9,10]. Recently, the corresponding (much more difficult) problem for $\dim R = 1$ was settled by Iyama in a series of papers [11] based on an improved version of ladders. This work strongly relies on an iterative descent $\Lambda\text{-CM} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots$ through suitable subcategories \mathcal{A}_i , ending up in a category $\mathcal{A}_n \approx \Gamma\text{-CM}$ for a non-singular Cohen–Macaulay order Γ . To carry out this induction, Iyama introduces the concept of τ -category which applies to the \mathcal{A}_i as well as to any $\Lambda\text{-CM}$ in case of an isolated singularity with $\dim R \leq 2$. Variations of the method led him to prove Auslander’s conjecture that artinian algebras have finite representation dimension [13], and Solomon’s second conjecture on ζ -functions of orders [14].

In [21] we introduce an adjoint pair of endofunctors L^{\pm} on the homotopy category $\mathcal{M}(\Lambda\text{-CM})$ of two-termed complexes over $\Lambda\text{-CM}$ together with a pair of natural transformations $L^+ \rightarrow 1 \rightarrow L^-$, such that ladders

$$\dots \rightarrow L^{+2}a \rightarrow L^+a \rightarrow a \rightarrow L^-a \rightarrow L^{-2}a \rightarrow \dots$$

with all desirable properties can be formed for any object $a: A_1 \rightarrow A_0$ of $\mathcal{M}(\Lambda\text{-CM})$. In particular, the existence of these *L-functors* L^{\pm} is guaranteed for isolated singularities Λ in any dimension of R . For example, in the special case of an almost split square (0) mentioned above, we have $L^+f \cong f'$. As a first application, the conjectured existence [12] of a criterion for finite Auslander–Reiten quivers $\mathbb{A}(\Lambda\text{-CM})$ with $\dim R = 1$ in terms of additive functions on the vertices of $\mathbb{A}(\Lambda\text{-CM})$ was proved by means of L-functors [21].

The purpose of the present article is to develop a theory of L-functors which clarifies their intimate relationship to almost split sequences and almost split triangles, and to improve the more complicated explicit definition given in [21]. Instead of $\Lambda\text{-CM}$ we will work more generally with an *Ext-category* [23], that is, an exact category \mathcal{A} with enough Ext-projectives and enough Ext-injectives such that every split epimorphism has a kernel. For such an \mathcal{A} we will show that $\mathcal{M}(\mathcal{A})$ behaves like a triangulated category. In $\mathcal{M}(\mathcal{A})$, however, the “triangles” (we call them *triads*)

$$Td \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d$$

are functorial. The morphism β of such a triad belongs to a class Σ of *regular* (i.e., monic and epic) morphisms related to the exact structure of \mathcal{A} [23, Theorem 1]. Let \mathcal{P} and \mathcal{I} be the largest full subcategories of $\mathcal{M}(\mathcal{A})$ such that every $\varepsilon \in \Sigma$ is \mathcal{P} -epic and \mathcal{I} -monic (see Section 1). Then Σ , \mathcal{P} , and \mathcal{I} determine each other, and α (respectively γ) is the (co-)kernel of β in the localization $\mathcal{M}(\mathcal{A})_{\mathcal{I}}$ (respectively $\mathcal{M}(\mathcal{A})_{\mathcal{P}}$). On the other hand, β is determined by $\gamma \in \mathcal{M}(\mathcal{A})_{\mathcal{P}}$ (respectively $\alpha \in \mathcal{M}(\mathcal{A})_{\mathcal{I}}$) as a *global (co-)kernel* (see Section 1). The functor T is an equivalence $T: \mathcal{M}(\mathcal{A})_{\mathcal{P}} \xrightarrow{\sim} \mathcal{M}(\mathcal{A})_{\mathcal{I}}$ compatible with morphisms of triads. An additive category with such properties (cf. Definition 1) will be called *triadic*. Thus $\mathcal{M}(\mathcal{A})$ is triadic, and its relationship to \mathcal{A} is somewhat closer than that of the derived category $D^b(\mathcal{A})$.¹

As $\mathcal{M}(\mathcal{A})$ is not far from a triangulated category, kernels and cokernels are rare in $\mathcal{M}(\mathcal{A})$. Surprisingly enough, $\mathcal{M}(\mathcal{A})$ carries a structure of an Ext-category. Therefore, unlike usual homotopy categories, $\mathcal{M}(\mathcal{A})$ admits an exact full embedding into an abelian category [17, 18]. On the other hand, since $\mathcal{M}(\mathcal{A})$ is triadic, the localizations $\mathcal{M}(\mathcal{A})_{\mathcal{P}} \approx \mathcal{M}(\mathcal{A})_{\mathcal{I}}$ are abelian. We will prove that they are equivalent to the homotopy category $\text{Ext}(\mathcal{A})$ of short exact sequences in \mathcal{A} . By this fact, the bridge between L-functors and almost split sequences becomes visible. Assume that \mathcal{A} has the Krull–Schmidt property. Then almost split sequences in \mathcal{A} are just the simple objects in $\text{Ext}(\mathcal{A})$. Furthermore, the semisimple objects in $\text{Ext}(\mathcal{A})$ form a reflective subcategory, i.e., every object a of $\mathcal{M}(\mathcal{A})$ admits a universal morphism $\pi_a: a \rightarrow Sa$ in $\mathcal{M}(\mathcal{A})_{\mathcal{P}} \approx \text{Ext}(\mathcal{A})$ with Sa semisimple. Thus we can extend π_a to a triad

$$TSa \xrightarrow{\sigma_a} L^+a \xrightarrow{\lambda_a^+} a \xrightarrow{\pi_a} Sa,$$

which yields a left L-functor L^+ . Conversely, the existence of L-functors (Definition 7) implies that \mathcal{A} has almost split sequences. Namely, every object B of \mathcal{A} can be regarded as an object $B^+: 0 \rightarrow B$ of $\mathcal{M}(\mathcal{A})$. If B is indecomposable and non-projective, the morphism $\lambda_{B^+}: L^+B^+ \rightarrow B^+$ in $\mathcal{M}(\mathcal{A})$ yields a commutative square (0) with $A = 0$, which gives an almost split sequence $A' \xrightarrow{f'} B' \rightarrow B$ in \mathcal{A} . We invite the reader to reprove the results of [21] in this new formalism. The objects B^+ with $B \in \text{Ob } \mathcal{A}$ form a full subcategory \mathcal{A}^+

¹ For a wide class of hereditary abelian categories \mathcal{A} , Reiten and van den Bergh [19] establish a correspondence between existence of Auslander–Reiten sequences in \mathcal{A} and Serre duality in $D^b(\mathcal{A})$. However, this does not work for $\text{Ext}_{\mathcal{A}}^2(-, -) \neq 0$.

of $\mathcal{M}(\mathcal{A})$ consisting of *right semisimple* objects (see Definition 4). Usually, there are other right semisimple objects in $\mathcal{M}(\mathcal{A})$.

In Section 2 we show that $\mathcal{M}(\mathcal{A})$ has a *polarization*, i.e., a weak torsion theory $(\mathcal{A}^+, \mathcal{A}^-)$ with a certain duality between \mathcal{A}^+ -epimorphisms and \mathcal{A}^- -monomorphisms (Definition 3). In particular, there are equivalences $\mathcal{M}(\mathcal{A})/[\mathcal{A}^-] \approx \mathbf{mod}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})/[\mathcal{A}^+] \approx \mathbf{mod}(\mathcal{A}^{\text{op}})^{\text{op}}$ which simplify the handling of triads. The polarization is needed, for example, to prove that $\mathcal{M}(\mathcal{A})$ is an Ext-category. Moreover, it provides a full embedding of \mathcal{A} into the class of left (respectively right) semisimple objects of $\mathcal{M}(\mathcal{A})$. Although it is not our present concern, we remark that every polarization gives rise to a stable equivalence (see Theorem 2 and its corollary). We do not know whether each stable equivalence (e.g., between artinian algebras) is of this form.

1. Triadic categories

Throughout this article, functors between additive categories are assumed to be additive. Let \mathcal{M} be an additive category with a full (additive) subcategory \mathcal{C} . A morphism $\varphi: a \rightarrow b$ in \mathcal{M} will be called *\mathcal{C} -epic* (*\mathcal{C} -monic*) if every morphism $c \rightarrow b$ (respectively $a \rightarrow c$) with $c \in \mathcal{C}$ factors through φ . For a class Σ of morphisms, let $\text{Pr } \Sigma$ (respectively $\text{In } \Sigma$) denote the largest full subcategory \mathcal{C} of \mathcal{M} such that every $\varphi \in \Sigma$ is \mathcal{C} -epic (\mathcal{C} -monic). If Σ is the class of cokernels (kernels), the objects in $\text{Pr } \Sigma$ (respectively $\text{In } \Sigma$) are said to be *projective* (*injective*). We say that \mathcal{M} has *enough projectives* (*enough injectives*) if for each $a \in \text{Ob } \mathcal{M}$ there is an epimorphism $p \rightarrow a$ with p projective (a monomorphism $a \rightarrow i$ with i injective). The ideal of \mathcal{M} generated by the identities 1_c , $c \in \mathcal{C}$, will be denoted by $[\mathcal{C}]$. By $\text{add } \mathcal{C}$ we denote the full subcategory of objects $c \in \text{Ob } \mathcal{M}$ with $1_c \in [\mathcal{C}]$. Obviously, $[\text{add } \mathcal{C}] = [\mathcal{C}]$. A monic and epic morphism in \mathcal{M} is said to be *regular*. If the class Σ_{reg} of regular morphisms in $\mathcal{M}/[\mathcal{C}]$ admits a calculus of left and right fractions [6], we say that $\mathcal{M}_{\mathcal{C}}$ *exists* [23] and define $\mathcal{M}_{\mathcal{C}} := (\mathcal{M}/[\mathcal{C}])[\Sigma_{\text{reg}}^{-1}]$ as the localization of \mathcal{M} with respect to \mathcal{C} (cf. [5, Chapter III]).

Assume that $\mathcal{M}_{\mathcal{C}}$ exists. For a morphism $\alpha \in \mathcal{M}$ we define the *local (co-)kernel* $\ker_{\mathcal{C}} \alpha$ (respectively $\text{cok}_{\mathcal{C}} \alpha$) as a (co-)kernel of α in $\mathcal{M}_{\mathcal{C}}$ when it exists. Similarly, we call $\alpha \in \mathcal{M}$ a *global kernel* of $\beta \in \mathcal{M}_{\mathcal{C}}$ if $\beta\alpha = 0$ holds in $\mathcal{M}_{\mathcal{C}}$, and for any $\alpha' \in \mathcal{M}$ with $\beta\alpha' = 0$ in $\mathcal{M}_{\mathcal{C}}$ there exists a unique $\gamma \in \mathcal{M}$ with $\alpha\gamma = \alpha'$. By definition, the global kernel and its dual, the *global cokernel*, are unique up to isomorphism. We write $\alpha = \ker^{\mathcal{C}} \beta$ (respectively $\text{cok}^{\mathcal{C}} \beta$) for the global (co-)kernel of β .

We call an additive category \mathcal{M} *left abelian* if every morphism in \mathcal{M} has a cokernel, and for each diagram

$$\begin{array}{ccccc} e & \xrightarrow{\delta} & d & & \\ \downarrow & & \downarrow \psi & & \\ a & \xrightarrow{\varphi} & b & \xrightarrow{\gamma} & c \end{array} \quad (1)$$

in \mathcal{M} with $\gamma = \text{cok } \varphi$ and $\gamma\psi = 0$, there exists a cokernel $\delta: e \rightarrow d$ such that $\psi\delta$ factors through φ . *Right abelian* categories are defined in a dual way.

Proposition 1. *Let \mathcal{M} be a left abelian category. Then every epimorphism is a cokernel, and every monomorphism is a kernel.*

Proof. Let $\varphi: a \rightarrow b$ be an epimorphism. Choosing $\psi = 1_b$ in (1), we find a cokernel $\delta: e \rightarrow b$ with $\delta = \varphi\alpha$ for some $\alpha: e \rightarrow a$. Assume that δ is a cokernel of $\beta: d \rightarrow e$. Since $\text{cok } \delta = 0$, diagram (1) with (φ, ψ) replaced by (δ, φ) yields a cokernel $\gamma: c \rightarrow a$ such that $\varphi\gamma = \delta\zeta$ for some $\zeta: c \rightarrow e$. Now it is easy to verify that φ is a cokernel of $(\gamma - \alpha\zeta, \alpha\beta): c \oplus d \rightarrow a$.

To show that a monomorphism $\varphi: a \rightarrow b$ is a kernel of its cokernel $\gamma: b \rightarrow c$, let $\psi: d \rightarrow b$ be a morphism with $\gamma\psi = 0$. So we have a diagram (1), and there is a cokernel $\delta: e \rightarrow d$ with $\psi\delta = \varphi\zeta$ for some $\zeta: e \rightarrow a$. If $\delta = \text{cok } \alpha$, then $\varphi\zeta\alpha = 0$, and thus $\zeta\alpha = 0$. So ζ factors through δ , and therefore, ψ factors through φ . \square

As an immediate consequence, we get the following corollary.

Corollary. *An additive category \mathcal{M} is abelian if and only if \mathcal{M} is left and right abelian.*

Definition 1. We call an additive category \mathcal{M} with a pair of full subcategories $\mathcal{P} := \mathcal{P}(\mathcal{M}) = \text{add } \mathcal{P}$ and $\mathcal{J} := \mathcal{J}(\mathcal{M}) = \text{add } \mathcal{J}$ *triadic* if the following are satisfied:

- (T1) $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{J}}$ exist, $\mathcal{M}_{\mathcal{P}}$ is left abelian with enough projectives, and $\mathcal{M}_{\mathcal{J}}$ is right abelian with enough injectives.
- (T2) Every morphism in $\mathcal{M}_{\mathcal{P}}$ (respectively $\mathcal{M}_{\mathcal{J}}$) has a global (co-)kernel.
- (T3) Every global kernel is a global cokernel, and vice versa.

We introduce the following derived concepts. A sequence of morphisms

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d \quad (2)$$

with $\alpha = \ker_{\mathcal{J}} \beta$, $\gamma = \text{cok}_{\mathcal{P}} \beta$, and $\beta = \text{cok}_{\mathcal{J}} \alpha = \ker^{\mathcal{P}} \gamma$ will be called a *triad*. Note that up to isomorphism, the morphisms $\alpha \in \mathcal{M}_{\mathcal{J}}$, $\beta \in \mathcal{M}$, and $\gamma \in \mathcal{M}_{\mathcal{P}}$ determine each other. A *morphism* of triads is a diagram

$$\begin{array}{ccccccc} a & \xrightarrow{\alpha} & b & \xrightarrow{\beta} & c & \xrightarrow{\gamma} & d \\ \downarrow \varphi & & \downarrow \chi & & \downarrow \psi & & \downarrow \omega \\ a' & \xrightarrow{\alpha'} & b' & \xrightarrow{\beta'} & c' & \xrightarrow{\gamma'} & d' \end{array} \quad (3)$$

with horizontal triads and commutative squares in $\mathcal{M}_{\mathcal{J}}$, \mathcal{M} , and $\mathcal{M}_{\mathcal{P}}$, respectively. If one of these commutative squares is given, the morphism (3) can be completed in a unique fashion. So in contrast to the triangles of a triangulated category, triads are distinguished by a functorial behaviour. The class of global kernels (= global cokernels by (T3)) will be denoted by $\Sigma_{\mathcal{M}}$ or simply Σ if no confusion is possible. The following theorem shows that the ends of triads are connected by a translation functor that provides an equivalence between $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{J}}$.

Theorem 1. *Let \mathcal{M} be a triadic category. There is an equivalence $T : \mathcal{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{J}}$ with the following property. For any triad (2), the morphism α can be replaced by $\tilde{\alpha} := \alpha\theta$ for a suitable isomorphism $\theta \in \mathcal{M}_{\mathcal{J}}$, so that every morphism (3) between such standard triads (i.e., with α, α' replaced by $\tilde{\alpha}, \tilde{\alpha}'$) satisfies $\varphi = T\omega$.*

Proof. Let (3) be a morphism between triads. A straightforward verification shows that $\varphi = 0 \Leftrightarrow \omega = 0$. Let us prove that φ is regular if and only if ω is so. To this end, let $\mu : a' \rightarrow e$ and $\nu : x \rightarrow a$ be morphisms in \mathcal{M} such that $\mu\varphi = 0$ and $\varphi\nu = 0$ holds in $\mathcal{M}_{\mathcal{J}}$. By (T1), there is a monomorphism $e \rightarrow i$ in $\mathcal{M}_{\mathcal{J}}$ with i injective. Replacing x, i by isomorphic objects in $\mathcal{M}_{\mathcal{J}}$, we may assume that there are morphisms of triads

$$\begin{array}{ccccccc}
 x & \xlongequal{\quad} & x & \longrightarrow & y & \longrightarrow & z \\
 \downarrow \nu & & \downarrow & & \downarrow & & \downarrow \\
 a & \xrightarrow{\alpha} & b & \xrightarrow{\beta} & c & \xrightarrow{\gamma} & d \\
 \downarrow \varphi & & \downarrow \chi & & \downarrow \psi & & \downarrow \omega \\
 a' & \xrightarrow{\alpha'} & b' & \xrightarrow{\beta'} & c' & \xrightarrow{\gamma} & d' \\
 \downarrow \mu & & \downarrow & & \downarrow & & \downarrow \mu' \\
 e & \longrightarrow & i & \longrightarrow & j & \longrightarrow & k.
 \end{array}$$

Now $\mu\varphi = 0$ implies $\mu'\omega = 0$. Thus if ω is epic, we get $\mu' = 0$, and therefore, $\mu = 0$. This shows that φ is epic if ω is epic. Similarly, φ is monic when ω is so.

Next we define a functor $U : \mathcal{M} \rightarrow \mathcal{M}_{\mathcal{J}}$ by the morphism of triads

$$\begin{array}{ccccccc}
 Ua & \xrightarrow{\sigma} & p & \xrightarrow{\varepsilon} & a & \xlongequal{\quad} & a \\
 \downarrow U\varphi & & \downarrow & & \downarrow \varphi & & \downarrow \varphi \\
 Ua' & \xrightarrow{\sigma'} & p' & \xrightarrow{\varepsilon'} & a' & \xlongequal{\quad} & a',
 \end{array} \tag{4}$$

where we assume, without loss of generality, that $\sigma, \sigma' \in \mathcal{M}$. By the above, $U\varphi$ is regular when φ is regular. Therefore, U induces a functor $T: \mathcal{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{J}}$. Dually, we get a functor $T': \mathcal{M}_{\mathcal{J}} \rightarrow \mathcal{M}_{\mathcal{P}}$ such that for each $a \in \text{Ob } \mathcal{M}$, we have a morphism of triads

$$\begin{array}{ccccccc} Ta & \xlongequal{\quad} & Ta & \longrightarrow & i & \longrightarrow & T'Ta \\ \downarrow 1 & & \downarrow \sigma & & \downarrow & & \downarrow \rho_a \\ Ta & \xrightarrow{\quad} & p & \xrightarrow{\quad \varepsilon} & a & \xlongequal{\quad} & a \end{array}$$

with $\rho_a \in \mathcal{M}_{\mathcal{P}}$ invertible. Hence $\rho: T'T \rightarrow 1$ is a natural isomorphism. By symmetry, this proves that T is an equivalence.

For an arbitrary triad (2) with $\gamma = \rho^{-1}\delta$ we get a morphism of triads

$$\begin{array}{ccccccc} a & \xrightarrow{\alpha} & b & \xrightarrow{\beta} & c & \xrightarrow{\gamma} & d \\ \downarrow \sigma & & \downarrow & & \downarrow \delta & & \downarrow \rho \\ Te & \xrightarrow{\quad} & p & \xrightarrow{\quad \varepsilon} & e & \xlongequal{\quad} & e \end{array} \quad (5)$$

with isomorphisms ρ and σ . Now we set $\tilde{\alpha} = \alpha\vartheta$ with $\vartheta := \sigma^{-1} \cdot T\rho$. This has the effect that if we replace α by $\tilde{\alpha}$, then σ turns into $T\rho$. For a morphism (3) between standard triads, we have a similar diagram (5) with $\rho': d' \rightarrow e'$ and $T\rho': Td' \rightarrow Te'$ as morphisms between the end terms. By the Ore condition, we can choose ρ' as a regular morphism in $\mathcal{M}/[\mathcal{P}]$ such that $\rho'\omega = \omega'\rho$ for a morphism $\omega': e \rightarrow e'$ in \mathcal{M} . Hence $T\rho' \cdot \varphi = T\omega' \cdot T\rho = T(\omega'\rho) = T\rho' \cdot T\omega$, and thus $\varphi = T\omega$. \square

Note. By Proposition 1, the theorem implies that $\mathcal{M}_{\mathcal{P}} \approx \mathcal{M}_{\mathcal{J}}$ is an abelian category with enough projectives and enough injectives.

Corollary. Let \mathcal{M} be a triadic category. A morphism $\alpha \in \mathcal{M}$ belongs to Σ if and only if α is monic and \mathcal{P} -epic.

Proof. By definition, a global kernel is monic and \mathcal{P} -epic. Conversely, let $\alpha: a \rightarrow b$ be monic and \mathcal{P} -epic in \mathcal{M} . Then α is monic in $\mathcal{M}_{\mathcal{P}}$. Therefore, if $\gamma := \text{cok}_{\mathcal{P}} \alpha$, then $\alpha = \ker \gamma$ in $\mathcal{M}_{\mathcal{P}}$. On the other hand, the global kernel $\beta: c \rightarrow b$ of γ satisfies $\beta = \ker \gamma$ in $\mathcal{M}_{\mathcal{P}}$, and $\alpha = \beta\rho$ for some $\rho: a \rightarrow c$ in \mathcal{M} . Hence ρ is invertible in $\mathcal{M}_{\mathcal{P}}$, and we have to prove that ρ is invertible in \mathcal{M} . Let $\varepsilon: p \rightarrow c$ be the global kernel of the identity $1_c \in \mathcal{M}_{\mathcal{P}}$. Then $\beta\varepsilon = \alpha\varepsilon'$ for some $\varepsilon': p \rightarrow a$. Since β is monic, this gives $\varepsilon = \rho\varepsilon'$. Moreover, ε' is \mathcal{P} -epic since α is monic. Hence $\varepsilon' = \ker^{\mathcal{P}} 1_a$, and we get a morphism of triads

$$\begin{array}{ccccccc} Ta & \xrightarrow{\quad} & p & \xrightarrow{\varepsilon'} & a & \xlongequal{\quad} & a \\ \downarrow T\rho & & \downarrow 1 & & \downarrow \rho & & \downarrow \rho \\ Tc & \xrightarrow{\quad} & p & \xrightarrow{\quad \varepsilon} & c & \xlongequal{\quad} & c. \end{array}$$

By Theorem 1, $T\rho$ is an isomorphism in $\mathcal{M}_{\mathcal{J}}$. Since $\varepsilon, \varepsilon'$ are global cokernels by (T3), it follows that $\rho \in \mathcal{M}$ is invertible. \square

Together with the following result, the corollary shows that \mathcal{P}, \mathcal{J} , and Σ determine each other.

Proposition 2. *Let \mathcal{M} be a triadic category. Then $\mathcal{P} = \text{Pr } \Sigma$ and $\mathcal{J} = \text{In } \Sigma$. Moreover, the following properties hold:*

- (E0) *Every $\varepsilon \in \Sigma$ is regular.*
- (E1) *If $\varepsilon \in \mathcal{M}$ is \mathcal{P} -epic and \mathcal{J} -monic, then $\varepsilon \in \Sigma$.*
- (E2) *For any $a \in \text{Ob } \mathcal{M}$ there are morphisms $p \rightarrow a \rightarrow i$ in Σ with $p \in \mathcal{P}$ and $i \in \mathcal{J}$.*
- (E3) *If $\varepsilon \in \Sigma$ is \mathcal{P} -monic or \mathcal{J} -epic, then ε is invertible.*

Proof. Every $\beta : b \rightarrow c$ in Σ is of the form $\beta = \ker^{\mathcal{P}} \gamma$. Hence β is monic. By (T3), this gives (E0). For any $\varphi : p \rightarrow c$ with $p \in \mathcal{P}$ we have $\gamma\varphi = 0$ in $\mathcal{M}_{\mathcal{P}}$. Therefore, φ factors through β , which gives $\mathcal{P} \subset \text{Pr } \Sigma$. For any $a \in \text{Ob } \mathcal{M}$, consider the global kernel $\varepsilon : p \rightarrow a$ of the identity $1_a \in \mathcal{M}_{\mathcal{P}}$. Then $\varepsilon = \ker 1_a$ in $\mathcal{M}_{\mathcal{P}}$. Hence $p \in \mathcal{P}$, which proves (E2). If $a \in \text{Pr } \Sigma$, then ε is split epic, hence invertible. Thus $\mathcal{P} = \text{Pr } \Sigma$. To prove (E1), let $\beta : b \rightarrow c$ be \mathcal{P} -epic and \mathcal{J} -monic. Then $\text{cok}^{\mathcal{J}} 1_b \in \Sigma$ factors through β . Hence β is monic. Therefore, the above corollary implies that $\beta \in \Sigma$.

Finally, let $\varepsilon : a \rightarrow b$ be a morphism in Σ . The cokernel of ε in $\mathcal{M}_{\mathcal{P}}$ can be represented by a morphism $\gamma : b \rightarrow c$ in \mathcal{M} . So there are morphisms $\alpha : a \rightarrow p$ and $\beta : p \rightarrow c$ in \mathcal{M} with $p \in \mathcal{P}$ and $\gamma\varepsilon = \beta\alpha$. Thus if ε is \mathcal{P} -monic, then $\alpha = \delta\varepsilon$ for some $\delta : b \rightarrow p$. Since ε is epic, we get $\gamma = \beta\delta \in [\mathcal{P}]$. Hence $\varepsilon = \ker^{\mathcal{P}} \gamma$ is invertible. \square

Remarks.

- (1) Proposition 2 shows that \mathcal{J} is a reflective and \mathcal{P} a coreflective subcategory of \mathcal{M} with common (co-)reflector $\mathcal{M} \rightarrow \mathcal{P} \approx \mathcal{J} \approx \mathcal{M}[\Sigma^{-1}]$. In particular, Σ admits a calculus of left and right fractions [6], and by (E1), Σ is saturated, i.e., a morphism $\varepsilon \in \mathcal{M}$ belongs to Σ if and only if ε is invertible in $\mathcal{M}[\Sigma^{-1}]$.
- (2) We will show in Section 2 (Corollary 2 of Theorem 3) that the homotopy category $\mathcal{M}(\mathcal{A})$ (see Introduction) is triadic for any Ext-category \mathcal{A} , thus in particular, for $\mathcal{A} = \Lambda\text{-CM}$ with a Cohen–Macaulay order Λ over a complete regular local ring.

Definition 2. Let \mathcal{M} be an additive category. As in [23], we call a class of morphisms $\Sigma \subset \mathcal{M}$ *exact* if the properties (E0)–(E2) hold for $\mathcal{P} = \text{Pr } \Sigma$ and $\mathcal{J} = \text{In } \Sigma$. We call Σ *fully exact* if, in addition,

- (E4) Every \mathcal{P} -epimorphism (\mathcal{J} -monomorphism) has a (co-)kernel.

Let \mathcal{M} be an additive category with an exact class $\Sigma \subset \mathcal{M}$, and let

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c \tag{6}$$

be a short exact sequence, i.e., $\alpha = \ker \beta$ and $\beta = \operatorname{cok} \alpha$. If α is $(\operatorname{In} \Sigma)$ -monic, then α will be called an *inflation*. If β is $(\operatorname{Pr} \Sigma)$ -epic, we call β a *deflation*. In what follows, we indicate inflations (deflations) by arrows of the form \rightharpoonup (respectively \rightarrow). If α is an inflation and β a deflation, we call (6) a *conflation*. The next result clarifies the rôle of conflations.

Proposition 3. *Let \mathcal{M} be an additive category with an exact class $\Sigma \subset \mathcal{M}$.*

- (a) Σ is fully exact if and only if the following are satisfied:
- (i) Every split epimorphism has a kernel.
 - (ii) Pullbacks and pushouts of any $\beta \in \Sigma$ (along an arbitrary morphism) exist and belong to Σ .
- (b) Assume that Σ is fully exact. Then the kernel κ of any $(\operatorname{Pr} \Sigma)$ -epimorphism $\varphi: a \rightarrow b$ is an inflation. Moreover, φ has a factorization $a \xrightarrow{\delta} e \xrightarrow{\varepsilon} b$ with $\delta = \operatorname{cok} \kappa$ and $\varepsilon \in \Sigma$.

Proof. (a) Assume first that Σ is fully exact. For a morphism $\varphi: a \rightarrow b$ and $\beta: d \rightarrow b$ in Σ , the morphism $(\varphi \beta): a \oplus d \rightarrow b$ is $(\operatorname{Pr} \Sigma)$ -epic. So the pullback

$$\begin{array}{ccc} c & \xrightarrow{\psi} & d \\ \downarrow \alpha & & \downarrow \beta \\ a & \xrightarrow{\varphi} & b \end{array} \quad (7)$$

exists, and $\alpha \in \Sigma$ since α is $(\operatorname{Pr} \Sigma)$ -epic and monic (see [23, Proposition 2]). Conversely, let (i) and (ii) be satisfied. For a $(\operatorname{Pr} \Sigma)$ -epimorphism $\varphi: a \rightarrow b$, consider a morphism $\beta: d \rightarrow b$ in Σ with $d \in \operatorname{Pr} \Sigma$. By (ii), there exists a pullback (7) with $\alpha \in \Sigma$. Since β factors through φ , the pullback property implies that ψ is split epic. By (i), ψ has a kernel $v: k \rightarrow c$ which is split monic. So we get $\alpha v = \ker \varphi$.

(b) As in the preceding argument, any $(\operatorname{Pr} \Sigma)$ -epimorphism $\varphi: a \rightarrow b$ gives rise to a pullback (7) with $\alpha, \beta \in \Sigma$ and $d \in \operatorname{Pr} \Sigma$. Again we infer that ψ is split epic, whence $v := \ker \psi$ is split monic. Therefore, $\kappa := \alpha v = \ker \varphi$ is $(\operatorname{In} \Sigma)$ -monic. By duality, there is a deflation $\delta = \operatorname{cok} \kappa$ which leads to a factorization $\varphi: a \xrightarrow{\delta} e \xrightarrow{\varepsilon} b$. To show that ε is monic, let $\xi: x \rightarrow e$ be a morphism with $\varepsilon \xi = 0$. Choose a morphism $\varepsilon': p \rightarrow x$ in Σ with $p \in \operatorname{Pr} \Sigma$. Then $\xi \varepsilon' = \delta \eta$ for some $\eta: p \rightarrow a$. Since $\varphi \eta = \varepsilon \xi \varepsilon' = 0$, η factors through κ . Consequently, $\delta \eta = 0$, and thus $\xi = 0$. So ε is monic and $(\operatorname{Pr} \Sigma)$ -epic, whence $\varepsilon \in \Sigma$ by [23, Proposition 2]. \square

Remark. More generally, we have proved the following assertion for $\Sigma \subset \mathcal{M}$ fully exact. If $\delta: a \twoheadrightarrow e$ is a deflation, and $\ker \delta = \ker(\varepsilon \delta)$, then ε is monic.

Proposition 4. *Let \mathcal{M} be a triadic category. Assume that every split epimorphism has a kernel. Then $\Sigma_{\mathcal{M}}$ is fully exact.*

Proof. Let $\varphi: a \rightarrow b$ be \mathcal{P} -epic. By Proposition 2, we only have to show that φ has a kernel. Consider the commutative diagram (7) with $\alpha = \ker^{\mathcal{P}} \varphi$ and $\beta = \ker^{\mathcal{P}} 1_b$. Since β

is monic, this is a pullback. As above, it follows that ψ is split epic. Thus by assumption, ψ has a kernel $v: k \rightarrow c$. Hence $\alpha v = \ker \varphi$. \square

Note that any additive category \mathcal{M} has a trivial triadic structure given by $\mathcal{P} = \mathcal{J} = \mathcal{M}$. This shows that the assumption that split epimorphisms have a kernel cannot be dropped.

2. Polarization

For an additive category \mathcal{A} , let $\text{Mor}(\mathcal{A})$ denote the category of two-termed complexes $0 \rightarrow A_1 \xrightarrow{a} A_0 \rightarrow 0$ over \mathcal{A} , and let $\mathcal{M}(\mathcal{A})$ be the corresponding homotopy category [23]. Thus objects of $\text{Mor}(\mathcal{A})$ are given by morphisms $a: A_1 \rightarrow A_0$ in \mathcal{A} , and morphisms in $\text{Mor}(\mathcal{A})$ are given by commutative squares in \mathcal{A} . There is a natural full embedding $\mathcal{A} \hookrightarrow \text{Mor}(\mathcal{A})$ which identifies objects A of \mathcal{A} with identities $1_A \in \text{Ob } \text{Mor}(\mathcal{A})$. Then a morphism $a \rightarrow b$ in $\text{Mor}(\mathcal{A})$ is homotopic to zero if and only if it belongs to the ideal $[\mathcal{A}]$, whence $\mathcal{M}(\mathcal{A}) = \text{Mor}(\mathcal{A})/[\mathcal{A}]$. Furthermore, there are two full subcategories \mathcal{A}^+ and \mathcal{A}^- of $\mathcal{M}(\mathcal{A})$ consisting of the objects $A^+: 0 \rightarrow A$ and $A^-: A \rightarrow 0$, respectively, and there are natural equivalences $(\)^\pm: \mathcal{A} \xrightarrow{\sim} \mathcal{A}^\pm$. The factor category $\mathbf{mod}(\mathcal{A}) := \mathcal{M}(\mathcal{A})/[\mathcal{A}^-]$ is equivalent to the category of finitely presented \mathcal{A} -modules (i.e., coherent functors $\mathcal{A}^{\text{op}} \rightarrow \mathbf{Ab}$; see [22, Section 1]). Dually, $\mathbf{com}(\mathcal{A}) := \mathcal{M}(\mathcal{A})/[\mathcal{A}^+] = \mathbf{mod}(\mathcal{A}^{\text{op}})^{\text{op}}$. By Proposition 1 and [22, Proposition 1] we have the following proposition.

Proposition 5. *An additive category \mathcal{M} is equivalent to $\mathbf{mod}(\mathcal{A})$ for some additive category \mathcal{A} if and only if \mathcal{M} is left abelian with enough projectives.*

Let \mathcal{M} be an additive category. Recall that a morphism $\alpha: a \rightarrow b$ is said to be a *weak kernel* of $\beta: b \rightarrow c$ if $\beta\alpha = 0$, and every $\gamma \in \mathcal{M}$ with $\beta\gamma = 0$ factors through α . If, in addition, β is a *weak cokernel* of α (the dual notion), then the complex (6) is said to be a *weak short exact sequence* in \mathcal{M} . We call a pair $(\mathcal{M}_+, \mathcal{M}_-)$ of full subcategories of \mathcal{M} which are closed with respect to isomorphisms a *(weak) torsion theory* [21] if $\text{Hom}_{\mathcal{M}}(\mathcal{M}_+, \mathcal{M}_-) = 0$, and each $a \in \text{Ob } \mathcal{M}$ admits a (weak) short exact sequence

$$a^+ \xrightarrow{\alpha_a} a \xrightarrow{\beta_a} a^- \quad (8)$$

with $a^+ \in \mathcal{M}^+$ and $a^- \in \mathcal{M}^-$. The notations in (8) will be retained throughout the sequel. If a weak short exact sequence (6) has the property that β is \mathcal{M}_+ -epic and α is \mathcal{M}_- -monic, we speak of a *polar sequence*. For a structural analysis of $\mathcal{M}(\mathcal{A})$, the following concept will be useful.

Definition 3. Let \mathcal{M} be an additive category. We define a *polarization* of \mathcal{M} as a pair $(\mathcal{M}_+, \mathcal{M}_-)$ of full subcategories with the following properties:

- (P1) $(\mathcal{M}_+, \mathcal{M}_-)$ is a weak torsion theory.
- (P2) Every \mathcal{M}_+ -epimorphism $\beta: b \rightarrow c$, and every \mathcal{M}_- -monomorphism $\alpha: a \rightarrow b$, can be completed to a polar sequence $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$.

Examples.

- (1) For an additive category \mathcal{A} , the pair $(\mathcal{A}^+, \mathcal{A}^-)$ is a polarization of $\mathcal{M}(\mathcal{A})$. By the proof of [21, Proposition 1], $(\mathcal{A}^+, \mathcal{A}^-)$ is a weak torsion theory. For an \mathcal{A}^+ -epimorphism $\beta: b \rightarrow c$, a polar sequence $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ can be obtained as follows:

$$\begin{array}{ccccc}
 B_1 & \xlongequal{\quad} & B_1 & \xrightarrow{g_1} & C_1 \\
 \downarrow \scriptstyle (b_{g_1}) & & \downarrow \scriptstyle b & & \downarrow \scriptstyle c \\
 B_0 \oplus C_1 & \xrightarrow{(1-sg_0, sc)} & B_0 & \xrightarrow{g_0} & C_0,
 \end{array} \tag{9}$$

Since β is \mathcal{A}^+ -epic, there are morphisms $s: C_0 \rightarrow B_0$ and $h: C_0 \rightarrow C_1$ with $ch + g_0s = 1$. A straightforward verification shows that (9) defines a polar sequence. ($\beta\alpha = 0$ by the homotopy $(hg_0 - 1 - hc): B_0 \oplus C_1 \rightarrow C_1$.)

If every split epimorphism in \mathcal{A} has a kernel, a more symmetric form of (9) can be given. The split epimorphism $(c \ g_0): C_1 \oplus B_0 \rightarrow C_0$ has a kernel $\begin{pmatrix} -d \\ e \end{pmatrix}: D \rightarrow C_1 \oplus B_0$, and β factors through the isomorphism $\gamma: e \rightarrow c$ given by

$$\begin{array}{ccc}
 D & \xrightarrow{d} & C_1 \\
 \downarrow \scriptstyle e & & \downarrow \scriptstyle c \\
 B_0 & \xrightarrow{g_0} & C_0.
 \end{array}$$

Thus if we replace β by $\gamma^{-1}\beta$, we find a polar sequence

$$\begin{array}{ccccc}
 B_1 & \xlongequal{\quad} & B_1 & \xrightarrow{f} & D \\
 \downarrow \scriptstyle f & & \downarrow \scriptstyle b & & \downarrow \scriptstyle e \\
 D & \xrightarrow{e} & B_0 & \xlongequal{\quad} & B_0.
 \end{array} \tag{10}$$

- (2) Let \mathcal{M} be an additive category. Then $(0, \mathcal{M})$ is a polarization if and only if idempotents split and \mathcal{M} is von-Neumann regular (i.e., for every $\varphi \in \mathcal{M}$ there is a morphism ψ with $\varphi\psi\varphi = \varphi$).

Let us call a morphism $\varphi: a \rightarrow b$ in an additive category *right tight* if every morphism $\psi: c \rightarrow b$ with $\gamma\varphi = 0 \Rightarrow \gamma\psi = 0$ for all $\gamma: b \rightarrow x$ factors through φ . Morphisms with the dual property will be called *left tight*.

Theorem 2. Let \mathcal{M} be an additive category. A pair $(\mathcal{M}_+, \mathcal{M}_-)$ of full subcategories of \mathcal{M} is a polarization if and only if the following are satisfied:

- (a) \mathcal{M}_+ and \mathcal{M}_- are closed under isomorphism, and $\text{Hom}_{\mathcal{M}}(\mathcal{M}_+, \mathcal{M}_-) = 0$.
- (b) The full embeddings $\mathcal{M}_+ \hookrightarrow \mathcal{M} \rightarrow \mathcal{M}/[\mathcal{M}_-]$ and $\mathcal{M}_- \hookrightarrow \mathcal{M} \rightarrow \mathcal{M}/[\mathcal{M}_+]$ induce equivalences $\mathcal{M}/[\mathcal{M}_-] \approx \mathbf{mod}(\mathcal{M}_+)$ and $\mathcal{M}/[\mathcal{M}_+] \approx \mathbf{com}(\mathcal{M}_-)$.
- (c) Every morphism in \mathcal{M}_+ (respectively \mathcal{M}_-) is right (left) tight in \mathcal{M} .

Proof. Assume first that $(\mathcal{M}_+, \mathcal{M}_-)$ is a polarization. Then (a) holds, and the full functor $\mathcal{M}_+ \hookrightarrow \mathcal{M} \rightarrow \mathcal{M}/[\mathcal{M}_-]$ is faithful. Let $\varphi: a \rightarrow b$ be a morphism in $\mathcal{M}/[\mathcal{M}_-]$. Then the \mathcal{M}_- -monomorphism $\begin{pmatrix} \varphi \\ \beta_a \end{pmatrix}: a \rightarrow b \oplus a^-$ can be completed to a polar sequence $a \rightarrow b \oplus a^- \rightarrow c$ which amounts to a commutative square

$$\begin{array}{ccc} a & \xrightarrow{\varphi} & b \\ \downarrow \beta_a & & \downarrow \gamma \\ a^- & \xrightarrow{\psi} & c. \end{array} \quad (11)$$

Since $(\gamma \ \psi): b \oplus a^- \rightarrow c$ is \mathcal{M}_+ -epic, $\text{Hom}_{\mathcal{M}}(\mathcal{M}_+, \mathcal{M}_-) = 0$ implies that γ is \mathcal{M}_+ -epic. Therefore, every $\xi: c \rightarrow x$ with $\xi\gamma \in [\mathcal{M}_-]$ satisfies $\xi\alpha_c = 0$, and thus $\xi \in [\mathcal{M}_-]$. Consequently, γ is epic in $\mathcal{M}/[\mathcal{M}_-]$, and (11) shows that $\gamma = \text{cok } \varphi$ in $\mathcal{M}/[\mathcal{M}_-]$. For any object c of \mathcal{M} , the \mathcal{M}_+ -epimorphism α_c extends to a polar sequence $a \xrightarrow{\varphi} c^+ \xrightarrow{\alpha_c} c$. Since φ is \mathcal{M}_- -monic, we get $\alpha_c = \text{cok } \varphi$ in $\mathcal{M}/[\mathcal{M}_-]$. Therefore, the objects of \mathcal{M}_+ are projective in $\mathcal{M}/[\mathcal{M}_-]$ and provide enough projectives. For a diagram (1) in $\mathcal{M}/[\mathcal{M}_-]$ with $\gamma = \text{cok } \varphi$ and $\gamma\psi = 0$, we get $\gamma\psi\alpha_d = 0$ in \mathcal{M} , and thus $\psi\alpha_d$ factors through φ by the weak pull-back property of (11). Hence $\mathcal{M}/[\mathcal{M}_-] \approx \mathbf{mod}(\mathcal{M}_+)$ by Proposition 5. By duality, this proves (b). Since every morphism in \mathcal{M}_+ is \mathcal{M}_- -monic as a morphism of \mathcal{M} , (c) follows by (P2).

Conversely, let (a)–(c) be satisfied. For each $a \in \text{Ob } \mathcal{M}$, there is an epimorphism $\pi: p \rightarrow a$ in $\mathbf{mod}(\mathcal{M}_+)$ with $p \in \mathcal{M}_+$ and a monomorphism $\sigma: a \rightarrow i$ in $\mathbf{com}(\mathcal{M}_-)$ with $i \in \mathcal{M}_-$. Then $p \xrightarrow{\pi} a \xrightarrow{\sigma} i$ defines a weak torsion theory $(\mathcal{M}_+, \mathcal{M}_-)$. As before, we write $\alpha_a := \pi$, $a^+ := p$, etc. Thus it remains to be shown that every \mathcal{M}_+ -epimorphism $\beta: b \rightarrow c$ can be completed to a polar sequence $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. First, let $\delta: d \rightarrow b$ be a kernel of β in $\mathcal{M}/[\mathcal{M}_+]$. Then $\beta\delta$ factors through α_c , and α_c factors through β . Therefore, we may assume that $\beta\delta = 0$. Since $\beta\alpha_b$ is \mathcal{M}_+ -epic, it is epic in $\mathcal{M}/[\mathcal{M}_-]$. Hence, by Proposition 1, there is a morphism $v: e \rightarrow b^+$ in \mathcal{M}_+ with $\beta\alpha_b = \text{cok } v$ in $\mathcal{M}/[\mathcal{M}_-]$. Define $\alpha := (\delta \ \alpha_b v): d \oplus e \rightarrow b$. Then $e \in \mathcal{M}_+$ implies that $\beta\alpha = 0$, and that α is \mathcal{M}_- -monic. To show that α is a weak kernel of β , let $\xi: x \rightarrow b$ be a morphism with $\beta\xi = 0$. Then we find a morphism $\varphi: x \rightarrow d$ with $\xi - \delta\varphi \in [\mathcal{M}_+]$, say, $\xi - \delta\varphi = \alpha_b\psi$. Hence $\beta\alpha_b\psi = 0$, and thus (c) implies that $\psi = v\omega$ for some $\omega: x \rightarrow e$. So we get $\alpha \begin{pmatrix} \varphi \\ \omega \end{pmatrix} = \delta\varphi + \alpha_b v\omega = \xi$. Finally, we show that β is a weak cokernel of α . Thus let $\eta: b \rightarrow y$ be a morphism with $\eta\alpha = 0$. Then $\eta\alpha_b \cdot v = 0$ implies that $\eta\alpha_b$ factors through $\beta\alpha_b$. Therefore, we may assume that $\eta\alpha_b = 0$, i.e., η factors through β_b . As $\beta_b\delta$ is \mathcal{M}_- -monic, the dual of the preceding argument yields an \mathcal{M}_+ -epic weak cokernel $\zeta: b^- \rightarrow z$ of $\beta_b\delta$. Hence $\alpha_z = 0$, and we may assume that $z \in \mathcal{M}_-$. Since $\delta = \ker \beta$ in $\mathcal{M}/[\mathcal{M}_+] \approx \mathbf{com}(\mathcal{M}_-)$, this implies that $\zeta\beta_b$ factors through β in $\mathcal{M}/[\mathcal{M}_+]$, hence in \mathcal{M} . As η factors through $\zeta\beta_b$, the proof is complete. \square

Let us call a pair of additive categories \mathcal{A}, \mathcal{B} *stably equivalent* if $\underline{\mathbf{mod}}(\mathcal{A}) := \mathbf{mod}(\mathcal{A})/[\mathcal{A}]$ is equivalent to $\underline{\mathbf{mod}}(\mathcal{B})$. For example, if $R\text{-proj}$ denotes the category of finitely generated projective left modules over a ring R , then $R\text{-proj}$ and $S\text{-proj}$ are stably equivalent if and only if R and S are stably equivalent. Note that $\mathbf{mod}(\mathcal{A})/[\mathcal{A}] \approx \mathbf{com}(\mathcal{A})/[\mathcal{A}]$. (This generalizes the Auslander–Bridger duality [3].) As an immediate consequence of Theorem 1, we have the following corollary.

Corollary. *If \mathcal{M} is an additive category with a polarization, then \mathcal{M}_+ and \mathcal{M}_- are stably equivalent.*

By definition, triadic categories involve the existence of localizations. We will see below that such structures become a lot more intelligible in the presence of a polarization.

Definition 4. An object s of an additive category \mathcal{M} will be called *left (right) semisimple* if every monomorphism $a \rightarrow s$ (epimorphism $s \rightarrow a$) in \mathcal{M} splits. We denote the full subcategory of left (right) semisimple objects by $S_l(\mathcal{M})$ (respectively $S_r(\mathcal{M})$). The objects of $S(\mathcal{M}) := S_l(\mathcal{M}) \cap S_r(\mathcal{M})$ will be called *semisimple*.

Examples.

- (1) For a module category \mathcal{M} , semisimple objects coincide with semisimple modules [1, Theorem 9.6], and there is no difference between left and right semisimplicity.
- (2) If Λ is an artinian algebra, there are two types of indecomposable left semisimple objects in $\mathcal{M} := \mathcal{M}(\Lambda\text{-mod})$, namely, the indecomposable objects of $(\Lambda\text{-mod})^-$, and the objects of the form $\text{Rad } P \hookrightarrow P$ with $P \in \Lambda\text{-mod}$ indecomposable and projective. A similar classification holds for orders Λ over a complete discrete valuation domain (see [22, Proposition 8]). For a Cohen–Macaulay order Λ over a complete regular local ring R of dimension 2, it can be shown that there are no left semisimple objects of $\mathcal{M}(\Lambda\text{-CM})$ other than those belonging to $(\Lambda\text{-CM})^-$.

Proposition 6. *Let \mathcal{M} be an additive category with a polarization.*

- (a) *If $\beta : b \rightarrow c$ is monic and \mathcal{M}_+ -epic, then β is split monic.*
- (b) *If, in addition, β is \mathcal{M}_- -epic, then β is invertible.*
- (c) *$\mathcal{M}_- \subset S_l(\mathcal{M})$ and $\mathcal{M}_+ \subset S_r(\mathcal{M})$.*

Proof. (a) There is a polar sequence $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ with $\alpha = 0$. So β is split monic.

(b) Let γ be a retraction of β . Then $(1 - \beta\gamma)\beta = 0$. If β is \mathcal{M}_+ -epic, this implies that $1 - \beta\gamma \in [\mathcal{M}_-]$. Thus if β is also \mathcal{M}_- -epic, we get $1 - \beta\gamma = \beta\gamma'$ for some $\gamma' : c \rightarrow b$, whence $1 = \beta(\gamma + \gamma')$.

(c) This follows by (a) and its dual. \square

Proposition 7. *Let \mathcal{M} be an additive category with an exact class $\Sigma \subset \mathcal{M}$ and a polarization. Then (E3) of Proposition 2 holds, and*

$$\mathcal{M}_- \subset \text{Pr } \Sigma; \quad \mathcal{M}_+ \subset \text{In } \Sigma. \quad (12)$$

If Σ is fully exact, then every \mathcal{M}_+ -epic $(\text{Pr } \Sigma)$ -epimorphism is a deflation.

Proof. The inclusions (12) follow by (E0), (E2), and Proposition 6(c). Therefore, Proposition 6(a), (b) yields (E3). Now let Σ be fully exact, and let $\varphi : a \rightarrow b$ be \mathcal{M}_+ -epic and $(\text{Pr } \Sigma)$ -epic. By Proposition 3, there is a factorization $\varphi : a \xrightarrow{\delta} e \xrightarrow{\varepsilon} b$ with a deflation δ and $\varepsilon \in \Sigma$. Hence ε is \mathcal{M}_+ -epic and thus invertible by Proposition 6. So φ is a deflation. \square

Property (E3) has the following consequences. Recall that a commutative square is said to be *exact* if it is a pullback and a pushout.

Proposition 8. *Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ such that (E3) holds.*

- (a) *Let $\varepsilon \in \Sigma$ be epic in $\mathcal{M}/[\text{Pr } \Sigma]$; then ε is invertible.*
- (b) *$\rho \in \mathcal{M}$ is regular in $\mathcal{M}/[\text{Pr } \Sigma]$ if and only if the following commutative square with $\alpha, \beta \in \Sigma$ and $p, q \in \text{Pr } \Sigma$ is exact.*

$$\begin{array}{ccc} p & \xrightarrow{\sigma} & q \\ \downarrow \alpha & & \downarrow \beta \\ a & \xrightarrow{\rho} & b. \end{array} \quad (13)$$

Proof. (a) For any $\varphi : a \rightarrow p \in \text{Pr } \Sigma$, Proposition 3 implies that there is a pushout

$$\begin{array}{ccc} a & \xrightarrow{\varepsilon} & b \\ \downarrow \varphi & & \downarrow \psi \\ p & \xrightarrow{\varepsilon'} & c \end{array} \quad (14)$$

with $\varepsilon' \in \Sigma$. Hence $\psi\varepsilon \in [\text{Pr } \Sigma]$ yields $\psi \in [\text{Pr } \Sigma]$, i.e., ψ factors through ε' . Since ε' is monic, this implies that φ factors through ε . Therefore, ε is invertible by (E3).

(b) Assume that ρ is regular in $\mathcal{M}/[\text{Pr } \Sigma]$. To show that (13) is a pullback, let $\varphi : x \rightarrow a$ and $\psi : x \rightarrow q$ be morphisms with $\rho\varphi = \beta\psi$. Then $\rho\varphi \in [\text{Pr } \Sigma]$, and so φ factors

through α . Since α and β are monic, this implies that (13) is a pullback. By Proposition 3, there is a pushout

$$\begin{array}{ccc} p & \xrightarrow{\sigma} & q \\ \downarrow \alpha & & \downarrow \beta' \\ a & \xrightarrow{\rho'} & b' \end{array}$$

with $\beta' \in \Sigma$. So there is a morphism $\gamma: b' \rightarrow b$ with $\rho = \gamma\rho'$ and $\beta = \gamma\beta'$. Hence γ is epic in $\mathcal{M}/[\text{Pr } \Sigma]$, and $\gamma \in \Sigma$ by Proposition 3. Thus γ is invertible by (a).

Conversely, let (13) be exact. Then the pullback property implies that ρ is monic in $\mathcal{M}/[\text{Pr } \Sigma]$. Let $\varphi: b \rightarrow c$ be a morphism with $\varphi\rho \in [\text{Pr } \Sigma]$. If $\gamma: q' \rightarrow c$ is a morphism in Σ with $q' \in \text{Pr } \Sigma$, then there is a $\psi: q \rightarrow q'$ with $\varphi\beta = \gamma\psi$, and a morphism $\delta: a \rightarrow q'$ with $\varphi\rho = \gamma\delta$. Hence $\psi\sigma = \delta\alpha$, and the pushout property of (13) implies that ψ factors through β . Since β is epic, we infer that φ factors through γ , i.e., $\varphi \in [\text{Pr } \Sigma]$. Thus ρ is epic in $\mathcal{M}/[\text{Pr } \Sigma]$. \square

Definition 5. Let \mathcal{M} be an additive category with a polarization. We call a fully exact class $\Sigma \subset \mathcal{M}$ *compatible* with the polarization if every deflation (inflation) is \mathcal{M}_+ -epic (\mathcal{M}_- -monic). (By Proposition 7, this means that deflations are characterized as \mathcal{M}_+ -epic ($\text{Pr } \Sigma$)-epimorphisms.) We call a triadic category \mathcal{M} *compatible* with a polarization if every split epimorphism has a kernel (see Proposition 4), and $\Sigma_{\mathcal{M}}$ is compatible with $(\mathcal{M}_+, \mathcal{M}_-)$.

Lemma 1. Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ and a polarization. Up to isomorphism, every $\varphi \in \mathcal{M}/[\text{Pr } \Sigma]$ is of the form $\varphi: a \xrightarrow{\pi} e \xrightarrow{\sigma} b$ with a a deflation π and an inflation σ in \mathcal{M} . Moreover, every deflation is epic in $\mathcal{M}/[\text{Pr } \Sigma]$.

Proof. For a morphism $\varphi: a \rightarrow b$, choose $\alpha: p \rightarrow b$ in Σ with $p \in \text{Pr } \Sigma$. Then $\varphi = (\varphi \alpha)$ in $\mathcal{M}/[\text{Pr } \Sigma]$. Hence we may assume that φ is $\text{Pr } \Sigma$ -epic. By Proposition 3, this implies that $\varphi = \varepsilon\pi$ with $\varepsilon \in \Sigma$ and a deflation $\pi: a \rightarrow e$. By (12), we have $e^- \in \text{Pr } \Sigma$. Hence $\varepsilon = \begin{pmatrix} \varepsilon \\ \beta_e \end{pmatrix}$ in $\mathcal{M}/[\text{Pr } \Sigma]$, and $\begin{pmatrix} \varepsilon \\ \beta_e \end{pmatrix}$ is an inflation by the dual of Proposition 7. To show that any deflation $\delta: a \rightarrow e$ is epic in $\mathcal{M}/[\text{Pr } \Sigma]$, let $\xi: e \rightarrow x$ be a morphism with $\xi\delta \in [\text{Pr } \Sigma]$. By (E2), there is a morphism $\beta: q \rightarrow x$ in Σ with $q \in \text{Pr } \Sigma$. Hence $\xi\delta = \beta\eta$ for some $\eta: a \rightarrow q$. Since $\beta\eta$ annihilates the kernel of δ , and β is monic, we infer that η factors through δ . As δ is epic, this implies that ξ factors through β , whence $\xi \in [\text{Pr } \Sigma]$. \square

Proposition 9. Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ and a polarization such that $p^+ \in \text{Pr } \Sigma$ for all $p \in \text{Pr } \Sigma$, and $i^- \in \text{In } \Sigma$ for all $i \in \text{In } \Sigma$. Then Σ is compatible with the polarization.

Proof. Let $\alpha: a \rightarrow b$ be a deflation. Then there is a polar sequence

$$a \xrightarrow{\begin{pmatrix} \alpha \\ \beta_a \end{pmatrix}} b \oplus a^- \xrightarrow{(\gamma \delta)} c.$$

By Lemma 1, α is epic in $\mathcal{M}/[\text{Pr } \Sigma]$. Hence $\gamma\alpha \in [\text{Pr } \Sigma]$ implies that $\gamma \in [\text{Pr } \Sigma]$. Thus let $\varepsilon: p \rightarrow c$ be a morphism in Σ with $p \in \text{Pr } \Sigma$. Then $(\gamma \delta)$ factors through ε . So we may assume without loss of generality that $c \in \text{Pr } \Sigma$. Since $(\gamma \delta)$ is \mathcal{M}_+ -epic, there is a morphism $\beta: c^+ \rightarrow b$ with $\alpha_c = \gamma\beta$. Therefore, the assumption $c^+ \in \text{Pr } \Sigma$ implies that β factors through α , and thus α_c factors through δ . Consequently, $\alpha_c = 0$, which gives $(\gamma \delta) \binom{\alpha_b}{0} = \gamma\alpha_b = 0$. Hence α_b factors through α . \square

As in [23], we define an *Ext-category* as an exact category \mathcal{A} with enough Ext-projectives and enough Ext-injectives such that every split epimorphism has a kernel. The distinguished short exact sequences in \mathcal{A} are called *conflations* [15], and they consist of an *inflation* followed by a *deflation*.

Example. Let Λ be a Cohen–Macaulay order over a complete regular local ring R . The embedding $\Lambda\text{-CM} \hookrightarrow \Lambda\text{-mod}$ induces an exact structure on $\Lambda\text{-CM}$ that makes $\Lambda\text{-CM}$ into an Ext-category. For any Ext-category \mathcal{A} , there is a natural exact class $\Sigma \subset \mathcal{M}(\mathcal{A})$ consisting of the morphisms $a \rightarrow b$ in $\mathcal{M}(\mathcal{A})$ given by a commutative square

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow a & & \downarrow b \\ A_0 & \xrightarrow{f_0} & B_0 \end{array} \quad (15)$$

in \mathcal{A} such that the induced sequence

$$A_1 \xrightarrow{\binom{-f_1}{a}} B_1 \oplus A_0 \xrightarrow{(b \ f_0)} B_0$$

is a conflation in \mathcal{A} (see [23, Theorem 1]). By [23, Lemma 1], an object $p: A \rightarrow P$ of $\mathcal{M}(\mathcal{A})$ belongs to $\text{Pr } \Sigma$ if and only if P is Ext-projective. Moreover, the sequence (8) for $a = p$ looks as follows:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xlongequal{\quad} & A \\ \downarrow & & \downarrow p & & \downarrow \\ P & \xlongequal{\quad} & P & \longrightarrow & 0. \end{array}$$

Hence $p^+ = P^+ \in \text{Pr } \Sigma$, and similarly, $i^- \in \text{In } \Sigma$ for all $i \in \text{In } \Sigma$. We will see below that $\mathcal{M}(\mathcal{A})$ is triadic with respect to $\mathcal{P} := \text{Pr } \Sigma$ and $\mathcal{I} := \text{In } \Sigma$. Hence $\Sigma = \Sigma_{\mathcal{M}(\mathcal{A})}$ by the corollary of Theorem 1 and [21, Proposition 2]. Thus Proposition 4 implies that Σ is even fully exact. By Proposition 9, Σ is compatible with the natural polarization $(\mathcal{A}^+, \mathcal{A}^-)$ of $\mathcal{M}(\mathcal{A})$.

For a $(\text{Pr } \Sigma)$ -epimorphism $\varphi: a \rightarrow b$ given by (15), it follows by [23, Lemma 1 and Proposition 1], that $(b \ f_0): B_1 \oplus A_0 \rightarrow B_0$ is a deflation in \mathcal{A} . So we get a commutative diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & C & \longrightarrow & B_1 \\ \downarrow a & & \downarrow & & \downarrow b \\ A_0 & \xlongequal{\quad} & A_0 & \xrightarrow{f_0} & B_0, \end{array}$$

where the right-hand square is in Σ . Therefore, we get a factorization $\varphi = \varepsilon\gamma$ with $\varepsilon \in \Sigma$ and an \mathcal{A}^+ -epimorphism γ . Since ε is monic, it follows that γ is a $(\text{Pr } \Sigma)$ -epimorphism, hence a deflation by Proposition 7. Consequently, the factorization $\varphi = \varepsilon\gamma$ coincides with that of Proposition 3(b). Furthermore, this shows that the deflations in $\mathcal{M}(\mathcal{A})$ are just the \mathcal{A}^+ -epic $(\text{Pr } \Sigma)$ -epimorphisms. In particular, the conflations in $\mathcal{M}(\mathcal{A})$ form a special class of polar sequences.

Our next result implies that the so determined conflations turn $\mathcal{M}(\mathcal{A})$ itself into an Ext-category.

Proposition 10. *Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ compatible with a polarization. Then the conflations make \mathcal{M} into an Ext-category.*

Proof. By Proposition 7 and Definition 5, the composition of deflations is a deflation. Furthermore, if $\varphi = \alpha\beta$ is a deflation, then α is a deflation. Now let $\rho: a \rightarrow b$ be a deflation and $\varphi: d \rightarrow b$ arbitrary. Then ρ has a factorization $\rho: a \xrightarrow{\binom{1}{0}} a \oplus d \xrightarrow{(\rho \ \varphi)} b$, whence $(\rho \ \varphi)$ is a deflation. Therefore, the pullback

$$\begin{array}{ccc} c & \xrightarrow{\rho'} & d \\ \downarrow & & \downarrow \varphi \\ a & \xrightarrow{\rho} & b \end{array} \quad (16)$$

exists, and ρ' is again a deflation. For any $a \in \text{Ob } \mathcal{M}$, there is a morphism $\varepsilon: p \rightarrow a$ in Σ with $p \in \text{Pr } \Sigma$. Hence $(\varepsilon \ \alpha_a): p \oplus a^+ \rightarrow a$ is a deflation. Since the objects of $\text{Pr } \Sigma$ and \mathcal{M}_+ are Ext-projective, this completes the proof. \square

Lemma 2. *Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ compatible with a polarization. Let $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ be a conflation in \mathcal{M} and $\varphi: b \rightarrow x$ a morphism with $\varphi\alpha = 0$ in $\mathcal{M}/[\text{Pr } \Sigma]$. Then there is a regular $\rho: x \rightarrow y$ in $\mathcal{M}/[\text{Pr } \Sigma]$ such that $\rho\varphi$ factors through β in $\mathcal{M}/[\text{Pr } \Sigma]$.*

Proof. Let $\varepsilon: p \rightarrow x$ be a morphism in Σ with $p \in \text{Pr } \Sigma$. Then $\varphi\alpha = \varepsilon\psi$ for some $\psi: a \rightarrow p$ in \mathcal{M} . Since p^- is Ext-injective, $\beta_p\psi$ factors through α , and we get a commutative diagram

$$\begin{array}{ccccc} a & \xrightarrow{\alpha} & b & \xrightarrow{\beta} & c \\ \downarrow \psi & & \downarrow (\varphi_\omega) & & \downarrow \\ p & \xrightarrow{(\varepsilon_{\beta_p})} & x \oplus p^- & \xrightarrow{(\rho\sigma)} & y \end{array}$$

with an inflation (ε_{β_p}) and $(\rho\sigma) = \text{cok}(\varepsilon_{\beta_p})$. By Proposition 3 and (12), $\sigma \in \Sigma$ and $p^- \in \text{Pr } \Sigma$. Therefore, Proposition 8 implies that ρ is regular in $\mathcal{M}/[\text{Pr } \Sigma]$. \square

Our next theorem provides a converse to Proposition 4. It determines the triadic categories which are compatible with a polarization.

Theorem 3. *Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ compatible with a polarization. Then Σ makes \mathcal{M} into a triadic category.*

Proof. We write $\mathcal{P} := \text{Pr } \Sigma$ and $\mathcal{I} := \text{In } \Sigma$ and show first that $\mathcal{M}_{\mathcal{P}}$ exists. Let $\rho: a \rightarrow b$ and $\varphi: d \rightarrow b$ be morphisms in \mathcal{M} with ρ regular in $\mathcal{M}/[\mathcal{P}]$. By Proposition 8, there is an exact square (13) with $\alpha, \beta \in \Sigma$ and $p, q \in \mathcal{P}$. Since $\rho = (\rho\beta)$ in $\mathcal{M}/[\mathcal{P}]$, we may assume that ρ is a deflation with $\text{Ker } \rho \in \mathcal{P}$. So there exists a pullback (16) with a deflation ρ' and $\text{Ker } \rho' = \text{Ker } \rho \in \mathcal{P}$. Hence, by Lemma 1, ρ' is regular in $\mathcal{M}/[\mathcal{P}]$. This proves the right Ore condition in $\mathcal{M}/[\mathcal{P}]$. To get the left Ore condition, let again $\rho: a \rightarrow b$ be a deflation with $p := \text{Ker } \rho \in \mathcal{P}$ and $\varphi: a \rightarrow x$ an arbitrary morphism in \mathcal{M} . By Lemma 2, there exists a regular $\rho': x \rightarrow y$ in $\mathcal{M}/[\mathcal{P}]$ such that $\rho'\varphi$ factors through ρ in $\mathcal{M}/[\mathcal{P}]$. Thus $\mathcal{M}_{\mathcal{P}}$ exists.

Next we show that every morphism $\varphi \in \mathcal{M}_{\mathcal{P}}$ has a cokernel. First, we may assume that φ is represented by a morphism $\varphi: a \rightarrow b$ in \mathcal{M} . By Lemma 1, we may further assume that φ is an inflation. Then Lemma 1 implies that the cokernel γ of φ in \mathcal{M} is epic in $\mathcal{M}_{\mathcal{P}}$. Hence $\gamma = \text{cok } \varphi$ holds in $\mathcal{M}_{\mathcal{P}}$ by Lemma 2.

To prove that $\mathcal{M}_{\mathcal{P}}$ is left abelian, consider a diagram (1) in $\mathcal{M}_{\mathcal{P}}$ with $\gamma = \text{cok } \varphi$ and $\gamma\psi = 0$. Using Lemma 1, we may assume that $\varphi, \psi \in \mathcal{M}$ and φ has a factorization $\varphi: a \xrightarrow{\pi} e \xrightarrow{\sigma} b$. Thus we can identify γ with the cokernel of σ in \mathcal{M} . So γ is \mathcal{P} -epic, and $\gamma\psi \in [\mathcal{P}]$. Therefore, we can replace ψ by a morphism ψ' with $\psi' - \psi \in [\mathcal{P}]$ and $\gamma\psi' = 0$. Hence $\psi' = \sigma\omega$ for some $\omega: d \rightarrow e$. By Proposition 10, taking the pullback of ω along π , we infer that $\mathcal{M}_{\mathcal{P}}$ is left abelian. Moreover, it follows that the objects of $\mathcal{M}_{\mathcal{P}}$ are projective in $\mathcal{M}_{\mathcal{P}}$. For any $a \in \text{Ob } \mathcal{M}$, the $\mathcal{M}_{\mathcal{P}}$ -epimorphism $\alpha_a: a^+ \rightarrow a$ is epic in $\mathcal{M}_{\mathcal{P}}$. In fact, there is a morphism $\varepsilon: p \rightarrow a$ in Σ with $p \in \mathcal{P}$, and $(\varepsilon\alpha_a): p \oplus a^+ \rightarrow a$ is a deflation by Proposition 7. Hence $\alpha_a = (\varepsilon\alpha_a) \in \mathcal{M}_{\mathcal{P}}$ is epic. By duality, this proves (T1) of Definition 1.

To get a global kernel of $\psi \in \mathcal{M}_{\mathcal{P}}$, we represent ψ as a morphism $\psi: b \rightarrow c$ in \mathcal{M} . Choose a morphism $\varepsilon': p \rightarrow c$ in Σ with $p \in \mathcal{P}$. By Proposition 3, there is a pullback (14) with $\varepsilon \in \Sigma$. Hence $\varepsilon = \ker^{\mathcal{P}} \psi$, which proves (T2). Finally, let $\varphi: a \rightarrow b$ be a global coker-

nel. Then the dual of the preceding construction yields $\varphi \in \Sigma$. Therefore, the pushout (11) yields $\gamma = \text{cok}_{\mathcal{P}} \varphi$. Since (11) is a pullback by the dual of Proposition 7, $\varphi = \ker^{\mathcal{P}} \gamma$. \square

Corollary 1. *Let \mathcal{M} be a triadic category compatible with a polarization. Let $\overline{\mathcal{M}}_+$ denote the factor category of \mathcal{M}_+ modulo the ideal of morphisms $\varphi \in \mathcal{M}_+$ which factor through some $p \in \mathcal{P}$. Then $\mathcal{M}_{\mathcal{P}} \approx \mathbf{mod}(\overline{\mathcal{M}}_+)$.*

Proof. We show that the natural functor $F: \mathcal{M}_+ \rightarrow \mathcal{M}_{\mathcal{P}}$ is full. For $a^+, b^+ \in \text{Ob } \mathcal{M}_+$, a morphism $a^+ \rightarrow b^+$ in $\mathcal{M}_{\mathcal{P}}$ is of the form $a^+ \xrightarrow{\varphi} c \xleftarrow{\rho} b^+$ with $\varphi, \rho \in \mathcal{M}$ and ρ regular in $\mathcal{M}/[\mathcal{P}]$. Thus $\varphi = \alpha_c \varphi'$ and $\rho = \alpha_c \rho'$ with $\varphi', \rho' \in \mathcal{M}$. Consider a morphism $\varepsilon: p \rightarrow c$ in Σ with $p \in \mathcal{P}$. Then $\binom{\varepsilon}{\rho}: p \oplus b^+ \rightarrow c$ is a deflation by Proposition 8. Hence α_c factors through $\binom{\varepsilon}{\rho}$, and thus $\alpha_c - \rho\beta \in [\mathcal{P}]$ for some $\beta: c^+ \rightarrow b^+$. Consequently, $\varphi - \rho\beta\varphi' = (\alpha_c - \rho\beta)\varphi' \in [\mathcal{P}]$, which gives $\rho^{-1}\varphi = \beta\varphi'$ in $\mathcal{M}_{\mathcal{P}}$. Thus F is full. Since the subcategory of projectives in $\mathcal{M}_{\mathcal{P}}$ is just $F(\mathcal{M}_+)$, we get the desired equivalence. \square

Corollary 2. *Let \mathcal{A} be an Ext-category. Then $\mathcal{M}(\mathcal{A})$ is triadic with respect to the class \mathcal{P} of objects $A \rightarrow P$ with P Ext-projective, and the class \mathcal{I} of objects $I \rightarrow B$ with I Ext-injective. Moreover, $\mathcal{M}(\mathcal{A})$ is compatible with its natural polarization $(\mathcal{A}^+, \mathcal{A}^-)$.*

Proof. By [23, Theorem 1], the class Σ of \mathcal{P} -epic \mathcal{I} -monomorphisms is exact. Σ consists of the morphisms $\varepsilon: a \rightarrow b$ given by commutative squares

$$\begin{array}{ccc} A_1 & \xrightarrow{e_1} & B_1 \\ \downarrow a & & \downarrow b \\ A_0 & \xrightarrow{e_0} & B_0 \end{array} \quad (17)$$

such that the sequence $A_1 \xrightarrow{\binom{-e_1}{a}} B_1 \oplus A_0 \xrightarrow{(b \ e_0)} B_0$ is a conflation. To show that Σ is fully exact, we use Proposition 3(a). First, let $\beta: b \rightarrow e$ be split epic in $\mathcal{M}(\mathcal{A})$. There is a polar sequence $f \xrightarrow{\alpha} b \xrightarrow{\beta} e$ which can be assumed to be of the form (10). Then the property of β to be split epic says that there are morphisms $g: D \rightarrow B_1$ and $h: B_0 \rightarrow D$ in \mathcal{A} with $fg + he = 1$. By symmetry, this implies that α is split monic. Next let (17) be a morphism $\varepsilon: a \rightarrow b$ in Σ , and let

$$\begin{array}{ccc} D_1 & \xrightarrow{f_1} & B_1 \\ \downarrow d & & \downarrow b \\ D_0 & \xrightarrow{f_0} & B_0 \end{array} \quad (18)$$

be an arbitrary morphism $\varphi : d \rightarrow b$ in $\mathcal{M}(\mathcal{A})$. Consider the pullbacks

$$\begin{array}{ccc} C_0 & \xrightarrow{p} & D_0 \\ \downarrow (h_g) & & \downarrow f_0 \\ B_1 \oplus A_0 & \xrightarrow{(be_0)} & B_0 \end{array} \quad \begin{array}{ccc} C_1 & \xrightarrow{q} & D_1 \\ \downarrow c & & \downarrow d \\ C_0 & \xrightarrow{p} & D_0 \end{array} \quad (19)$$

in \mathcal{A} . Then the right-hand square in (19) gives a morphism $\varepsilon' : c \rightarrow d$ in $\mathcal{M}(\mathcal{A})$. By [23, Proposition 1], the factorization $p : C_0 \xrightarrow{\binom{1}{0}} C_0 \oplus D_1 \xrightarrow{(p \ d)} D_0$ shows that $(p \ d)$ is a deflation. Hence $\varepsilon' \in \Sigma$. By [23, Lemma 1], the composition $\varphi \varepsilon'$ factors through ε . So we get a commutative square

$$\begin{array}{ccc} c & \xrightarrow{\varepsilon'} & d \\ \downarrow & & \downarrow \varphi \\ a & \xrightarrow{\varepsilon} & b \end{array} \quad (20)$$

in $\mathcal{M}(\mathcal{A})$, and it is easy to verify that (20) is in fact a pullback. This proves that Σ is fully exact. By Proposition 9, Σ is compatible with $(\mathcal{A}^+, \mathcal{A}^-)$. \square

3. The homotopy category of short exact sequences

Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ and a polarization. Then (E2) implies that every $a \in \text{Ob } \mathcal{M}$ gives rise to morphisms $p \rightarrow a \rightarrow i$ in Σ with $p \in \text{Pr } \Sigma$ and $i \in \text{In } \Sigma$. We call the object a an *inflation* (*deflation*) if $i \in \text{add } \mathcal{M}_+$ (respectively $p \in \text{add } \mathcal{M}_-$). The terminology will be explained after the following proposition which implies that inflations (deflations) are invariant under isomorphism in $\mathbf{com}(\mathcal{M}_-)$ (respectively $\mathbf{mod}(\mathcal{M}_+)$). Therefore, the classes of inflations and deflations in \mathcal{M} induce full subcategories $\text{inf } \mathcal{M}_- \subset \mathbf{com}(\mathcal{M}_-)$ and $\text{def } \mathcal{M}_+ \subset \mathbf{mod}(\mathcal{M}_+)$, respectively. Let $\overline{\text{Pr}} \Sigma$ (respectively $\overline{\text{In}} \Sigma$) denote the full subcategory of $\mathbf{mod}(\mathcal{M}_+)$ (respectively $\mathbf{com}(\mathcal{M}_-)$) induced by $\text{Pr } \Sigma \subset \mathcal{M}$ (respectively $\text{In } \Sigma \subset \mathcal{M}$). Thus $\overline{\text{Pr}} \Sigma \approx \text{Pr } \Sigma / [\mathcal{M}_-]$ and $\overline{\text{In}} \Sigma \approx \text{In } \Sigma / [\mathcal{M}_+]$.

Proposition 11. *Let \mathcal{M} be an additive category with a fully exact class $\Sigma \subset \mathcal{M}$ and a polarization. Then $(\overline{\text{Pr}} \Sigma, \text{def } \mathcal{M}_+)$ is a torsion theory in $\mathbf{mod}(\mathcal{M}_+)$, and $(\text{inf } \mathcal{M}_-, \overline{\text{In}} \Sigma)$ is a torsion theory in $\mathbf{com}(\mathcal{M}_-)$.*

Proof. For any $a \in \text{Ob } \mathcal{M}$, there is a sequence

$$p \xrightarrow{\varepsilon} a \xrightarrow{\gamma} c \quad (21)$$

of morphisms in \mathcal{M} with $\varepsilon \in \Sigma$, $p \in \text{Pr } \Sigma$, and $\gamma = \text{cok } \varepsilon$ in $\mathbf{mod}(\mathcal{M}_+)$. By (12), ε is monic in $\mathcal{M}/[\mathcal{M}_-]$. Therefore, since $\mathbf{mod}(\mathcal{M}_+)$ is left abelian by Proposition 5, Proposition 1

implies that (21) is a short exact sequence in $\mathbf{mod}(\mathcal{M}_+)$. Hence a is a deflation if and only if γ is invertible in $\mathbf{mod}(\mathcal{M}_+)$. Thus if $b \in \text{Ob } \mathcal{M}$ is a deflation and $\rho: a \rightarrow b$ an isomorphism in $\mathbf{mod}(\mathcal{M}_+)$, the commutative diagram

$$\begin{array}{ccccc} p & \xrightarrow{\varepsilon} & a & \xrightarrow{\gamma} & c \\ \downarrow & & \downarrow \rho & & \downarrow \\ 0 & \longrightarrow & b & \xrightarrow{\sim} & c' \end{array}$$

in $\mathbf{mod}(\mathcal{M}_+)$ shows that a is a deflation. By Proposition 6(a), $\overline{\text{Pr}} \Sigma$ is closed with respect to isomorphisms. For an arbitrary $a \in \text{Ob } \mathcal{M}$, there is a polar sequence $p \xrightarrow{(\varepsilon, \beta_p)} a \oplus p^- \xrightarrow{\delta} d$. Hence $\delta = \mu \cdot \text{cok} \begin{pmatrix} \varepsilon \\ \beta_p \end{pmatrix}$ with μ monic by the remark following Proposition 3. Therefore, the morphism γ in the pushout

$$\begin{array}{ccc} p & \xrightarrow{\varepsilon} & a \\ \downarrow \beta_p & \text{PO} & \downarrow \gamma \\ p^- & \longrightarrow & c \end{array}$$

is \mathcal{M}_+ -epic, whence $\gamma = \text{cok } \varepsilon$ in $\mathbf{mod}(\mathcal{M}_+)$. Thus Proposition 3(a) yields $c \in \text{def } \mathcal{M}_+$. Since every morphism $q \rightarrow a$ with $q \in \text{Pr } \Sigma$ factors through ε , we infer that $(\overline{\text{Pr}} \Sigma, \text{def } \mathcal{M}_+)$ is a torsion theory in $\mathbf{mod}(\mathcal{M}_+)$. \square

When \mathcal{A} is an Ext-category, $\text{def } \mathcal{A}^+$ consists of the objects $a^+: A_1^+ \rightarrow A_0^+$ in $\mathbf{mod}(\mathcal{A}^+)$ such that $a \in \mathcal{A}$ is a deflation, and $\text{inf } \mathcal{A}^-$ consists of the objects $a^-: A_0^- \rightarrow A_1^-$ in $\mathbf{com}(\mathcal{A}^-)$ such that $a \in \mathcal{A}$ is an inflation. We define $\text{Ext}(\mathcal{A})$ as the category of conflations $A \rightarrowtail B \twoheadrightarrow C$ in \mathcal{A} , regarded as complexes $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, modulo homotopy. Then there are natural equivalences

$$\text{inf } \mathcal{A}^- \approx \text{Ext}(\mathcal{A}) \approx \text{def } \mathcal{A}^+ \quad (22)$$

according to the relationship between inflations, conflations, and deflations.

Note. If \mathcal{A} is left and right coherent, e.g., if \mathcal{A} is a category $\Lambda\text{-CM}$ of maximal Cohen–Macaulay modules over a Cohen–Macaulay order Λ representing an isolated singularity [2, Proposition 11] and (22) yield a quasi-tilting ([20, §3], [4]) between $\mathbf{mod}(\mathcal{A})$ and $\mathbf{com}(\mathcal{A})$.

Proposition 12. Let \mathcal{M} be a triadic category with a polarization such that every split epimorphism has a kernel. Then the functor $\mathbf{mod}(\mathcal{M}_+) \approx \mathcal{M}/[\mathcal{M}_-] \rightarrow \mathcal{M}_{\mathcal{P}}$ respects cokernels of morphisms.

Proof. We show first that every \mathcal{M}_+ -epimorphism $\gamma: b \rightarrow c$ in \mathcal{M} is epic in $\mathcal{M}_{\mathcal{P}}$. Choose a morphism $\varepsilon: p \rightarrow c$ in Σ with $p \in \mathcal{P}$. Then $(\gamma \varepsilon): b \oplus p \rightarrow c$ is a deflation by Proposition 7. Hence $\gamma = (\gamma \varepsilon) \in \mathcal{M}_{\mathcal{P}}$ is epic by Lemma 1. Assume further that $\alpha: q \rightarrow b$ is a morphism in \mathcal{M} such that $q \xrightarrow{\alpha} b \xrightarrow{\gamma} c$ is a short exact sequence in $\mathbf{mod}(\mathcal{M}_+)$, and $q \in \mathcal{P}$. We show that γ is invertible in $\mathcal{M}_{\mathcal{P}}$. Consider the pullback

$$\begin{array}{ccccc} & & a & \xrightarrow{\delta} & p \\ & & \downarrow \varepsilon' & \text{PB} & \downarrow \varepsilon \\ q & \xrightarrow{\alpha} & b & \xrightarrow{\gamma} & c \end{array} \quad (23)$$

in \mathcal{M} . Thus $\varepsilon' = \ker^{\mathcal{P}} \gamma$, and by Proposition 8, it suffices to show that $a \in \mathcal{P}$. Since $\mathcal{M}_- \subset \mathcal{P}$, it follows that (23) is a pullback in $\mathbf{mod}(\mathcal{M}_+)$. Hence $\alpha = \varepsilon' \beta$ with $\beta = \ker \delta$ in $\mathbf{mod}(\mathcal{M}_+)$. By the pullback property of (23), δ is \mathcal{M}_+ -epic. Therefore, we get a short exact sequence $q \xrightarrow{\beta} a \xrightarrow{\delta} p$ in $\mathbf{mod}(\mathcal{M}_+)$. Let $\beta': q' \rightarrow a$ be a morphism in Σ with $q' \in \mathcal{P}$. Then β factors through β' . So the cokernel $\delta': a \rightarrow d$ of β' in $\mathbf{mod}(\mathcal{M}_+)$ factors through δ . Since $d \in \text{def } \mathcal{M}_+$ by Proposition 11, we get $\delta' = 0$, whence $a \in \mathcal{P}$.

Now let $\varphi: a \rightarrow b$ be a morphism in \mathcal{M} with $\gamma = \text{cok } \varphi$ in $\mathbf{mod}(\mathcal{M}_+)$, and let $\xi: b \rightarrow x$ be a morphism with $\xi \varphi \in [\mathcal{P}]$. Consider an $\varepsilon: p \rightarrow x$ in Σ with $p \in \mathcal{P}$. Then $\xi \varphi = \varepsilon \psi$ for some $\psi: a \rightarrow p$. We form the pushout

$$\begin{array}{ccccc} a & \xrightarrow{\varphi} & b & \xrightarrow{\gamma} & c \\ \downarrow \psi & \text{PO} & \downarrow \nu & & \parallel \\ p & \xrightarrow{\mu} & d & \xrightarrow{\delta} & c \end{array}$$

in $\mathbf{mod}(\mathcal{M}_+)$ with $\delta = \text{cok } \mu$ in $\mathbf{mod}(\mathcal{M}_+)$. Then ε factors through μ in $\mathbf{mod}(\mathcal{M}_+)$, whence μ is monic in $\mathbf{mod}(\mathcal{M}_+)$. By the above, this implies that δ is invertible in $\mathcal{M}_{\mathcal{P}}$. Since ξ factors through ν in $\mathbf{mod}(\mathcal{M}_+)$, we get $\gamma = \text{cok}_{\mathcal{P}} \varphi$. \square

Proposition 13. Let \mathcal{M} be a triadic category with a polarization such that every split epimorphism has a kernel, and let $\gamma: b \rightarrow c$ be an epimorphism in $\mathbf{mod}(\mathcal{M}_+)$ with $b \in \text{def } \mathcal{M}_+$. Then $c \in \text{def } \mathcal{M}_+$.

Proof. We may assume that γ is an \mathcal{M}_+ -epimorphism in \mathcal{M} . Choose a morphism $\varphi: a \rightarrow b$ in \mathcal{M} with $\gamma = \text{cok } \varphi$ in $\mathbf{mod}(\mathcal{M}_+)$, and a morphism $\alpha: p \rightarrow b$ in Σ with $p \in \mathcal{P}$. By Proposition 3, the \mathcal{P} -epimorphism $(\varphi \alpha): a \oplus p \rightarrow b$ admits a factorization $(\varphi \alpha): a \oplus p \xrightarrow{\delta} e \xrightarrow{\varepsilon} b$ with a deflation δ and $\varepsilon \in \Sigma$. Since $\gamma \cdot (\varphi \alpha) \in [\mathcal{P}]$ and $c \in \text{def } \mathcal{M}_+$, Lemma 1 implies that $\gamma \varepsilon \in [\mathcal{M}_-]$. Consequently, $\gamma = \text{cok } \varepsilon$ holds in $\mathbf{mod}(\mathcal{M}_+)$, hence in $\mathcal{M}_{\mathcal{P}}$ by Proposition 12. Now let $\beta: q \rightarrow c$ be a morphism in Σ with $q \in \mathcal{P}$. Then $\beta \alpha_q = \gamma \zeta$ for some $\zeta: q^+ \rightarrow b$. Since $\varepsilon = \ker^{\mathcal{P}} \gamma$ and $\gamma \zeta \in [\mathcal{P}]$, it follows that ζ factors through ε ,

whence $\gamma\zeta \in [\mathcal{M}_-]$. Therefore, $\beta\alpha_q = 0$. So we get $\beta \in [\mathcal{M}_-]$, and thus $c \in \text{def } \mathcal{M}_+$ by Proposition 11. \square

The equivalence $\inf \mathcal{M}_- \approx \text{def } \mathcal{M}_+$ for $\mathcal{M} = \mathcal{M}(\mathcal{A})$ remains valid under the assumption of Proposition 13. Furthermore, we get a new interpretation of the localizations $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{J}}$.

Theorem 4. *Let \mathcal{M} be a triadic category with a polarization such that every split epimorphism has a kernel. Then there are natural equivalences*

$$\inf \mathcal{M}_- \approx \mathcal{M}_{\mathcal{P}} \approx \mathcal{M}_{\mathcal{J}} \approx \text{def } \mathcal{M}_+. \quad (24)$$

Proof. Since $\mathcal{M}_- \subset \mathcal{P}$ by Proposition 7, the functor $\mathcal{M} \rightarrow \mathcal{M}_{\mathcal{P}}$ factors through $\mathcal{M} \rightarrow \mathbf{mod}(\mathcal{M}_+)$. So we get a functor

$$F : \text{def } \mathcal{M}_+ \hookrightarrow \mathbf{mod}(\mathcal{M}_+) \longrightarrow \mathcal{M}_{\mathcal{P}}. \quad (25)$$

Let a, b be objects in $\text{def } \mathcal{M}_+$. If $\varphi : a \rightarrow b$ is a morphism in $\mathbf{mod}(\mathcal{M}_+)$ with $F(\varphi) = 0$, then $\varphi \in [\overline{\text{Pr}} \Sigma]$. Since $b \in \text{def } \mathcal{M}_+$, this yields $\varphi = 0$. So F is faithful. Next, consider a short exact sequence (21). By Proposition 12, γ is invertible in $\mathcal{M}_{\mathcal{P}}$. Hence F is dense. Finally, let $\rho^{-1}\varphi : a \rightarrow c \leftarrow b$ be a morphism in $\mathcal{M}_{\mathcal{P}}$, i.e., $\varphi, \rho \in \mathcal{M}$ with ρ regular in $\mathcal{M}/[\mathcal{P}]$. Then the preceding argument gives a morphism $\rho' : c \rightarrow c'$ in \mathcal{M} with $c' \in \text{def } \mathcal{M}_+$ such that ρ' is regular in $\mathcal{M}/[\mathcal{P}]$. Therefore, we may assume without loss of generality that $c \in \text{def } \mathcal{M}_+$, and it remains to be shown that $\rho \in \mathbf{mod}(\mathcal{M}_+)$ is invertible. Since ρ is monic in $\mathcal{M}/[\mathcal{P}]$, and $b \in \text{def } \mathcal{M}_+$, we infer that ρ is monic in $\mathbf{mod}(\mathcal{M}_+)$. Let $\varepsilon : p \rightarrow c$ be a morphism in Σ with $p \in \mathcal{P}$. Then Proposition 8 implies that $(\rho \varepsilon) : b \oplus p \rightarrow c$ is a deflation. By Lemma 1, this implies that $(\rho \varepsilon)$ is epic in $\mathcal{M}/[\mathcal{P}]$. Therefore, if $\gamma : c \rightarrow d$ is the cokernel of ρ in $\mathbf{mod}(\mathcal{M}_+)$, then $\gamma \in [\mathcal{P}]$. Since $c \in \text{def } \mathcal{M}_+$, Proposition 13 yields $d \in \text{def } \mathcal{M}_+$. Hence $\gamma = 0$ in $\mathbf{mod}(\mathcal{M}_+)$. Thus ρ is invertible in $\mathbf{mod}(\mathcal{M}_+)$. \square

For an Ext-category \mathcal{A} , Theorem 4 implies that

$$\mathcal{M}(\mathcal{A})_{\mathcal{P}} \approx \mathcal{M}(\mathcal{A})_{\mathcal{J}} \approx \text{Ext}(\mathcal{A}). \quad (26)$$

In particular, this shows that $\text{Ext}(\mathcal{A})$ is abelian. Let us determine the semisimple objects in $\text{Ext}(\mathcal{A})$.

Definition 6. Let \mathcal{A} be an Ext-category with the *Krull–Schmidt property*, i.e., every object of \mathcal{A} is a finite direct sum of objects with local endomorphism rings. The ideal $\text{Rad } \mathcal{A}$ of \mathcal{A} generated by the non-invertible morphisms between indecomposable objects is called the *radical* of \mathcal{A} . A morphism $f : A \rightarrow B$ in $\text{Rad } \mathcal{A}$ is said to be *right (left) almost split* if every morphism $A' \rightarrow B$ (respectively $A \rightarrow B'$) in $\text{Rad } \mathcal{A}$ factors through f . If such a morphism f exists for each object B (respectively A) of \mathcal{A} , we will say that \mathcal{A} has *right (left) almost split morphisms*. We call a conflation

$$A \xrightarrow{a} B \xrightarrow{b} C \quad (27)$$

of \mathcal{A} *almost split* [23] if $a, b \in \text{Rad } \mathcal{A}$, and b is right almost split. (We shall see below that this property is self-dual.) We say that \mathcal{A} *has almost split sequences* if an almost split conflation (27) exists for each non-Ext-projective indecomposable object C and for each non-Ext-injective indecomposable object A of \mathcal{A} .

For a Krull–Schmidt category \mathcal{A} , the objects $a: A_1 \rightarrow A_0$ in $\mathcal{M}(\mathcal{A})$ with $a \in \text{Rad } \mathcal{A}$ form a full subcategory $\mathcal{M}(\mathcal{A})$ which is equivalent to $\mathcal{M}(\mathcal{A})$ [21]. Similarly, if (27) is a conflation, then $b = \begin{pmatrix} b' & 0 \\ 0 & e \end{pmatrix}$ with $b' \in \text{Rad } \mathcal{A}$ and e invertible. Hence $a = \begin{pmatrix} a' \\ 0 \end{pmatrix}$, where a' is an inflation by [23, Proposition 1]. Therefore, the object (27) in $\text{Ext}(\mathcal{A})$ is isomorphic to $A \xrightarrow{a'} B' \xrightarrow{b'} C'$. By symmetry, it follows that the conflations (27) with $a, b \in \text{Rad } \mathcal{A}$ form a full subcategory $\text{Ext}(\mathcal{A})$ of $\text{Ext}(\mathcal{A})$ equivalent to $\text{Ext}(\mathcal{A})$.

Proposition 14. *Let \mathcal{A} be an Ext-category with the Krull–Schmidt property. An object (27) of $\text{Ext}(\mathcal{A})$ is an almost split conflation if and only if it belongs to $\mathcal{S}(\text{Ext}(\mathcal{A}))$.*

Proof. By (22) and [22, Proposition 3], an almost split conflation (27) corresponds to an object $b \in \text{def } \mathcal{A}^+$ which is semisimple in $\mathbf{mod}(\mathcal{A}^+)$. Therefore, we have to show that $b \in \mathcal{S}(\mathbf{mod}(\mathcal{A}^+)) \Leftrightarrow b \in \mathcal{S}(\text{def } \mathcal{A}^+)$. Assume first that $b \in \mathcal{S}(\mathbf{mod}(\mathcal{A}^+))$. Since $\text{def } \mathcal{A}^+$ is abelian, it suffices to prove that $b \in \mathcal{S}_l(\text{def } \mathcal{A}^+)$. Thus let $\mu: a \rightarrow b$ be a monomorphism in $\text{def } \mathcal{A}^+$, and let $\xi: x \rightarrow a$ be a morphism in $\mathbf{mod}(\mathcal{A}^+)$ with $\mu\xi = 0$. By Proposition 11, there is a short exact sequence $p \xrightarrow{\varepsilon} x \xrightarrow{\gamma} y$ in $\mathbf{mod}(\mathcal{A}^+)$ with $p \in \mathcal{P}$ and $y \in \text{def } \mathcal{A}^+$. Then ξ factors through γ , which implies that $\xi = 0$. Hence μ is monic in $\mathbf{mod}(\mathcal{A}^+)$, and thus μ is split monic. Conversely, assume that $b \in \mathcal{S}(\text{def } \mathcal{A}^+)$. Then $b \in \mathcal{S}_r(\mathbf{mod}(\mathcal{A}^+))$ by Proposition 13. Hence $b \in \mathcal{S}(\mathbf{mod}(\mathcal{A}^+))$ by [22, Proposition 2]. \square

Using the symmetry of $\text{Ext}(\mathcal{A})$, we immediately obtain (cf. [7, Proposition 1.5]) the following corollary.

Corollary. *Let \mathcal{A} be an Ext-category with the Krull–Schmidt property. A conflation (27) is almost split in \mathcal{A} if and only if it is almost split in \mathcal{A}^{op} .*

4. Almost split sequences versus L-functors

The foregoing theory enables us to give a very simple definition of L-functors [21, 23]. Recall that a *pointed functor* [16] of an additive category \mathcal{M} is defined as a functor $L^-: \mathcal{M} \rightarrow \mathcal{M}$ together with a natural transformation $\lambda^-: 1 \rightarrow L^-$. For example, if \mathcal{C} is a reflective full subcategory of \mathcal{M} , i.e., if the inclusion $I: \mathcal{C} \hookrightarrow \mathcal{M}$ has a left adjoint $T: \mathcal{M} \rightarrow \mathcal{C}$, then IT together with the unit $\eta: 1 \rightarrow IT$ is a pointed functor. We call IT the *reflection* of \mathcal{C} . Dually, an *augmented functor* of \mathcal{M} is given by an endofunctor L^+ together with a natural transformation $\lambda^+: L^+ \rightarrow 1$. In particular, every coreflective full subcategory \mathcal{C} of \mathcal{M} gives rise to an augmented functor of \mathcal{M} , the *coreflection* of \mathcal{C} . We call \mathcal{M} *(co-)semilocal* if the subcategory $\mathcal{S}(\mathcal{M})$ is (co-)reflective. If every non-zero object a of \mathcal{M} admits non-zero morphisms $s \rightarrow a \rightarrow s'$ with $s, s' \in \mathcal{S}(\mathcal{M})$, we will say that \mathcal{M} *has enough semisimple objects*. For an augmented or pointed functor L^\pm we define $\text{Pr } L^\pm$ (respectively $\text{In } L^\pm$) as the

largest full subcategory of \mathcal{M} such that λ_a^\pm is $(\text{Pr } L^\pm)$ -epic (respectively $(\text{In } L^\pm)$ -monic) for each $a \in \text{Ob } \mathcal{M}$. Note that a left adjoint of a pointed functor is augmented, and vice versa [23, §2].

Definition 7. Let \mathcal{M} be a triadic category. We define a *left triadic* functor of \mathcal{M} as an augmented functor $L^+ : \mathcal{M} \rightarrow \mathcal{M}$ with $\lambda_a^+ \in \Sigma_{\mathcal{M}}$ for all $a \in \text{Ob } \mathcal{M}$. Thus every $a \in \text{Ob } \mathcal{M}$ gives rise to a standard triad

$$T Sa \xrightarrow{\sigma_a} L^+ a \xrightarrow{\lambda_a^+} a \xrightarrow{\pi_a} Sa \quad (28)$$

such that every morphism $a \rightarrow b$ induces a morphism of standard triads. We call L^+ a *left L-functor* if, in addition, Sa is semisimple for each $a \in \text{Ob } \mathcal{M}$, $\text{Pr } L^+ \subset \mathcal{P}$, and $\text{In } L^+ \subset \mathcal{J}$. *Right L-functors*, and *right triadic* functors, are defined in a dual way. If \mathcal{M} admits a left L-functor L^+ and a right L-functor L^- , we simply say that \mathcal{M} has *L-functors*.

Remark. For a triadic category \mathcal{M} , the left L-functors defined here are left L-functors in the sense of [23]. In fact, the properties (L1)–(L7) given in [23, Definition 6], are easily checked except (L3) and its dual (L4), which need a moment's reflection. For any $a \in \text{Ob } \mathcal{M}_{\mathcal{P}}$, there exists a triad $Ta = Ta \rightarrow c \xrightarrow{\gamma} a$ with $c \in \mathcal{J}$. Hence γ satisfies (L3) of [23]. Conversely, let $L : \mathcal{M} \rightarrow \mathcal{M}$ be an L-functor in the sense of [23]. Then the $(\text{Pr } L)$ -epic $(\text{In } L)$ -monomorphisms form an exact class $\Sigma \subset \mathcal{M}$. If Σ makes \mathcal{M} into a triadic category, then L is a left L-functor.

Lemma 3. Let \mathcal{M} be a triadic category with a left triadic functor L^+ such that $\text{Pr } L^+ \subset \mathcal{P}$. For an object a of \mathcal{M} , let π_a be split epic. Then π_a is invertible.

Proof. Since $\lambda_a^+ = \ker \pi_a$ in $\mathcal{M}_{\mathcal{P}}$, there are morphisms $\varphi : a \rightarrow b$ and $\rho : L^+ a \rightarrow b$ in \mathcal{M} with ρ regular in $\mathcal{M}/[\mathcal{P}]$ such that $\rho^{-1} \varphi \cdot \lambda_a^+ = 1$ in $\mathcal{M}_{\mathcal{P}}$. Hence $\rho - \lambda_b^+ \cdot L^+ \varphi = \rho - \varphi \lambda_a^+ \in [\mathcal{P}]$, and thus λ_b^+ is invertible in $\mathcal{M}_{\mathcal{P}}$. Therefore, $\pi_b = 0$, which gives $b \in \text{Pr } L^+ \subset \mathcal{P}$. Consequently, $L^+ a \in \mathcal{P}$, whence π_a is invertible. \square

Proposition 15. Let \mathcal{M} be a triadic category.

- (a) A left L-functor exists if and only if $\mathcal{M}_{\mathcal{P}}$ is semilocal with enough semisimple objects.
- (b) A left L-functor is unique up to isomorphism.
- (c) If L^+ is a left L-functor, then an endofunctor of \mathcal{M} is right adjoint to L^+ if and only if it is a right L-functor.

Proof. (a) Let L^+ be a left L-functor. For $a \in \text{Ob } \mathcal{M}$, let $\varphi : a \rightarrow s$ be a morphism in \mathcal{M} with $s \in \text{S}(\mathcal{M}_{\mathcal{P}})$. Then π_s is split epic, and Lemma 3 implies that π_s is invertible. Hence φ factors through π_a in $\mathcal{M}_{\mathcal{P}}$. This shows that $\mathcal{M}_{\mathcal{P}}$ is semilocal. By Theorem 1, $T Sa \in \text{S}(\mathcal{M}_{\mathcal{J}})$ for all $a \in \text{Ob } \mathcal{M}$. Therefore, if $\text{Hom}_{\mathcal{M}_{\mathcal{J}}}(T Sa, x) = 0$ for some $x \in \text{Ob } \mathcal{M}$, then each morphism $L^+ a \rightarrow x$ factors through λ_a^+ . Since $\text{Pr } L^+ \subset \mathcal{P}$ and $\text{In } L^+ \subset \mathcal{J}$, it follows that $\mathcal{M}_{\mathcal{P}} \approx \mathcal{M}_{\mathcal{J}}$ has enough semisimple objects. Conversely, let $\mathcal{M}_{\mathcal{P}}$ be semilocal with reflection $S : \mathcal{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$ and unit $\pi : 1 \rightarrow S$. Assume that $\mathcal{M}_{\mathcal{P}}$ has enough semisimple

objects. For an object a of \mathcal{M} , let $\gamma : Sa \rightarrow c$ be the cokernel of π_a . Then $\pi_c \gamma \pi_a = 0$ implies that $\pi_c \gamma = 0$. Hence $\pi_c = 0$, and thus $c = 0$ in $\mathcal{M}_{\mathcal{P}}$. Therefore, π_a is epic, and $\lambda_a^+ := \ker^{\mathcal{P}} \pi_a$ yields a left triadic functor L^+ with $\text{Pr } L^+ \subset \mathcal{P}$. Assume that $a \in \text{In } L^+$. To prove that $a \in \mathcal{J}$, we have to show that $\text{Hom}_{\mathcal{M}_{\mathcal{J}}}(\mathcal{S}(\mathcal{M}_{\mathcal{J}}), a) = 0$. Thus let $\varphi : s \rightarrow a$ be a morphism in \mathcal{M} with $s \in \mathcal{S}(\mathcal{M}_{\mathcal{J}})$. Then there is a triad $s = s \rightarrow i \xrightarrow{\omega} T^{-1}s$ with $i \in \mathcal{J}$. Since ω factors through π_i , we get a morphism of triads

$$\begin{array}{ccccccc} TSi & \xrightarrow{\sigma_i} & L^+i & \xrightarrow{\lambda_i^+} & i & \xrightarrow{\pi_i} & Si \\ \downarrow T\zeta & & \downarrow \psi & & \downarrow 1 & & \downarrow \zeta \\ s & \xlongequal{\quad} & s & \longrightarrow & i & \xrightarrow{\omega} & T^{-1}s \end{array}$$

with ζ epic in $\mathcal{M}_{\mathcal{P}}$. Hence $T\zeta$ is epic in $\mathcal{M}_{\mathcal{J}}$. Since $\varphi\psi$ factors through λ_i^+ , we have $\varphi \cdot T\zeta = 0$, and thus $\varphi = 0$ in $\mathcal{M}_{\mathcal{J}}$. The uniqueness (b) follows by the uniqueness of the reflection S of $\mathcal{S}(\mathcal{M}_{\mathcal{P}})$.

(c) Let L^+ be a left L-functor. Then $\text{Pr } L^+ = \mathcal{P}$ and $\text{In } L^+ = \mathcal{J}$. Assume that L^- is right adjoint to L^+ . Then [23, Proposition 7], implies that $\text{Pr } L^- = \mathcal{P}$ and $\text{In } L^- = \mathcal{J}$. Hence L^- is right triadic. By [23, Proposition 4], we have a morphism of triads

$$\begin{array}{ccccccc} TS(L^-a) & \longrightarrow & L^+L^-a & \xrightarrow{\lambda_{L^-a}^+} & L^-a & \longrightarrow & S(L^-a) \\ \downarrow & & \downarrow \varepsilon_a & & \downarrow 1 & & \downarrow \\ S^-a & \longrightarrow & a & \xrightarrow{\lambda_a^-} & L^-a & \longrightarrow & T^{-1}S^-a \end{array}$$

for each $a \in \text{Ob } \mathcal{M}$, where $\varepsilon : L^+L^- \rightarrow 1$ denotes the counit of the adjunction $L^+ \dashv L^-$. Since $S(L^-a) \in \mathcal{S}(\mathcal{M}_{\mathcal{P}})$, it follows that $T^{-1}S^-a \in \mathcal{S}_r(\mathcal{M}_{\mathcal{P}}) = \mathcal{S}(\mathcal{M}_{\mathcal{P}})$, and thus $S^-a \in \mathcal{S}(\mathcal{M}_{\mathcal{J}})$. So L^- is a right L-functor. Conversely, let L^- be a right L-functor. By the above, every morphism $\varphi : a \rightarrow L^-b$ in \mathcal{M} induces a morphism of triads

$$\begin{array}{ccccccc} TSa & \longrightarrow & L^+a & \xrightarrow{\lambda_a^+} & a & \xrightarrow{\pi_a} & Sa \\ \downarrow & & \downarrow \varphi^+ & & \downarrow \varphi & & \downarrow \\ S^-b & \longrightarrow & b & \xrightarrow{\lambda_b^-} & L^-b & \longrightarrow & T^{-1}S^-b \end{array}$$

with a unique φ^+ . By symmetry, $\varphi \mapsto \varphi^+$ defines an adjunction $L^+ \dashv L^-$. \square

Proposition 15 provides the link between L-functors and almost split sequences. For an Ext-category \mathcal{A} , let $\mathcal{P}(\mathcal{A})$ (respectively $\mathcal{J}(\mathcal{A})$) denote the full subcategory of Ext-projectives (Ext-injectives).

Theorem 5. Let \mathcal{A} be an Ext-category with the Krull–Schmidt property. The following are equivalent:

- (a) \mathcal{A} has almost split sequences.
- (b) $\text{Ext}(\mathcal{A})$ is semilocal and cosemilocal with enough semisimple objects.
- (c) $\mathcal{M}(\mathcal{A})$ has L -functors.
- (d) $\mathcal{A}/[\mathcal{P}(\mathcal{A})]$ has right almost split morphisms, and $\mathcal{A}/[\mathcal{I}(\mathcal{A})]$ has left almost split morphisms.

Proof. The equivalence (b) \Leftrightarrow (c) follows by (26) and Proposition 15. The category $\overline{\mathcal{M}}(\mathcal{A})_+$ in Corollary 1 of Theorem 3 is equivalent to $\mathcal{A}/[\mathcal{P}(\mathcal{A})]$. Therefore, the equivalence (b) \Leftrightarrow (d) follows by [22, Proposition 4].

(a) \Rightarrow (b). For an object $A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0$ in $\text{Ext}(\mathcal{A})$, let $A_0 = C \oplus P$ be a decomposition with $P \in \mathcal{P}(\mathcal{A})$, such that C has no non-zero Ext-projective direct summands. So there is an almost split conflation $A \twoheadrightarrow B \twoheadrightarrow C$ in \mathcal{A} , and we get a morphism

$$\begin{array}{ccccc} A_2 & \twoheadrightarrow & A_1 & \twoheadrightarrow & A_0 = C \oplus P \\ \downarrow & & \downarrow & & \downarrow (10) \\ A & \twoheadrightarrow & B & \twoheadrightarrow & C \end{array} \quad (29)$$

in $\text{Ext}(\mathcal{A})$. Using Proposition 14, a straightforward verification shows that (29) defines a left adjoint of the inclusion $S(\text{Ext}(\mathcal{A})) \hookrightarrow \text{Ext}(\mathcal{A})$. Hence $\text{Ext}(\mathcal{A})$ is semilocal. If the lower short exact sequence in (29) splits, then $C = 0$, and so the upper sequence in (29) also splits. By duality, this proves (b).

(b) \Rightarrow (a). Let E be an indecomposable non-Ext-projective object of \mathcal{A} . Then there exists an object $D \twoheadrightarrow P \twoheadrightarrow E$ in $\text{Ext}(\mathcal{A})$ with P Ext-projective. By assumption, there exists a morphism

$$\begin{array}{ccccc} D & \twoheadrightarrow & P & \xrightarrow{p} & E \\ \downarrow & & \downarrow & & \downarrow f \\ A & \twoheadrightarrow & B & \twoheadrightarrow & C \end{array} \quad (30)$$

in $\text{Ext}(\mathcal{A})$ which represents the unit of the reflection of $S(\text{Ext}(\mathcal{A})) \hookrightarrow \text{Ext}(\mathcal{A})$. By Proposition 14, the lower short exact sequence in (30) is an almost split conflation. Since $\text{Ext}(\mathcal{A})$ has enough semisimple objects and p does not split, it follows that the morphism (30) is non-zero. Hence $f \notin \text{Rad } \mathcal{A}$, and so f is split monic. Since (30) is epic by virtue of (c), we infer that f is invertible. By symmetry, this proves (a). \square

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