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Products of finite abelian groups as profinite groups

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Abstract

We investigate products of finite abelian groups of bounded exponent as profinite structures in the sense of Newelski. In such groups we describe orbits under the action of the standard structural group of automorphisms. Then we conclude that such groups are small, m-normal and m-stable. Let X be a product of countably many finite abelian groups. We also investigate the influence of modifications of the standard structural group of X on its smallness, m-normality and m-stability.

Keywords: Small profinite groups; Products of finite groups

0. Introduction

A profinite structure is a profinite topological space X with a distinguished structural group $Aut^*(X)$ which is a closed subgroup of the group of all homeomorphisms of X respecting the appropriate inverse system. A profinite group in this context is an inverse limit of finite groups with structural group preserving the group action. We say that a structural group of a profinite structure (group) X is standard if it is the group of all homeomorphisms (topological automorphisms) of X respecting the appropriate inverse system. We say that X is small if for every natural number $n > 0$, on the set $X^n = X \times \cdots \times X$ there

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are countably many orbits under the action of $Aut^*(X)$. Profinite structures and groups in this sense have been introduced in [N2] and [N3]. Small profinite groups occur naturally in model theory as profinite groups interpretable in small theories [N2]. Newelski has developed the model theory of small profinite structures. Many results from stable model theory have been proven in this context. m-normality and m-stability (see Definitions 1.2 and 1.3) play the prominent role in all these considerations.

Unfortunately, there have been very little explicit examples of small profinite groups so far. Wagner has proved [W] that every small m-stable profinite group has an open abelian subgroup and has finite exponent. On the other hand, it is easy to see (Example 1) that not every abelian profinite group (even with the standard structural group) of finite exponent is small.

The main aim of this paper is to find new classes of examples of small profinite groups. The main result is a classification of small products of finite groups with the standard structural group (Remark 1.4, Corollary 3.2.2 and Theorem 4.2).

More precisely, in this paper we deal with products of countably many finite groups. Such products can be naturally considered as profinite groups (see Section 1). Newelski has pointed out [N2] that if a product $X = \prod_{i \in \omega} X_i$ of finite groups is small, then almost all X_i are abelian and X has finite exponent. In this paper we show the converse: if X is a product of finitely many finite groups and countably many finite abelian groups of bounded exponent with the standard structural group, then X is small. This yields a new class of examples of small profinite groups. We prove also that these groups are m-normal and m-stable.

Let X be a product of countably many finite abelian groups of bounded exponent. To get more examples of profinite groups we consider some modifications (by changing the group $Aut^*(X)$) of the profinite structure of X. As a result we obtain a family of closed subgroups of the standard structural group of X such that X with any group of this family as a new structural group is m-normal and m-stable. We also prove that if X is a product of finite abelian p-groups (where p is a prime number) of bounded exponent and we replace the standard structural group of X by its Sylow p-subgroup, then the arising profinite structure is still small, m-normal and m-stable. This is relevant to interpreting small profinite groups in fields [K]. Finally we consider products of finite groups as inverse limits of arbitrary inverse systems of finite subproducts. We describe when such products are small, and we show that if such a product is small, then it is also m-normal.

Our results yield new classes of examples of small profinite groups. Moreover, they show that in all classes of groups which we consider (i.e. in products of finite groups with the standard structural group and with some non-standard structural groups) there is no small but not m-normal profinite group. This means that the answer to the open question, if there exists a small profinite group (structure) which is not m-normal (see [N2,N3]), is negative in all these classes of groups.

1. Preliminaries

We present here all necessary definitions and basic facts on profinite structures and groups in the sense of Newelski. For the proofs and more details about profinite structures and groups see [N2,N3,W]. We start from the general definition of profinite structure. A profinite space is an inverse limit of finite discrete spaces with topology inherited from the product of finite spaces of our inverse system.

Definition 1.1. A profinite structure is a profinite topological space X with a distinguished structural group $Aut^*(X)$ which is a closed subgroup of the group of all homeomorphisms of X respecting the inverse system defining X.

A profinite group in this context is an inverse limit of finite groups with a structural group preserving the group action. We say that a structural group of a profinite structure (group) X is standard if it is the group of all homeomorphisms (topological automorphisms) of X respecting the appropriate inverse system. We denote a profinite structure by $(X, Aut^*(X))$. When it is clear what the structural group is we just write X. It turns out that $Aut^*(X)$ is always a profinite group acting continuously on X.

The simplest examples of profinite groups are products of countably many finite groups. Let $X = \prod_{i \leqslant \omega} X_i$ be such a product. We consider it as the inverse limit of finite groups $X \restriction n = \prod_{i<n} X_i$, $n > 0$, with the natural projections. The standard structural group consists here of these automorphisms of X which induce automorphisms of each $X \restriction n$ for $n > 0$. Any other structural group of X can be chosen as a closed subgroup of the standard one. In the whole paper, for $\eta \in X$, by $\eta \restriction n$ we denote the first n coordinates of η.

Let X be a profinite structure, e.g. a profinite group. Let $A \subseteq X$ be finite. By $Aut^*(X/A)$ we denote the set of elements of $Aut^*(X)$ fixing A pointwise. We say that $V \subseteq X$ is A-invariant if $f[V] = V$ for every $f \in Aut^*(X/A)$. If V is additionally closed, then we say that V is A-definable. An $a \in X$ is a name of V when for every $f \in Aut^*(X)$ we have that $f[V] = V$ iff $f(a) = a$. It is easy (see [N2]) that every definable set V has a canonical name denoted by $\ulcorner V \urcorner$. This name belongs not necessarily to X, but is of the form a/E, where $a \in X^n$ and E is a $\emptyset$-definable equivalence relation on X^n.

For $a \in X^n$ and $A \subseteq X$ we define $o(a/A) = \{f(a)\colon f \in Aut^*(X/A)\}$ (the orbit of a over A). Let $O_n(A) = \{o(a/A)\colon a \in X^n\}$. Each orbit is always a closed subset of X. From now on $A, B, \ldots$ denote finite subsets of X and $a, b, \ldots$ denote elements or finite tuples of elements of X.

We say that a profinite structure X is small if $|O_n(\emptyset)| \leqslant \omega$ for every natural number $n > 0$. Equivalently $O_1(A)$ is countable for every finite set $A \subseteq X$.

Every small profinite structure can be enlarged to X^{eq} by adding so called imaginary elements, i.e. elements of the form a/E, where $a \in X^n$ and E is a $\emptyset$-definable equivalence relation on X^n. Then, for every such E, X^n/E is still a profinite structure, where the structural group is induced by $Aut^*(X)$ acting on X/E. Formally, X^{eq} is a disjoint union of all spaces X/E, where E is a $\emptyset$-definable equivalence relation on X^n, equipped with the disjoint union topology. Then $Aut^*(X)$ acts continuously on X^{eq} and we consider $(X^{\mathrm{eq}}, Aut^*(X))$ in the same way as $(X, Aut^*(X))$. We see that canonical names belong to X^{eq}. For more details see [N2].

Two profinite structures $(X, Aut^*(X))$ and $(Y, Aut^*(Y))$ are isomorphic if there is a homeomorphism $f : X \to Y$ such that the pullback function f^* maps $Aut^*(Y)$ onto $Aut^*(X)$. We say that a profinite structure X is interpretable in a profinite structure Y if there is a continuous 1–1 mapping f of X onto a set $f(X)$ definable in Y^{eq} over some finite set A, such that the pullback function maps $Aut^*(Y/A)$ onto a closed subgroup of $Aut^*(X)$. It is easy to see that any profinite structure interpretable in a small one is also small.

For a finite $A \subseteq X$ by $acl(A)$ we denote the algebraic closure of A, i.e. the set of these elements of X^{eq} which have finitely many conjugates under $Aut^*(X/A)$.

For every finite $A, B \subseteq X$ we have that $o(a/AB)$ (AB denotes $A \cup B$) is open or nowhere dense in $o(a/A)$. In the first case we say that a is m-independent of B over A and we write $a \overset{m}{\mathop{\smile}\limits^{\vert}}_A B$, otherwise a is m-dependent on B over A and we write $a \overset{m}{\not\mathop{\smile}\limits^{\vert}}_A B$.

In small profinite structures m-independence $\overset{m}{\mathop{\smile}\limits^{\vert}}$ has similar properties as forking independence in stable theories.

1. (Symmetry) For finite $A, B, C \subseteq X$ we have that $A \overset{m}{\mathop{\smile}\limits^{\vert}}_C B$ iff $B \overset{m}{\mathop{\smile}\limits^{\vert}}_C A$.
2. (Transitivity) For finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$ we have that $a \overset{m}{\mathop{\smile}\limits^{\vert}}_A C$ iff $a \overset{m}{\mathop{\smile}\limits^{\vert}}_B C$ and $a \overset{m}{\mathop{\smile}\limits^{\vert}}_A B$.
3. (Extensions) For every finite $a, A, B \subseteq X$ there is some $a' \in o(a/A)$ with $a' \overset{m}{\mathop{\smile}\limits^{\vert}}_A B$.
4. $a \in acl(A)$ implies $a \overset{m}{\mathop{\smile}\limits^{\vert}}_A B$ for every finite $B \subseteq X$.

Definition 1.2. The rank $\mathcal{M}$ is the function from the collection of orbits over finite sets to the ordinals together with ∞ satisfying

$$\mathcal{M}(a/A) \geqslant \alpha + 1 \quad \text{iff} \quad \text{there is a } B \supseteq A \text{ with } a \overset{m}{\not\mathop{\smile}\limits^{\vert}}_A B \text{ and } \mathcal{M}(a/B) \geqslant \alpha.$$

X is m-stable if every orbit has ordinal $\mathcal{M}$-rank. Equivalently there is no infinite sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite subsets of X and $a \in X$ such that $o(a/A_{i+1})$ is nowhere dense in $o(a/A_i)$ for every i. We say that X has $\mathcal{M}$-rank n if the supremum of $\mathcal{M}$-ranks of 1-orbits in X equals n.

Definition 1.3. A profinite structure X is m-normal if for every finite $a, A \subseteq X$, there is a clopen $U \ni a$, such that $U \cap o(a/A)$ has finitely many conjugates under $Aut^*(X/a)$.

In the above definition we can choose as U a canonical open neighbourhood of a, where by a canonical open set in

$$X = \varprojlim X_i \subseteq \prod X_i$$

we mean the set of elements of X with the ith coordinate fixed (i is arbitrary). A canonical open set in X^n is a product of canonical open sets in X.

It is worth noticing that m-normality and m-stability have been investigated so far only under the assumption of smallness [N2,N3]. This is because under the assumption of smallness these notions have good model-theoretic properties.

We recall here the remark of Newelski [N2].

Remark 1.4. If a product $X = \prod_{i \in \omega} X_i$ of countably many finite groups is small, then almost all X_i are abelian and X has finite exponent.

Denote by $Aut^*(X)$ the standard structural group of a profinite structure X. Let G_1 and G_2 be closed subgroups of $Aut^*(X)$. By a simple calculation we obtain

Remark 1.5. If G_1 and G_2 are conjugate in $Aut^*(X)$, then (X, G_1) is small (m-normal, m-stable) iff (X, G_2) is small (m-normal, m-stable).

We end this general part by a remark that m-normality is invariant under fixing finite subsets of X. More precisely, for a finite subset B of a small profinite structure X ($Aut^*(X)$ is not necessarily standard), we have

Remark 1.6. $(X, Aut^*(X))$ is m-normal iff $(X, Aut^*(X/B))$ is m-normal.

Proof. $(\rightarrow)$ is trivial.

$(\leftarrow)$ In the proof we use properties of $\overset{m}{\perp}$ listed earlier. Consider finite $a, A \subseteq X$ as in the definition of m-normality. Without loss of generality we can assume that $aA \overset{m}{\perp} B$. Now from the assumption we can choose a canonical open neighbourhood of a in X^n such that $o(a/AB) \cap U = o(a/A) \cap U$ and $o(a/AB) \cap U$ has finitely many conjugates under $Aut^*(X/aB)$. So $\ulcorner o(a/AB) \cap U \urcorner \in acl(aB)$.

To finish our proof it is enough to show that $\ulcorner o(a/A) \cap U \urcorner \in acl(a)$. Suppose for a contradiction that $\ulcorner o(a/A) \cap U \urcorner \notin acl(a)$. Of course we have that $\ulcorner o(a/A) \cap U \urcorner \in acl(aA)$. Using this and $A \overset{m}{\perp}_a B$ we get $\ulcorner o(a/A) \cap U \urcorner \overset{m}{\perp}_a B$, so $\ulcorner o(a/A) \cap U \urcorner \notin acl(aB)$. This is a contradiction. $\square$

The same is true for m-stability. Namely, for a small profinite structure X we have

Remark 1.7. $(X, Aut^*(X))$ is m-stable iff $(X, Aut^*(X/B))$ is m-stable.

Proof. $(\rightarrow)$ is trivial.

$(\leftarrow)$ Suppose for a contradiction that there are finite sets $A_0 \subseteq A_1 \subseteq \cdots \subseteq X$ and $a \in X$ such that $a \overset{m}{\not\perp}_{A_i} A_{i+1}$ for all $i \in \omega$. Using properties of $\overset{m}{\perp}$ we can find $B' \in o(B)$ such that $B' \overset{m}{\perp} aA_{<i}$ for all $i \in \omega$. This implies that $a \overset{m}{\not\perp}_{B'A_i} B'A_{i+1}$ for all $i \in \omega$, so by the automorphism we can find an a' and $A'_0 \subseteq A'_1 \subseteq \cdots \subseteq X$ such that $a' \overset{m}{\not\perp}_{BA'_i} BA'_{i+1}$ for all $i \in \omega$. This is a contradiction. $\square$

Notice that Remark 1.7 follows directly from the standard fact that $\mathcal{M}$-rank is invariant on m-independent extensions, i.e. if $a \overset{m}{\perp}_A B$, then $\mathcal{M}(a/A) = \mathcal{M}(a/AB)$. This also gives the following remark.

Remark 1.8. The $\mathcal{M}$-rank of $(X, Aut^*(X))$ equals the $\mathcal{M}$-rank of $(X, Aut^*(X/B))$.

2. The main technical lemma

The following group theoretic lemma is essential in the paper.

Lemma 2.1. *Let $X = X_0 \times X_1 \times \cdots \times X_w$ be a finite product of finite abelian groups, e be the exponent of X and $n \leqslant w$. If we have finite subgroups A and B of X, an isomorphism f between them and $\alpha \in X$, $\beta_n \in X \restriction n$ such that*

$$(\forall a \in A)(\forall k \in \mathbb{Z}_e)(\forall m \leqslant w+1) \quad \bigl(k \mid a \restriction m \Leftrightarrow k \mid f(a) \restriction m\bigr)$$

and

$$(\forall a \in A)(\forall k, l \in \mathbb{Z}_e)(\forall m \leqslant n) \quad \bigl(k \mid (l\alpha \restriction m - a \restriction m) \Leftrightarrow k \mid \bigl(l\beta_n \restriction m - f(a) \restriction m\bigr)\bigr),$$

then there exists a $\beta_{n+1} \in X \restriction n+1$ extending β_n, such that

$$(\forall a \in A)(\forall k, l \in \mathbb{Z}_e) \quad \bigl(k \mid (l\alpha \restriction n+1 - a \restriction n+1) \Leftrightarrow k \mid \bigl(l\beta_{n+1} - f(a) \restriction n+1\bigr)\bigr).$$

For $n = 0$ there is no β_0 and then Lemma 2.1 says that there exists a $\beta_1 \in X_0$ satisfying the last condition.

Proof. At the beginning we reduce the lemma to the case when all X_i are p-groups for a prime number p. So let us assume that the lemma is true in this case. We write each X_i in the form $Y_{i1} \oplus \cdots \oplus Y_{ij_i}$, where Y_{ij} is a p_{ij}-group for a prime number p_{ij} ($p_{ij_1} \neq p_{ij_2}$ for $j_1 \neq j_2$). Let $P = \{p_1, \ldots, p_u\} = \{p_{ij}\colon\ i \leqslant w,\ j \leqslant j_i\}$. For each $i \leqslant w$ and $p \in P$ let Y_{ii_p} be the p-group if such an $i_p \leqslant j_i$ exists, otherwise we put $Y_{ii_p} = 0$. Then for each $p \in P$ we use our lemma for the product $\prod_{i \leqslant w} Y_{ii_p}$ and we get that the lemma is true without any extra assumptions. More precisely, denote by π_p the projection from X onto $\prod_{i \leqslant w} Y_{ii_p}$. Of course f induces an isomorphism $f_p : \pi_p[A] \to \pi_p[B]$ and we easily see that $\pi_p[X] = \prod_{i \leqslant w} Y_{ii_p}$, $\pi_p[A]$, $\pi_p[B]$, f_p, $\pi_p(\alpha)$ and the projection of β_n onto $\prod_{i \leqslant n-1} Y_{ii_p}$ satisfy the assumptions of our lemma. So we can find a $\beta_{n+1,p} \in \prod_{i \leqslant n} Y_{ii_p}$ satisfying the conclusion of the lemma. Now $\beta_{n+1} := \sum_{p \in P} \beta_{n+1,p}$ does the job.

Assume that X is a p-group for a prime number p. Then $X_n = \langle \xi_1 \rangle \oplus \cdots \oplus \langle \xi_k \rangle$, where $\langle \xi_i \rangle \cong \mathbb{Z}_{p^{l_i}}$. Let $e = p^g$ be the exponent of X. Without loss of generality we can assume that $n = w$. Then $X \restriction n+1 = X$. We have to show that there is a $\beta_{n+1} \supseteq \beta_n$ in X such that

$$(\forall a \in A)(\forall s, r \leqslant g) \quad \bigl(p^s \bigm| \bigl(p^r\alpha - a\bigr) \Leftrightarrow p^s \bigm| \bigl(p^r\beta_{n+1} - f(a)\bigr)\bigr).$$

So it is enough to show that

$$(\forall a \in A)(\forall s, r \leqslant g) \quad \bigl(p^s \bigm| \bigl(p^r\alpha - a\bigr) \Rightarrow p^s \bigm| \bigl(p^r\beta_{n+1} - f(a)\bigr)\bigr) \tag{A}$$

and

$$(\forall a \in A)(\forall s, r \leqslant g) \quad \bigl(p^s \parallel \bigl(p^r\alpha - a\bigr) \Rightarrow p^s \parallel \bigl(p^r\beta_{n+1} - f(a)\bigr)\bigr), \tag{B}$$

where $p^s \parallel x$ means that p^s is the highest power of p dividing x.

First we will find a β_{n+1} satisfying (A), and then we will modify it to satisfy (B). For $\eta \in X \restriction m$ and $m' < m$ by $\eta_{(m')}$ we denote the m'th coordinate of η. Let

$$m_i = \max\bigl\{m\colon\ m = s - r,\ \text{where } s, r \leqslant l_i \text{ and } p^s \bigm| \bigl(p^r\alpha - a\bigr) \text{ for some } a \in A\bigr\}.$$

We choose $r_i, s_i \leqslant l_i$ such that $m_i = s_i - r_i$ and $p^{s_i} \mid (p^{r_i}\alpha - a_i)$ for some $a_i \in A$. Of course we have that $m_i \geqslant 0$, so $r_i \leqslant s_i$.

Claim 1. There exists a $\beta_{n+1} \supseteq \beta_n$ such that $p^{r_i}(\beta_{n+1})_{(n)} - f(a_i)_{(n)} \in \bigoplus_{j \neq i}\langle \xi_j \rangle$ for all $i = 1, \dots, k$.

Proof. $p^{s_i} \mid (p^{r_i}\alpha - a_i)$ and $r_i \leqslant s_i$, so $p^{r_i} \mid a_i$. From the assumptions of the lemma we get $p^{r_i} \mid f(a_i)$, so the existence of appropriate β_{n+1} is obvious. □

Let β_{n+1} be as in the above claim. Let $1 \leqslant i \leqslant k$.

Claim 2. If $p^s \mid (p^r\alpha - a)$ for some $a \in A$, then $p^r(\beta_{n+1})_{(n)} - f(a)_{(n)} \in \langle p^s\xi_i \rangle + \bigoplus_{j \neq i}\langle \xi_j \rangle$.

Proof. Without loss of generality we can assume that $s \leqslant l_i$. There are three cases.

Case 1. $r \geqslant s$. Then we have successively

$$p^s \mid a \quad \Longrightarrow \quad p^s \mid f(a) \quad \Longrightarrow \quad p^s \bigm| \bigl(p^r(\beta_{n+1})_{(n)} - f(a)_{(n)}\bigr)$$

and finally $p^r(\beta_{n+1})_{(n)} - f(a)_{(n)} \in \langle p^s\xi_i \rangle + \bigoplus_{j \neq i}\langle \xi_j \rangle$.

Case 2. $r \geqslant r_i$. We have that $p^{s_i} \mid (p^{r_i}\alpha - a_i)$ and $p^s \mid (p^r\alpha - a)$, so $p^{\min(s, r - r_i + s_i)} \mid (p^{r - r_i}a_i - a)$ and finally

$$p^{\min(s, r - r_i + s_i)} \bigm| \bigl(p^{r - r_i}f(a_i) - f(a)\bigr).$$

On the other hand, using the assumption about β_{n+1} (Claim 1), we obtain

$$p^r(\beta_{n+1})_{(n)} - p^{r - r_i}f(a_i)_{(n)} \in \bigoplus_{j \neq i}\langle \xi_j \rangle.$$

From these two statements we get that

$$p^r(\beta_{n+1})_{(n)} - f(a)_{(n)} \in \bigl\langle p^{\min(s, r - r_i + s_i)}\xi_i \bigr\rangle + \bigoplus_{j \neq i}\langle \xi_j \rangle.$$

Since $r + s_i - r_i = r + m_i \geqslant s$ we get what we need.

Case 3. $r < s$ and $r_i > r$. Similarly as in the previous case we get $p^{\min(s_i, r_i - r + s)} \mid (p^{r_i - r} a - a_i)$. We have also that $s_i - r_i = m_i \geqslant s - r$. As a consequence we get $p^{s + r_i - r} \mid (p^{r_i - r} a - a_i)$, so

$$p^{s+r_i-r} \mid \left(p^{r_i-r} f(a) - f(a_i)\right).$$

We know that $p^s \mid (p^r \alpha - a)$ and $r < s$, so $p^r \mid a$, which implies that $p^r \mid f(a)$. Hence there is a $\beta'_{n+1} \supseteq \beta_n$ such that

$$p^r \left(\beta'_{n+1}\right)_{(n)} = f(a)_{(n)}.$$

Combining the last two statements we get that $p^{s+r_i-r} \mid (p^{r_i}(\beta'_{n+1})_{(n)} - f(a_i)_{(n)})$.

Using this we can find a $(\beta''_{n+1})_{(n)}$ satisfying $p^{s-r} \mid ((\beta''_{n+1})_{(n)} - (\beta'_{n+1})_{(n)})$ and

$$p^{r_i} \left(\beta''_{n+1}\right)_{(n)} = f(a_i)_{(n)}. \tag{†}$$

Then we have also that

$$p^s \mid \left(p^r \left(\beta''_{n+1}\right)_{(n)} - f(a)_{(n)}\right). \tag{††}$$

From (†) and the assumption that $p^{r_i}(\beta_{n+1})_{(n)} - f(a_i)_{(n)} \in \bigoplus_{j \neq i} \langle \xi_j \rangle$ we obtain that $p^{r_i}((\beta_{n+1})_{(n)} - (\beta''_{n+1})_{(n)}) \in \bigoplus_{j \neq i} \langle \xi_j \rangle$. We know that $r_i \leqslant l_i$, hence

$$(\beta_{n+1})_{(n)} - \left(\beta''_{n+1}\right)_{(n)} \in \left\langle p^{l_i - r_i} \xi_i \right\rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle.$$

However, $l_i - r_i \geqslant s_i - r_i = m_i \geqslant s - r$, so

$$(\beta_{n+1})_{(n)} - \left(\beta''_{n+1}\right)_{(n)} \in \left\langle p^{s-r} \xi_i \right\rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle.$$

Using this together with (††) we end the proof of the claim. □

From Claim 2 we conclude that our β_{n+1} satisfies condition (A).

Now we take a β_{n+1} satisfying condition (A) and we are going to modify it to satisfy (B). When $p^s \parallel (p^r \alpha - a)$ and $p^{s+1} \nmid (p^r \alpha \upharpoonright n - a \upharpoonright n)$, then we have that $p^s \parallel (p^r \beta_{n+1} - f(a))$ without any modifications of β_{n+1}. If $p^s \parallel (p^r \alpha - a)$ and $p^{s+1} \mid (p^r \alpha \upharpoonright n - a \upharpoonright n)$, then there is an $i \leqslant k$ such that $p^r \alpha_{(n)} - a_{(n)} \notin \langle p^{s+1} \xi_i \rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle$. Then, of course, $s < l_i$.

Claim 3. If we additionally assume that $s - r < m_i$, then $p^s \parallel (p^r \beta_{n+1} - f(a))$.

Proof. There are two cases.

Case 1. $r > r_i$. We have $p^s \parallel (p^r\alpha - a)$, $p^{r-r_i+s_i} \mid (p^r\alpha - p^{r-r_i}a_i)$ and $r + s_i - r_i = r + m_i > s$, so $p^s \parallel (p^{r-r_i}a_i - a)$. Then $p^s \parallel (p^{r-r_i}f(a_i) - f(a))$. On the other hand, $p^{r-r_i+s_i} \mid (p^r\beta_{n+1} - p^{r-r_i}f(a_i))$. Finally we get $p^s \parallel (p^r\beta_{n+1} - f(a))$.

Case 2. $r \leqslant r_i$. From the assumptions we have $p^r\alpha_{(n)} - a_{(n)} \notin \langle p^{s+1}\xi_i\rangle + \bigoplus_{j\neq i}\langle\xi_j\rangle$ and $r_i - r + s < r_i + m_i = s_i \leqslant l_i$, so we get that

$$p^{r_i}\alpha_{(n)} - p^{r_i-r}a_{(n)} \notin \left\langle p^{r_i-r+s+1}\xi_i\right\rangle + \bigoplus_{j\neq i}\langle\xi_j\rangle.$$

On the other hand, from $p^s \mid (p^r\alpha - a)$ we get $p^{r_i-r+s} \mid (p^{r_i}\alpha - p^{r_i-r}a)$. Hence

$$p^{r_i-r+s} \parallel \left(p^{r_i}\alpha - p^{r_i-r}a\right).$$

Using this and the assumptions $p^{s_i} \mid (p^{r_i}\alpha - a_i)$ and $s_i > r_i - r + s$ we get $p^{r_i+s-r} \parallel (p^{r_i-r}a - a_i)$, so

$$p^{r_i+s-r} \parallel \left(p^{r_i-r}f(a) - f(a_i)\right).$$

We know also that $p^{s_i} \mid (p^{r_i}\beta_{n+1} - f(a_i))$, therefore

$$p^{r_i-r+s} \parallel \left(p^{r_i-r}\left(p^r\beta_{n+1} - f(a)\right)\right).$$

However, $p^s \mid (p^r\beta_{n+1} - f(a))$, hence $p^s \parallel (p^r\beta_{n+1} - f(a))$. $\square$

Let B consist of these $(r,s,a) \in \{0,1,\ldots,g\} \times \{0,1,\ldots,g\} \times A$ for which

1. $p^s \parallel (p^r\alpha - a)$.
2. $p^{s+1} \mid (p^r\alpha \restriction n - a \restriction n)$.
3. For every $i \leqslant k$, if $p^r\alpha_{(n)} - a_{(n)} \notin \langle p^{s+1}\xi_i\rangle + \bigoplus_{j\neq i}\langle\xi_j\rangle$, then $s - r = m_i$.

From Claim 3 and the observations just before it, it is enough to modify β_{n+1} to satisfy condition (B) only for triples $(r,s,a) \in B$ and preserve condition (A) in its full generality.

We define an equivalence relation $\sim$ on B as the transitive closure of the relation $\sim_0$ defined as follows:

$$(r,s,a) \sim_0 (r',s',a') \iff$$
$$\exists i \leqslant k\colon\quad p^r\alpha_{(n)} - a_{(n)} \notin \left\langle p^{s+1}\xi_i\right\rangle + \bigoplus_{j\neq i}\langle\xi_j\rangle \text{ and } p^{r'}\alpha_{(n)} - a'_{(n)} \notin \left\langle p^{s'+1}\xi_i\right\rangle + \bigoplus_{j\neq i}\langle\xi_j\rangle.$$

For i in the above definition we have $s, s' < l_i$. We present B as the disjoint union $B_1 \cup \cdots \cup B_S$ of $\sim$-equivalence classes. Let

$$C_h = \left\{\xi_i\colon \left(\exists(r,s,a) \in B_h\right)\left(p^r\alpha_{(n)} - a_{(n)} \notin \left\langle p^{s+1}\xi_i\right\rangle + \bigoplus_{j\neq i}\langle\xi_j\rangle\right)\right\}$$

for $h \leqslant S$. Then the sets $C_h, h \leqslant S$, are pairwise disjoint. From the definition of B and relation $\sim$ we have that $(\forall \xi_{i_1}, \xi_{i_2} \in C_h)(m_{i_1} = m_{i_2})$. So let us denote by M_h this common value of m_i for all $\xi_i \in C_h$. Of course, $M_h \geqslant 0$.

For $(r,s,a) \in B_h$ let

$$R_{r,s,a} = \max\bigl\{R\colon\ p^{R-r+s} \parallel p^R \alpha_{(n)} - p^{R-r} a_{(n)}\bigr\}.$$

For $h \leqslant S$ let

$$R_h = \min\bigl\{R_{r,s,a}\colon\ (r,s,a) \in B_h\bigr\}.$$

Let us notice that for $(r,s,a) \in B_h$ and $r \leqslant R \leqslant R_{r,s,a}$ we have that $(R, R-r+s, p^{R-r}a) \in B_h$.

We define $T_h = \{a \in A\colon\ (R_h, s, a) \in B_h\}$, here s uniquely determined by $s = R_h + M_h$. For $(R_h, s, a) \in B_h$ let

$$C_{R_h,s,a} = \Biggl\{\xi_i\colon\ p^{R_h}\alpha_{(n)} - a_{(n)} \notin \langle p^{s+1}\xi_i\rangle + \bigoplus_{j\neq i}\langle \xi_j\rangle\Biggr\}.$$

Then, of course,

$$\bigcup_{(R_h,s,a)\in B_h} C_{R_h,s,a} \subseteq C_h \tag{$\flat$}$$

and for each $(R_h, s, a) \in B_h$ we have that

$$C_{R_h,s,a} \neq \emptyset. \tag{$\flat\flat$}$$

On the other hand, by the definition of B we have that all elements in the set $\{a_{(n)}\colon\ a \in T_h\}$ are equal modulo $p^s X_n$. This together with ($\flat$) and ($\flat\flat$) implies that in $\{a_{(n)}\colon\ a \in T_h\}$ there are less than $p^{|C_h|}$ elements modulo $p^{s+1}X_n$. From the definition of B we also know that in the set $\{a \upharpoonright n\colon\ a \in T_h\}$ all elements are equal modulo $p^{s+1}X \upharpoonright n$. So in T_h there are less than $p^{|C_h|}$ elements modulo $p^{s+1}X$. Hence the same is true for the set $f[T_h]$. This gives us that

$$\text{in the set } \bigl\{f(a)_{(n)}\colon\ a \in T_h\bigr\} \text{ there are less than } p^{|C_h|} \text{ elements modulo } p^{s+1}X_n. \tag{$*$}$$

Now we can already modify β_{n+1}. Fix an $h \leqslant S$.

If for every $(\beta'_{n+1})_{(n)} \in X_n$ satisfying $(\beta'_{n+1})_{(n)} - (\beta_{n+1})_{(n)} \in \bigoplus_{i\in C_h}\langle p^{M_h}\xi_i\rangle$ we could find $(R_h, s, a) \in B_h$ such that

$$p^{R_h}\bigl(\beta'_{n+1}\bigr)_{(n)} - f(a)_{(n)} \in \bigoplus_{i\in C_h}\bigl\langle p^{s+1}\xi_i\bigr\rangle + \bigoplus_{j\notin C_h}\langle \xi_j\rangle,$$

then we would get a contradiction with $(*)$. Hence we conclude that there exists $\beta_{n+1}^{(h)} \supseteq \beta_n$ such that

$$\big(\beta_{n+1}^{(h)}\big)_{(n)} - (\beta_{n+1})_{(n)} \in \bigoplus_{i \in C_h} \big\langle p^{M_h} \xi_i \big\rangle \tag{$\diamond_h$}$$

and

$$\big(\forall (R_h, s, a) \in B_h\big)(\exists i \in C_h) \quad \left(p^{R_h} \big(\beta_{n+1}^{(h)}\big)_{(n)} - f(a)_{(n)} \notin \big\langle p^{s+1} \xi_i \big\rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle \right). \tag{$\diamond\diamond_h$}$$

We choose such a $\beta_{n+1}^{(h)}$ for every $h \leqslant S$. Let us define a $\beta'_{n+1} \supseteq \beta_n$ by

$$\big(\beta'_{n+1}\big)_{(n)} = (\beta_{n+1})_{(n)} + \sum_{h=1}^{S} \Big(\big(\beta_{n+1}^{(h)}\big)_{(n)} - (\beta_{n+1})_{(n)} \Big).$$

Using $(\diamond_h)$ for all $h \leqslant S$ we obtain

$$\big(\beta'_{n+1}\big)_{(n)} - (\beta_{n+1})_{(n)} \in \bigoplus_{h \leqslant S} \bigoplus_{i \in C_h} \big\langle p^{M_h} \xi_i \big\rangle. \tag{$\diamond$}$$

The sets $C_h, h \leqslant S$, are pairwise disjoint, so using $(\diamond_h)$ and $(\diamond\diamond_h)$ for all $h \leqslant S$ we conclude that

$$\begin{aligned} &(\forall h \leqslant S)\big(\forall (R_h, s, a) \in B_h\big)(\exists i \in C_h) \\ &\quad \left(p^{R_h} \big(\beta'_{n+1}\big)_{(n)} - f(a)_{(n)} \notin \big\langle p^{s+1} \xi_i \big\rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle \right). \end{aligned} \tag{$\diamond\diamond$}$$

For such a choice of β'_{n+1} condition (A) is still satisfied. Indeed, assume that $p^{s'} \mid (p^r \alpha - a)$. Then

$$p^{s'} \mid \big(p^r \beta_{n+1} - f(a)\big).$$

We have that $p^r \beta'_{n+1} - f(a) = p^r(\beta'_{n+1} - \beta_{n+1}) + (p^r \beta_{n+1} - f(a))$. Our aim is to show that for all $i \leqslant k$ we have $p^r(\beta'_{n+1})_{(n)} - f(a)_{(n)} \in \langle p^{s'} \xi_i \rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle$. So let $i \leqslant k$. If $\xi_i \in C_h$ for an $h \leqslant S$, then applying the inequality $r + M_h = r + m_i \geqslant \min(s', l_i)$ and $(\diamond)$ we obtain that $p^r(\beta'_{n+1})_{(n)} - f(a)_{(n)} \in \langle p^{s'} \xi_i \rangle + \bigoplus_{j \neq i} \langle \xi_j \rangle$. Otherwise the same conclusion is obvious from $(\diamond)$.

Condition (A) for β'_{n+1} together with $(\diamond\diamond)$ imply that

$$(\forall h \leqslant S)\big(\forall (R_h, s, a) \in B_h\big) \quad \Big(p^s \parallel \Big(p^{R_h} \big(\beta'_{n+1}\big)_{(n)} - f(a)_{(n)}\Big)\Big). \tag{$\sharp$}$$

We have obtained β'_{n+1} as a modification of β_{n+1}. Denote β'_{n+1} once again by β_{n+1}. We will show that this new β_{n+1} is appropriate.

We have showed that our new β_{n+1} satisfies condition (A), so it is enough to prove that it satisfies condition (B) for all triples $(r, s', a) \in B$.

Let $(r, s', a) \in B_h$ and $s = R_h + M_h$ for an $h \leqslant S$. Then $s' - r = s - R_h = M_h$. There are two cases.

Case 1. $r \leqslant R_h$.

By the definition of R_h we have $r \leqslant R_h \leqslant R_{r,s',a}$, so $(R_h, R_h - r + s', p^{R_h - r} a) \in B_h$. From ($\sharp$) we get $p^s \parallel (p^{R_h} \beta_{n+1} - p^{R_h - r} f(a))$. Using condition (A) and $s = s' + R_h - r$ we obtain $p^{s'} \parallel (p^r \beta_{n+1} - f(a))$.

Case 2. $r > R_h$.

By the definition of R_h we can find a $(r_1, s_1, a_1) \in B_h$ such that $R_h = R_{r_1,s_1,a_1}$. So $(R_h, R_h - r_1 + s_1, p^{R_h - r_1} a) \in B_h$. Then $R_h - r_1 + s_1 = s$ and let $b = p^{R_h - r_1} a$. From the definition of R_{r_1,s_1,a_1} we get $p^{r - R_h + s + 1} \mid (p^r \alpha - p^{r - R_h} b)$. But $r - R_h + s + 1 = r + M_h + 1$, so

$$p^{r + M_h + 1} \mid \bigl(p^r \alpha - p^{r - R_h} b\bigr). \tag{!}$$

We have also that $p^{s'} \parallel (p^r \alpha - a)$, so $p^{r + M_h} \parallel (a - p^{r - R_h} b)$, hence

$$p^{r + M_h} \parallel \bigl(f(a) - p^{r - R_h} f(b)\bigr). \tag{!!}$$

Suppose for a contradiction that $p^{s'+1} \mid (p^r \beta_{n+1} - f(a))$. Then using (!!) we get $p^{r + M_h} \parallel (p^r \beta_{n+1} - p^{r - R_h} f(b))$. On the other hand, from (!) we get $p^{r + M_h + 1} \mid (p^r \beta_{n+1} - p^{r - R_h} f(b))$. This is a contradiction. $\square$

3. Products as inverse limits of initial subproducts

In this section we consider products of countably many finite groups as profinite groups as it was described in Section 1, i.e. as inverse limits of the systems of all initial subproducts with the natural projections.

3.1. *Description of orbits*

Using Lemma 2.1 we will describe orbits in products of finite abelian groups under the action of the standard structural group.

Let $X = \prod_{i \in \omega} X_i$, where all X_i are finite abelian groups and let the exponent e of X be finite. So X is a module over $\mathbb{Z}_e = \mathbb{Z}/e\mathbb{Z}$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a tuple from X and A be a finite subset of X. We are going to investigate orbits over A in the profinite group X with the standard structural group. It is easy to see that $o(\alpha/A) = o(\alpha/Lin(A))$, where $Lin(A)$ denotes the submodule of X generated by A. So without loss of generality we can assume that A is a submodule of X. From now on, every time when we investigate orbits over a finite set A we can and do assume that A is a submodule of X. Recall that for $\eta \in X$, by $\eta \restriction n$ we denote the first n coordinates of η.

Lemma 3.1.1 (Description of orbits). *$o(\alpha/A) = U$, where U consists of elements $\beta \in X^m$ such that for all $a \in A$, $k \in \mathbb{Z}_e$, $l_1, \ldots, l_m \in \mathbb{Z}_e$ and $n \geqslant 1$ we have*

$$k \,\Big|\, \left(\sum_{i=1}^{m} l_i\alpha_i \restriction n - a \restriction n\right) \iff k \,\Big|\, \left(\sum_{i=1}^{m} l_i\beta_i \restriction n - a \restriction n\right).$$

The above lemma is enough to prove that products of finite abelian groups of bounded exponent, considered with the standard structural group, are small, m-normal and m-stable (see Theorem 3.2.1). However, to get similar results for products with some non-standard structural groups we need a more general description of orbits. Namely, let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a tuple from $X \restriction m_1 \times \cdots \times X \restriction m_m$ for some $m_1, \ldots, m_m \in \omega \cup \{\omega\}$ and A be a subset of $\bigcup_{n \in \omega\cup\{\omega\}} X \restriction n$. Since we are going to investigate orbits over A in the profinite group X with the standard structural group, we can assume that A is closed on restrictions (i.e. $A \cap X \restriction n = \{\eta \restriction n\colon \eta \in A\}$, for $n \in \omega$) and $A \cap X \restriction n$ is a submodule of $X \restriction n$ for every $n \in \omega \cup \{\omega\}$.

Lemma 3.1.2 (Generalized description of orbits). *$o(\alpha/A) = U$, where U consists of elements $\beta \in X \restriction m_1 \times \cdots \times X \restriction m_m$ such that for all $n \geqslant 1$, $a \in A \cap X \restriction n$, $k \in \mathbb{Z}$, $l_1, \ldots, l_m \in \mathbb{Z}$ we have*

$$k \,\Big|\, \left(\sum_{i=1}^{m} l_i\alpha_i \restriction n - a\right) \iff k \,\Big|\, \left(\sum_{i=1}^{m} l_i\beta_i \restriction n - a\right),$$

where in the case of $n > m_i$ we do not have the ith summand in both sums $\sum_{i=1}^{m} l_i\alpha_i \restriction n$ and $\sum_{i=1}^{m} l_i\beta_i \restriction n$.

Proof. Only the inclusion $o(\alpha/A) \supseteq U$ requires an explanation.

Take a $\beta \in U$. It is enough to show that for each $w \geqslant 1$ there exists an automorphism $f \in Aut^*(X \restriction w)$ such that

(1) For every $j \leqslant w$, if $j \leqslant m_i$, then $f(\alpha_i \restriction j) = \beta_i \restriction j$ for $i = 1, \ldots, m$.
(2) $(\forall i \leqslant w)(\forall \eta \in A \cap X \restriction i)\ (f(\eta) = \eta)$.

We prove this by induction on w.

Assume that for a $w \geqslant 0$ we have an $f \in Aut^*(X \restriction w)$ satisfying (1) and (2). We will show that there is an $f' \in Aut^*(X \restriction w + 1)$ satisfying (1) and (2) (for $w + 1$ instead of w), such that $f' \restriction w = f$ (if $w = 0$, then we do not have f and we just want to find an $f' \in Aut^*(X \restriction 1)$ satisfying (1) and (2)).

The assumption that $\beta \in U$ implies that there is an isomorphism f_0 between submodules C_0 and D_0 of $X \upharpoonright w+1$ generated by $(A \cap X \upharpoonright w+1) \cup \{\alpha_1 \upharpoonright w+1, \ldots, \alpha_m \upharpoonright w+1\}$ and $(A \cap X \upharpoonright w+1) \cup \{\beta_1 \upharpoonright w+1, \ldots, \beta_m \upharpoonright w+1\}$, respectively, which is defined by

$$f_0\left(\sum_{i=1}^{m} l_i(\alpha_i \upharpoonright w+1) - (a \upharpoonright w+1)\right) = \sum_{i=1}^{m} l_i(\beta_i \upharpoonright w+1) - (a \upharpoonright w+1),$$

for $a \in A \cap X \upharpoonright w+1, l_1, \ldots, l_m \in \mathbb{Z}$ (if $m_i < w+1$, then we do not have the ith summand in both sums $\sum_{i=1}^{m} l_i(\alpha_i \upharpoonright w+1)$ and $\sum_{i=1}^{m} l_i(\beta_i \upharpoonright w+1)$). Then

$$(\forall a \in C_0)(\forall k \in \mathbb{Z})(\forall i \leqslant w+1) \quad \big(k \mid a \upharpoonright i \;\Leftrightarrow\; k \mid f_0(a) \upharpoonright i\big).$$

We have also that $f_0 \upharpoonright (X \upharpoonright w)$ coincides with f, wherever the former function is defined.

If f_0 is an automorphism of $X \upharpoonright w+1$, then using the definition of f_0 and properties of f we are done. Otherwise there is an $\alpha' \in X \upharpoonright w+1$ outside C_0. In virtue of Lemma 2.1 we have that there exists $\beta' \in X \upharpoonright w+1$ such that $\beta' \upharpoonright w = f(\alpha' \upharpoonright w)$ and

$$(\forall a \in C_0)(\forall k, l \in \mathbb{Z})(\forall i \leqslant w+1) \quad \big(k \mid (l\alpha' \upharpoonright i - a \upharpoonright i) \;\Leftrightarrow\; k \mid (l\beta' \upharpoonright i - f_0(a) \upharpoonright i)\big).$$

As above the isomorphism f_0 has an extension to isomorphism f_1 between submodules C_1, D_1 of $X \upharpoonright w+1$ generated by $C_0\alpha'$ and $D_0\beta'$ respectively, such that

$$(\forall a \in C_1)(\forall k \in \mathbb{Z})(\forall i \leqslant w+1) \quad \big(k \mid a \upharpoonright i \;\Leftrightarrow\; k \mid f_1(a) \upharpoonright i\big)$$

and $f_1 \upharpoonright (X \upharpoonright w)$ coincides with f, wherever the former function is defined. If f_1 is an automorphism of $X \upharpoonright w+1$, then we are done. Otherwise we continue this process until we get an automorphism f' of $X \upharpoonright w+1$ extending f_0, satisfying

$$(\forall a \in X \upharpoonright w+1)(\forall k \in \mathbb{Z})(\forall i \leqslant w+1) \quad \big(k \mid a \upharpoonright i \;\Leftrightarrow\; k \mid f'(a) \upharpoonright i\big)$$

and such that $f' \upharpoonright (X \upharpoonright w) = f$. So $f' \in Aut^*(X \upharpoonright w+1)$. By the definition of f_0 and properties of f we get that f' satisfies (1) and (2). □

Of course, Lemma 3.1.2 implies Lemma 3.1.1. Let us notice that Lemma 3.1.2 easily implies Lemma 2.1.

Proof of Lemma 2.1 from Lemma 3.1.2. By Lemma 3.1.2 we get that there is a standard structural automorphism h of X extending f and mapping $\alpha \upharpoonright n$ on β_n. So $\beta_{n+1} := h(\alpha \upharpoonright n+1)$ is appropriate. □

At the end we would like to say that descriptions of orbits in products of finite abelian groups can also be obtained by means of model theory of suitable abelian structures. We will use this approach in another paper, in which we are going to deal with arbitrary abelian profinite groups.

3.2. Consequences of the description of orbits

First we work with products with the standard structural group. Theorem 3.2.1 or rather Corollary 3.2.2 is the main result of Section 3.

Let X be a product of countably many finite abelian groups and assume that X has finite exponent e. Let A be a finite $\mathbb{Z}_e$-submodule of X and $\alpha = (\alpha_1, \ldots, \alpha_m) \in X^m$. For $k \in \mathbb{Z}_e$, $l = (l_1, \ldots, l_m) \in \mathbb{Z}_e^m$ and $a \in A$ we define

$$n_{k,l,a} = \begin{cases} \max\left\{n \in \omega\colon\ k \,\middle|\, \left(\sum_{i=1}^{m} l_i\alpha_i \restriction n - a \restriction n\right)\right\}, & \text{when such a maximal } n \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Finally we define the natural number

$$N_{\alpha,A} = \max\bigl\{n_{k,l,a}\colon\ (k,l,a) \in \mathbb{Z}_e \times \mathbb{Z}_e^m \times A\bigr\} + 1,$$

which will be useful in several proofs below.

Theorem 3.2.1. *Let X be a product of countably many finite abelian groups of bounded exponent with the standard structural group. Then X is small, m-normal and m-stable.*

Proof. (1) Smallness. Let A be a finite subset of X. We have to show that $O_1(A)$ is countable. Without loss of generality A is a submodule of X. For $\alpha \in X$ we define

$$\begin{aligned} A_\alpha = \Bigl\{&(k,l,a,n) \in \mathbb{Z}_e \times \mathbb{Z}_e \times A \times \bigl(\omega \cup \{\infty\}\bigr)\colon \\ &n = \max\bigl\{m \in \omega \cup \{\infty\}\colon\ m = 0 \text{ or } k \mid (l\alpha \restriction m - a \restriction m)\bigr\}\Bigr\}. \end{aligned}$$

By Lemma 3.1.1 we have that $o(\alpha/A) = o(\beta/A) \iff A_\alpha = A_\beta$. Since there are only countably many possibilities for the A_α, we get smallness.

(2) m-normality. Let A be as above and α be a finite tuple of elements of X. For simplicity we assume that α is a single element.

Let $n = N_{\alpha,A}$ and $U = \{\eta \in X\colon\ \eta \restriction n = \alpha \restriction n\}$. Then $U \cap o(\alpha/A) = o(\alpha/A\alpha \restriction n)$. By Lemma 3.1.1 and by the choice of n we have that

$$\begin{aligned} o(\alpha/A\alpha \restriction n) = \bigl\{&\beta \in X\colon\ \alpha \restriction n \subseteq \beta \text{ and} \\ &(\forall a \in A)(\forall k,l \in \mathbb{Z}_e)\bigl(k \mid (l\alpha - a) \Rightarrow k \mid (l\beta - a)\bigr)\bigr\}. \end{aligned} \qquad (*)$$

To complete our proof it is enough to show the following claim.

Claim. The set $U \cap o(\alpha/A)$ is fixed setwise by any $f \in Aut^*(X/\alpha)$.

Proof. Let $f \in Aut^*(X/\alpha)$. We need to show that for each $\beta \in o(\alpha/A\alpha \restriction n)$ we have that $f(\beta) \in o(\alpha/A\alpha \restriction n)$. Let $k \mid (l\alpha - a)$. Then $k \mid (l\alpha - f(a))$, so $k \mid (a - f(a))$. Now $\beta \in o(\alpha/A\alpha \restriction n)$, so $k \mid (l\beta - a)$, hence $k \mid (lf(\beta) - f(a))$ and finally $k \mid (lf(\beta) - a)$. Using $(*)$ we end our proof. $\square$

(3) m-stability. We have to show that there is no sequence $A_0 \subseteq A_1 \subseteq \cdots$ of finite subsets of X and no element $\alpha \in X$ such that $o(\alpha/A_{i+1})$ is nowhere dense in $o(\alpha/A_i)$ for every i. Suppose that such a sequence exists. Without loss of generality we can assume that each A_i is submodule of X. Let $C_i = \{(k,l) \in \mathbb{Z}_e^2\colon (\exists a \in A_i)(k \mid (l\alpha - a))\}$ and $n_i = N_{\alpha,A_i}$. We have that $n_{i+1} \geqslant n_i$ for $i \in \omega$. Then for every $n_i' \geqslant n_i$ we have

$$o(\alpha/A_i\alpha \restriction n_i') = \{\beta \in X\colon \alpha \restriction n_i' \subseteq \beta \text{ and } (\forall a \in A_i)(\forall k,l \in \mathbb{Z}_e)(k \mid (l\alpha - a) \Rightarrow k \mid (l\beta - a))\}.$$

From the assumption that $o(\alpha/A_{i+1})$ is nowhere dense in $o(\alpha/A_i)$ we get that $o(\alpha/A_{i+1}\alpha \restriction n_{i+1})$ is nowhere dense in $o(\alpha/A_i\alpha \restriction n_{i+1})$.

Claim. $C_i \subsetneq C_{i+1}$ for each $i \in \omega$.

Proof. Suppose for a contradiction that $C_i = C_{i+1}$. We will show $o(\alpha/A_{i+1}\alpha \restriction n_{i+1}) = o(\alpha/A_i\alpha \restriction n_{i+1})$, which is an obvious contradiction.

Take a $\beta \in o(\alpha/A_i\alpha \restriction n_{i+1})$. Suppose that

$$k \mid l\alpha - a, \tag{†}$$

for some $k,l \in \mathbb{Z}_e$ and $a \in A_{i+1}$. We have to show that $k \mid l\beta - a$.

By the equality $C_i = C_{i+1}$ we can find an $a' \in A_i$ so that $k \mid l\alpha - a'$. Then $k \mid l\beta - a'$, hence $k \mid l\beta - l\alpha$. So by (†) we get $k \mid l\beta - a$. $\square$

So $C_1 \subsetneq C_2 \subsetneq \cdots \subseteq \mathbb{Z}_e \times \mathbb{Z}_e$. This is a contradiction. $\square$

The next corollary is the converse to Remark 1.4 for products with the standard structural group.

Corollary 3.2.2. *Let $X = \prod_{i\in\omega} X_i$ be a product of finite groups such that almost all X_i are abelian and assume that X has finite exponent. Then X, with its standard structural group, is small, m-normal and m-stable.*

Proof. We can present X as $Y_{-1} \times \prod_{i\in\omega} Y_i$, where every Y_i is abelian and finite and Y_{-1} is finite. More precisely, there is a $k \in \omega$ such that $Y_{-1} = X_0 \times \cdots \times X_{k-1}$ and $Y_i = X_{i+k}$ for $i \in \omega$. It is easy to see that X is interpretable in $Y = \prod_{i\in\omega} Y_i$, so using the previous theorem for Y we obtain the smallness of X.

The situation is a little bit more complicated for m-normality and m-stability. Let $A = \{\eta \in X\colon (\forall i \geqslant k)(\eta(i) = 0)\}$. This set is, of course, finite. It is easy to see that

$(X, Aut^*(X/A)) \cong (|A| \times Y, Aut^*(Y))$, where $|A| \times Y$ is considered as a disjoint union of $|A|$-many copies of Y and $Aut^*(Y)$ acts on each summand of this union as on Y. So once again using the previous theorem we obtain m-normality and m-stability of $(X, Aut^*(X/A))$. The last step in the proof is getting rid of the set A. For m-normality we use Remark 1.6, for m-stability Remark 1.7. □

Let X be a product of countably many finite abelian groups of bounded exponent. By Theorem 3.2.1 we know that X is m-stable and, moreover, from the proof of m-stability we see that $\mathcal{M}(X) < \omega$.

Proposition 3.2.3. *Let X be a product of countably many finite abelian groups of bounded exponent and e be the exponent of X. Then $\mathcal{M}(X) = M - 1$, where M is the maximum of lengths of all descending sequences $H_0 > H_1 > \cdots > H_n$ of subgroups of X, which are defined by conjunctions of formulas of the form $k \mid lx$, where $k, l \in \mathbb{Z}_e$, and such that H_{i+1} is nowhere dense in H_i for every $0 \leqslant i < n$.*

Proof. We have to prove $\mathcal{M}(X) = M - 1$.

($\leqslant$) Let $\mathcal{M}(X) = m$. There is a sequence $A_0 \subseteq \cdots \subseteq A_m$ of finite submodules of X and an element $\alpha \in X$ such that $o(\alpha/A_{i+1})$ is nowhere dense in $o(\alpha/A_i)$ for every $0 \leqslant i < m$. Let

$$C_i = \bigl\{(k,l) \in \mathbb{Z}_e^2\colon\ (\exists a \in A_i)\bigl(k \mid (l\alpha - a)\bigr)\bigr\}$$

and

$$G_i = \bigl\{x \in X\colon\ \bigl(\forall (k,l) \in C_i\bigr)(k \mid lx)\bigr\} < X.$$

Now we take $n_i = N_{\alpha, A_i}$, $0 \leqslant i \leqslant m$. Let $n = \max\{n_i\colon\ 0 \leqslant i \leqslant m\}$ and let $U = \{\eta \in X\colon \eta \restriction n = \alpha \restriction n\}$. By the description of orbits we get

$$o(\alpha/A_i\alpha \restriction n) = \bigl\{\beta \in X\colon\ \alpha \restriction n \subseteq \beta \text{ and } (\forall a \in A_i)(\forall k, l \in \mathbb{Z}_e)\ \bigl(k \mid (l\alpha - a)\ \Rightarrow\ k \mid (l\beta - a)\bigr)\bigr\}.$$

As a consequence we get that

$$o(\alpha/A_i\alpha \restriction n) = U \cap (\alpha + G_i),$$

for $0 \leqslant i \leqslant m$. So $G_0 > G_1 > \cdots > G_m$ is a descending sequence of subgroups of X, each G_i is defined by a conjunction of formulas of the form $k \mid lx$, where $k, l \in \mathbb{Z}_e$, and G_{i+1} is nowhere dense in G_i for every $0 \leqslant i < m$. Hence $m \leqslant M - 1$.

($\geqslant$) It follows from the fact (see [N2, Lemma 2.6]) that if $H_1 < H_2$ are groups definable in a small m-stable profinite structure and H_1 is nowhere dense in H_2, then $\mathcal{M}(H_1) < \mathcal{M}(H_2)$. □

Corollary 3.2.4. *Let $n \in \omega$ and p be a prime number. Then $\mathcal{M}(\mathbb{Z}_{p^n}^{\omega}) = n$.*

If we want to calculate $\mathcal{M}(X)$ for $X = \prod_{i\in\omega} X_i$ being a product of countably many finite groups of bounded exponent such that almost all X_i are abelian, then arguing similarly as in the proof of Corollary 3.2.2 and using Remark 1.8 we see that it is enough to choose an $n \in \omega$ such that X_i is abelian for $i > n$ and to calculate $\mathcal{M}(\prod_{i>n} X_i)$.

Let $X = \prod_{i\leqslant\omega} X_i$ be a product of finite groups. Then any permutation or grouping of X_i, $i \in \omega$, changes the profinite structure of X, i.e. it changes the standard structural group of X. So the question arises if permutations or grouping of X_i have an effect on smallness, m-normality and m-stability of X. Using Corollary 3.2.2 and Remark 1.4 we get the following answer to this question.

Corollary 3.2.5. *Smallness of $X = \prod_{i\in\omega} X_i$ (with the standard structural group) does not depend on permutation and grouping of X_i. If such a product is small, then it is m-normal and m-stable and remains such after any permutation and grouping of X_i.*

When we have a profinite topological space (group) we can treat it as the inverse limit of different inverse systems. We give now an example showing that smallness of a profinite group depends on its presentation as an inverse limit of finite groups.

Example 1. Let $X = \mathbb{Z}_2^\omega$. We consider X as the inverse limit of all its finite subproducts with natural projections. This induces a profinite structure on X with the trivial standard structural group. Hence X is not small.

On the other hand, if we consider X as the inverse limit of the system

$$\mathbb{Z}_2 \longleftarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longleftarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longleftarrow \cdots$$

of initial finite subproducts of X with natural projections, then by Theorem 3.2.1 we get that X is small.

We investigate the above phenomenon in Section 4. Now we are going to consider products of countably many finite groups with some non-standard structural groups. This is the place where we need the generalized description of orbits given in Lemma 3.1.2. The results of this part of Section 3 yield new examples of small profinite groups, also interpretable in fields [K].

Proposition 3.2.6. *Let p be a prime number and $X = \prod_{i\in\omega} X_i$, where each X_i is a finite abelian p-group. Let X have finite exponent $e = p^g$ and S be a Sylow p-subgroup of the standard structural group $Aut^*(X)$. Then (X, S) is still small, m-normal and m-stable.*

Proof. We define $G = Aut^*(X/A_0)$, where

$$A_0 = \big\{\eta \in X \restriction n\colon\ n \geqslant 1,\ \eta \restriction n-1 = (0, \ldots, 0)\big\}.$$

Then G is a closed subgroup of $Aut^*(X)$. Let $\alpha \in X$ and A be a finite submodule of X. Everywhere below we consider orbits under the action of $Aut^*(X)$.

By Lemma 3.1.2 we have that for every $n \in \omega$

$$o(\alpha/A_0A\alpha\restriction n)$$
$$=\bigl\{\beta\in X\colon\ \alpha\restriction n\subseteq\beta \text{ and } (\forall i\in\omega)(\forall a\in A)(\forall\eta\in A_0\cap X\restriction i)(\forall k,l\in\mathbb{Z}_e)$$
$$\bigl(k\mid(l\alpha\restriction i-a\restriction i-\eta)\ \Leftrightarrow\ k\mid(l\beta\restriction i-a\restriction i-\eta)\bigr)\bigr\}.\tag{$\dagger$}$$

Let $n=N_{\alpha,A}$.

Claim. $o(\alpha/A\alpha\restriction n)=o(\alpha/A_0A\alpha\restriction n)$.

Proof. ($\supseteq$) is obvious.

($\subseteq$) Let us take a $\beta\in o(\alpha/A\alpha\restriction n)$. Suppose for a contradiction that $\beta\notin o(\alpha/A_0A\alpha\restriction n)$. From ($\dagger$) we get that there is an $a_0\in A_0$ (more precisely $a_0\in X\restriction m$ for some $m>n$), $a\in A$ and $k,l\in\mathbb{Z}_e$ such that one of the two following cases holds.

Case 1. $k\mid(l\alpha\restriction m-a\restriction m-a_0)$ and $k\nmid(l\beta\restriction m-a\restriction m-a_0)$.

Case 2. $k\nmid(l\alpha\restriction m-a\restriction m-a_0)$ and $k\mid(l\beta\restriction m-a\restriction m-a_0)$.

We will show how to get a contradiction in the first case. The second case can be checked similarly. Since $a_0\restriction n=(0,\dots,0)$, the assumption of the first case gives that $k\mid(l\alpha\restriction n-a\restriction n)$. This together with the definition of n gives that $k\mid(l\alpha-a)$. As a consequence $k\mid(l\alpha\restriction m-a\restriction m)$, so $k\mid a_0$ and $k\mid(l\beta\restriction m-a\restriction m)$. Finally $k\mid(l\beta\restriction m-a\restriction m-a_0)$. □

By the claim and Theorem 3.2.1 one can conclude that for any $H\geqslant G$, the structure (X,H) is small, m-normal and m-stable.

If we show that G is a pro-p-group, then we will get that there is a Sylow p-subgroup G_p of $Aut^*(X)$ containing G. So we will obtain that (X,G_p) is small, m-normal and m-stable. Using Remark 1.5 and the fact that all Sylow p-subgroups of $Aut^*(X)$ are conjugate our proof will be done.

So let us show that G is a pro-p-group. Let $G_i=G\restriction(X\restriction i)$ for $i\geqslant 1$. Then

$$G=\varprojlim G_i,$$

so we have to show that every G_i is a p-group. We do this by induction on i.

For $i=1$ there is nothing to do, because $|G_1|=1$.

Assume that G_i is a p-group. Let us consider $H_{i+1}=\{g\in G_{i+1}\colon\ g\restriction(X\restriction i)=id_{X\restriction i}\}$. Then $G_{i+1}/H_{i+1}\cong G_i$, so it is enough to show that H_{i+1} is a p-group. For every $k\leqslant i$ we can write $X_k=\langle\xi_{k1}\rangle\oplus\cdots\oplus\langle\xi_{kl_k}\rangle$, where each $\langle\xi_{kj}\rangle$ is a cyclic p-group. Let E consist of these elements $\varepsilon\in X\restriction i+1$ which have only one non-zero coordinate and this coordinate has a form $\varepsilon(k)=\xi_{kj}$ for some $j\leqslant l_k$. For $\varepsilon\in E$ let E_ε be the set of elements of X_i with the exponent less or equal to the exponent of ε. We will show that

$$f\in H_{i+1}\quad\Longrightarrow\quad(\forall\varepsilon\in E)\quad f(\varepsilon)=\begin{cases}\varepsilon, & \text{when } \varepsilon(i)\neq 0,\\ (\varepsilon\restriction i)^\frown\theta & \text{for some } \theta\in E_\varepsilon,\ \text{otherwise},\end{cases}\tag{$*$}$$

and when we choose $\theta_\varepsilon \in E_\varepsilon$ for each $\varepsilon \in E$, then

$$(\exists! f \in H_{i+1})(\forall \varepsilon \in E) \quad f(\varepsilon) = \begin{cases} \varepsilon, & \text{when } \varepsilon(i) \neq 0, \\ (\varepsilon \restriction i)^\frown \theta_\varepsilon, & \text{otherwise.} \end{cases} \tag{$**$}$$

Condition $(*)$ is obvious.

To show $(**)$ let us point that E generates $X \restriction i+1$, hence the uniqueness is clear. Now we have to find an appropriate f. We define it by

$$f\left(\sum_{\varepsilon \in E} l_\varepsilon \varepsilon\right) = \sum_{\varepsilon \in E} l_\varepsilon (\varepsilon \restriction i)^\frown \theta_\varepsilon$$

(for ε such that $\varepsilon(i) \neq 0$ we assume that $\theta_\varepsilon = \varepsilon(i)$). To complete the proof of $(**)$ it is enough to check that this definition does not depend on the presentation of an element of $X \restriction i+1$ as a combination of elements $\varepsilon \in E$ and that f is $1-1$. Both things are easy, so we check only the first one. Let $\sum_{\varepsilon \in E} l_\varepsilon \varepsilon = \sum_{\varepsilon \in E} l'_\varepsilon \varepsilon$. For $\eta \in X \restriction i+1$ by $e(\eta)$ we denote the exponent of η. Then for $\varepsilon \in E$ we have $e(\varepsilon) \mid l_\varepsilon - l'_\varepsilon$, so $\sum_{\varepsilon \in E} l_\varepsilon (\varepsilon \restriction i)^\frown \theta_\varepsilon = \sum_{\varepsilon \in E} l'_\varepsilon (\varepsilon \restriction i)^\frown \theta_\varepsilon$.

Finally, each $|E_\varepsilon|$ is some power of p, so using $(*)$ and $(**)$ we get that H_{i+1} is a p-group. □

Corollary 3.2.7. *Let p be a prime number. Let $X = \prod_{i \in \omega} X_i$ be a product of finite p-groups such that almost all X_i are abelian and assume that X has finite exponent. Let S be a Sylow p-subgroup of the standard structural group $Aut^*(X)$. Then (X, S) is small, m-normal and m-stable.*

Proof. Similarly as in the proof of Corollary 3.2.2 we can find a $k \in \omega$ and present X as $Y_{-1} \times \prod_{i \in \omega} Y_i$, where $Y_{-1} := X_0 \times \cdots \times X_{k-1}$ and for every $i \in \omega$, $Y_i := X_{i+k}$ is abelian.

Let $A = \{\eta \in X\colon (\forall i \geqslant k)(\eta(i) = 0)\}$. In the proof of Corollary 3.2.2 we noticed that $(X, Aut^*(X/A)) \cong (|A| \times Y, Aut^*(Y))$, where $|A| \times Y$ is considered as a disjoint union of $|A|$-many copies of Y and $Aut^*(Y)$ acts on each summand of this union as on Y. We can identify $Aut^*(X/A)$ with $Aut^*(Y)$. Let S_Y be a Sylow p-subgroup of $Aut^*(Y)$. It can be enlarged to a Sylow p-subgroup S_X of $Aut^*(X)$. Then $S_X \cap Aut^*(X/A) = S_Y$.

From Proposition 3.2.6 we obtain that (X, S_Y) is small, m-normal and m-stable. Hence (X, S_X) is small. To show m-normality and m-stability of (X, S_X) we use Remarks 1.6 and 1.7, respectively.

By Remark 1.5 and since every Sylow p-subgroup of $Aut^*(X)$ is conjugate with S_X, the proof is completed. □

It is worth noticing that if X is not necessarily a pro-p-group and if we define G in the same way as it was defined in the proof of Proposition 3.2.6, then the claim appearing in this proof is still true and as a consequence (X, G) is small, m-normal and m-stable. Even more generally, we can define G as a closed subgroup of $Aut^*(X)$ fixing pointwise only a subset of A_0 (A_0 is defined in the same way as in the proof of Proposition 3.2.6) to get the same result. Considering only m-normality and m-stability we can show even more.

Proposition 3.2.8. *Let $X = \prod_{i\in\omega} X_i$ be a product of finite abelian groups and let X have finite exponent e. $Aut^*(X)$ denotes the standard structural group of X.*

(i) *If A_0 is any family of canonical open sets in X, then $(X, Aut^*(X/A_0))$ is m-normal and m-stable.*
(ii) *If A_0 is any subset of $X \cup \bigcup_{i\geqslant 1} X \restriction i$, then $(X, Aut^*(X/A_0))$ is m-normal and m-stable.*
(iii) *Under the weaker assumption that almost all X_i are abelian we have that if A_0 is the same as in* (i) *or* (ii) *and if $(X, Aut^*(X/A_0))$ is small, then it is m-normal and m-stable.*

Recall that in the case of product $X = \prod_{i\in\omega} X_i$, a canonical open set consists of elements with fixed first i coordinates ($i \geqslant 1$).

Proof. (i) We identify canonical open sets in X with elements from $X \restriction i$ for $i \geqslant 1$. Without loss of generality we can assume that for every $i \geqslant 1$ we have that $\{\eta \restriction i\colon \eta \in A_0\}$ is a submodule of $X \restriction i$ and $A_0 \cap X \restriction i = \{\eta \restriction i\colon \eta \in A_0\}$.

The proof is an elaboration of the proofs of m-normality and m-stability in Theorem 3.2.1.

Let A be a finite submodule of X and $\alpha \in X$. We consider orbits under the action of $Aut^*(X)$. By Lemma 3.1.2 we obtain that for every $n \in \omega$

$$\begin{aligned} &o(\alpha/A_0A\alpha \restriction n) \\ &\quad = \big\{\beta \in X\colon \alpha \restriction n \subseteq \beta \text{ and } (\forall i \in \omega)(\forall a \in A)(\forall \eta \in A_0 \cap X \restriction i)(\forall k, l \in \mathbb{Z}_e) \\ &\qquad \big(k \mid (l\alpha \restriction i - a \restriction i - \eta) \Leftrightarrow k \mid (l\beta \restriction i - a \restriction i - \eta)\big)\big\}. \end{aligned} \tag{$*$}$$

(1) *m-normality*. Let A be a finite submodule of X and α be a finite tuple of elements of X. We assume for simplicity that α is a single element. For $k, l \in \mathbb{Z}_e$ and $a \in A$ we define

$$n_{k,l,a} = \max\big\{n \in \omega\colon (\exists \eta \in A_0 \cap X \restriction n)\big(k \mid (l\alpha \restriction n - a \restriction n - \eta)\big)\big\},$$

when such a maximal n exists, or $n_{k,l,a} = 0$, otherwise. Finally we define

$$N_{\alpha,A,A_0} = \max\big\{n_{k,l,a}\colon (k, l, a) \in \mathbb{Z}_e \times \mathbb{Z}_e \times A\big\} + 1.$$

Let $n = N_{\alpha,A,A_0}$ and $U = \{\eta \in X\colon \eta \restriction n = \alpha \restriction n\}$. So $U \cap o(\alpha/A_0A) = o(\alpha/A_0A\alpha \restriction n)$. From $(*)$ and the choice of n we get

$$\begin{aligned} &o(\alpha/A_0A\alpha \restriction n) \\ &\quad = \big\{\beta \in X\colon \alpha \restriction n \subseteq \beta \text{ and } (\forall i \in \omega)(\forall a \in A)(\forall \eta \in A_0 \cap X \restriction i)(\forall k, l \in \mathbb{Z}_e) \\ &\qquad \big(k \mid (l\alpha \restriction i - a \restriction i - \eta) \Rightarrow k \mid (l\beta \restriction i - a \restriction i - \eta)\big)\big\}. \end{aligned} \tag{$**$}$$

To show $(**)$ choose a β in the right-hand side of $(**)$. Assume that $k \mid (l\beta \restriction i - a \restriction i - \eta)$. In virtue of $(*)$ we have to show that $k \mid (l\alpha \restriction i - a \restriction i - \eta)$. If $i \leqslant n$, then $\alpha \restriction i = \beta \restriction i$, so $k \mid (l\alpha \restriction i - a \restriction i - \eta)$. Otherwise, we have $k \mid (l\beta \restriction n - a \restriction n - \eta \restriction n)$, so $k \mid (l\alpha \restriction n - a \restriction n - \eta \restriction n)$. From the definition of n we conclude that there is an $\eta' \in A_0 \cap X \restriction i$ such that $k \mid (l\alpha \restriction i - a \restriction i - \eta')$. Hence $k \mid (l\beta \restriction i - a \restriction i - \eta')$. By the assumption, this implies that $k \mid (\eta' - \eta)$. Finally we have $k \mid (l\alpha \restriction i - a \restriction i - \eta)$ and this shows $(**)$.

The proof of m-normality boils down to the following claim, whose proof uses $(**)$ and is an obvious generalization of the proof of the claim formulated in the proof of m-normality in Theorem 3.2.1.

Claim. The set $U \cap o(\alpha/A_0A)$ is fixed setwise by any $f \in Aut^*(X/A_0\alpha)$.

(2) *m-stability*. Suppose for a contradiction that there is a sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite submodules of X and an element $\alpha \in X$ such that $o(\alpha/A_0A_{i+1})$ is nowhere dense in $o(\alpha/A_0A_i)$ for every $i \geqslant 1$.

Let $n_i = N_{\alpha,A_i,A_0}$. Then $n_{i+1} \geqslant n_i$ for $i \geqslant 1$. Let

$$C_i = \bigl\{(k,l) \in \mathbb{Z}_e^2\colon\ (\exists a \in A_i)(\exists \eta \in A_0 \cap X \restriction n_i)\bigl(k \mid (l\alpha \restriction n_i - a \restriction n_i - \eta)\bigr)\bigr\}.$$

From $(*)$ and the choice of n_i we get that for any $n_i' \geqslant n_i$

$$\begin{aligned} &o\bigl(\alpha/A_0A_i\alpha \restriction n_i'\bigr) \\ &\quad = \bigl\{\beta \in X\colon\ \alpha \restriction n_i' \subseteq \beta \text{ and } (\forall j \in \omega)(\forall a \in A_i)(\forall \eta \in A_0 \cap X \restriction j)(\forall k,l \in \mathbb{Z}_e) \\ &\qquad\qquad \bigl(k \mid (l\alpha \restriction j - a \restriction j - \eta) \Rightarrow k \mid (l\beta \restriction j - a \restriction j - \eta)\bigr)\bigr\}. \end{aligned}$$

By the assumption that $o(\alpha/A_0A_{i+1})$ is nowhere dense in $o(\alpha/A_0A_i)$ we have that $o(\alpha/A_0A_{i+1}\alpha \restriction n_{i+1})$ is nowhere dense in $o(\alpha/A_0A_i\alpha \restriction n_{i+1})$. One can check that then we have $C_i \subsetneq C_{i+1}$. So $C_1 \subsetneq C_2 \subsetneq \cdots \subseteq \mathbb{Z}_e \times \mathbb{Z}_e$. This is a contradiction.

(ii) This follows from (i).

(iii) The proof is analogous to the proof of Corollary 3.2.2. It uses (i) and Remarks 1.6 and 1.7. $\square$

Let X be a product of countably many finite abelian groups of bounded exponent and $\mathcal{G}$ be the family of all groups of the form $Aut^*(X/A_0)$, where A_0 is an arbitrary family of canonical open sets in X. Then (X,G) is m-normal for any $G \in \mathcal{G}$. So the question arises, whether each such product with an arbitrary structural group is m-normal (and small or not). The example below yields the negative answer to this question.

Example 2. We treat $Y = \mathbb{Z}_3 \times \mathbb{Z}_3$ as the inverse limit of the system $\mathbb{Z}_3 \leftarrow \mathbb{Z}_3 \times \mathbb{Z}_3$ with the natural projection on the first coordinate. Let $Aut^*(Y)$ be the standard structural group of Y. We consider orbits on Y under the action of $Aut^*(Y)$. For $\alpha_0 = \langle 1,0\rangle$, $\alpha_1 = \langle 1,1\rangle$, $\alpha_2 = \langle 1,2\rangle$ and $\beta = \langle 2,0\rangle$ we have $o(\alpha_1/\beta) = \{\alpha_1,\alpha_2\}$ and $o(\alpha_2/\alpha_1) = \{\alpha_2,\alpha_0\}$.

Let $X = Y^\omega$. We take $Aut^*(X) = Aut^*(Y)^\omega$ as a structural group of X, where for $f = (f_0, f_1, \ldots) \in Aut^*(Y)^\omega$ and $\eta = (\eta_0, \eta_1, \ldots) \in X$ we define $f(\eta) = (f_0(\eta_0), f_1(\eta_1), \ldots)$. Let $a_1 = (\alpha_1, \alpha_1, \ldots)$ and $b = (\beta, \beta, \ldots)$. Then

$$o(a_1/b) = \big\{\eta \in X\colon (\forall i \in \omega)\big(\eta(i) = \alpha_1 \text{ or } \eta(i) = \alpha_2\big)\big\}.$$

For an $n \in \omega$ let $U = \{\eta \in X\colon a_1 \restriction n \subseteq \eta\}$ be a canonical open neighbourhood of a_1. For every $A \subseteq \omega \setminus \{0, \ldots, n-1\}$ we can find an automorphism $f_A \in Aut^*(X/a_1)$ such that for each $\eta \in o(a_1/b)$ and $i \in \omega$ we have

$$f_A(\eta)(i) = \begin{cases} \alpha_1, & \text{when } \eta(i) = \alpha_1, \\ \eta(i), & \text{when } i \notin A, \\ \alpha_0, & \text{when } \eta(i) = \alpha_2 \text{ and } i \in A. \end{cases}$$

Then for all $A \neq A'$ we have $f_A[U \cap o(a_1/b)] \neq f_{A'}[U \cap o(a_1/b)]$ and we get that $(X, Aut^*(X))$ is not m-normal. One can check that $(X, Aut^*(X))$ is not small and not m-stable.

4. Changing the inverse system

Now we are going to consider a product $X = \prod_{i\in\omega} X_i$ of countably many finite groups with structural groups arising in some another special way. Namely, for a directed set $\mathcal{S} \subseteq [\omega]^{<\omega}$ such that $\bigcup \mathcal{S} = \omega$ we consider the profinite group $(X_{\mathcal{S}}, Aut^*_{\mathcal{S}}(X))$, which is just the group X regarded as the inverse limit of the system $\{X_S\colon S \in \mathcal{S}\}$, where $X_S = \prod_{i\in S} X_i$, $S \in \mathcal{S}$, with the standard structural group (denoted by $Aut^*_{\mathcal{S}}(X)$). So the universe $X_{\mathcal{S}}$ of our profinite group can be identified with X. For simplicity, from now on we will write $X_{\mathcal{S}}$ instead of $(X_{\mathcal{S}}, Aut^*_{\mathcal{S}}(X))$.

In this section we will characterize these directed sets $\mathcal{S} \subseteq [\omega]^{<\omega}$ for which $X_{\mathcal{S}}$ is small, and we will show that if $X_{\mathcal{S}}$ is small, then it is also m-normal.

First of all without loss of generality we can assume that $\mathcal{S}$ satisfies

(i) $S_1 \in \mathcal{S}$ and $S_2 \in \mathcal{S} \Rightarrow S_1 \cup S_2 \in \mathcal{S}$ and $S_1 \cap S_2 \in \mathcal{S}$.

For each $S_0 \in \mathcal{S}$ we can consider $X_{S_0} = \prod_{i\in S_0} X_i$ as the inverse limit of the system $\{X_S\colon S \in \mathcal{S} \text{ and } S \subseteq S_0\}$. Let $\mathcal{S} \restriction S_0 = \{S \in \mathcal{S}\colon S \subseteq S_0\}$. Then $Aut^*_{\mathcal{S}\restriction S_0}(X_{S_0})$ is the standard structural group of X_{S_0} regarded as the inverse limit as above.

Now we give the description of orbits in $X_{\mathcal{S}}$ under the assumption that all X_i are abelian. So let $\alpha = (\alpha_1, \ldots, \alpha_m) \in X^m$ and A be a submodule (over $\mathbb{Z}$) of X.

Lemma 4.1 (Description of orbits). *$o(\alpha/A) = U$, where U consists of elements $\beta \in X^m$ such that for all $a \in A$, $k \in \mathbb{Z}$, $l_1, \ldots, l_m \in \mathbb{Z}$ and $S \in \mathcal{S}$ we have*

$$k \,\Big|\, \left(\sum_{i=1}^m l_i\alpha_i \restriction S - a \restriction S\right) \iff k \,\Big|\, \left(\sum_{i=1}^m l_i\beta_i \restriction S - a \restriction S\right).$$

Proof. ($\subseteq$) is obvious.

($\supseteq$) Take a $\beta \in U$. We have to show that $\beta \in o(\alpha/A)$. Let $T_0 \subseteq T_1 \subseteq \cdots \subseteq \omega$ be a sequence of sets from $\mathcal{S}$ cofinal in ω. We will construct sequences $S_0 \subseteq S_1 \subseteq \cdots \subseteq \omega$ and $f_0, f_1, \ldots$ such that for all $i \in \omega$ we have:

(1) $S_i \in \mathcal{S}$ and S_{i+1} is a minimal set in $\mathcal{S}$ containing properly S_i and contained in T_{j_i}, where j_i is the minimal natural number such that $S_i \subsetneq T_{j_i}$.
(2) $f_i \in Aut^*_{\mathcal{S}\restriction S_i}(X_{S_i}/A \restriction S_i)$.
(3) $f_{i+1} \restriction X_{S_i} = f_i$.
(4) $f_i(\alpha \restriction S_i) = \beta \restriction S_i$.

We define S_0 as a minimal non-empty set from $\mathcal{S}$. By the assumption that $\beta \in U$ and a simple application of Lemma 2.1, there is an automorphism of X_{S_0} fixing the set $A \restriction S_0$ pointwise and satisfying (4). As f_0 we choose an arbitrary such automorphism.

Now assume that we have chosen $(S_i)_{i \leqslant n}$ and $(f_i)_{i \leqslant n}$. We define S_{n+1} as a minimal set from $\mathcal{S}$ containing properly S_n and contained in T_{j_n}, where j_n is the minimal natural number such that $S_n \subsetneq T_{j_n}$. Let S' be the smallest set from $\mathcal{S}$ such that $S_{n+1} \setminus S_n \subseteq S' \subseteq S_{n+1}$. Of course, $f_n \restriction X_{S_n \cap S'} \in Aut^*_{\mathcal{S}\restriction S_n \cap S'}(X_{S_n \cap S'}/A \restriction S_n \cap S')$. So by the assumption that $\beta \in U$ and by Lemma 2.1 we get that there is an $f'_n \in Aut^*_{\mathcal{S}\restriction S'}(X_{S'}/A \restriction S')$ such that $f'_n \restriction X_{S_n \cap S'} = f_n \restriction X_{S_n \cap S'}$ and $f'_n(\alpha \restriction S') = \beta \restriction S'$. Now we can already define $f_{n+1} \in Aut^*_{\mathcal{S}\restriction S_{n+1}}(X_{S_{n+1}}/A \restriction S_{n+1})$ by $f_{n+1}((x_i)_{i \in S_{n+1}}) = (y_i)_{i \in S_{n+1}}$, where y_j is the projection of $f_n((x_i)_{i \in S_n})$ on the jth coordinate, when $j \in S_n$, y_j is the projection of $f'_n((x_i)_{i \in S'})$ on the jth coordinate, when $j \in S'$.

One can easily check that for $j \in S' \cap S_n$ both lines above agree and that f_{n+1} satisfies (2)–(4). Now item (1) implies that $\bigcup_{i \in \omega} S_i = \omega$, so automorphisms $f_0, f_1, \ldots$ yield an automorphism $f \in Aut^*_{\mathcal{S}}(X/A)$ for which $f(\alpha) = \beta$. □

We say that $\mathcal{I} \subseteq \mathcal{S}$ is an ideal in $\mathcal{S}$ if:

- $I_1 \in \mathcal{I}$ and $I_2 \in \mathcal{I} \Rightarrow I_1 \cup I_2 \in \mathcal{I}$,
- $I \in \mathcal{I}$, $J \subseteq I$ and $J \in \mathcal{S} \Rightarrow J \in \mathcal{I}$.

By Remark 1.4 we know that if $X_{\mathcal{S}}$ is small, then almost all X_i are abelian and X has finite exponent e. So assume this in the next theorem and, moreover, that each X_i is a non-trivial group.

Theorem 4.2. *$X_{\mathcal{S}}$ is small iff there are only countably many ideals in $\mathcal{S}$.*

Proof. First we show the theorem in the case when all X_i are abelian. Let A be a finite submodule of X. For $\alpha \in X$, $a \in A$, $k, l \in \mathbb{Z}_e$ we define the ideal

$$\mathcal{I}(\alpha, a, k, l) = \{S \in \mathcal{S}\colon\ k \mid (l\alpha \restriction S - a \restriction S)\}.$$

By Lemma 4.1 we see that an element $\beta \in X$ belongs to $o(\alpha/A)$ iff $\mathcal{I}(\alpha, a, k, l) = \mathcal{I}(\beta, a, k, l)$ for all $k, l \in \mathbb{Z}_e$ and $a \in A$. Hence we get ($\leftarrow$) in Theorem 4.2. On the other hand, for an arbitrary ideal $\mathcal{I}$ of $\mathcal{S}$ we can find an $\alpha \in X$ for which $\mathcal{I}(\alpha, 0, 0, 1) = \mathcal{I}$. So the number of 1-orbits over $\emptyset$ is not less than the number of ideals in $\mathcal{S}$.

To finish the proof we have to consider the case when not all X_i are abelian. Let $S_0 \in \mathcal{S}$ be such that each X_i, $i \notin S_0$, is abelian. Let $\mathcal{S}' = \{S \setminus S_0\colon\ S \in \mathcal{S}\}$. It is easy to see that $X_{\mathcal{S}}$ is interpretable in $(\prod_{i \in \omega \setminus S_0} X_i)_{\mathcal{S}'}$. On the other hand, $(\prod_{i \in \omega \setminus S_0} X_i)_{\mathcal{S}'}$ is a $\emptyset$-definable subgroup of $X_{\mathcal{S}}$. Hence

$$X_{\mathcal{S}} \text{ is small} \quad \text{iff} \quad \left(\prod_{i \in \omega \setminus S_0} X_i\right)_{\mathcal{S}'} \text{ is small.} \tag{$*$}$$

Let λ and λ' be the number of ideals in $\mathcal{S}$ and $\mathcal{S}'$, respectively. One can check that

$$\lambda \leqslant \omega \quad \text{iff} \quad \lambda' \leqslant \omega. \tag{$**$}$$

Now $(*)$, $(**)$ and the fact that the theorem is true for $(\prod_{i \in \omega \setminus S_0} X_i)_{\mathcal{S}'}$ finish the proof. $\square$

Proposition 4.3. *If $X_{\mathcal{S}}$ is small, then it is also m-normal.*

Proof. Assume that $X_{\mathcal{S}}$ is small. Then we have that almost all X_i are abelian and X has finite exponent e. Arguing similarly as in the proof of Corollary 3.2.2 without loss of generality we can assume that all X_i are non-trivial and abelian. Now suppose for a contradiction that $X_{\mathcal{S}}$ is not m-normal. Hence there is a finite tuple α from X and a finite submodule A of X such that for an arbitrary canonical open neighbourhood U of α we have that the set $\{f[o(\alpha/A) \cap U]\colon\ f \in Aut^*_{\mathcal{S}}(X/\alpha)\}$ has at least two elements. In other words for an arbitrary canonical open neighbourhood U of α

$$\big(\exists \beta \in o(\alpha/A) \cap U\big)\big(\exists f \in Aut^*_{\mathcal{S}}(X/\alpha)\big) \quad \big(f(\beta) \notin o(\alpha/A)\big). \tag{$*$}$$

For simplicity assume that α is a single element.

Claim 1. For all $\beta \in o(\alpha/A)$, $a \in A$, $k, l \in \mathbb{Z}_e$, $S \in \mathcal{S}$ and $f \in Aut^*_{\mathcal{S}}(X/\alpha)$ we have

$$k \mid (l\alpha \restriction S - a \restriction S) \quad \Longrightarrow \quad k \mid \big(lf(\beta) \restriction S - a \restriction S\big).$$

Proof. Let $k \mid (l\alpha \restriction S - a \restriction S)$. Then $k \mid (l\alpha \restriction S - f(a) \restriction S)$, so $k \mid (a \restriction S - f(a) \restriction S)$. Now $\beta \in o(\alpha/A)$, so $k \mid (l\beta \restriction S - a \restriction S)$, hence $k \mid (lf(\beta) \restriction S - f(a) \restriction S)$ and, finally, $k \mid (lf(\beta) \restriction S - a \restriction S)$. $\square$

Claim 2. There are sets $S_1, S_2, \ldots \in \mathcal{S}$ such that $S_i \setminus \bigcup_{j \neq i} S_j \neq \emptyset$ for every $i \in \omega$.

Proof. In the proof of Theorem 4.2 we defined ideals $\mathcal{I}(\eta, a, k, l)$ for $\eta \in X$, $a \in A$ and $k, l \in \mathbb{Z}_e$. Let $\mathcal{A} = A \times \mathbb{Z}_e \times \mathbb{Z}_e$.

We construct $r_i \in \mathcal{A}$, $\beta_i \in X$, $S_{ir_i} \in \mathcal{S}$ and $f_i \in Aut^*_{\mathcal{S}}(X/\alpha)$, $i \in \omega$, such that for all i we have:

(1) $\beta_i \in o(\alpha/A) \cap U_i$, where $U_i = \{\eta \in X\colon \eta \upharpoonright \bigcup_{j<i} S_{jr_j} = \alpha \upharpoonright \bigcup_{j<i} S_{jr_j}\}$.
(2) $S_{ir_i} \in \mathcal{I}(f_i(\beta_i), r_i)$ but $S_{ir_i} \notin \mathcal{I}(\alpha, r_i)$.

The fact that orbits are determined by ideals (see the proof of Theorem 4.2) together with Claim 1 and $(*)$ show that this construction is possible.

There is an $r \in \mathcal{A}$ such that there are infinitely many indices $i \in \omega$ for which $r_i = r$. Choose such an r and let $(S_i)_{i\in\omega}$ be the subsequence of the sequence $(S_{ir_i})_{i\in\omega}$ consisting of elements S_{ir_i} for which $r_i = r$.

We will show that the sequence $(S_i)_{i\in\omega}$ satisfies our demands. Of course, $r = (a, k, l)$ for some $a \in A$ and $k, l \in \mathbb{Z}_e$. Let

$$T = \big\{i \in \omega\colon k \mid \big(l\alpha(i) - a(i)\big)\big\}.$$

To finish the proof it is enough to show the following statement.

The family of sets $\{S_i \setminus T\colon i \in \omega\}$ consists of nonempty pairwise disjoint sets. $(**)$

By induction on n we will show that the sets $S_0 \setminus T, \ldots, S_n \setminus T$ are nonempty and pairwise disjoint. For $n = 0$, if we had $S_0 \subseteq T$, then we would get that $S_0 \in \mathcal{I}(\alpha, r)$, a contradiction with (2). Suppose now that $S_0 \setminus T, \ldots, S_n \setminus T$ satisfy our demands. The fact that $S_{n+1} \setminus T$ is nonempty follows as above. So suppose for a contradiction that $(S_{n+1} \setminus T) \cap (S_k \setminus T)$ is nonempty for some $0 \leqslant k \leqslant n$. Let $i \in (S_{n+1} \setminus T) \cap (S_k \setminus T)$. We have $S_k = S_{jr_j}$ and $S_{n+1} = S_{j'r_{j'}}$ for some $j < j'$. By the construction we have that $\beta_{j'} \upharpoonright S_k = \alpha \upharpoonright S_k$, hence $f_{j'}(\beta_{j'}) \upharpoonright S_k = \alpha \upharpoonright S_k$. But $i \in (S_k \setminus T) \cap S_{n+1}$, so $k \nmid (lf(\beta_{j'}) \upharpoonright S_{n+1} - a \upharpoonright S_{n+1})$ and finally $S_{n+1} \notin \mathcal{I}(f(\beta_{j'}), r_{j'})$, a contradiction. □

From Claim 2 we obtain uncountably many ideals in $\mathcal{S}$, a contradiction with Theorem 4.2. □

At the beginning of [N1] there is an example of a small but not m-stable first order theory. Here we recall it in the context of profinite groups. The example is of the form $X_{\mathcal{S}}$ for some $\mathcal{S} \subseteq \omega$ and $X = \mathbb{Z}_2^{\omega}$. This example shows that the counterpart of Proposition 4.3 with m-normality replaced by m-stability does not hold.

Example 3. Let $X = \mathbb{Z}_2^{\omega\times\omega}$ (before we were considering countable products indexed by ω, here we index it by $\omega \times \omega$ for convenience). We define

$$\mathcal{S}' = \big\{\big\{(i, j) \in \omega \times \omega\colon i \leqslant n, j \leqslant m\big\}\colon n \leqslant m < \omega\big\}.$$

To satisfy condition (i) from the beginning of Section 4 we define $\mathcal{S}$ as the closure of $\mathcal{S}'$ on finite unions.

It is easy to see that there are countably many ideals in $\mathcal{S}$, hence by Theorem 4.2 and Proposition 4.3 we get that $X_{\mathcal{S}}$ is small and m-normal.

Let $H_n = \{\eta \in X\colon \eta \restriction n \times \omega = 0\}$ and $H_{n,m} = \{\eta \in X\colon \eta \restriction n \times m = 0\}$ for $n \leqslant m \in \omega$. Since $H_n = \bigcap_{m \geqslant n} H_{n,m}$, we get that H_n is a definable subgroup of $X_{\mathcal{S}}$. Moreover, H_{n+1} is nowhere dense in H_n for every $n \in \omega$. So by the fact (see [N2, Lemma 2.6]) that in a small m-stable profinite group there is no descending sequence of definable subgroups $(G_n)_{n \in \omega}$ such that G_{n+1} is nowhere dense in G_n, $n \in \omega$, we get that $X_{\mathcal{S}}$ is not m-stable.

We see that the profinite group from Example 2 is also of the form $X_{\mathcal{S}}$ for $X = \mathbb{Z}_3^{\omega}$ and some $\mathcal{S} \subseteq \omega$. So we see that abelian profinite groups of finite exponent and of the form $X_{\mathcal{S}}$ can have very different model theoretic properties. However, there is no group of the form $X_{\mathcal{S}}$ which is small but not m-normal.

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