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Supporting degrees of multi-graded local cohomology modules

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A B S T R A C T

For a finitely generated graded module M over a positively-graded commutative Noetherian ring R, the second author established in 1999 some restrictions, which can be formulated in terms of the Castelnuovo regularity of M or the so-called a^*-invariant of M, on the supporting degrees of a graded-indecomposable graded-injective direct summand, with associated prime ideal containing the irrelevant ideal of R, of any term in the minimal graded-injective resolution of M. Earlier, in 1995, T. Marley had established connections between finitely graded local cohomology modules of M and local behaviour of M across $\operatorname{Proj}(R)$.

The purpose of this paper is to present some multi-graded analogues of the above-mentioned work.

0. Introduction

Very briefly, the purpose of this paper is to explore multi-graded analogues of some results in the algebra of modules, and particularly local cohomology modules, over a commutative Noetherian ring that is graded by the additive semigroup $\mathbb{N}_0$ of non-negative integers.

To describe the results that we plan to generalize, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be such a 'positively-graded' commutative Noetherian ring. Any unexplained notation in this Introduction will be as in Chapters 12

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and 13 of our book [4]. In particular, the *injective envelope of a graded R-module M will be denoted by ${}^*E(M)$ (see [4, §13.2]), and, for $t \in \mathbb{Z}$, the tth shift functor (on the category ${}^*\mathcal{C}(R)$ of all graded R-modules and homogeneous R-homomorphisms) will be denoted by $(\bullet)(t)$ (see [4, §12.1]).

Let $\mathbb{N}$ denote the set of positive integers; set $R_+ := \bigoplus_{n\in\mathbb{N}} R_n$, the irrelevant ideal of R. For a graded R-module M and $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ (the set of homogeneous prime ideals of R), we use $M_{(\mathfrak{p})}$ to denote the homogeneous localization of M at $\mathfrak{p}$. For $i \in \mathbb{N}_0$, the ordinary Bass number $\mu^i(\mathfrak{p}, M)$ is equal to the rank of the homogeneous localization $({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M))_{(\mathfrak{p})}$ as a (free) module over $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ (see R. Fossum and H.-B. Foxby [6, Corollary 4.9]).

Let $i \in \mathbb{N}_0$, and consider a direct decomposition given by a homogeneous isomorphism

$$^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha\in\Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-n_\alpha),$$

for an appropriate family $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda_i}$ of graded prime ideals of R and an appropriate family $(n_\alpha)_{\alpha\in\Lambda_i}$ of integers. (See [4, §13.2].)

Suppose that the graded prime ideal $\mathfrak{p}$ contains the irrelevant ideal R_+. In this case, the graded ring $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is concentrated in degree 0, and its 0th component is a field isomorphic to $k_{R_0}(\mathfrak{p}_0)$, the residue field of the local ring $(R_0)_{\mathfrak{p}_0}$. Thus,

$$\mu^i(\mathfrak{p}, M) = \dim_{k_{R_0}(\mathfrak{p}_0)}\big({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\big)_{(\mathfrak{p})} = \sum_{t\in\mathbb{Z}} \dim_{k_{R_0}(\mathfrak{p}_0)}\big(\big({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\big)_{(\mathfrak{p})}\big)_t.$$

In [15], it was shown that the graded $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$-module $({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M))_{(\mathfrak{p})}$ carries information about the shifts '$-n_\alpha$' for those $\alpha \in \Lambda_i$ for which $\mathfrak{p}_\alpha = \mathfrak{p}$. One has

$$^*E(R/\mathfrak{p})(n) \ncong {}^*E(R/\mathfrak{p})(m) \quad \text{in } {}^*\mathcal{C}(R) \text{ for } m, n \in \mathbb{Z} \text{ with } m \neq n,$$

and, for a given $t \in \mathbb{Z}$, the cardinality of the set $\{\alpha \in \Lambda_i\colon\ \mathfrak{p}_\alpha = \mathfrak{p} \text{ and } n_\alpha = t\}$ is equal to

$$\dim_{k_{R_0}(\mathfrak{p}_0)}\big(\big({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\big)_{(\mathfrak{p})}\big)_t,$$

the dimension of the tth component of $({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M))_{(\mathfrak{p})}$.

Let ${}^*\operatorname{Var}(R_+) := \{\mathfrak{q} \in {}^*\operatorname{Spec}(R)\colon\ \mathfrak{q} \supseteq R_+\}$. Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_+)$, let $i \in \mathbb{N}_0$ and let $t \in \mathbb{Z}$. We say that t is an *ith level anchor point of $\mathfrak{p}$ for M* if

$$\big(\big({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)\big)_{(\mathfrak{p})}\big)_t \neq 0;$$

the set of all ith level anchor points of $\mathfrak{p}$ for M is denoted by $\operatorname{anch}^i(\mathfrak{p}, M)$; also, we write

$$\operatorname{anch}(\mathfrak{p}, M) = \bigcup_{j\in\mathbb{N}_0} \operatorname{anch}^j(\mathfrak{p}, M),$$

and refer to this as the set of *anchor points of $\mathfrak{p}$ for M*. Thus $\operatorname{anch}^i(\mathfrak{p}, M)$ is the set of integers h for which, when we decompose

$$^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha\in\Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-n_\alpha)$$

by means of a homogeneous isomorphism, there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $n_\alpha = h$. Note that $\operatorname{anch}^i(\mathfrak{p}, M) = \emptyset$ if $\mu^i(\mathfrak{p}, M) = 0$, and that $\operatorname{anch}^i(\mathfrak{p}, M)$ is a finite set when M is finitely generated.

It was also shown in [15] that, when the graded R-module M is non-zero and finitely generated, the Castelnuovo regularity $\operatorname{reg}(M)$ of M is an upper bound for the set

$$\bigcup_{\mathfrak{p}\in^* \operatorname{Var}(R_+)} \operatorname{anch}(\mathfrak{p}, M)$$

of all anchor points of M. Consequently, for each $i \geqslant 0$, every *indecomposable *injective direct summand F of ${}^*E^i(M)$ with associated prime containing R_+ must have $F_j = 0$ for all $j > \operatorname{reg}(M)$.

In Sections 2, 3 we shall present an analogue of this theory for a standard multi-graded commutative Noetherian ring $S = \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} S_{\mathbf{n}}$ (where $r \in \mathbb{N}$ with $r \geqslant 2$). There is a satisfactory generalization of anchor point theory to the multi-graded case, but we must stress now that we have not uncovered any links between our multi-graded anchor point theory and the fast-developing theory of multi-graded Castelnuovo regularity (see, for example, Huy Tài Hà [9] and D. Maclagan and G.G. Smith [12]). This may be because our multi-graded anchor point theory only yields information about multi-graded local cohomology modules with respect to $\mathbb{N}_0^r$-graded ideals of S that contain one of the components $S_{(0,\dots,0,1,0,\dots,0)}$, whereas the ideal $S_+ := \bigoplus_{\mathbf{n}\in\mathbb{N}^r} S_{\mathbf{n}}$, which is relevant to multi-graded Castelnuovo regularity, normally does not have that property.

The short Section 4 provides some motivation for our work in Section 5, where we provide multi-graded analogues of work of T. Marley [14] about finitely graded local cohomology modules. We say that a graded R-module $L = \bigoplus_{n\in\mathbb{Z}} L_n$ is *finitely graded* precisely when $L_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. In [14], Marley defined, for a finitely generated graded R-module M,

$$g_{\mathfrak{a}}(M) := \sup\bigl\{k \in \mathbb{N}_0\colon\ H^i_{\mathfrak{a}}(M) \text{ is finitely graded for all } i < k\bigr\},$$

and he modified ideas of N.V. Trung and S. Ikeda in [16, Lemma 2.2] to prove that

$$g_{\mathfrak{a}}(M) := \sup\Bigl\{k \in \mathbb{N}_0\colon\ R_+ \subseteq \sqrt{\bigl(0 :_R H^i_{\mathfrak{a}}(M)\bigr)} \text{ for all } i < k\Bigr\};$$

he then used Faltings' Annihilator Theorem for local cohomology (see [5] and [4, Theorem 9.5.1]). In Section 5 below, we shall obtain some multi-graded analogues of some of Marley's results in this area.

1. Background results in multi-graded commutative algebra

Let $R = \bigoplus_{g\in G} R_g$ be a commutative Noetherian ring graded by a finitely generated, additively-written, torsion-free Abelian group G. Some aspects of the G-graded analogue of the theory of Bass numbers have been developed by S. Goto and K.-i. Watanabe [8, §§1.2, 1.3], and it is appropriate for us to review some of those here.

We shall denote by ${}^*\mathcal{C}^G(R)$ (or sometimes by ${}^*\mathcal{C}(R)$ when the grading group G is clear) the category of all G-graded R-modules and G-homogeneous R-homomorphisms of degree 0_G between them. Projective (respectively injective) objects in the category ${}^*\mathcal{C}^G(R)$ will be referred to as *projective (respectively *injective) G-graded R-modules. Similarly, the attachment of '*' to other concepts indicates that they refer to the obvious interpretations of those concepts in the category ${}^*\mathcal{C}^G(R)$, although we shall sometimes use 'G' instead of '*' in order to emphasize the grading group. However, the following comments about ${}^*\operatorname{Hom}_R$ and the ${}^*\operatorname{Ext}^i_R$ $(i \geqslant 0)$ may be helpful.

1.1. Reminders. Let $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ be G-graded R-modules.

(i) Let $a \in G$. We say that an R-homomorphism $f : M \to N$ is *G-homogeneous of degree a* precisely when $f(M_g) \subseteq N_{g+a}$ for all $g \in G$. Such a G-homogeneous homomorphism of degree 0_G is simply called *G-homogeneous*. We denote by ${}^*\operatorname{Hom}_R(M,N)_a$ the R_{0_G}-submodule of $\operatorname{Hom}_R(M,N)$

consisting of all G-homogeneous R-homomorphisms from M to N of degree a. Then the sum $\sum_{a \in G} {}^*\operatorname{Hom}_R(M,N)_a$ is direct, and we set

$$ {}^*\operatorname{Hom}_R(M,N) := \sum_{a \in G} {}^*\operatorname{Hom}_R(M,N)_a = \bigoplus_{a \in G} {}^*\operatorname{Hom}_R(M,N)_a. $$

This is an R-submodule of $\operatorname{Hom}_R(M,N)$, and the above direct decomposition provides it with a structure as G-graded R-module. It is straightforward to check that

$$ {}^*\operatorname{Hom}_R(\bullet,\bullet) : {}^*\mathcal{C}^G(R) \times {}^*\mathcal{C}^G(R) \longrightarrow {}^*\mathcal{C}^G(R) $$

is a left exact, additive functor.

(ii) If M is finitely generated, then $\operatorname{Hom}_R(M,N)$ is actually equal to ${}^*\operatorname{Hom}_R(M,N)$ with its G-grading forgotten.

(iii) For $i \in \mathbb{N}_0$, the functor ${}^*\operatorname{Ext}^i_R$ is the ith right derived functor in ${}^*\mathcal{C}^G(R)$ of ${}^*\operatorname{Hom}_R$. We make two comments here about the case where M is finitely generated. In that case $\operatorname{Ext}^i_R(M,N)$ is actually equal to ${}^*\operatorname{Ext}^i_R(M,N)$ with its G-grading forgotten, and, second, one can calculate the ${}^*\operatorname{Ext}^i_R(M,N)$ by applying the functor ${}^*\operatorname{Hom}_R(M,\bullet)$ to a (deleted) *injective resolution of N in the category ${}^*\mathcal{C}^G(R)$ and then taking cohomology of the resulting complex.

For $a \in G$, we shall denote the *ath shift functor* by $(\bullet)(a)$: ${}^*\mathcal{C}^G(R) \to {}^*\mathcal{C}^G(R)$: thus, for a G-graded R-module $M = \bigoplus_{g \in G} M_g$, we have $(M(a))_g = M_{g+a}$ for all $g \in G$; also, $f(a)\lceil_{(M(a))_g} = f\lceil_{M_{g+a}}$ for each morphism f in ${}^*\mathcal{C}^G(R)$ and all $g \in G$.

1.2. Theorem. *(See S. Goto and K.-i. Watanabe [8, §1.3].) Let M be a G-graded R-module, and denote by ${}^*\operatorname{Spec}(R)$ the set of G-graded prime ideals of R. We denote by ${}^*E(M)$ or ${}^*E_R(M)$ 'the' *injective envelope of M, and by ${}^*E^i(M)$ or ${}^*E^i_R(M)$ 'the' ith term in 'the' minimal *injective resolution of M (for each $i \geqslant 0$).*

(i) $\operatorname{Ass}_R {}^*E_R(M) = \operatorname{Ass}_R M$.

(ii) *We have that M is a *indecomposable *injective G-graded R-module if and only if M is isomorphic (in the category ${}^*\mathcal{C}^G(R)$) to ${}^*E(R/\mathfrak{q})(a)$ for some $\mathfrak{q} \in {}^*\operatorname{Spec}(R)$ and $a \in G$. In this case, $\operatorname{Ass}_R M = \{\mathfrak{q}\}$ and $\mathfrak{q}$ is uniquely determined by M.*

(iii) *Let $(M_\lambda)_{\lambda \in \Lambda}$ be a non-empty family of G-graded R-modules. Then $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is *injective if and only if M_λ is *injective for all $\lambda \in \Lambda$.*

(iv) *Each *injective G-graded R-module M is a direct sum of *indecomposable *injective G-graded submodules, and this decomposition is uniquely determined by M up to isomorphisms.*

(v) *Let i be a non-negative integer. In view of part* (iv) *above, there is a family $(\mathfrak{p}_\alpha)_{\alpha \in \Lambda_i}$ of G-graded prime ideals of R and a family $(g_\alpha)_{\alpha \in \Lambda_i}$ of elements of G for which there is a G-homogeneous isomorphism*

$$ {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-g_\alpha). $$

Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$. Then the cardinality of the set $\{\alpha \in \Lambda_i \colon \mathfrak{p}_\alpha = \mathfrak{p}\}$ is equal to the ordinary Bass number $\mu^i(\mathfrak{p}, M)$ (that is, to $\dim_{k(\mathfrak{p})} \operatorname{Ext}^i_R(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$, where $k(\mathfrak{p})$ denotes the residue field of the local ring $R_\mathfrak{p}$).

A significant part of Section 2 of this paper is concerned with the shifts '$-g_\alpha$' in the statement of part (v) of Theorem 1.2. (The minus signs are inserted for notational convenience.) In [15], the second author obtained some results about such shifts in the special case in which R is graded by the semigroup $\mathbb{N}_0$ of non-negative integers, and in Section 2 below, we shall establish some multi-graded analogues.

We shall employ the following device used by Huy Tài Hà [9, §2].

1.3. Definition. Let $\phi : G \to H$ be a homomorphism of finitely generated torsion-free Abelian groups, and let $R = \bigoplus_{g\in G} R_g$ be a G-graded commutative Noetherian ring.

For each $h \in H$, set $R^{\phi}_h := \bigoplus_{g\in\phi^{-1}(\{h\})} R_g$; then

$$R^{\phi} := \bigoplus_{h\in H} R^{\phi}_h = \bigoplus_{h\in H}\Bigl(\bigoplus_{g\in\phi^{-1}(\{h\})} R_g\Bigr)$$

provides an H-grading on R, and we denote R by R^{ϕ} when considering it as an H-graded ring in this way.

Furthermore, for each G-graded R-module $M = \bigoplus_{g\in G} M_g$, set $M^{\phi}_h := \bigoplus_{g\in\phi^{-1}(\{h\})} M_g$ and $M^{\phi} := \bigoplus_{h\in H} M^{\phi}_h$; then M^{ϕ} is an H-graded R^{ϕ}-module. Also, if $f : M \to N$ is a G-homogeneous homomorphism of G-graded R-modules, then the same map f becomes an H-homogeneous homomorphism of H-graded R^{ϕ}-modules $f^{\phi} : M^{\phi} \to N^{\phi}$.

In this way, $(\bullet)^{\phi}$ becomes an exact additive covariant functor from ${}^*\mathcal{C}^G(R)$ to ${}^*\mathcal{C}^H(R)$.

1.4. Notation. We shall use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive and non-negative integers, respectively, and r will denote a fixed positive integer. Throughout the remainder of the paper, $R := \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} R_{\mathbf{n}}$ will denote a commutative Noetherian ring, graded by the additively-written finitely generated free Abelian group $\mathbb{Z}^r$ (with its usual addition). For $\mathbf{n} = (n_1,\dots,n_r)$, $\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{Z}^r$, we shall write

$$\mathbf{n} \leqslant \mathbf{m} \quad \text{if and only if} \quad n_i \leqslant m_i \quad \text{for all } i = 1,\dots,r;$$

furthermore, $\mathbf{n} < \mathbf{m}$ will mean that $\mathbf{n} \leqslant \mathbf{m}$ and $\mathbf{n} \neq \mathbf{m}$. The zero element of $\mathbb{Z}^r$ will be denoted by $\mathbf{0}$, and, for each $i = 1,\dots,r$, we shall use $\mathbf{e}_i$ to denote the element of $\mathbb{Z}^r$ which has 1 in the ith spot and all other components zero. Also, $\mathbf{1}$ will denote $(1,\dots,1) \in \mathbb{Z}^r$. Thus $\mathbf{1} = \sum_{i=1}^{r} \mathbf{e}_i$, and $R_{\mathbf{e}_1} R_{\mathbf{e}_2} \dots R_{\mathbf{e}_r} \subseteq R_{\mathbf{1}}$.

We shall sometimes denote the ith component of a general member $\mathbf{w}$ of $\mathbb{Z}^r$ by w_i without additional explanation.

Comments made above that apply to the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ will be used without further comment. For example, we shall say that a graded ideal of R is **maximal* if it is maximal among the set of proper $\mathbb{Z}^r$-graded ideals of R, and that R is **local* if it has a unique *maximal ideal. We shall use ${}^*\operatorname{Max}(R)$ to denote the set of *maximal ideals of R.

We shall use ${}^*\operatorname{Spec}(R)$ to denote the set of $\mathbb{Z}^r$-graded prime ideals of R; for a $\mathbb{Z}^r$-graded ideal $\mathfrak{a}$ of R, we shall set ${}^*\operatorname{Var}(\mathfrak{a}) := \{\mathfrak{p} \in {}^*\operatorname{Spec}(R) \colon\ \mathfrak{p} \supseteq \mathfrak{a}\}$.

The next three lemmas are multi-graded analogues of preparatory results in [15, §1].

1.5. Lemma. *Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ and let a be a $\mathbb{Z}^r$-homogeneous element of degree $\mathbf{n}$ in $R \setminus \mathfrak{p}$. Then multiplication by a provides a $\mathbb{Z}^r$-homogeneous automorphism of degree $\mathbf{n}$ of ${}^*E(R/\mathfrak{p})$. Also, each element of ${}^*E(R/\mathfrak{p})$ is annihilated by some power of $\mathfrak{p}$.*

Consequently, if S is a multiplicatively closed subset of $\mathbb{N}_0^r$-homogeneous elements of R such that $S \cap \mathfrak{p} \neq \emptyset$, then $S^{-1}({}^*E(R/\mathfrak{p})) = 0$.

Proof. Multiplication by a provides a $\mathbb{Z}^r$-homogeneous R-homomorphism

$$\mu_a : {}^*E(R/\mathfrak{p}) \longrightarrow {}^*E(R/\mathfrak{p})(\mathbf{n}).$$

Since $\operatorname{Ker}\mu_a$ has zero intersection with $R/\mathfrak{p}$, it follows that μ_a is injective. In view of Theorem 1.2(ii), $\operatorname{Im}\mu_a$ is a non-zero *injective $\mathbb{Z}^r$-graded submodule of the *indecomposable *injective $\mathbb{Z}^r$-graded R-module ${}^*E(R/\mathfrak{p})(\mathbf{n})$. Hence μ_a is surjective.

The fact that each element of ${}^*E(R/\mathfrak{p})$ is annihilated by some power of $\mathfrak{p}$ follows from Theorem 1.2(i), which shows that $\mathfrak{p}$ is the only associated prime ideal of each non-zero cyclic submodule of ${}^*E(R/\mathfrak{p})$. The final claim is then immediate. $\square$

The next two lemmas below can be proved by making obvious modifications to the proofs of the (well-known) 'ungraded' analogues.

1.6. Lemma. *Let $f : L \to M$ be a $\mathbb{Z}^r$-homogeneous homomorphism of $\mathbb{Z}^r$-graded R-modules such that M is a *essential extension of $\operatorname{Im} f$. Let S be a multiplicatively closed subset of $\mathbb{Z}^r$-homogeneous elements of R. Then $S^{-1}M$ is a *essential extension of its $\mathbb{Z}^r$-graded submodule $\operatorname{Im}(S^{-1}f)$.*

Proof. Modify the proof of [4, 11.1.5] in the obvious way. $\square$

1.7. Lemma. *Let S be a multiplicatively closed subset of $\mathbb{Z}^r$-homogeneous elements of R, and let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ be such that $\mathfrak{p} \cap S = \emptyset$. Then*

(i) *the natural map ${}^*E_R(R/\mathfrak{p}) \to S^{-1}({}^*E_R(R/\mathfrak{p}))$ is a $\mathbb{Z}^r$-homogeneous R-isomorphism, so that ${}^*E_R(R/\mathfrak{p})$ has a natural structure as a $\mathbb{Z}^r$-graded $S^{-1}R$-module;*
(ii) *there is a $\mathbb{Z}^r$-homogeneous isomorphism (in ${}^*\mathcal{C}(S^{-1}R)$)*

$$ {}^*E_R(R/\mathfrak{p}) \cong {}^*E_{S^{-1}R}\big(S^{-1}R/S^{-1}\mathfrak{p}\big); $$

(iii) ${}^*E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p})$, *when considered as a $\mathbb{Z}^r$-graded R-module by means of the natural homomorphism $R \to S^{-1}R$, is $\mathbb{Z}^r$-homogeneously isomorphic to ${}^*E_R(R/\mathfrak{p})$;*
(iv) *for each $\mathbf{n} \in \mathbb{Z}^r$, there is a $\mathbb{Z}^r$-homogeneous isomorphism (in ${}^*\mathcal{C}(S^{-1}R)$)*

$$ S^{-1}\big({}^*E_R(R/\mathfrak{p})(\mathbf{n})\big) \cong {}^*E_{S^{-1}R}\big(S^{-1}R/S^{-1}\mathfrak{p}\big)(\mathbf{n}); $$

(v) *if I is a *injective $\mathbb{Z}^r$-graded R-module, then the $\mathbb{Z}^r$-graded $S^{-1}R$-module $S^{-1}I$ is *injective.*

Proof. (i) This is immediate from 1.5.

(ii) One can make the obvious modifications to the proof of [4, 10.1.11] to see that, as a $\mathbb{Z}^r$-graded $S^{-1}R$-module, ${}^*E_R(R/\mathfrak{p})$ is *injective; it is also $\mathbb{Z}^r$-homogeneously isomorphic, as a $\mathbb{Z}^r$-graded $S^{-1}R$-module, to $S^{-1}({}^*E_R(R/\mathfrak{p}))$. One can use 1.6 to see that $S^{-1}({}^*E_R(R/\mathfrak{p}))$ is a *essential extension of $S^{-1}R/S^{-1}\mathfrak{p}$. The claim follows.

(iii), (iv) These are now easy.

(v) This can now be proved by making the obvious modifications to the proof of [4, 10.1.13(ii)]. $\square$

2. A multi-graded analogue of anchor point theory

2.1. Definition. We shall say that R is *positively graded* precisely when $R_{\mathbf{n}} = 0$ for all $\mathbf{n} \not\geqslant \mathbf{0}$. When that is the case, we say that R (as in 1.4) is *standard* precisely when $R = R_{\mathbf{0}}[R_{\mathbf{e}_1}, \ldots, R_{\mathbf{e}_r}]$.

The main results of this paper will concern the case where R is positively graded and standard.

2.2. Lemma. *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. If $\mathfrak{a}$ is an $\mathbb{N}_0^r$-graded ideal of R such that $\mathfrak{a} \supseteq R_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{N}_0^r$, then $\mathfrak{a} \supseteq R_{\mathbf{n}}$ for each $\mathbf{n} \in \mathbb{N}_0^r$ with $\mathbf{n} \geqslant \mathbf{t}$.*

Proof. Since R is standard, $R_{\mathbf{n}} = R_{\mathbf{t}}R_{\mathbf{n}-\mathbf{t}}$, and so is contained in $\mathfrak{a}$. $\square$

2.3. Definition. Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$. The set $\{j \in \{1,\dots,r\}\colon R_{\mathbf{e}_j} \subseteq \mathfrak{p}\}$ will be called *the set of* $\mathfrak{p}$*-directions* and will be denoted by $\operatorname{dir}(\mathfrak{p})$.

Observe that, if $i \in \operatorname{dir}(\mathfrak{p})$, then $\mathfrak{p} \supseteq R_{\mathbf{1}}$ by 2.2. Conversely, if $\mathfrak{p} \supseteq R_{\mathbf{1}}$, then, since $R_{\mathbf{1}} = R_{\mathbf{e}_1}\dots R_{\mathbf{e}_r}$, there exists $i \in \{1,\dots,r\}$ such that $R_{\mathbf{e}_i} \subseteq \mathfrak{p}$, and $i \in \operatorname{dir}(\mathfrak{p})$. Thus $\operatorname{dir}(\mathfrak{p}) \neq \emptyset$ if and only if $\mathfrak{p} \supseteq R_{\mathbf{1}}$.

More generally, let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R. We define *the set of* $\mathfrak{b}$*-directions* to be

$$\operatorname{dir}(\mathfrak{b}) := \left\{j \in \{1,\dots,r\}\colon R_{\mathbf{e}_j} \subseteq \sqrt{\mathfrak{b}}\right\}.$$

The members of the set $\{1,\dots,r\} \setminus \operatorname{dir}(\mathfrak{b})$ are called the *non-*$\mathfrak{b}$*-directions.* It is easy to see that $\operatorname{dir}(\mathfrak{b}) = \bigcap_{\mathfrak{p}\in\operatorname{Min}(\mathfrak{b})} \operatorname{dir}(\mathfrak{p})$, where $\operatorname{Min}(\mathfrak{b})$ denotes the set of minimal prime ideals of $\mathfrak{b}$.

2.4. Remark. It follows from Lemma 2.2 that, in the situation of Definition 2.3, each $\mathbb{N}_0^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree with ith component 0 for all $i \in \operatorname{dir}(\mathfrak{p})$.

2.5. Proposition. *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$. For notational convenience, suppose that $\operatorname{dir}(\mathfrak{p}) = \{1,\dots,m\}$, where $0 < m \leqslant r$. For each $i \in \{1,\dots,r\} \setminus \operatorname{dir}(\mathfrak{p}) = \{m+1,\dots,r\}$, select $u_i \in R_{\mathbf{e}_i} \setminus \mathfrak{p}$.*

Let $\mathbf{a} = (a_1,\dots,a_m) \in \mathbb{Z}^m$. For $\mathbf{c} = (c_{m+1},\dots,c_r) \in \mathbb{Z}^{r-m}$, we shall denote by $\mathbf{a}|\mathbf{c}$ the element $(a_1,\dots,a_m,c_{m+1},\dots,c_r)$ of $\mathbb{Z}^r$ obtained by juxtaposition.

(i) *For all choices of $\mathbf{c},\mathbf{d} \in \mathbb{Z}^{r-m}$, there is an isomorphism of $R_{\mathbf{0}}$-modules*

$$\left({}^*E_R(R/\mathfrak{p})\right)_{\mathbf{a}|\mathbf{c}} \cong \left({}^*E_R(R/\mathfrak{p})\right)_{\mathbf{a}|\mathbf{d}}.$$

(*Note that this does not say anything of interest if $m = r$.*)

(ii) *If $({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{c}} \neq 0$ for any $\mathbf{c} \in \mathbb{Z}^{r-m}$, then $\mathbf{a} \leqslant \mathbf{0}$.*

(iii) *Let $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, where $R_{(\mathfrak{p})}$ is the $\mathbb{Z}^r$-homogeneous localization of R at $\mathfrak{p}$. Then*
  (a) *T is a simple $\mathbb{Z}^r$-graded ring in the sense of* [8, Definition 1.1.1];
  (b) *$T_{\mathbf{0}}$ is a field;*
  (c) *for each $\mathbf{c} = (c_{m+1},\dots,c_r) \in \mathbb{Z}^{r-m}$,*

$$T_{\mathbf{a}|\mathbf{c}} = \begin{cases} 0 & \text{if } \mathbf{a} \neq \mathbf{0}, \\ T_{\mathbf{0}}(\overline{u_{m+1}/1})^{c_{m+1}}\dots(\overline{u_r/1})^{c_r} & \text{if } \mathbf{a} = \mathbf{0} \end{cases}$$

  (*where '$\overline{}$' is used to denote natural images of elements of $R_{(\mathfrak{p})}$ in T*); *and*
  (d) *every $\mathbb{Z}^r$-graded T-module is free.*

(iv) *We have $(0 :_{{}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})} \mathfrak{p}R_{(\mathfrak{p})}) = R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$.*

(v) *If $\mathbf{a},\mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{c},\mathbf{d} \in \mathbb{Z}^{r-m}$, and there is a $\mathbb{Z}^r$-homogeneous isomorphism*

$$\left({}^*E_R(R/\mathfrak{p})\right)(\mathbf{a}|\mathbf{c}) \cong \left({}^*E_R(R/\mathfrak{p})\right)(\mathbf{b}|\mathbf{d}),$$

then $\mathbf{a} = \mathbf{b}$.

Note. The obvious interpretation of the above statement is to be made in the case where $m = r$.

Proof. It will be convenient to write $\mathbf{v}$ for a general member of $\mathbb{Z}^m$ and $\mathbf{w}$ for a general member of $\mathbb{Z}^{r-m}$, and to use $\mathbf{v}|\mathbf{w}$ to indicate the element of $\mathbb{Z}^r$ obtained by juxtaposition.

(i) By Lemma 1.5, for each $i = m+1,\dots,r$, multiplication by u_i provides a $\mathbb{Z}^r$-homogeneous automorphism of ${}^*E_R(R/\mathfrak{p})$ of degree $\mathbf{e}_i$; the claim follows from this.

(ii) Set $\Delta := \{\mathbf{v} \in \mathbb{Z}^m\colon\ v_i > 0 \text{ for some } i \in \{1,\ldots,m\}\}$. Since $R_{\mathbf{e}_i} \subseteq \mathfrak{p}$ for all $i = 1,\ldots,m$, it follows from Lemma 2.2 that the $\mathbb{Z}^r$-graded R-module $R/\mathfrak{p}$ has $(R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}} = 0$ for all choices of $\mathbf{v}|\mathbf{w} \in \mathbb{Z}^r$ with $\mathbf{v} \in \Delta$. Therefore the $\mathbb{Z}^r$-graded submodule

$$\bigoplus_{\substack{\mathbf{v}\in\Delta\\ \mathbf{w}\in\mathbb{Z}^{r-m}}} (R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}}$$

of $R/\mathfrak{p}$ is zero. Since ${}^*E_R(R/\mathfrak{p})$ is a *essential extension of $R/\mathfrak{p}$, it follows that

$$\bigoplus_{\substack{\mathbf{v}\in\Delta\\ \mathbf{w}\in\mathbb{Z}^{r-m}}} \bigl({}^*E_R(R/\mathfrak{p})\bigr)_{\mathbf{v}|\mathbf{w}} = 0.$$

(iii) By Remark 2.4, each $\mathbb{N}_0^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree $\mathbf{v}|\mathbf{w}$ with $\mathbf{v} = \mathbf{0}$. Also, $(R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}} = 0$ for all $\mathbf{v} \in \mathbb{Z}^m$ with $\mathbf{v} > \mathbf{0}$. Now every non-zero $\mathbb{Z}^r$-homogeneous element of T is a unit of T, so that T is a simple $\mathbb{Z}^r$-graded ring. Furthermore, the subgroup

$$G := \bigl\{\mathbf{n} \in \mathbb{Z}^r\colon\ T_{\mathbf{n}} \text{ contains a unit of } T\bigr\}$$

is equal to $\{(n_1,\ldots,n_m,n_{m+1},\ldots,n_r) \in \mathbb{Z}^r\colon\ n_1 = \cdots = n_m = 0\}$. The claims in parts (b), (c) and (d) now follow from [8, Lemma 1.1.2, Corollary 1.1.3 and Theorem 1.1.4].

(iv) Recall that $T = R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$. Now the $\mathbb{Z}^r$-graded T-module $(0 :_{{}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})} \mathfrak{p}R_{(\mathfrak{p})})$ contains its $\mathbb{Z}^r$-graded T-submodule $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, and cannot be strictly larger, by *essentiality and the fact (see part (iii)) that every $\mathbb{Z}^r$-graded T-module is free.

(v) By Lemma 1.7(iv), there is a $\mathbb{Z}^r$-homogeneous isomorphism of $\mathbb{Z}^r$-graded $R_{(\mathfrak{p})}$-modules

$$\bigl({}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})\bigr)(\mathbf{a}|\mathbf{c}) \cong \bigl({}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})\bigr)(\mathbf{b}|\mathbf{d}).$$

Abbreviate ${}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})$ by F. It follows from part (iv) that

$$\begin{aligned} T(\mathbf{a}|\mathbf{c}) &= (0 :_F \mathfrak{p}R_{(\mathfrak{p})})(\mathbf{a}|\mathbf{c}) = (0 :_{F(\mathbf{a}|\mathbf{c})} \mathfrak{p}R_{(\mathfrak{p})}) \\ &\cong (0 :_{F(\mathbf{b}|\mathbf{d})} \mathfrak{p}R_{(\mathfrak{p})}) = (0 :_F \mathfrak{p}R_{(\mathfrak{p})})(\mathbf{b}|\mathbf{d}) \\ &= T(\mathbf{b}|\mathbf{d}), \end{aligned}$$

where the isomorphism is $\mathbb{Z}^r$-homogeneous. But, for $\mathbf{n} = (n_1,\ldots,n_m,n_{m+1},\ldots,n_r) \in \mathbb{Z}^r$, we have

$$T(\mathbf{a}|\mathbf{c})_{\mathbf{n}} \neq 0 \quad \text{if and only if} \quad (n_1,\ldots,n_m) = -\mathbf{a}$$

(by part (iii)). Therefore $\mathbf{a} = \mathbf{b}$. $\square$

2.6. Remark. Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R for which $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$.

Write $\operatorname{dir}(\mathfrak{b}) = \{i_1,\ldots,i_m\}$, where $0 < m \leqslant r$ and $i_1 < \cdots < i_m$. Let $\phi(\mathfrak{b})\colon \mathbb{Z}^r \to \mathbb{Z}^m$ be the epimorphism of Abelian groups defined by

$$\phi(\mathfrak{b})((n_1,\ldots,n_r)) = (n_{i_1},\ldots,n_{i_m}) \quad \text{for all } (n_1,\ldots,n_r) \in \mathbb{Z}^r.$$

We can think of $\phi(\mathfrak{b})\colon \mathbb{Z}^r \to \mathbb{Z}^m$ as the homomorphism which ‘forgets the co-ordinates in the non-$\mathfrak{b}$-directions’.

Now let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$. The above defines an Abelian group homomorphism $\phi(\mathfrak{p})\colon \mathbb{Z}^r \to \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{p})}$. (For a finite set Y, the notation $\#Y$ denotes the cardinality of the set Y.) In the case where $\mathfrak{b} \subseteq \mathfrak{p}$, we have $\operatorname{dir}(\mathfrak{b}) \subseteq \operatorname{dir}(\mathfrak{p})$, and we define the Abelian group homomorphism $\phi(\mathfrak{p};\mathfrak{b}) : \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{p})} \to \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ to be the unique $\mathbb{Z}$-homomorphism such that $\phi(\mathfrak{p};\mathfrak{b}) \circ \phi(\mathfrak{p}) = \phi(\mathfrak{b})$.

Now let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ and $\#\operatorname{dir}(\mathfrak{p}) = m$; we use the notation of 1.3. Let $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, and let L be a $\mathbb{Z}^r$-graded T-module.

(i) By Proposition 2.5(iii), for each $\mathbf{a} \in \mathbb{Z}^m$ and each $\mathbf{n} \in \mathbb{Z}^r$,

$$\left(T(-\mathbf{n})^{\phi(\mathfrak{p})}\right)_{\mathbf{a}} = \begin{cases} 0 & \text{if } \phi(\mathfrak{p})(\mathbf{n}) \neq \mathbf{a}, \\ (T^{\phi(\mathfrak{p})})_{\mathbf{0}} & \text{if } \phi(\mathfrak{p})(\mathbf{n}) = \mathbf{a}. \end{cases}$$

In particular, the $\mathbb{Z}^m$-graded ring $T^{\phi(\mathfrak{p})}$ is concentrated in degree $\mathbf{0} \in \mathbb{Z}^m$.

(ii) Each component of the $\mathbb{Z}^m$-graded $T^{\phi(\mathfrak{p})}$-module $L^{\phi(\mathfrak{p})}$ is a free $(T^{\phi(\mathfrak{p})})_{\mathbf{0}}$-submodule of $L^{\phi(\mathfrak{p})}$.

(iii) If L is finitely generated, then

$$\operatorname{rank}_{T^{\phi(\mathfrak{p})}} L^{\phi(\mathfrak{p})} = \sum_{\mathbf{a} \in \mathbb{Z}^m} \operatorname{rank}_{(T^{\phi(\mathfrak{p})})_{\mathbf{0}}} \left(L^{\phi(\mathfrak{p})}\right)_{\mathbf{a}};$$

since the left-hand side of the above equation is finite, all except finitely many of the terms on the right-hand side are zero.

2.7. Theorem. *Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a $\mathbb{Z}^r$-graded R-module, and let*

$$I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots$$

*be the minimal *injective resolution of M. For each $i \in \mathbb{N}_0$, let*

$$\theta_i : {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a $\mathbb{Z}^r$-homogeneous isomorphism, where $\mathfrak{p}_\alpha \in {}^*\operatorname{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ and use the notation $\phi(\mathfrak{p}) : \mathbb{Z}^r \to \mathbb{Z}^m$ and $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ of Remark 2.6, where m is the number of $\mathfrak{p}$-directions.

Let $i \in \mathbb{N}_0$ and let $\mathbf{a} \in \mathbb{Z}^m$. Then the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p} \text{ and } \phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}\}$ is equal to

$$\operatorname{rank}_{(T^{\phi(\mathfrak{p})})_{\mathbf{0}}}\left(\left(\left({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)^{\phi(\mathfrak{p})}\right)_{\mathbf{a}}\right).$$

Proof. By Lemmas 1.5, 1.6 and 1.7, there are $\mathbb{Z}^r$-homogeneous isomorphisms of graded $R_{(\mathfrak{p})}$-modules

$${}^*E^i_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) \cong \left({}^*E^i_R(M)\right)_{(\mathfrak{p})} \cong \bigoplus_{\substack{\alpha \in \Lambda_i \\ \mathfrak{p}_\alpha \subseteq \mathfrak{p}}} {}^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})(-\mathbf{n}_\alpha).$$

One can calculate ${}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})$ (up to isomorphism in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{p})})$) by taking the ith cohomology module of the complex $(0 :_{(I^\bullet)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})})$. Note that, by Lemma 1.6, for each $j \in \mathbb{N}_0$, the inclusion $\operatorname{Ker}(d^j_{(\mathfrak{p})}) \subseteq {}^*E^j(M)_{(\mathfrak{p})}$ is *essential, so that the inclusion

$$\operatorname{Ker}\left(d^j_{(\mathfrak{p})}\right) \cap (0 :_{{}^*E^j(M)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})}) \subseteq (0 :_{{}^*E^j(M)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})})$$

is also *essential. Because, by Proposition 2.5(iii)(d), each $\mathbb{Z}^r$-graded T-module is free, it follows that all the 'differentiation' maps in the complex $(0 :_{(I^\bullet)_{(\mathfrak{p})}} \mathfrak{p}R_{(\mathfrak{p})})$ are zero. Hence

$$^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \cong \bigoplus_{\substack{\alpha\in\Lambda_i \\ \mathfrak{p}_\alpha\subseteq\mathfrak{p}}} (0 :_{^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)} \mathfrak{p}R_{(\mathfrak{p})}) \quad \text{in } ^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{p})}).$$

For $\alpha \in \Lambda_i$ such that $\mathfrak{p}_\alpha \subset \mathfrak{p}$ (the symbol '$\subset$' is reserved to denote strict inclusion), there exists an $\mathbb{N}_0^r$-homogeneous element $u \in \mathfrak{p} \setminus \mathfrak{p}_\alpha$, and the fact (see Lemma 1.5) that multiplication by $u/1 \in R_{(\mathfrak{p})}$ provides an automorphism of $^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})$ ensures that

$$(0 :_{^*E(R_{(\mathfrak{p})}/\mathfrak{p}_\alpha R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)} \mathfrak{p}R_{(\mathfrak{p})}) = 0.$$

If $\alpha \in \Lambda_i$ is such that $\mathfrak{p}_\alpha = \mathfrak{p}$, then, by Proposition 2.5(iv),

$$(0 :_{^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)} \mathfrak{p}R_{(\mathfrak{p})}) = (R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)$$

and, by Proposition 2.5(iii)(d), this is a free $\mathbb{Z}^r$-graded T-module.

Therefore there is a $\mathbb{Z}^r$-homogeneous isomorphism of $\mathbb{Z}^r$-graded T-modules

$$^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \cong \bigoplus_{\substack{\alpha\in\Lambda_i \\ \mathfrak{p}_\alpha=\mathfrak{p}}} (R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})(-\mathbf{n}_\alpha).$$

Now apply the functor $(\bullet)^{\phi(\mathfrak{p})}$ to obtain a $\mathbb{Z}^m$-homogeneous isomorphism of $\mathbb{Z}^m$-graded $T^{\phi(\mathfrak{p})}$-modules

$$\left(^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)^{\phi(\mathfrak{p})} \cong \bigoplus_{\substack{\alpha\in\Lambda_i \\ \mathfrak{p}_\alpha=\mathfrak{p}}} \left((R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})(-\mathbf{n}_\alpha)\right)^{\phi(\mathfrak{p})}.$$

But, by Remark 2.6(i), for an $\alpha \in \Lambda_i$,

$$\left(\left(T(-\mathbf{n}_\alpha)\right)^{\phi(\mathfrak{p})}\right)_{\mathbf{a}} = \begin{cases} 0 & \text{if } \phi(\mathfrak{p})(\mathbf{n}_\alpha) \neq \mathbf{a}, \\ (T^{\phi(\mathfrak{p})})_{\mathbf{0}} & \text{if } \phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}. \end{cases}$$

The desired result now follows from Remark 2.6(iii). □

2.8. Definitions. Let the situation and notation be as in Theorem 2.7, so that, in particular, $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ and m denotes the number of $\mathfrak{p}$-directions.

Let $i \in \mathbb{N}_0$. We say that $\mathbf{a} \in \mathbb{Z}^m$ is an *ith level anchor point of $\mathfrak{p}$ for M* if

$$\left(\left(^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\right)^{\phi(\mathfrak{p})}\right)_{\mathbf{a}} \neq 0;$$

the set of all ith level anchor points of $\mathfrak{p}$ for M is denoted by $\operatorname{anch}^i(\mathfrak{p}, M)$; also, we write

$$\operatorname{anch}(\mathfrak{p}, M) = \bigcup_{j\in\mathbb{N}_0} \operatorname{anch}^j(\mathfrak{p}, M),$$

and refer to this as the set of *anchor points of $\mathfrak{p}$ for M*.

Thus $\operatorname{anch}^i(\mathfrak{p}, M)$ is the set of m-tuples $\mathbf{a} \in \mathbb{Z}^m$ for which, when we decompose

$$^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha\in\Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

by means of a $\mathbb{Z}^r$-homogeneous isomorphism, there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $\phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}$. Note that $\operatorname{anch}^i(\mathfrak{p}, M) = \emptyset$ if $\mu^i(\mathfrak{p}, M) = 0$, and that, if M is finitely generated, then $\operatorname{anch}^i(\mathfrak{p}, M)$ is a finite set, by Remark 2.6(iii).

The details in our present multi-graded situation are more complicated (and therefore more interesting!) than in the singly-graded situation studied in [15] because there might exist a $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$ for which the set of $\mathfrak{p}$-directions is a proper subset of $\{1, \dots, r\}$. This cannot happen when $r = 1$. It is worthwhile for us to draw attention to the simplifications that occur in the above theory when $\operatorname{dir}(\mathfrak{p}) = \{1, \dots, r\}$, for that case provides a more-or-less exact analogue of the anchor point theory for the singly-graded case developed in [15].

2.9. Example. Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a $\mathbb{Z}^r$-graded R-module, and let

$$I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots$$

be the minimal *injective resolution of M. For each $i \in \mathbb{N}_0$, let

$$\theta_i : {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a $\mathbb{Z}^r$-homogeneous isomorphism, where $\mathfrak{p}_\alpha \in {}^*\operatorname{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ be such that $\mathfrak{p} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} > \mathbf{0}$, so that $\operatorname{dir}(\mathfrak{p}) = \{1, \dots, r\}$. In this case, $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is concentrated in degree $\mathbf{0}$, and $T_{\mathbf{0}}$ is a field isomorphic to $k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})$.

Let $i \in \mathbb{N}_0$. Then $\operatorname{anch}^i(\mathfrak{p}, M)$ is the set of r-tuples $\mathbf{a} \in \mathbb{Z}^r$ for which there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $\mathbf{n}_\alpha = \mathbf{a}$. The cardinality of the set of such αs is

$$\dim_{k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})}\big(\big({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\big)_{\mathbf{a}}\big),$$

and we have

$$\sum_{\mathbf{a} \in \mathbb{Z}^r} \dim_{k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})}\big(\big({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})\big)_{\mathbf{a}}\big) = \mu^i(\mathfrak{p}, M).$$

In particular, if M is finitely generated, then there are only finitely many ith level anchor points of $\mathfrak{p}$ for M.

This reflects rather well the singly-graded anchor point theory studied in [15].

Our next aim is to extend (in a sense) the final result in Example 2.9 (namely that, when M (as in the example) is a finitely generated $\mathbb{Z}^r$-graded R-module and $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$ is such that $\mathfrak{p} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} > \mathbf{0}$, then, for each $i \in \mathbb{N}_0$, there are only finitely many ith level anchor points of $\mathfrak{p}$ for M) to *all* $\mathbb{N}_0^r$-graded primes of R that contain $R_{\mathbf{1}}$.

2.10. Remark. Let S be a multiplicatively closed set of $\mathbb{Z}^r$-homogeneous elements of R, and let M, N be $\mathbb{Z}^r$-graded R-modules with M finitely generated. Then, for each $i \in \mathbb{N}_0$, there is a $\mathbb{Z}^r$-homogeneous $S^{-1}R$-isomorphism

$$S^{-1}\big({}^*\operatorname{Ext}^i_R(M, N)\big) \cong {}^*\operatorname{Ext}^i_{S^{-1}R}\big(S^{-1}M, S^{-1}N\big).$$

2.11. Theorem. *Assume that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let M be a $\mathbb{Z}^r$-graded R-module. Let $i \in \mathbb{N}_0$, and let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$. Then*

$$\operatorname{anch}^i(\mathfrak{p}, M) = \operatorname{anch}^i\big(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}\big),$$

and so is finite if M is finitely generated.

Proof. Suppose, for ease of notation, that $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$, where $0 < m \leqslant r$. Note that $\mathfrak{p}^{\phi(\mathfrak{p})}$ is a $\mathbb{Z}^m$-graded prime ideal of the $\mathbb{Z}^m$-graded ring $R^{\phi(\mathfrak{p})}$, and that $\operatorname{dir}(\mathfrak{p}^{\phi(\mathfrak{p})}) = \{1, \ldots, m\}$ (by Lemma 2.2).

Set $E := {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)$. Let $\mathbf{a} \in \mathbb{Z}^m$. In view of 2.10, the m-tuple $\mathbf{a}$ is an ith level anchor point of $\mathfrak{p}$ for M if and only if $((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}} \neq 0$. Our initial task in this proof is to show that this is the case if and only if

$$\big(\big({}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}\big(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}\big)\big)_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\big)_{\mathbf{a}} \neq 0.$$

Now the $\mathbb{Z}^r$-graded R-module E can be constructed by application of the functor ${}^*\operatorname{Hom}_R(\bullet, M)$ to a (deleted) *free resolution of $R/\mathfrak{p}$ by finitely generated *free $\mathbb{Z}^r$-graded modules in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ and then taking cohomology of the resulting complex. It follows that there is a $\mathbb{Z}^m$-homogeneous isomorphism of $\mathbb{Z}^m$-graded $R^{\phi(\mathfrak{p})}$-modules

$$E^{\phi(\mathfrak{p})} \cong {}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}\big(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}\big).$$

Suppose that $((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}} \neq 0$. Thus there exists $\mathbf{n} \in \mathbb{Z}^r$ such that $\phi(\mathfrak{p})(\mathbf{n}) = \mathbf{a}$ and $\xi \in (E_{(\mathfrak{p})})_{\mathbf{n}}$ such that $\xi \neq 0$. By Remark 2.4, there exists $\mathbf{n}' \in \mathbb{Z}^r$ such that $\phi(\mathfrak{p})(\mathbf{n}') = \mathbf{a}$ and $e \in E_{\mathbf{n}'}$ which is not annihilated by any $\mathbb{Z}^r$-homogeneous element of $R \setminus \mathfrak{p}$. Now any $\mathbb{Z}^m$-homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$ will, when written as a sum of $\mathbb{Z}^r$-homogeneous elements of R, have at least one component outside $\mathfrak{p}$, and so $0 \neq e/1 \in (E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})}$. Hence $((E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})})_{\mathbf{a}} \neq 0$, so that

$$\big(\big({}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}\big(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}\big)\big)_{(\mathfrak{p}^{\phi(\mathfrak{p})})}\big)_{\mathbf{a}} \neq 0.$$

Now suppose that $(({}^*\operatorname{Ext}^i_{R^{\phi(\mathfrak{p})}}(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})})_{\mathbf{a}} \neq 0$. Then $((E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})})_{\mathbf{a}} \neq 0$. Since every $\mathbb{Z}^m$-homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$ has degree $\mathbf{0} \in \mathbb{Z}^m$, it follows that there exists $e \in (E^{\phi(\mathfrak{p})})_{\mathbf{a}}$ that is not annihilated by any $\mathbb{Z}^m$-homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$. In particular, e is not annihilated by any $\mathbb{Z}^r$-homogeneous element of $R \setminus \mathfrak{p}$. Therefore $0 \neq e/1 \in ((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}}$.

This proves that $\operatorname{anch}^i(\mathfrak{p}, M) = \operatorname{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})})$. Finally, since $\operatorname{dir}(\mathfrak{p}^{\phi(\mathfrak{p})}) = \{1, \ldots, m\}$, it follows from Example 2.9 that $\operatorname{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})})$ is finite when M is finitely generated. $\square$

The aim of the remainder of this section is to establish a multi-graded analogue of a result of Bass [1, Lemma 3.1]. However, there are some subtleties which mean that our generalization of [15, Lemma 1.8] is not completely straightforward.

2.12. Theorem. *Assume that* $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ *is positively graded and standard, and let M be a finitely generated* $\mathbb{Z}^r$*-graded R-module. Let* $\mathfrak{p}, \mathfrak{q} \in {}^*\operatorname{Spec}(R)$ *be such that* $R_{\mathbf{1}}R \subseteq \mathfrak{p} \subset \mathfrak{q}$ (*we reserve the symbol '*$\subset$*' to denote strict inclusion*) *and that there is no* $\mathbb{Z}^r$*-graded prime ideal strictly between* $\mathfrak{p}$ *and* $\mathfrak{q}$. *Note that* $\operatorname{dir}(\mathfrak{p}) \subseteq \operatorname{dir}(\mathfrak{q})$: *suppose, for ease of notation, that* $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$ *and* $\operatorname{dir}(\mathfrak{q}) = \{1, \ldots, m, m+1, \ldots, h\}$, *where* $0 < m \leqslant h \leqslant r$.

Let $i \in \mathbb{N}_0$. *Then, for each* $\mathbf{a} = (a_1, \ldots, a_m) \in \operatorname{anch}^i(\mathfrak{p}, M)$, *there exists*

$$\mathbf{b} = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_h) \in \operatorname{anch}^{i+1}(\mathfrak{q}, M)$$

such that $(b_1, \ldots, b_m) = (a_1, \ldots, a_m) = \mathbf{a}$.

Proof. There exists an $\mathbb{N}_0^r$-homogeneous element $b \in \mathfrak{q} \setminus \mathfrak{p}$. By Remark 2.4, each $\mathbb{N}_0^r$-homogeneous element of $R \setminus \mathfrak{p}$ has degree with first m components 0. In particular, $\deg(b) = \mathbf{0}|\mathbf{v} \in \mathbb{Z}^m \times \mathbb{Z}^{r-m}$ for some $\mathbf{v} \in \mathbb{Z}^{r-m}$.

Since $\mathbf{a} \in \operatorname{anch}^i(\mathfrak{p}, M)$, there exists $\mathbf{w} \in \mathbb{Z}^{r-m}$ such that $({}^*\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}))_{\mathbf{a}|\mathbf{w}} \neq 0$. Set $E := {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}, M)$. In view of Remark 2.10, we must have $(E_{(\mathfrak{p})})_{\mathbf{a}|\mathbf{w}} \neq 0$. Since each $\mathbb{N}^r_0$-homogeneous element of $R \setminus \mathfrak{p}$ has degree with first m components 0, this means that there exists a homogeneous element $e \in E$, with $\deg(e) = \mathbf{a}|\mathbf{w}'$ for some $\mathbf{w}' \in \mathbb{Z}^{r-m}$, that is not annihilated by any $\mathbb{N}^r_0$-homogeneous element of $R \setminus \mathfrak{p}$. But $R \setminus \mathfrak{q} \subseteq R \setminus \mathfrak{p}$, and so it follows that $(E_{(\mathfrak{q})})_{\mathbf{a}|\mathbf{w}'} \neq 0$. By Remark 2.10 again, $({}^*\operatorname{Ext}^i_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}, M_{(\mathfrak{q})}))_{\mathbf{a}|\mathbf{w}'} \neq 0$. Write $F := {}^*\operatorname{Ext}^i_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}, M_{(\mathfrak{q})})$.

There is an exact sequence

$$0 \longrightarrow (R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})})\bigl(-(\mathbf{0}|\mathbf{v})\bigr) \xrightarrow{b/1} R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})} \longrightarrow R_{(\mathfrak{q})}/\bigl(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}\bigr) \longrightarrow 0$$

in ${}^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{q})})$, and this induces an exact sequence

$$F \xrightarrow{b/1} F(\mathbf{0}|\mathbf{v}) \longrightarrow {}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}\bigl(R_{(\mathfrak{q})}/\bigl(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}\bigr), M_{(\mathfrak{q})}\bigr).$$

Recall that $\deg(b) = \mathbf{0}|\mathbf{v}$. We claim that there exists $\mathbf{y} \in \mathbb{Z}^{r-m}$ such that $F_{\mathbf{a}|\mathbf{y}} \neq (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$. To see this, note that $b/1 \in \mathfrak{q}R_{(\mathfrak{q})}$, the unique *maximal ideal of the homogeneous localization $R_{(\mathfrak{q})}$, and if we had $F_{\mathbf{a}|\mathbf{y}} = (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$ for every $\mathbf{y} \in \mathbb{Z}^{r-m}$, then we should have $F_{\mathbf{a}|\mathbf{w}'} \subseteq \bigcap_{n\in\mathbb{N}}(b/1)^n F$, which is zero by the multi-graded version of Krull's Intersection Theorem. (One can show that $G := \bigcap_{n\in\mathbb{N}}(b/1)^n F$ satisfies $G = (b/1)G$, and then use the multi-graded version of Nakayama's Lemma.) Thus there exists $\mathbf{y} \in \mathbb{Z}^{r-m}$ such that $F_{\mathbf{a}|\mathbf{y}} \neq (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$, and therefore, in view of the last exact sequence,

$$\bigl({}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}\bigl(R_{(\mathfrak{q})}/\bigl(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}\bigr), M_{(\mathfrak{q})}\bigr)\bigr)_{\mathbf{a}|\mathbf{y}} \neq 0.$$

Now $R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})})$ is concentrated in $\mathbb{Z}^r$-degrees whose first m components are all zero. Therefore all its $\mathbb{Z}^r$-graded R-homomorphic images and all its $\mathbb{Z}^r$-graded submodules are also concentrated in $\mathbb{Z}^r$-degrees whose first m components are all zero.

The only $\mathbb{Z}^r$-graded prime ideal of $R_{(\mathfrak{q})}$ that contains the ideal $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ is $\mathfrak{q}R_{(\mathfrak{q})}$, and so $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ is $\mathfrak{q}R_{(\mathfrak{q})}$-primary. It follows that there is a chain of $\mathbb{Z}^r$-graded ideals of $R_{(\mathfrak{q})}$ from $\mathfrak{q}R_{(\mathfrak{q})}$ to $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ with the property that each subquotient is $R_{(\mathfrak{q})}$-isomorphic to $(R_{(\mathfrak{q})}/\mathfrak{q}R_{(\mathfrak{q})})(\mathbf{0}|\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{r-m}$. It therefore follows from the half-exactness of ${}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}$ that there exists $\mathbf{y}' \in \mathbb{Z}^{r-m}$ such that

$$\bigl({}^*\operatorname{Ext}^{i+1}_{R_{(\mathfrak{q})}}(R_{(\mathfrak{q})}/\mathfrak{q}R_{(\mathfrak{q})}, M_{(\mathfrak{q})})\bigr)_{\mathbf{a}|\mathbf{y}'} \neq 0.$$

The claim then follows from Theorem 2.7. □

2.13. Corollary. *Assume that $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^r_0} R_{\mathbf{n}}$ is positively graded and standard, and let M be a finitely generated $\mathbb{Z}^r$-graded R-module. Let $\mathfrak{p} \in {}^*\operatorname{Var}(R_{\mathbf{1}}R)$, and suppose, for ease of notation, that $\operatorname{dir}(\mathfrak{p}) = \{1, \ldots, m\}$.*

Let $\mathbf{a} \in \operatorname{anch}(\mathfrak{p}, M)$. Then there exists $\mathfrak{q} \in {}^*\operatorname{Spec}(R)$ such that $\mathfrak{q} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}^r_0$ with $\mathbf{n} > \mathbf{0}$ and $\mathbf{b} = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_r) \in \operatorname{anch}(\mathfrak{q}, M)$ such that $\mathbf{a} = (b_1, \ldots, b_m)$.

Proof. There exists a saturated chain $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{q}$ of $\mathbb{Z}^r$-graded prime ideals of R such that $\mathfrak{q}$ is *maximal. Since $\mathfrak{q}$ is contained in the $\mathbb{Z}^r$-graded prime ideal

$$(\mathfrak{q} \cap R_{\mathbf{0}}) \bigoplus \bigoplus_{\mathbf{n}>\mathbf{0}} R_{\mathbf{n}},$$

these two $\mathbb{Z}^r$-graded prime ideals must be the same; we therefore see that $\mathfrak{q} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}^r_0$ with $\mathbf{n} > \mathbf{0}$. The claim is now immediate from Theorem 2.12. □

3. The ends of certain multi-graded local cohomology modules

We begin with a combinatorial lemma.

3.1. Lemma. *Let* $\mathbf{a} := (a_1, \dots, a_r) \in \mathbb{Z}^r$ *and let* Σ *be a non-empty subset of* $\mathbb{Z}^r$ *such that* $\mathbf{n} \leqslant \mathbf{a}$ *for all* $\mathbf{n} \in \Sigma$. *Then* Σ *has only finitely many maximal elements.*

Note. We are grateful to the referee for drawing our attention to the following proof, which is shorter than our original.

Proof. The set $\Delta := \mathbf{a} - \Sigma := \{\mathbf{a} - \mathbf{n}\colon\ \mathbf{n} \in \Sigma\}$ is a non-empty subset of $\mathbb{N}_0^r$. Now $\mathbb{N}_0^r$ is a Noetherian monoid with respect to addition, by [11, Proposition 1.3.5], for example. (All terminology concerning monoids in this proof is as in [11, Chapter 1].) Therefore the monoideal (Δ) of $\mathbb{N}_0^r$ generated by Δ can be generated by finitely many elements of Δ, say by $\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(s)} \in \Delta$. Therefore

$$\Delta \subseteq (\Delta) = \bigl(\mathbf{m}^{(1)} + \mathbb{N}_0^r\bigr) \cup \cdots \cup \bigl(\mathbf{m}^{(s)} + \mathbb{N}_0^r\bigr),$$

from which it follows that any minimal member of Δ must belong to the set $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(s)}\}$. Therefore any maximal member of Σ must belong to the set $\{\mathbf{a} - \mathbf{m}^{(1)}, \dots, \mathbf{a} - \mathbf{m}^{(s)}\}$. □

3.2. Notation. Let $\Sigma, \Delta \subseteq \mathbb{Z}^r$. We shall denote by $\max(\Sigma)$ the set of maximal members of Σ. (If Σ has no maximal member, then we interpret $\max(\Sigma)$ as the empty set.)

We shall write $\Sigma \preccurlyeq \Delta$ to indicate that, for each $\mathbf{n} \in \Sigma$, there exists $\mathbf{m} \in \Delta$ such that $\mathbf{n} \leqslant \mathbf{m}$; moreover, we shall describe this situation by the terminology 'Δ *dominates* Σ'. We shall use obvious variants of this terminology. Observe that, if $\Sigma \preccurlyeq \Delta$ and $\Delta \preccurlyeq \Sigma$, then $\max(\Sigma) = \max(\Delta)$, and $\Sigma \preccurlyeq \max(\Sigma)$ if and only if $\Delta \preccurlyeq \max(\Delta)$.

3.3. Remark. (See Huy Tài Hà [9, §2].) Let $\phi : \mathbb{Z}^r \to \mathbb{Z}^m$, where m is a positive integer, be a homomorphism of Abelian groups. We use the notation R^ϕ, *etcetera*, of Definition 1.3. Let $\mathfrak{a}$ be a $\mathbb{Z}^r$-graded ideal of R. Then $((H^i_{\mathfrak{a}}(\bullet))^\phi)_{i \in \mathbb{N}_0}$ and $((H^i_{\mathfrak{a}^\phi}(\bullet^\phi)))_{i \in \mathbb{N}_0}$ are both negative strongly connected sequences of covariant functors from ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ to ${}^*\mathcal{C}^{\mathbb{Z}^m}(R^\phi)$; moreover, the 0th members of these two connected sequences are the same functor, and, whenever, I is a *injective $\mathbb{Z}^r$-graded R-module and $i > 0$, we have $H^i_{\mathfrak{a}}(I) = 0$ when all gradings are forgotten, so that $(H^i_{\mathfrak{a}}(I))^\phi = 0$ and $H^i_{\mathfrak{a}^\phi}(I^\phi) = 0$. Consequently, the two above-mentioned connected sequences are isomorphic. Hence, for each $\mathbb{Z}^r$-graded R-module M, there is a $\mathbb{Z}^m$-homogeneous isomorphism of $\mathbb{Z}^m$-graded R^ϕ-modules

$$\bigl(H^i_{\mathfrak{a}}(M)\bigr)^\phi \cong H^i_{\mathfrak{a}^\phi}\bigl(M^\phi\bigr) \quad \text{for each } i \in \mathbb{N}_0.$$

3.4. Notation. Throughout this section, we shall be concerned with the situation where

$$R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$$

is positively graded; we shall only assume that R is standard when this is explicitly stated.

We shall be greatly concerned with the $\mathbb{N}_0^r$-graded ideal

$$\mathfrak{c} := \mathfrak{c}(R) := \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}_0^r \\ \mathbf{n} > \mathbf{0}}} R_{\mathbf{n}}.$$

We shall accord R_+ its usual meaning (see E. Hyry [10, p. 2215]), so that

$$R_+ := \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0^r \\ \mathbf{n}\geqslant\mathbf{1}}} R_{\mathbf{n}} = \bigoplus_{\mathbf{n}\in\mathbb{N}^r} R_{\mathbf{n}}.$$

Observe that, when $r=1$, we have $\mathfrak{c}=R_+$. However, in general this is not the case when $r>1$.

3.5. Definition. Suppose that $R=\bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard; let $M=\bigoplus_{\mathbf{n}\in\mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded R-module, and let $j\in\mathbb{N}_0$.

Let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal such that $\operatorname{dir}(\mathfrak{b})\neq\emptyset$, and let $i\in\operatorname{dir}(\mathfrak{b})$; consider the Abelian group homomorphism $\phi_i:\mathbb{Z}^r\to\mathbb{Z}$ for which $\phi_i((n_1,\dots,n_r))=n_i$ for all $(n_1,\dots,n_r)\in\mathbb{Z}^r$, which is just the ith co-ordinate function.

By Lemma 2.2, since $R_{\mathbf{e}_i}\subseteq\sqrt{\mathfrak{b}}$, we have

$$\left(R^{\phi_i}\right)_+ = \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0^r \\ n_i>0}} R_{\mathbf{n}} \subseteq \sqrt{\mathfrak{b}}^{\phi_i}.$$

It therefore follows from [15, Corollary 2.5], with the notation of that paper, that the $\mathbb{N}_0$-graded R^{ϕ_i}-module $(H^j_{\mathfrak{b}}(M))^{\phi_i}\cong H^j_{\mathfrak{b}^{\phi_i}}(M^{\phi_i})$, if non-zero, has finite end satisfying

$$\operatorname{end}\left(\left(H^j_{\mathfrak{b}}(M)\right)^{\phi_i}\right)\leqslant a^*\left(M^{\phi_i}\right)=\sup\left\{\operatorname{end}\left(H^k_{(R^{\phi_i})_+}\left(M^{\phi_i}\right)\right):\ k\in\mathbb{N}_0\right\}=\sup\left\{a^k_{(R^{\phi_i})_+}\left(M^{\phi_i}\right):\ k\in\mathbb{N}_0\right\}.$$

(Note that, in these circumstances, the invariant $a^*(M^{\phi_i})$ is an integer.) Thus, if $\mathbf{n}:=(n_1,\dots,n_r)\in\mathbb{Z}^r$ is such that $H^j_{\mathfrak{b}}(M)_{\mathbf{n}}\neq 0$, then $n_i\leqslant a^*(M^{\phi_i})$. Thus there exists $\mathbf{a}\in\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ such that, for all $\mathbf{n}:=(n_1,\dots,n_r)\in\mathbb{Z}^r$ with $H^j_{\mathfrak{b}}(M)_{\mathbf{n}}\neq 0$, we have $\phi(\mathfrak{b})(\mathbf{n})\leqslant\mathbf{a}$. We define the *end of* $H^j_{\mathfrak{b}}(M)$ by

$$\operatorname{end}\left(H^j_{\mathfrak{b}}(M)\right):=\max\left\{\phi(\mathfrak{b})(\mathbf{n}):\ \mathbf{n}\in\mathbb{Z}^r \text{ and } H^j_{\mathfrak{b}}(M)_{\mathbf{n}}\neq 0\right\}.$$

By Lemma 3.1, if $H^j_{\mathfrak{b}}(M)\neq 0$ and $\operatorname{dir}(\mathfrak{b})\neq\emptyset$, then this end is a non-empty finite set of points of $\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$. Note that the end of $H^j_{\mathfrak{b}}(M)$ dominates $\phi(\mathfrak{b})(\mathbf{n})$ for every $\mathbf{n}\in\mathbb{Z}^r$ for which $H^j_{\mathfrak{b}}(M)_{\mathbf{n}}\neq 0$.

We draw the reader's attention to the fact that, when $r>1$ and $R_{\mathbf{e}_i}\neq 0$ for all $i\in\{1,\dots,r\}$, the ideal $R_+=\bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0^r \\ \mathbf{n}\geqslant\mathbf{1}}} R_{\mathbf{n}}$ has empty set of directions; consequently, we have not defined the end of the ith local cohomology module $H^i_{R_+}(M)$ of M with respect to R_+. Thus we are not, in this paper, making any contribution to the theory of multi-graded Castelnuovo regularity, and, in particular, we are not proposing an alternative definition of a-invariant or a^*-invariant (see [9, Definitions 3.1.1 and 3.1.2]).

With this definition of the ends of (certain) multi-graded local cohomology modules, we can now establish multi-graded analogues of some results in [15, §2].

3.6. Theorem. *Suppose that $R:=\bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a finitely generated $\mathbb{Z}^r$-graded R-module, and let*

$$I^\bullet: 0\longrightarrow {}^*E^0(M)\xrightarrow{d^0}{}^*E^1(M)\longrightarrow\cdots\longrightarrow{}^*E^i(M)\xrightarrow{d^i}{}^*E^{i+1}(M)\longrightarrow\cdots$$

be the minimal *injective resolution of M.

Let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal such that $\operatorname{dir}(\mathfrak{b})\neq\emptyset$, *and let $j\in\mathbb{N}_0$. Then*

$$\max\left(\bigcup_{i=0}^{j}\operatorname{end}\left(H_{\mathfrak{b}}^{j}(M)\right)\right)=\max\left\{\phi(\mathfrak{b})(\mathbf{n})\colon\ \mathbf{n}\in\mathbb{Z}^r \textit{ and } \left(\Gamma_{\mathfrak{b}}\left({}^*E^i(M)\right)\right)_{\mathbf{n}}\neq 0 \textit{ for some } i\in\{0,\dots,j\}\right\}$$
$$=\max\left(\bigcup_{i=0}^{j}\ \bigcup_{\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})}\phi(\mathfrak{p};\mathfrak{b})\left(\operatorname{anch}^i(\mathfrak{p},M)\right)\right).$$

Proof. Let $i\in\mathbb{N}_0$ and set

$$\Delta_i:=\left\{\phi(\mathfrak{b})(\mathbf{n})\colon\ \mathbf{n}\in\mathbb{Z}^r \text{ and } H_{\mathfrak{b}}^{i}(M)_{\mathbf{n}}\neq 0\right\},\qquad \Sigma_i:=\left\{\phi(\mathfrak{b})(\mathbf{n})\colon\ \mathbf{n}\in\mathbb{Z}^r \text{ and } \left(\Gamma_{\mathfrak{b}}\left({}^*E^i(M)\right)\right)_{\mathbf{n}}\neq 0\right\}$$

and

$$\Phi_i:=\bigcup_{\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})}\phi(\mathfrak{p};\mathfrak{b})\left(\operatorname{anch}^i(\mathfrak{p},M)\right).$$

Also, let

$$\theta_i:{}^*E^i(M)\xrightarrow{\cong}\bigoplus_{\alpha\in\Lambda_i}{}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a $\mathbb{Z}^r$-homogeneous isomorphism, where $\mathfrak{p}_\alpha\in{}^*\operatorname{Spec}(R)$ and $\mathbf{n}_\alpha\in\mathbb{Z}^r$ for all $\alpha\in\Lambda_i$.

We shall first show that $\Delta_i\preccurlyeq\Sigma_i\preccurlyeq\Phi_i$. Now $H_{\mathfrak{b}}^{i}(M)$ is a homomorphic image, by a $\mathbb{Z}^r$-homogeneous epimorphism, of

$$\operatorname{Ker}\left(\Gamma_{\mathfrak{b}}\left(d^i\right):\Gamma_{\mathfrak{b}}\left({}^*E^i(M)\right)\longrightarrow\Gamma_{\mathfrak{b}}\left({}^*E^{i+1}(M)\right)\right).$$

Therefore, if $\mathbf{n}\in\mathbb{Z}^r$ is such that $H_{\mathfrak{b}}^{i}(M)_{\mathbf{n}}\neq 0$, then $(\Gamma_{\mathfrak{b}}({}^*E^i(M)))_{\mathbf{n}}\neq 0$. This proves that $\Delta_i\subseteq\Sigma_i$, so that $\Delta_i\preccurlyeq\Sigma_i$.

Furthermore, given $\mathbf{n}\in\mathbb{Z}^r$ such that $(\Gamma_{\mathfrak{b}}({}^*E^i(M)))_{\mathbf{n}}\neq 0$, we can see from the isomorphism θ_i that there must exist $\alpha\in\Lambda_i$ such that $\mathfrak{b}\subseteq\mathfrak{p}_\alpha$ and $({}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}}\neq 0$. It now follows from Proposition 2.5(ii) that $\phi(\mathfrak{p}_\alpha)(\mathbf{n})\leqslant\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$, so that

$$\phi(\mathfrak{p}_\alpha;\mathfrak{b})\left(\phi(\mathfrak{p}_\alpha)(\mathbf{n})\right)\leqslant\phi(\mathfrak{p}_\alpha;\mathfrak{b})\left(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)\right).$$

Now $\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ is an ith level anchor point of $\mathfrak{p}_\alpha$ for M, and $\phi(\mathfrak{p}_\alpha;\mathfrak{b})\circ\phi(\mathfrak{p}_\alpha)=\phi(\mathfrak{b})$. This is enough to prove that $\Sigma_i\preccurlyeq\Phi_i$.

In particular, we have proved that $\Delta_0\preccurlyeq\Sigma_0\preccurlyeq\Phi_0$. We shall prove the desired result by induction on j. We show next that $\Phi_0\preccurlyeq\Delta_0$, and this, together with the above, will prove the claim in the case where $j=0$. Let $\mathbf{m}\in\Phi_0$. Thus $\mathbf{m}\in\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ and there exists $\alpha\in\Lambda_0$ such that $\mathfrak{p}_\alpha\in{}^*\operatorname{Var}(\mathfrak{b})$ and $\mathbf{m}=\phi(\mathfrak{p}_\alpha;\mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$. Now the image of

$$\bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r\\ \phi(\mathfrak{p}_\alpha)(\mathbf{n})\geqslant\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)}}\left({}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)\right)_{\mathbf{n}}$$

under θ_0^{-1} is a non-zero $\mathbb{Z}^r$-graded submodule of $\Gamma_{\mathfrak{b}}({}^*E^0(M))$; as the latter is a *essential extension of $\Gamma_{\mathfrak{b}}(M)$, it follows that there exists $\mathbf{n}\in\mathbb{Z}^r$ with $\phi(\mathfrak{p}_\alpha)(\mathbf{n})\geqslant\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ such that $(\Gamma_{\mathfrak{b}}(M))_{\mathbf{n}}\neq 0$. Moreover,

$$\phi(\mathfrak{b})(\mathbf{n})=\phi(\mathfrak{p}_\alpha;\mathfrak{b})\left(\phi(\mathfrak{p}_\alpha)(\mathbf{n})\right)\geqslant\phi(\mathfrak{p}_\alpha;\mathfrak{b})\left(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)\right)=\mathbf{m}.$$

It follows that $\Phi_0 \preccurlyeq \Delta_0$, so that $\max(\Delta_0) = \max(\Sigma_0) = \max(\Phi_0)$, and the desired result has been proved when $j = 0$.

Now suppose that $j > 0$ and make the obvious inductive assumption. As we have already proved that $\Delta_i \preccurlyeq \Sigma_i$ and $\Sigma_i \preccurlyeq \Phi_i$ for all $i = 0, \ldots, j$, it will be enough, in order to complete the inductive step, for us to prove that $\Phi_j \preccurlyeq \bigcup_{k=0}^{j} \Delta_k$. So consider $\alpha \in \Lambda_j$ such that $\mathfrak{p}_\alpha \in {}^*\mathrm{Var}(\mathfrak{b})$; we shall show that $\phi(\mathfrak{p}_\alpha; \mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$ is dominated by a member of $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{j-1} \cup \Delta_j$.

Now the image of

$$\bigoplus_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ \phi(\mathfrak{p}_\alpha)(\mathbf{n}) \geqslant \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)}} \big({}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)\big)_{\mathbf{n}}$$

under θ_j^{-1} is a non-zero $\mathbb{Z}^r$-graded submodule of $\Gamma_{\mathfrak{b}}({}^*E^j(M))$; as the latter is a *essential extension of $\operatorname{Ker}\Gamma_{\mathfrak{b}}(d^j)$, it follows that there exists $\mathbf{n} \in \mathbb{Z}^r$ with $\phi(\mathfrak{p}_\alpha)(\mathbf{n}) \geqslant \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ such that $(\operatorname{Ker}\Gamma_{\mathfrak{b}}(d^j))_{\mathbf{n}} \neq 0$. There is an exact sequence

$$0 \longrightarrow \operatorname{Im}\Gamma_{\mathfrak{b}}\big(d^{j-1}\big) \longrightarrow \operatorname{Ker}\Gamma_{\mathfrak{b}}\big(d^j\big) \longrightarrow H^j_{\mathfrak{b}}(M) \longrightarrow 0$$

of graded $\mathbb{Z}^r$-modules and homogeneous homomorphisms. Therefore either $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0$ or

$$\big(\operatorname{Im}\Gamma_{\mathfrak{b}}\big(d^{j-1}\big)\big)_{\mathbf{n}} \neq 0.$$

In the first case, $\phi(\mathfrak{p}_\alpha; \mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n})) = \phi(\mathbf{b})(\mathbf{n}) \in \Delta_j$. In the second case, $(\Gamma_{\mathfrak{b}}({}^*E^{j-1}(M)))_{\mathbf{n}} \neq 0$, whence $\phi(\mathbf{b})(\mathbf{n}) \in \Sigma_{j-1}$, so that, by the inductive hypothesis, $\phi(\mathbf{b})(\mathbf{n})$ is dominated by an element of $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{j-1}$; thus, in this case also, $\phi(\mathfrak{p}_\alpha; \mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$ is dominated by an element of $\bigcup_{k=0}^{j} \Delta_k$. This is enough to complete the inductive step. $\square$

3.7. Notation. Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let M be a finitely generated $\mathbb{Z}^r$-graded R-module. Let $\mathcal{Q}$ be a non-empty subset of $\{1, \ldots, r\}$. Define $\mathfrak{c}^{\mathcal{Q}} := \sum_{i\in\mathcal{Q}} R_{\mathbf{e}_i}R$. Then $\mathrm{dir}(\mathfrak{c}^{\mathcal{Q}}) \supseteq \mathcal{Q}$, and $\mathfrak{c}^{\mathcal{Q}}$ is the smallest ideal (up to radical) with set of directions containing $\mathcal{Q}$. We also define the *$\mathcal{Q}$-bound* $\mathrm{bnd}^{\mathcal{Q}}(M)$ of M by

$$\mathrm{bnd}^{\mathcal{Q}}(M) := \max\left(\bigcup_{i\in\mathbb{N}_0} \mathrm{end}\big(H^i_{\mathfrak{c}^{\mathcal{Q}}}(M)\big)\right).$$

Observe that $\mathrm{bnd}^{\mathcal{Q}}(M)$ is a finite set of points in $\mathbb{Z}^{\#\,\mathrm{dir}(\mathfrak{c}^{\mathcal{Q}})}$, because $H^i_{\mathfrak{c}^{\mathcal{Q}}}(M) = 0$ whenever i exceeds the arithmetic rank of $\mathfrak{c}^{\mathcal{Q}}$.

For consistency with our earlier notation in 3.4, we abbreviate $\mathfrak{c}^{\{1,\ldots,r\}} = \sum_{\mathbf{n}>\mathbf{0}} R_{\mathbf{n}}$ by $\mathfrak{c}$. Note that $\mathrm{bnd}^{\{1,\ldots,r\}}(M) = \max\big(\bigcup_{i\in\mathbb{N}_0} \mathrm{end}(H^i_{\mathfrak{c}}(M))\big)$ is a finite set of points in $\mathbb{Z}^r$.

The following corollaries, which are multi-graded analogues of [15, Corollaries 2.5, 2.6], can now be deduced immediately from Theorem 3.6.

3.8. Corollary. *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a finitely generated $\mathbb{Z}^r$-graded R-module, and let*

$$I^\bullet : 0 \longrightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \longrightarrow \cdots \longrightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \longrightarrow \cdots$$

be the minimal *injective resolution of M.

Let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R such that $\mathrm{dir}(\mathfrak{b}) \neq \emptyset$, *and let $j \in \mathbb{N}_0$. Then*

$$\max\left(\bigcup_{i=0}^{j}\operatorname{end}\left(H^j_{\mathfrak{b}}(M)\right)\right) \preccurlyeq \max\left(\bigcup_{i=0}^{j}\ \bigcup_{\mathfrak{p}\in^*\operatorname{Var}(\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})})}\phi\left(\mathfrak{p};\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})}\right)\left(\operatorname{anch}^i(\mathfrak{p},M)\right)\right)$$

$$= \max\left\{\phi(\mathfrak{b})(\mathbf{n})\colon\ \mathbf{n}\in\mathbb{Z}^r \text{ and } \left(\Gamma_{\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})}}\left({}^*E^i(M)\right)\right)_{\mathbf{n}} \neq 0 \text{ for an } i\in\{0,\ldots,j\}\right\}$$

$$= \max\left(\bigcup_{i=0}^{j}\operatorname{end}\left(H^j_{\mathfrak{c}^{\operatorname{dir}(\mathfrak{b})}}(M)\right)\right) \preccurlyeq \operatorname{bnd}^{\operatorname{dir}(\mathfrak{b})}(M).$$

3.9. Corollary. *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a finitely generated $\mathbb{Z}^r$-graded R-module.*

Let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R of arithmetic rank t such that $\operatorname{dir}(\mathfrak{b}) \neq \emptyset$, and let $k\in\mathbb{N}$ with $k>t$. Then

$$\max\left(\bigcup_{i=0}^{t}\ \bigcup_{\mathfrak{p}\in^*\operatorname{Var}(\mathfrak{b})}\phi(\mathfrak{p};\mathfrak{b})\left(\operatorname{anch}^i(\mathfrak{p},M)\right)\right) = \max\left(\bigcup_{i=0}^{t}\operatorname{end}\left(H^i_{\mathfrak{b}}(M)\right)\right) = \max\left(\bigcup_{i=0}^{k}\operatorname{end}\left(H^i_{\mathfrak{b}}(M)\right)\right)$$

$$= \max\left(\bigcup_{i=0}^{k}\ \bigcup_{\mathfrak{p}\in^*\operatorname{Var}(\mathfrak{b})}\phi(\mathfrak{p};\mathfrak{b})\left(\operatorname{anch}^i(\mathfrak{p},M)\right)\right).$$

Consequently, for a $\mathfrak{p}\in{}^*\operatorname{Var}(\mathfrak{b})$ and $\mathbf{a}\in\operatorname{anch}(\mathfrak{p},M)$, we can conclude that $\phi(\mathfrak{p};\mathfrak{b})(\mathbf{a})$ is dominated by

$$\max\left(\bigcup_{i=0}^{t}\ \bigcup_{\mathfrak{p}\in^*\operatorname{Var}(\mathfrak{b})}\phi(\mathfrak{p};\mathfrak{b})\left(\operatorname{anch}^i(\mathfrak{p},M)\right)\right),$$

a set of points of $\mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ which arises from consideration of just the 0th, 1st, $\ldots$, $(t-1)$th and tth terms of the minimal *injective resolution of M.

Our next aim is the establishment of multi-graded analogues of [15, Corollaries 3.1 and 3.2].

3.10. Lemma. *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded, and let $\mathfrak{m}$ be a *maximal ideal of R. Then $\mathfrak{m}_{\mathbf{0}} := \mathfrak{m}\cap R_{\mathbf{0}}$ is a maximal ideal of $R_{\mathbf{0}}$ and $\mathfrak{m} = \mathfrak{m}_{\mathbf{0}}\oplus\mathfrak{c}$, where $\mathfrak{c}$ is as defined in Notation 3.4.*

Proof. Recall that

$$\mathfrak{c} := \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0^r\\ \mathbf{n}>\mathbf{0}}} R_{\mathbf{n}}.$$

Since $\mathfrak{m}_{\mathbf{0}}\in\operatorname{Spec}(R_{\mathbf{0}})$, it follows that $R\supset\mathfrak{m}_{\mathbf{0}}\oplus\mathfrak{c}\supseteq\mathfrak{m}$, so that $\mathfrak{m}=\mathfrak{m}_{\mathbf{0}}\oplus\mathfrak{c}$. Furthermore, $\mathfrak{m}_{\mathbf{0}}$ must be a maximal ideal of $R_{\mathbf{0}}$. $\square$

3.11. Corollary. *Suppose that $R := \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a finitely generated $\mathbb{Z}^r$-graded R-module; let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R such that $\operatorname{dir}(\mathfrak{b})\neq\emptyset$. Then*

$$\max\left(\bigcup_{i\in\mathbb{N}_0}\operatorname{end}\left(H^i_{\mathfrak{b}}(M)\right)\right) = \max\left(\bigcup_{\mathfrak{m}\in^*\operatorname{Var}(\mathfrak{b})\cap^*\operatorname{Max}(R)}\ \bigcup_{i\in\mathbb{N}_0}\phi(\mathfrak{m};\mathfrak{b})\operatorname{end}\left(H^i_{\mathfrak{m}}(M)\right)\right).$$

Proof. Let $\mathfrak{m} \in {}^*\operatorname{Var}(\mathfrak{b}) \cap {}^*\operatorname{Max}(R)$. By Lemma 3.10, $\operatorname{dir}(\mathfrak{m}) = \{1, \dots, r\}$; therefore, by Theorem 3.6, $\max(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{m}}(M))) = \max(\bigcup_{i\in\mathbb{N}_0} \operatorname{anch}^i(\mathfrak{m}, M))$. Another use of Theorem 3.6 therefore shows that

$$\begin{aligned}
\max\Bigl(\bigcup_{i\in\mathbb{N}_0} \phi(\mathfrak{m};\mathfrak{b})\bigl(\operatorname{end}\bigl(H^i_{\mathfrak{m}}(M)\bigr)\bigr)\Bigr) &= \max\Bigl(\bigcup_{i\in\mathbb{N}_0} \phi(\mathfrak{m};\mathfrak{b})\bigl(\operatorname{anch}^i(\mathfrak{m}, M)\bigr)\Bigr)\\
&\preccurlyeq \max\Bigl(\bigcup_{i\in\mathbb{N}_0} \bigcup_{\mathfrak{p}\in {}^*\operatorname{Var}(\mathfrak{b})} \phi(\mathfrak{p};\mathfrak{b})\bigl(\operatorname{anch}^i(\mathfrak{p}, M)\bigr)\Bigr)\\
&= \max\Bigl(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}\bigl(H^i_{\mathfrak{b}}(M)\bigr)\Bigr).
\end{aligned}$$

We have thus proved that

$$\max\Bigl(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}\bigl(H^i_{\mathfrak{b}}(M)\bigr)\Bigr) \succcurlyeq \max\Bigl(\bigcup_{\mathfrak{m}\in {}^*\operatorname{Var}(\mathfrak{b})\cap {}^*\operatorname{Max}(R)} \bigcup_{i\in\mathbb{N}_0} \phi(\mathfrak{m};\mathfrak{b})\bigl(\operatorname{end}\bigl(H^i_{\mathfrak{m}}(M)\bigr)\bigr)\Bigr).$$

Now let $\mathbf{n} \in \mathbb{Z}^{\#\operatorname{dir}(\mathfrak{b})}$ be a maximal member of $\bigcup_{i\in\mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{b}}(M))$. By Theorem 3.6, there exist $s \in \mathbb{N}_0$ and $\mathfrak{p} \in {}^*\operatorname{Var}(\mathfrak{b})$ such that $\mathbf{n} = \phi(\mathfrak{p};\mathfrak{b})(\mathbf{w})$ for some sth level anchor point $\mathbf{w}$ of $\mathfrak{p}$ for M. Now use Theorem 2.12 repeatedly, in conjunction with a saturated chain (of length t say) of $\mathbb{N}^r_0$-graded prime ideals of R with $\mathfrak{p}$ as its smallest term and a *maximal ideal $\mathfrak{m}$ as its largest term: the conclusion is that there exists $\mathbf{v} \in \operatorname{anch}^{s+t}(\mathfrak{m}, M)$ such that $\phi(\mathfrak{m};\mathfrak{p})(\mathbf{v}) = \mathbf{w}$. Now

$$\mathbf{n} = \phi(\mathfrak{p};\mathfrak{b})(\mathbf{w}) = \phi(\mathfrak{p};\mathfrak{b})\bigl(\phi(\mathfrak{m};\mathfrak{p})(\mathbf{v})\bigr) = \phi(\mathfrak{m};\mathfrak{b})(\mathbf{v}).$$

But, by Theorem 3.6 again, $\mathbf{v}$ is dominated by $\max(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}(H^i_{\mathfrak{m}}(M)))$; it follows that

$$\max\Bigl(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}\bigl(H^i_{\mathfrak{b}}(M)\bigr)\Bigr) \preccurlyeq \max\Bigl(\bigcup_{\mathfrak{m}\in {}^*\operatorname{Var}(\mathfrak{b})\cap {}^*\operatorname{Max}(R)} \bigcup_{i\in\mathbb{N}_0} \phi(\mathfrak{m};\mathfrak{b})\bigl(\operatorname{end}\bigl(H^i_{\mathfrak{m}}(M)\bigr)\bigr)\Bigr).$$

The desired conclusion follows. □

3.12. Corollary. *Let the situation be as in Corollary* 3.11, *but assume in addition that* $(R_{\mathbf{0}}, \mathfrak{m}_{\mathbf{0}})$ *is local and that* $\mathfrak{b}$ *is proper*; *set* $\mathfrak{m} := \mathfrak{m}_{\mathbf{0}} \oplus \mathfrak{c}$, *where* $\mathfrak{c}$ *is as defined in Notation* 3.4. *Then*

$$\max\Bigl(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}\bigl(H^i_{\mathfrak{b}}(M)\bigr)\Bigr) = \max\Bigl(\bigcup_{i\in\mathbb{N}_0} \phi(\mathfrak{m};\mathfrak{b})\bigl(\operatorname{end}\bigl(H^i_{\mathfrak{m}}(M)\bigr)\bigr)\Bigr).$$

In particular,

$$\max\Bigl(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}\bigl(H^i_{\mathfrak{c}}(M)\bigr)\Bigr) = \max\Bigl(\bigcup_{i\in\mathbb{N}_0} \operatorname{end}\bigl(H^i_{\mathfrak{m}}(M)\bigr)\Bigr).$$

4. Some vanishing results for multi-graded components of local cohomology modules

It is well known that, when $r = 1$, if M is a finitely generated $\mathbb{Z}$-graded R-module, then there exists $t \in \mathbb{Z}$ such that $H^i_{R_+}(M)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geqslant t$; it then follows from [15, Corollary 2.5] that, if $\mathfrak{b}$ is any graded ideal of R with $\mathfrak{b} \supseteq R_+$, then $H^i_{\mathfrak{b}}(M)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geqslant t$. One of the aims of this section is to establish a multi-graded analogue of this result.

4.1. Notation. Throughout this section, we shall be concerned with the situation where

$$R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$$

is positively graded; we shall only assume that R is standard when this is explicitly stated.

We shall be concerned with the $\mathbb{N}_0^r$-graded ideal R_+ of R given (see Notation 3.4) by

$$R_+ := \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}_0^r \\ \mathbf{n} \geqslant \mathbf{1}}} R_{\mathbf{n}}.$$

Although it is well known (see Hyry [10, Theorem 1.6]) that, if M is a finitely generated $\mathbb{Z}^r$-graded R-module, then $H^i_{R_+}(M)_{(n_1,\dots,n_r)} = 0$ for all $n_1, \dots, n_r \gg 0$, we have not been able to find in the literature a proof of the corresponding statement with R_+ replaced by an $\mathbb{N}_0^r$-graded ideal $\mathfrak{b}$ that contains R_+. We present such a proof below, because we think it is of interest in its own right.

4.2. Theorem. *Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded; let M be a finitely generated $\mathbb{Z}^r$-graded R-module. Let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R such that $\mathfrak{b} \supseteq R_+$. Then there exists $t \in \mathbb{Z}$ such that*

$$H^i_{\mathfrak{b}}(M)_{\mathbf{n}} = 0 \quad \textit{for all } i \in \mathbb{N}_0 \textit{ and all } \mathbf{n} \geqslant (t, t, \dots, t).$$

Proof. We shall prove this by induction on r. In the case where $r = 1$ the result follows from [15, Corollary 2.5], as was explained in the introduction to this section.

Now suppose that $r > 1$ and that the claim has been proved for smaller values of r. We define three more $\mathbb{N}_0^r$-graded ideals $\mathfrak{a}$, $\mathfrak{c}$ and $\mathfrak{d}$ of R, as follows. Set

$$\mathfrak{a} := \bigoplus_{\mathbf{n}=(n_1,\dots,n_r) \in \mathbb{N}_0^r} \mathfrak{a}_{\mathbf{n}} \quad \text{where } \mathfrak{a}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } n_r = 0, \\ R_{\mathbf{n}} & \text{if } n_r > 0; \end{cases}$$

$$\mathfrak{c} := \bigoplus_{\mathbf{n}=(n_1,\dots,n_r) \in \mathbb{N}_0^r} \mathfrak{c}_{\mathbf{n}} \quad \text{where } \mathfrak{c}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } (n_1, \dots, n_{r-1}) \ngeqslant (1, \dots, 1), \\ R_{\mathbf{n}} & \text{if } (n_1, \dots, n_{r-1}) \geqslant (1, \dots, 1); \end{cases}$$

and $\mathfrak{d} := \mathfrak{a} + \mathfrak{c}$.

Consider $\mathfrak{a} \cap \mathfrak{c}$: for each $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, the $\mathbf{n}$th component $(\mathfrak{a} \cap \mathfrak{c})_{\mathbf{n}}$ satisfies

$$(\mathfrak{a} \cap \mathfrak{c})_{\mathbf{n}} = \mathfrak{a}_{\mathbf{n}} \cap \mathfrak{c}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } n_r = 0 \text{ or } (n_1, \dots, n_{r-1}) \ngeqslant (1, \dots, 1), \\ R_{\mathbf{n}} & \text{if } n_r > 0 \text{ and } (n_1, \dots, n_{r-1}) \geqslant (1, \dots, 1). \end{cases}$$

Since $\mathfrak{b} \supseteq R_+$, we see that $\mathfrak{a} \cap \mathfrak{c} = \mathfrak{b}$.

Let $\sigma : \mathbb{Z}^r \to \mathbb{Z}^{r-1}$ be the group homomorphism defined by

$$\sigma((n_1, \dots, n_r)) = (n_1 + n_r, \dots, n_{r-1} + n_r) \quad \text{for all } (n_1, \dots, n_r) \in \mathbb{Z}^r.$$

Note that, for $(n_1, \dots, n_r) \in \mathbb{N}_0^r$, we have $(n_1 + n_r, \dots, n_{r-1} + n_r) \geqslant \mathbf{1}$ in $\mathbb{Z}^{r-1}$ if and only if $n_r \geqslant 1$ or $(n_1, \dots, n_{r-1}) \geqslant \mathbf{1}$; furthermore, if $n_r \geqslant 1$, then $\mathfrak{a}_{\mathbf{n}} = R_{\mathbf{n}}$, and if $(n_1, \dots, n_{r-1}) \geqslant \mathbf{1}$, then $\mathfrak{c}_{\mathbf{n}} = R_{\mathbf{n}}$. Let $\mathbf{m} \in \mathbb{Z}^{r-1}$ with $\mathbf{m} \geqslant \mathbf{1}$. Therefore, in the $\mathbb{N}_0^{r-1}$-graded ring R^σ, we have

$$\left(\mathfrak{d}^\sigma\right)_{\mathbf{m}} = \bigoplus_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ \sigma(\mathbf{n}) = \mathbf{m}}} (\mathfrak{a}_{\mathbf{n}} + \mathfrak{c}_{\mathbf{n}}) = \bigoplus_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ \sigma(\mathbf{n}) = \mathbf{m}}} R_{\mathbf{n}} = \left(R^\sigma\right)_{\mathbf{m}}.$$

Thus $\mathfrak{d}^\sigma \supseteq \bigoplus_{\mathbf{m}\geqslant \mathbf{1}} (R^\sigma)_{\mathbf{m}} = (R^\sigma)_+$.

It therefore follows from the inductive hypothesis that there exists $\tilde{t} \in \mathbb{Z}$ such that $(H^j_{\mathfrak{d}^\sigma}(M^\sigma))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geqslant (\tilde{t}, \dots, \tilde{t})$ in $\mathbb{Z}^{r-1}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{d}}(M))^\sigma)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geqslant (\tilde{t}, \dots, \tilde{t})$ in $\mathbb{Z}^{r-1}$, so that, for all $j \in \mathbb{N}_0$,

$$H^j_{\mathfrak{d}}(M)_{(n_1,\dots,n_r)} = 0 \quad \text{whenever } (n_1, \dots, n_{r-1}, n_r) \geqslant \left(\tfrac{1}{2}\tilde{t}, \dots, \tfrac{1}{2}\tilde{t}, \tfrac{1}{2}\tilde{t}\right) \text{ in } \mathbb{Z}^r.$$

We now give two similar, but simpler, arguments. Let $\pi : \mathbb{Z}^r \to \mathbb{Z}$ be the group homomorphism given by projection onto the rth co-ordinate. Note that, for $\mathbf{n} \in \mathbb{N}_0^r$, if $\pi(\mathbf{n}) \geqslant 1$, then $\mathfrak{a}_{\mathbf{n}} = R_{\mathbf{n}}$. Therefore $\mathfrak{a}^\pi \supseteq (R^\pi)_+$. It therefore follows from the case where $r = 1$ that there exists $\bar{t} \in \mathbb{Z}$ such that $(H^j_{\mathfrak{a}^\pi}(M^\pi))_n = 0$ for all $j \in \mathbb{N}_0$ and all $n \geqslant \bar{t}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{a}}(M))^\pi)_n = 0$ for all $j \in \mathbb{N}_0$ and all $n \geqslant \bar{t}$, that is,

$$H^j_{\mathfrak{a}}(M)_{(n_1,\dots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } n_r \geqslant \bar{t}.$$

Next, let $\theta : \mathbb{Z}^r \to \mathbb{Z}^{r-1}$ be the group homomorphism defined by

$$\theta((n_1, \dots, n_r)) = (n_1, \dots, n_{r-1}) \quad \text{for all } (n_1, \dots, n_r) \in \mathbb{Z}^r.$$

Note that, if $\mathbf{n} \in \mathbb{Z}^r$ has $\theta(\mathbf{n}) \geqslant \mathbf{1}$ in $\mathbb{Z}^{r-1}$, then $\mathfrak{c}_{\mathbf{n}} = R_{\mathbf{n}}$. Therefore, for $\mathbf{m} \in \mathbb{Z}^{r-1}$ with $\mathbf{m} \geqslant \mathbf{1}$, we have $(\mathfrak{c}^\theta)_{\mathbf{m}} = (R^\theta)_{\mathbf{m}}$. This means that, in the $\mathbb{N}_0^{r-1}$-graded ring R^θ, we have $\mathfrak{c}^\theta \supseteq \bigoplus_{\mathbf{m}\geqslant \mathbf{1}} (R^\theta)_{\mathbf{m}} = (R^\theta)_+$.

It therefore follows from the inductive hypothesis that there exists $\hat{t} \in \mathbb{Z}$ such that $(H^j_{\mathfrak{c}^\theta}(M^\theta))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geqslant (\hat{t}, \dots, \hat{t})$ in $\mathbb{Z}^{r-1}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{c}}(M))^\theta)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geqslant (\hat{t}, \dots, \hat{t})$ in $\mathbb{Z}^{r-1}$, so that

$$H^j_{\mathfrak{c}}(M)_{(n_1,\dots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \dots, n_{r-1}) \geqslant (\hat{t}, \dots, \hat{t}).$$

We recall that $\mathfrak{a} \cap \mathfrak{c} = \mathfrak{b}$. There is an exact Mayer–Vietoris sequence (in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$)

$$\begin{aligned}
0 \longrightarrow H^0_{\mathfrak{d}}(M) &\longrightarrow H^0_{\mathfrak{c}}(M) \oplus H^0_{\mathfrak{a}}(M) \longrightarrow H^0_{\mathfrak{b}}(M) \\
\longrightarrow H^1_{\mathfrak{d}}(M) &\longrightarrow H^1_{\mathfrak{c}}(M) \oplus H^1_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathfrak{b}}(M) \\
\longrightarrow \cdots & \\
\longrightarrow H^i_{\mathfrak{d}}(M) &\longrightarrow H^i_{\mathfrak{c}}(M) \oplus H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{b}}(M) \\
\longrightarrow H^{i+1}_{\mathfrak{d}}(M) &\longrightarrow \cdots.
\end{aligned}$$

It now follows from this Mayer–Vietoris sequence that, if we set $t := \max\{\frac{1}{2}\tilde{t}, \hat{t}, \bar{t}\}$, then

$$H^j_{\mathfrak{b}}(M)_{(n_1,\dots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \dots, n_r) \geqslant (t, \dots, t).$$

This completes the inductive step, and the proof. $\square$

We can deduce from the above Theorem 4.2 a vanishing result for multi-graded components of local cohomology modules with respect to a multi-graded ideal that has both directions and non-directions.

4.3. Corollary. *Suppose that $R = \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard; let M be a finitely generated $\mathbb{Z}^r$-graded R-module. Let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R that has some directions and some non-directions: to be* *precise, and for ease of notation, suppose that* $\operatorname{dir}(\mathfrak{b}) = \{m+1,\dots,r\}$*, where* $1 \leqslant m < r$*. Then there exists* $t \in \mathbb{Z}$ *such that, for all* $j \in \mathbb{N}_0$*, and for all* $\mathbf{n} = (n_1,\dots,n_r) \in \mathbb{Z}^r$ *for which* $(n_1,\dots,n_m,n_{m+1}+\dots+n_r) \geqslant (t,\dots,t)$ *in* $\mathbb{Z}^{m+1}$*, we have* $H^j_{\mathfrak{b}}(M)_{\mathbf{n}} = 0$*.*

Note. As $\mathfrak{b}$ has some directions and R is standard, it follows from Lemma 2.2 that $R_+ \subseteq \mathfrak{b}$, so that Theorem 4.2 yields a $t' \in \mathbb{Z}$ such that $H^i_{\mathfrak{b}}(M)_{\mathbf{n}} = 0$ for all $\mathbf{n} \geqslant (t',\dots,t')$. Thus, when $m = r-1$, the conclusion of Corollary 4.3 already follows from Theorem 4.2.

Proof. Without loss of generality, we can, and do, assume that $\mathfrak{b} = \sqrt{\mathfrak{b}}$.

Let $\phi : \mathbb{Z}^r \to \mathbb{Z}^{m+1}$ be the group homomorphism defined by

$$\phi((n_1,\dots,n_r)) = (n_1,\dots,n_m,n_{m+1}+\dots+n_r) \quad \text{for all } (n_1,\dots,n_r) \in \mathbb{Z}^r.$$

Let $\mathbf{n} = (n_1,\dots,n_r) \in \mathbb{N}_0^r$ be such that $\phi(\mathbf{n}) \geqslant \mathbf{1}$ in $\mathbb{Z}^{m+1}$. Then $n_{m+1}+\dots+n_r \geqslant 1$, so that one of $n_{m+1},\dots,n_r$ is positive. Now $R_{\mathbf{e}_i} \subseteq \sqrt{\mathfrak{b}} = \mathfrak{b}$ for all $i = m+1,\dots,r$, and since $\mathbf{n} \geqslant \mathbf{e}_i$ for one of these is, it follows from Lemma 2.2 that $\mathfrak{b} \supseteq R_{\mathbf{n}}$. It therefore follows that, in the $\mathbb{N}_0^{m+1}$-graded ring R^ϕ, we have $\mathfrak{b}^\phi \supseteq \bigoplus_{\mathbf{m}\geqslant\mathbf{1}} (R^\phi)_{\mathbf{m}} = (R^\phi)_+$.

We can now appeal to Theorem 4.2 to deduce that there exists $t \in \mathbb{Z}$ such that $(H^j_{\mathfrak{b}^\phi}(M^\phi))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geqslant (t,\dots,t)$ in $\mathbb{Z}^{m+1}$. In view of Remark 3.3, this means that $((H^j_{\mathfrak{b}}(M))^\phi)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geqslant (t,\dots,t)$ in $\mathbb{Z}^{m+1}$, so that

$$H^j_{\mathfrak{b}}(M)_{(n_1,\dots,n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1,\dots,n_m,n_{m+1}+\dots+n_r) \geqslant (t,\dots,t). \qquad \square$$

One of the reasons why we consider that Theorem 4.2 is of interest in its own right concerns the structure of the (multi-)graded components $H^i_{\mathfrak{b}}(M)_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{Z}^r$) as modules over $R_{\mathbf{0}}$ (the hypotheses and notation here are as in Theorem 4.2). The example in [4, Exercise 15.1.7] shows that these graded components need not be finitely generated $R_{\mathbf{0}}$-modules; however, it is always the case that (for a finitely generated $\mathbb{Z}^r$-graded R-module M) the (multi-)graded components $H^i_{R_+}(M)_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{Z}^r$) of the ith local cohomology module of M with respect to R_+ are finitely generated $R_{\mathbf{0}}$-modules (for all $i \in \mathbb{N}_0$), as we now show.

4.4. Theorem. *Suppose that* $R = \bigoplus_{\mathbf{n}\in\mathbb{N}_0^r} R_{\mathbf{n}}$ *is positively graded; let* M *be a finitely generated* $\mathbb{Z}^r$*-graded* R*-module. Then* $H^i_{R_+}(M)_{\mathbf{n}}$ *is a finitely generated* $R_{\mathbf{0}}$*-module, for all* $i \in \mathbb{N}_0$ *and all* $\mathbf{n} \in \mathbb{Z}^r$*.*

Note. In the case where $r = 1$, this result is well known: see [4, Proposition 15.1.5].

Proof. We use induction on i. When $i = 0$, the claim is immediate from the fact that $H^0_{R_+}(M)$ is isomorphic to a submodule of M, and so is finitely generated. So suppose that $i > 0$ and that the claim has been proved for smaller values of i, for all finitely generated $\mathbb{Z}^r$-graded R-modules.

Recall that all the associated prime ideals of M are $\mathbb{N}_0^r$-graded. Set $B(M) := \operatorname{Ass}_R(M) \setminus {}^*\operatorname{Var}(R_+)$, and denote $\#B(M)$ by $b(M)$; we shall argue by induction on $b(M)$. If $b(M) = 0$, then M is R_+-torsion, so that $H^i_{R_+}(M) = 0$ and the desired result is clear in this case.

Now suppose that $b(M) = 1$: let $\mathfrak{p}$ be the unique member of $B(M)$. Set $\overline{M} := M/\Gamma_{R_+}(M)$. We can use the long exact sequence of local cohomology modules induced by the exact sequence

$$0 \longrightarrow \Gamma_{R_+}(M) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0,$$

together with the fact that $H^j_{R_+}(\Gamma_{R_+}(M)) = 0$ for all $j \in \mathbb{N}$, to see that, in order to complete the proof in this case, it is sufficient for us to prove the result for $\overline{M}$. Now $\overline{M}$ is R_+-torsion-free, and $\operatorname{Ass}(\overline{M}) = \{\mathfrak{p}\}$. (See [4, Exercise 2.1.12].) There exists a $\mathbb{Z}^r$-homogeneous element $a \in R_+ \setminus \mathfrak{p}$; note that

a is a non-zero-divisor on $\overline{M}$. Let the degree of a be $\mathbf{v} = (v_1, \dots, v_r)$, and note that $v_j > 0$ for all $j = 1, \dots, r$. By Theorem 4.2, there exists $t \in \mathbb{Z}$ such that $H^j_{R_+}(\overline{M})_{\mathbf{n}} = 0$ for all $\mathbf{n} \geqslant (t, t, \dots, t)$.

Let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$. Since $v_j > 0$ for all $j = 1, \dots, r$, there exists $w \in \mathbb{N}$ such that $n_j + v_j w \geqslant t$ for all $j = 1, \dots, r$. The exact sequence

$$0 \longrightarrow \overline{M} \xrightarrow{a^w} \overline{M}(w\mathbf{v}) \longrightarrow \big(\overline{M}/a^w\overline{M}\big)(w\mathbf{v}) \longrightarrow 0$$

induces an exact sequence of $R_{\mathbf{0}}$-modules

$$H^{i-1}_{R_+}\big(\overline{M}/a^w\overline{M}\big)_{\mathbf{n}+w\mathbf{v}} \longrightarrow H^i_{R_+}(\overline{M})_{\mathbf{n}} \longrightarrow H^i_{R_+}(\overline{M})_{\mathbf{n}+w\mathbf{v}},$$

and since w was chosen to ensure that the rightmost term in this sequence is zero, it follows from the inductive hypothesis that $H^i_{R_+}(\overline{M})_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$-module. This completes the proof in the case where $b(M) = 1$.

Now suppose that $b(M) = b > 1$ and that it has been proved that all the graded components of $H^i_{R_+}(L)$ are finitely generated $R_{\mathbf{0}}$-modules for all choices of finitely generated $\mathbb{Z}^r$-graded R-module L with $b(L) < b$. Let $\mathfrak{p}, \mathfrak{q} \in B(M)$ with $\mathfrak{p} \neq \mathfrak{q}$: suppose, for the sake of argument, that $\mathfrak{p} \not\subseteq \mathfrak{q}$. Consider the $\mathfrak{p}$-torsion submodule $\Gamma_{\mathfrak{p}}(M)$ of M. By [4, Exercise 2.1.12], $\operatorname{Ass}(\Gamma_{\mathfrak{p}}(M))$ and $\operatorname{Ass}(M/\Gamma_{\mathfrak{p}}(M))$ are disjoint and $\operatorname{Ass} M = \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M)) \cup \operatorname{Ass}(M/\Gamma_{\mathfrak{p}}(M))$. Now $\mathfrak{p} \in \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M))$ and $\mathfrak{q} \notin \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M))$; hence $b(\Gamma_{\mathfrak{p}}(M)) < b$ and $b(M/\Gamma_{\mathfrak{p}}(M)) < b$. Therefore, by the inductive hypothesis, both $H^i_{R_+}(\Gamma_{\mathfrak{p}}(M))_{\mathbf{n}}$ and $H^i_{R_+}(M/\Gamma_{\mathfrak{p}}(M))_{\mathbf{n}}$ are finitely generated $R_{\mathbf{0}}$-modules, for all $\mathbf{n} \in \mathbb{Z}^r$. We can now use the long exact sequence of local cohomology modules (with respect to R_+) induced from the exact sequence

$$0 \to \Gamma_{\mathfrak{p}}(M) \to M \to M/\Gamma_{\mathfrak{p}}(M) \to 0$$

to deduce that $H^i_{R_+}(M)_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$-module for all $\mathbf{n} \in \mathbb{Z}^r$. The result follows. □

5. A multi-graded analogue of Marley's work on finitely graded local cohomology modules

As was mentioned in the Introduction, the purpose of this section is to obtain some multi-graded analogues of results about finitely graded local cohomology modules that were proved, in the case where $r = 1$, by Marley in [14]. We shall present a multi-graded analogue of one of Marley's results and some extensions of that analogue.

5.1. Notation. Throughout this section, we shall be concerned with the situation where $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and we shall let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a $\mathbb{Z}^r$-graded R-module. Also, $\mathfrak{b}$ will always denote an $\mathbb{N}_0^r$-graded ideal of R.

For $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, we shall denote $\{i \in \{1, \dots, r\}\colon\ n_i \neq 0\}$ by $\mathcal{P}(\mathbf{n})$.

5.2. Definition. An r-tuple $\mathbf{n} \in \mathbb{Z}^r$ is called a *supporting degree of* M precisely when $M_{\mathbf{n}} \neq 0$; we denote the set of all supporting degrees of M by $\mathcal{S}(M)$.

Note that Theorem 4.2 imposes substantial restrictions on $\mathcal{S}(H^i_{\mathfrak{b}}(M))$ when ($i \in \mathbb{N}_0$ and) $\mathfrak{b} \supseteq R_+$. The example below is included as motivation for the introduction of some notation.

5.3. Example. Let k be an algebraically closed field and let

$$A = k \oplus A_1 \oplus \cdots \oplus A_m \oplus \cdots \quad \text{and} \quad B = k \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots$$

be two normal Noetherian standard $\mathbb{N}_0$-graded k-algebra domains with $w := \dim A > 1$ and $v := \dim B > 1$. We consider the $\mathbb{N}_0^2$-graded k-algebra

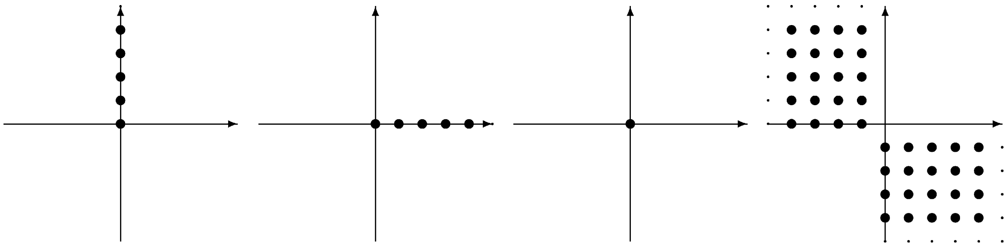

Fig. 1. $\mathcal{S}(H^i_{R_+}(R))$ for $i = 2, 3, 4, 5$ respectively.

$$R := A \otimes_k B = \bigoplus_{(m,n)\in\mathbb{N}_0^2} A_m \otimes_k B_n.$$

Clearly $R = k[R_{(1,0)}, R_{(0,1)}]$ is positively graded and standard, and, as a finitely generated k-algebra, is Noetherian. By [17, Chapter III, §15, Theorem 40, Corollary 1], R is again an integral domain. Observe that $R_+ = R_{(1,1)}R = A_+ \otimes_k B_+$. As A and B are normal and their dimensions exceed 1, we have $H^i_{A_+}(A) = H^i_{B_+}(B) = 0$ for $i = 0, 1$. The Künneth relations for tensor products (see [7] or [13, Theorem 10.1]) yield, for each $i \in \mathbb{N}_0$, an isomorphism of $\mathbb{Z}^2$-graded R modules

$$H^i_{R_+}(R) \cong \big(A \otimes_k H^i_{B_+}(B)\big) \oplus \big(H^i_{A_+}(A) \otimes_k B\big) \oplus \Bigg(\bigoplus_{\substack{j,l\in\mathbb{N}\setminus\{1\}\\ j+l=i+1}} \big(H^j_{A_+}(A) \otimes_k H^l_{B_+}(B)\big)\Bigg).$$

As $\mathcal{S}(A) = \mathcal{S}(B) = \mathbb{N}_0$, it follows that, for each $i \in \mathbb{N}_0$,

$$\mathcal{S}\big(H^i_{R_+}(R)\big) = \big(\mathbb{N}_0 \times \mathcal{S}\big(H^i_{B_+}(B)\big)\big) \cup \big(\mathcal{S}\big(H^i_{A_+}(A)\big) \times \mathbb{N}_0\big) \cup \Bigg(\bigcup_{\substack{j,l\in\mathbb{N}\setminus\{1\}\\ j+l=i+1}} \big(\mathcal{S}\big(H^j_{A_+}(A)\big) \times \mathcal{S}\big(H^l_{B_+}(B)\big)\big)\Bigg).$$

Observe, in particular, that $H^i_{R_+}(R) = 0$ for $i = 0, 1$ and for all $i \geqslant w + v$.

Appropriate choices for A and B yield many examples for R. We shall just concentrate on a class of examples obtained by this procedure when A and B are chosen in a particular way, which we now describe. We can use [2, Proposition (2.13)], in conjunction with the Serre–Grothendieck correspondence (see [4, 20.4.4]), to choose the algebra A (as above) so that, for a prescribed set $W \subseteq \{2, \dots, w-1\}$, we have

$$\mathcal{S}\big(H^i_{A_+}(A)\big) = \begin{cases} \emptyset & \text{for all } i \in \mathbb{N}_0 \setminus \big(W \cup \{w\}\big), \\ \{0\} & \text{for all } i \in W, \\ \{k \in \mathbb{Z}\colon\ k < 0\} & \text{for } i = w. \end{cases}$$

Similarly, for a prescribed set $V \subseteq \{2, \dots, v-1\}$, we choose B (as above) so that

$$\mathcal{S}\big(H^i_{B_+}(B)\big) = \begin{cases} \emptyset & \text{for all } i \in \mathbb{N}_0 \setminus \big(V \cup \{v\}\big), \\ \{0\} & \text{for all } i \in V, \\ \{k \in \mathbb{Z}\colon\ k < 0\} & \text{for } i = v. \end{cases}$$

With such a choice of A for $w = 5$ and $W = \{2\}$, and such a choice of B for $v = 5$ and $V = \{3\}$, the sets of supporting degrees $\mathcal{S}(H^i_{R_+}(R))$ for $i = 2, 3, 4, 5$ are as in Fig. 1.

In view of Theorem 4.2, the supporting set $\mathcal{S}(H^5_{R_+}(R))$ seems unremarkable. The local cohomology module $H^4_{R_+}(R)$ is finitely graded. Although neither $H^3_{R_+}(R)$ nor $H^2_{R_+}(R)$ is finitely graded, both have sets of supporting degrees that are quite restricted.

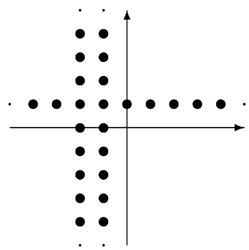

Fig. 2. The set $\mathbb{X}((-2,1),(0,2))$ in $\mathbb{Z}^2$.

We now return to the general situation described in Notation 5.1. In the case where $r=1$, one way of recording that a local cohomology module $H^i_{\mathfrak{b}}(M)$ is finitely graded is to state that there exist $s,t \in \mathbb{Z}$ with $s<t$ such that

$$\mathcal{S}\big(H^i_{\mathfrak{b}}(M)\big) = \big\{n \in \mathbb{Z}\colon\ H^i_{\mathfrak{b}}(M)_n \neq 0\big\} \subseteq \{n \in \mathbb{Z}\colon\ s \leqslant n < t\}.$$

One might expect the natural generalization to our multi-graded situation to involve conditions such as

$$\mathcal{S}\big(H^i_{\mathfrak{b}}(M)\big) = \big\{\mathbf{n} \in \mathbb{Z}^r\colon\ H^i_{\mathfrak{b}}(M)_{\mathbf{n}} \neq 0\big\} \subseteq \big\{\mathbf{n} = (n_1,\ldots,n_r) \in \mathbb{Z}^r\colon\ s_i \leqslant n_i < t_i \text{ for all } i = 1,\ldots,r\big\},$$

where $\mathbf{s} = (s_1,\ldots,s_r)$, $\mathbf{t} = (t_1,\ldots,t_r) \in \mathbb{Z}^r$ satisfy $\mathbf{s} \leqslant \mathbf{t}$. However, in the light of evidence like that provided by Example 5.3 above, and other examples, we introduce the following.

5.4. Notation. Let $\mathbf{s} = (s_1,\ldots,s_r)$, $\mathbf{t} = (t_1,\ldots,t_r) \in \mathbb{Z}^r$ with $\mathbf{s} \leqslant \mathbf{t}$. We set

$$\mathbb{X}(\mathbf{s},\mathbf{t}) := \big\{\mathbf{n} = (n_1,\ldots,n_r) \in \mathbb{Z}^r\colon\ \text{there exists } i \in \{1,\ldots,r\} \text{ such that } s_i \leqslant n_i < t_i\big\}.$$

5.5. Example. Fig. 2 illustrates, in the case where $r=2$, the set $\mathbb{X}((-2,1),(0,2))$.

5.6. Remark. Let $\mathbf{s},\mathbf{s}',\mathbf{s}'',\mathbf{t},\mathbf{t}',\mathbf{t}'' \in \mathbb{Z}^r$ with $\mathbf{s} \leqslant \mathbf{t}$, $\mathbf{s}' \leqslant \mathbf{t}'$ and $\mathbf{s}'' \leqslant \mathbf{t}''$. Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$.

(i) Clearly $(\mathbf{s} + \mathbb{N}_0^r) \setminus (\mathbf{t} + \mathbb{N}_0^r) \subseteq \mathbb{X}(\mathbf{s},\mathbf{t})$.

(ii) Suppose that $\mathcal{P}(\mathbf{t}-\mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$. Let $\mathbf{w} \in \mathbb{Z}^r$ be such that $\mathcal{P}(\mathbf{w}) \subseteq \{1,\ldots,r\} \setminus \mathcal{P}(\mathbf{m})$. Then

$$\mathbb{X}(\mathbf{s}+\mathbf{w},\mathbf{t}+\mathbf{w}) = \mathbb{X}(\mathbf{s},\mathbf{t}) = \big\{\mathbf{n} \in \mathbb{Z}^r\colon\ \text{there exists } i \in \mathcal{P}(\mathbf{m}) \text{ such that } s_i \leqslant n_i < t_i\big\}.$$

(iii) Clearly $\mathbb{X}(\mathbf{s}',\mathbf{t}') \cup \mathbb{X}(\mathbf{s}'',\mathbf{t}'') \subseteq \mathbb{X}((\min\{s'_1,s''_1\},\ldots,\min\{s'_r,s''_r\}),(\max\{t'_1,t''_1\},\ldots,\max\{t'_r,t''_r\}))$.

(iv) Assume that $\mathcal{P}(\mathbf{t}'-\mathbf{s}') \subseteq \mathcal{P}(\mathbf{m})$ and $\mathcal{P}(\mathbf{t}''-\mathbf{s}'') \subseteq \mathcal{P}(\mathbf{m})$. For each $i \in \{1,\ldots,r\}$, set

$$\tilde{s}_i := \min\big\{s'_i, s''_i\big\} \quad \text{and} \quad \tilde{t}_i := \begin{cases} \max\{t'_i, t''_i\} & \text{if } i \in \mathcal{P}(\mathbf{m}), \\ \tilde{s}_i & \text{if } i \in \{1,\ldots,r\} \setminus \mathcal{P}(\mathbf{m}). \end{cases}$$

Set $\tilde{\mathbf{s}} := (\tilde{s}_1,\ldots,\tilde{s}_r)$ and $\tilde{\mathbf{t}} := (\tilde{t}_1,\ldots,\tilde{t}_r)$. Then

$$\tilde{\mathbf{s}} \leqslant \tilde{\mathbf{t}}, \qquad \mathcal{P}(\tilde{\mathbf{t}}-\tilde{\mathbf{s}}) \subseteq \mathcal{P}(\mathbf{m}) \quad \text{and} \quad \mathbb{X}(\mathbf{s}',\mathbf{t}') \cup \mathbb{X}(\mathbf{s}'',\mathbf{t}'') \subseteq \mathbb{X}(\tilde{\mathbf{s}},\tilde{\mathbf{t}}).$$

The next lemma provides a small hint about the importance of the sets $\mathbb{X}(\mathbf{s},\mathbf{t})$ of Notation 5.4 for our work.

5.7. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$. Assume that M is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Then there exist $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ such that $\mathbf{s} \leqslant \mathbf{t}$, $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$ and $\mathcal{S}(M) \subseteq (\mathbf{s} + \mathbb{N}_0^r) \setminus (\mathbf{t} + \mathbb{N}_0^r)$, so that $\mathcal{S}(M) \subseteq \mathbb{X}(\mathbf{s}, \mathbf{t})$ in view of Remark* 5.6(i).

Proof. As M is finitely generated, there exist $\mathbf{s}, \mathbf{w} \in \mathbb{Z}^r$ such that $\mathbf{s} \leqslant \mathbf{w}$ and $M = \sum_{\mathbf{s} \leqslant \mathbf{n} \leqslant \mathbf{w}} RM_{\mathbf{n}}$. In particular, $\mathcal{S}(M) \subseteq \mathbf{s} + \mathbb{N}_0^r$.

Moreover, there exists $u \in \mathbb{N}$ such that $(R_{\mathbf{m}})^u \subseteq (0 :_R M)$; since R is standard, $(R_{\mathbf{m}})^u = R_{u\mathbf{m}}$; hence $R_{u\mathbf{m}} M_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^r$.

Let $\mathbf{t} = \mathbf{s} + \sum_{i \in \mathcal{P}(\mathbf{m})} (w_i - s_i + um_i)\mathbf{e}_i$. Now, let $\mathbf{h} = (h_1, \ldots, h_r) \in \mathbf{t} + \mathbb{N}_0^r$. Our proof will be complete once we have shown that $M_{\mathbf{h}} = 0$. For each $i \in \mathcal{P}(\mathbf{m})$, we have $h_i \geqslant t_i = w_i + um_i$. Moreover,

$$M_{\mathbf{h}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}} M_{\mathbf{n}}, \quad \text{where } \mathcal{T} = \{\mathbf{n} \in \mathbb{Z}^r \colon\ \mathbf{s} \leqslant \mathbf{n} \leqslant \mathbf{w},\ \mathbf{n} \leqslant \mathbf{h}\}.$$

Let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathcal{T}$. If $i \in \mathcal{P}(\mathbf{m})$, then $n_i + um_i \leqslant w_i + um_i \leqslant h_i$; if $i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{m})$, then $n_i + um_i = n_i \leqslant h_i$. Consequently $\mathbf{n} + u\mathbf{m} \leqslant \mathbf{h}$. Therefore $u\mathbf{m} \leqslant \mathbf{h} - \mathbf{n}$ for all $\mathbf{n} \in \mathcal{T}$, and hence

$$M_{\mathbf{h}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}} M_{\mathbf{n}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}-u\mathbf{m}} R_{u\mathbf{m}} M_{\mathbf{n}} = 0. \qquad \square$$

5.8. Definition. Let $\mathcal{Q} \subseteq \{1, \ldots, r\}$. By a *$\mathcal{Q}$-domain in $\mathbb{Z}^r$* we mean a set of the form

$$\mathbb{X}(\mathbf{s}, \mathbf{t}) \quad \text{with } \mathbf{s}, \mathbf{t} \in \mathbb{Z}^r,\ \mathbf{s} \leqslant \mathbf{t} \text{ and } \mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{Q}.$$

5.9. Remarks. The following statements are immediate from the definition.

(i) A $\emptyset$-domain in $\mathbb{Z}^r$ is empty.
(ii) If $\mathcal{Q} \subseteq \mathcal{Q}' \subseteq \{1, \ldots, r\}$ and if $\mathbb{X}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$, then $\mathbb{X}$ is a $\mathcal{Q}'$-domain in $\mathbb{Z}^r$.
(iii) If $\mathbb{X}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$ and $\mathbf{w} \in \mathbb{Z}^r$, then $\mathbf{w} + \mathbb{X} := \{\mathbf{w} + \mathbf{n} \colon\ \mathbf{n} \in \mathbb{X}\}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$.
(iv) If $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ with $\mathbf{s} \leqslant \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{Q}$, then $(\mathbf{s} + \mathbb{N}_0^r) \setminus (\mathbf{t} + \mathbb{N}_0^r)$ is contained in a $\mathcal{Q}$-domain in $\mathbb{Z}^r$, by Remark 5.6(i).
(v) If $\mathbb{X}$ is a $\mathcal{Q}$-domain in $\mathbb{Z}^r$ and $\mathbf{w} \in \mathbb{Z}^r$ is such that $\mathcal{P}(\mathbf{w}) \cap \mathcal{Q} = \emptyset$, then $\mathbb{X} = \mathbf{w} + \mathbb{X}$, by Remark 5.6(ii).
(vi) By Remark 5.6(iv), the union of finitely many $\mathcal{Q}$-domains in $\mathbb{Z}^r$ is contained in a $\mathcal{Q}$-domain in $\mathbb{Z}^r$.

5.10. Lemma. *Let $\mathbf{m}, \mathbf{k} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$, and let T be a $\mathbb{Z}^r$-graded R-module such that $R_{\mathbf{m}} T = 0$. Let $y \in R_{\mathbf{k}}$, and let K denote the kernel of the homogeneous R-homomorphism $T \to T(\mathbf{k})$ given by multiplication by y.*

(i) *If $\mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$, then there exists $v \in \mathbb{N}_0$ such that $\mathcal{S}(T) \subseteq \bigcup_{j=0}^{v} (\mathcal{S}(K) - j\mathbf{k})$.*
(ii) *If $\mathcal{P}(\mathbf{m}) \nsubseteq \mathcal{P}(\mathbf{k})$, if multiplication by y provides an isomorphism $T \xrightarrow{\cong} T(\mathbf{k})$, and if T considered as an R_y-module is finitely generated, then $\mathcal{S}(T)$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain in $\mathbb{Z}^r$.*

Proof. Write $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mathbf{k} = (k_1, \ldots, k_r)$. Let $u \in \mathbb{N}$ be such that $m_i \leqslant uk_i$ for all $i \in \mathcal{P}(\mathbf{k})$. Set $\mathbf{h} := \sum_{i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{k})} m_i \mathbf{e}_i$. Then, if $i \in \mathcal{P}(\mathbf{k})$, we have $(u\mathbf{k} + \mathbf{h})_i = uk_i \geqslant m_i$, whereas, if $i \in \{1, \ldots, r\} \setminus \mathcal{P}(\mathbf{k})$, we have $(u\mathbf{k} + \mathbf{h})_i = uk_i + m_i \geqslant m_i$. Therefore $\mathbf{m} \leqslant u\mathbf{k} + \mathbf{h}$.

Now, let $z \in R_{\mathbf{h}}$. Then, because R is standard, $y^u z \in R_{u\mathbf{k}+\mathbf{h}} = R_{u\mathbf{k}+\mathbf{h}-\mathbf{m}} R_{\mathbf{m}}$. As $R_{\mathbf{m}} T = 0$, it follows that $y^u z T = 0$. Therefore $y^u R_{\mathbf{h}} T = 0$.

(i) Assume that $\mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$. Then $\mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}) = \emptyset$, so that $\mathbf{h} = \mathbf{0}$. Hence $y^u T = y^u R_{\mathbf{0}} T = 0$.

Now let $\mathbb{K} := \bigcup_{j=0}^{u-1} (\mathcal{S}(K) - j\mathbf{k})$, and let $\mathbf{n} \in \mathbb{Z}^r \setminus \mathbb{K}$. If we show that $T_{\mathbf{n}} = 0$, then we shall have proved part (i). Now $\mathbf{n} + j\mathbf{k} \notin \mathcal{S}(K)$ for all $j \in \{0, \ldots, u-1\}$, and so the $R_{\mathbf{0}}$-homomorphism $y^u : T_{\mathbf{n}} \to T_{\mathbf{n}+u\mathbf{k}}$, which is the composition of the $R_{\mathbf{0}}$-homomorphisms $y : T_{\mathbf{n}+j\mathbf{k}} \to T_{\mathbf{n}+(j+1)\mathbf{k}}$ for $j = 0, \ldots, u-1$, is injective. But $y^u T_{\mathbf{n}} = 0$, and so $T_{\mathbf{n}} = 0$.

(ii) Now assume that $\mathcal{P}(\mathbf{m}) \nsubseteq \mathcal{P}(\mathbf{k})$, that multiplication by y provides an isomorphism $T \xrightarrow{\cong} T(\mathbf{k})$, and that T considered as an R_y-module is finitely generated. As $y^u R_{\mathbf{h}} T = 0$, it follows that $R_{\mathbf{h}} T = 0$.

As T is finitely generated over R_y, there are finitely many r-tuples $\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(q)} \in \mathbb{Z}^r$ such that $T = \sum_{j=1}^{q} R_y T_{\mathbf{g}^{(j)}}$. Now, for $i \in \{1, \ldots, r\}$, set

$$s_i := \begin{cases} 0 & \text{if } i \notin \mathcal{P}(\mathbf{h}), \\ \min\{g_i^{(j)} : j = 1, \ldots, q\} & \text{if } i \in \mathcal{P}(\mathbf{h}), \end{cases} \qquad t_i := \begin{cases} 0 & \text{if } i \notin \mathcal{P}(\mathbf{h}), \\ \max\{g_i^{(j)} : j = 1, \ldots, q\} + h_i & \text{if } i \in \mathcal{P}(\mathbf{h}), \end{cases}$$

and put $\mathbf{s} = (s_1, \ldots, s_r)$, $\mathbf{t} = (t_1, \ldots, t_r)$. Then $\mathbf{s} \leqslant \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) = \mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k})$. Let $\mathbf{n} \in \mathbb{Z}^r \setminus \mathbb{X}(\mathbf{s}, \mathbf{t})$. If we show that $T_{\mathbf{n}} = 0$, then we shall have proved part (ii). Let $\alpha \in T_{\mathbf{n}}$. There exist integers $v_1, \ldots, v_q$ such that $\alpha \in \sum_{j=1}^{q} y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} T_{\mathbf{g}^{(j)}}$.

Note that, for each $i \in \mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k})$, we have either $n_i < s_i$ or $t_i \leqslant n_i$ (because $\mathbf{n} \notin \mathbb{X}(\mathbf{s}, \mathbf{t})$).

Assume first that there is some $i \in \mathcal{P}(\mathbf{h})$ with $n_i < s_i$. As $i \notin \mathcal{P}(\mathbf{k})$, it follows that

$$\big(\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}\big)_i = n_i - v_j k_i - g_i^{(j)} = n_i - g_i^{(j)} < s_i - g_i^{(j)} \leqslant 0,$$

for all $j \in \{1, \ldots, q\}$, so that $R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} = 0$ and $\alpha = 0$.

Therefore, we can, and do, assume that $t_i \leqslant n_i$ for all $i \in \mathcal{P}(\mathbf{h})$. In this case, for each $i \in \mathcal{P}(\mathbf{h})$ and each $j \in \{1, \ldots, q\}$, we have

$$\big(\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}\big)_i = n_i - v_j k_i - g_i^{(j)} = n_i - g_i^{(j)} \geqslant t_i - g_i^{(j)} \geqslant h_i.$$

Therefore, for each $j \in \{1, \ldots, q\}$, either $\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} \geqslant \mathbf{h}$, or $\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}$ has a negative component and $R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} = 0$. This means that

$$\alpha \in \sum_{j=1}^{q} y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} T_{\mathbf{g}^{(j)}} = \sum_{\substack{j=1 \\ \mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} \geqslant \mathbf{0}}}^{q} y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} - \mathbf{h}} R_{\mathbf{h}} T_{\mathbf{g}^{(j)}} = 0.$$

It follows that $T_{\mathbf{n}} = 0$, as required. $\square$

5.11. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ and $\mathbf{k} \in \mathbb{N}_0^r$. Assume that M is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Let $y \in R_{\mathbf{k}}$. Then there exists a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain $\mathbb{X}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^1_{yR}(M)) \subseteq \mathbb{X}$.*

Proof. Assume first that $\mathbf{k} = \mathbf{0}$. Then $\mathcal{P}(\mathbf{k}) = \emptyset$ and, by the multi-graded analogue of [4, Lemma 13.1.10], there are $R_{\mathbf{0}}$-isomorphisms $H^1_{yR}(M)_{\mathbf{n}} \cong H^1_{yR_{\mathbf{0}}}(M_{\mathbf{n}})$ for all $\mathbf{n} \in \mathbb{Z}^r$. Therefore $\mathcal{S}(H^1_{yR}(M)) \subseteq \mathcal{S}(M)$, and the claim follows in this case from Lemma 5.7.

We now deal with the remaining case, where $\mathbf{k} \neq \mathbf{0}$. Since (by the multi-graded analogue of [4, 12.4.2]) there is a $\mathbb{Z}^r$-homogeneous epimorphism of $\mathbb{Z}^r$-graded R-modules $D_{yR}(M) \to H^1_{yR}(M)$, it suffices for us to show that $\mathcal{S}(D_{yR}(M))$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain in $\mathbb{Z}^r$.

Recall that there is a homogeneous isomorphism $D_{yR}(M) \cong M_y$, and so the multiplication map $y : D_{yR}(M) \to D_{yR}(M)(\mathbf{k})$ is an isomorphism, and $D_{yR}(M)$ is finitely generated as an R_y-module. Since $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$, there exists $u \in \mathbb{N}$ such that $R_{u\mathbf{m}} M = 0$, so that $R_{u\mathbf{m}} M_y = 0$ and $R_{u\mathbf{m}} D_{yR}(M) = 0$. Observe that $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m})$. We now apply Lemma 5.10, with $D_{yR}(M)$ as the module T and $u\mathbf{m}$ in the rôle of $\mathbf{m}$: if $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$, then part (i) of Lemma 5.10 yields that $\mathcal{S}(D_{yR}(M)) = \emptyset$, while if $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m}) \nsubseteq \mathcal{P}(\mathbf{k})$, then it follows from part (ii) of Lemma 5.10 that $\mathcal{S}(D_{yR}(M))$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$-domain in $\mathbb{Z}^r$. $\square$

5.12. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$. Assume that M is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Then there exists a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}$ for all $i \in \mathbb{N}_0$.*

Proof. Since $H^i_{\mathfrak{b}}(M) = 0$ for all $i > \operatorname{ara}(\mathfrak{b})$, it follows from Remark 5.9(vi) that it is sufficient for us to show that, for each $i \in \mathbb{N}_0$, there exists a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_i$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}_i$. For $i = 0$, this is immediate from Lemma 5.7.

Let $y_1, \ldots, y_s$ be $\mathbb{N}_0^r$-homogeneous elements of R that generate $\mathfrak{b}$. We argue by induction on s. When $s = 1$ and $i = 1$, the desired result follows from Lemma 5.11; as we have already dealt, in the preceding paragraph, with the case where $i = 0$, and as $H^i_{y_1 R}(M) = 0$ for all $i > 1$, we have established the desired result in all cases when $s = 1$.

So suppose now that $s > 1$ and that the desired result has been proved in all cases where $\mathfrak{b}$ can be generated by fewer than s $\mathbb{N}_0^r$-homogeneous elements. Again, we have already dealt with the case where $i = 0$. For $i \in \mathbb{N}$, there is an exact Mayer–Vietoris sequence (in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$)

$$\cdots \longrightarrow H^{i-1}_{(y_1 y_s, \ldots, y_{s-1} y_s)R}(M) \longrightarrow H^i_{\mathfrak{b}}(M) \longrightarrow H^i_{(y_1, \ldots, y_{s-1})R}(M) \oplus H^i_{y_s R}(M) \longrightarrow \cdots.$$

By the inductive hypothesis, there exist $\mathcal{P}(\mathbf{m})$-domains $\mathbb{X}'_i, \mathbb{X}''_i, \mathbb{X}'''_i$ in $\mathbb{Z}^r$ such that

$$\mathcal{S}\big(H^{i-1}_{(y_1 y_s, \ldots, y_{s-1} y_s)R}(M)\big) \subseteq \mathbb{X}'_i, \qquad \mathcal{S}\big(H^i_{(y_1, \ldots, y_{s-1})R}(M)\big) \subseteq \mathbb{X}''_i \quad \text{and} \quad \mathcal{S}\big(H^i_{y_s R}(M)\big) \subseteq \mathbb{X}'''_i.$$

Therefore $\mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}'_i \cup \mathbb{X}''_i \cup \mathbb{X}'''_i$, and so the desired result follows from Remark 5.9(vi). $\square$

5.13. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of R such that $R_{\mathbf{m}} \not\subseteq \mathfrak{p}_i$ for each $i = 1, \ldots, n$. Then there exists $u \in \mathbb{N}$ such that $R_{u\mathbf{m}} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$.*

Proof. Consider the (Noetherian) $\mathbb{N}_0$-graded ring $R_{\mathbf{0}}[R_{\mathbf{m}}] = \bigoplus_{j \in \mathbb{N}_0} R_{j\mathbf{m}}$ (in which $R_{j\mathbf{m}}$ is the component of degree j, for all $j \in \mathbb{N}_0$). Apply the ordinary Homogeneous Prime Avoidance Lemma (see [4, Lemma 15.1.2]) to the graded ideal $R_{\mathbf{m}} R_{\mathbf{0}}[R_{\mathbf{m}}] = \bigoplus_{j \in \mathbb{N}} R_{j\mathbf{m}}$ and the prime ideals $\mathfrak{p}_i \cap R_{\mathbf{0}}[R_{\mathbf{m}}]$ ($i = 1, \ldots, n$). $\square$

5.14. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ and let $\mathbb{X}$ be a $\mathcal{P}(\mathbf{m})$-domain in $\mathbb{Z}^r$. Then there exists $u \in \mathbb{N}$ such that, for each $\mathbf{w} \in \mathbb{Z}^r$, there is some $j \in \{0, \ldots, \#\mathcal{P}(\mathbf{m})\}$ with $\mathbf{w} + ju\mathbf{m} \notin \mathbb{X}$.*

Proof. There exist $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ with $\mathbf{s} \leqslant \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$ for which $\mathbb{X} = \mathbb{X}(\mathbf{s}, \mathbf{t})$. Choose $u \in \mathbb{N}$ such that $u\mathbf{m} \geqslant \mathbf{t} - \mathbf{s}$.

For an arbitrary $\mathbf{w} \in \mathbb{Z}^r$, set $\mathcal{I}(\mathbf{w}) = \{i \in \{1, \ldots, r\}\colon\ s_i \leqslant w_i < t_i\}$, and observe that $\mathcal{I}(\mathbf{w}) \subseteq \mathcal{P}(\mathbf{m})$, and that $\mathbf{w} \in \mathbb{X}$ if and only if $\mathcal{I}(\mathbf{w}) \neq \emptyset$. Note also that, for $i \in \mathcal{I}(\mathbf{w})$ and $j \in \mathbb{N}$, we have

$$(\mathbf{w} + ju\mathbf{m})_i = w_i + jum_i \geqslant s_i + um_i \geqslant s_i + t_i - s_i = t_i,$$

so that $i \notin \mathcal{I}(\mathbf{w} + ju\mathbf{m})$. So, for each $i \in \mathcal{P}(\mathbf{m})$, if there is a $j' \in \mathbb{N}_0$ with $i \in \mathcal{I}(\mathbf{w} + j'u\mathbf{m})$, then $i \notin \mathcal{I}(\mathbf{w} + ju\mathbf{m})$ for all $j > j'$. This means that, for each $i \in \mathcal{P}(\mathbf{m})$, there is at most one $j' \in \mathbb{N}_0$ with $i \in \mathcal{I}(\mathbf{w} + j'u\mathbf{m})$. By the pigeon-hole principle, it is therefore possible to choose a $j \in \{0, \ldots, \#\mathcal{P}(\mathbf{m})\}$ for which $\mathcal{I}(\mathbf{w} + ju\mathbf{m}) \cap \mathcal{P}(\mathbf{m}) = \emptyset$, and then $\mathbf{w} + ju\mathbf{m} \notin \mathbb{X}$. $\square$

The concept introduced in the next definition can be regarded as a multi-graded analogue of one defined by Marley in [14, §2].

5.15. Definition. Let $\mathcal{Q} \subseteq \{1, \ldots, r\}$, and let $\mathfrak{b}$ be an $\mathbb{N}_0^r$-graded ideal of R. We define the *$\mathcal{Q}$-finiteness dimension $g^{\mathcal{Q}}_{\mathfrak{b}}(M)$ of M with respect to* $\mathfrak{b}$ by

$$g^{\mathcal{Q}}_{\mathfrak{b}}(M) := \sup\big\{k \in \mathbb{N}_0\colon \text{ for all } i < k, \text{ there exists a } \mathcal{Q}\text{-domain } \mathbb{X}_i \text{ in } \mathbb{Z}^r \text{ with } \mathcal{S}\big(H^i_{\mathfrak{b}}(M)\big) \subseteq \mathbb{X}_i\big\},$$

if this supremum exists, and ∞ otherwise.

5.16. Example. For R as in Example 5.3, we have

$$g_{R_+}^{\emptyset}(R)=2, \qquad g_{R_+}^{\{1\}}(R)=3, \qquad g_{R_+}^{\{2\}}(R)=2, \qquad g_{R_+}^{\{1,2\}}(R)=5.$$

5.17. Remarks. The first three of the statements below are immediate from Remarks 5.9(i)–(iii) respectively.

(i) In the case where $\mathcal{Q}=\emptyset$, we have $g_{\mathfrak{b}}^{\emptyset}(M)=\inf\{i\in\mathbb{N}_0\colon\ H_{\mathfrak{b}}^i(M)\neq 0\}$ (with the usual convention that the infimum of the empty set of integers is interpreted as ∞).
(ii) If $\mathcal{Q}\subseteq\mathcal{Q}'\subseteq\{1,\dots,r\}$, then $g_{\mathfrak{b}}^{\mathcal{Q}}(M)\leqslant g_{\mathfrak{b}}^{\mathcal{Q}'}(M)$.
(iii) For $\mathbf{n}\in\mathbb{Z}^r$, we have $g_{\mathfrak{b}}^{\mathcal{Q}}(M(\mathbf{n}))=g_{\mathfrak{b}}^{\mathcal{Q}}(M)$.
(iv) Let $(\mathcal{Q}_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of $\{1,\dots,r\}$. Set

$$\Omega:=\left\{\bigcap_{\lambda\in\Lambda}\mathbb{X}_\lambda\colon\ \mathbb{X}_\lambda \text{ is a } \mathcal{Q}_\lambda\text{-domain in } \mathbb{Z}^r \text{ for all } \lambda\in\Lambda\right\}.$$

It is straightforward to check that

$$\inf\bigl\{g_{\mathfrak{b}}^{\mathcal{Q}_\lambda}(M)\colon\ \lambda\in\Lambda\bigr\}=\sup\bigl\{k\in\mathbb{N}_0\colon \text{ for all } i<k,\ \text{ there exists } \mathbb{Y}_i\in\Omega \text{ with } \mathcal{S}\bigl(H_{\mathfrak{b}}^i(M)\bigr)\subseteq\mathbb{Y}_i\bigr\}.$$

(v) Since a subset of $\mathbb{Z}^r$ is finite if and only if it is contained in a set of the form $\bigcap_{j=1}^r\mathbb{X}_j$, where $\mathbb{X}_j$ is a $\{j\}$-domain in $\mathbb{Z}^r$ for all $j\in\{1,\dots,r\}$, it therefore follows from part (iv) that

$$\min\bigl\{g_{\mathfrak{b}}^{\{1\}}(M),\dots,g_{\mathfrak{b}}^{\{r\}}(M)\bigr\}=\sup\bigl\{k\in\mathbb{N}_0\colon\ \mathcal{S}\bigl(H_{\mathfrak{b}}^i(M)\bigr) \text{ is finite for all } i<k\bigr\}.$$

Thus we can say that $\min\{g_{\mathfrak{b}}^{\{1\}}(M),\dots,g_{\mathfrak{b}}^{\{r\}}(M)\}$ identifies the smallest integer i (if there be any) for which $H_{\mathfrak{b}}^i(M)$ is not finitely graded.

5.18. Proposition. *Let $\mathbf{m}\in\mathbb{N}_0^r\setminus\{\mathbf{0}\}$, and let $f\in\mathbb{N}$. Assume that M is finitely generated. The following statements are equivalent*:

(i) $R_{\mathbf{m}}\subseteq\sqrt{(0:_R H_{\mathfrak{b}}^i(M))}$ *for all integers $i<f$*;
(ii) *for each integer $i<f$, there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_i$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M))\subseteq\mathbb{X}_i$, that is $f\leqslant g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)$*;
(iii) *there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H_{\mathfrak{b}}^i(M))\subseteq\mathbb{X}$ for all integers $i<f$.*

Proof. (ii) $\Leftrightarrow$ (iii) This is immediate from Remark 5.9(vi).

(iii) $\Rightarrow$ (i) Assume that statement (iii) holds. By Lemma 5.14, there exist $u, v:=\#\mathcal{P}(\mathbf{m})\in\mathbb{N}$ such that, for each $\mathbf{n}\in\mathbb{Z}^r$, there exists $j(\mathbf{n})\in\{0,\dots,v\}$ with $\mathbf{n}+j(\mathbf{n})u\mathbf{m}\notin\mathbb{X}$. So, for each $\mathbf{n}\in\mathbb{Z}^r$ and each integer $i<f$, we have $H_{\mathfrak{b}}^i(M)_{\mathbf{n}+j(\mathbf{n})u\mathbf{m}}=0$ and

$$R_{vu\mathbf{m}}H_{\mathfrak{b}}^i(M)_{\mathbf{n}}=R_{vu\mathbf{m}-j(\mathbf{n})u\mathbf{m}}R_{j(\mathbf{n})u\mathbf{m}}H_{\mathfrak{b}}^i(M)_{\mathbf{n}}\subseteq R_{vu\mathbf{m}-j(\mathbf{n})u\mathbf{m}}H_{\mathfrak{b}}^i(M)_{\mathbf{n}+j(\mathbf{n})u\mathbf{m}}=0.$$

Therefore $R_{vu\mathbf{m}}H_{\mathfrak{b}}^i(M)=0$ for all integers $i<f$, and hence

$$(R_{\mathbf{m}})^{vu}\subseteq R_{vu\mathbf{m}}\subseteq\bigl(0:_R H_{\mathfrak{b}}^i(M)\bigr)\quad \text{for all } i<f.$$

(i) $\Rightarrow$ (ii) Assume that statement (i) holds. We argue by induction on f. When $f=1$, the desired conclusion is immediate from Lemma 5.7 (applied to $H_{\mathfrak{b}}^0(M)$).

So assume now that $f > 1$ and that statement (ii) has been proved for smaller values of f. This inductive hypothesis implies that there exist $\mathcal{P}(\mathbf{m})$-domains $\mathbb{X}_0, \dots, \mathbb{X}_{f-2}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^i_{\mathfrak{b}}(M)) \subseteq \mathbb{X}_i$ for all $i \in \{0, \dots, f-2\}$. It thus remains to find a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_{f-1}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^{f-1}_{\mathfrak{b}}(M)) \subseteq \mathbb{X}_{f-1}$.

Set $\overline{M} := M/\Gamma_{R_{\mathbf{m}}R}(M)$, and observe that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R \Gamma_{R_{\mathbf{m}}R}(M))}$. It therefore follows from Lemma 5.12 that there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}'$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^{f-1}_{\mathfrak{b}}(\Gamma_{R_{\mathbf{m}}R}(M))) \subseteq \mathbb{X}'$. In view of the exact sequence of $\mathbb{Z}^r$-graded R-modules

$$H^{f-1}_{\mathfrak{b}}\big(\Gamma_{R_{\mathbf{m}}R}(M)\big) \longrightarrow H^{f-1}_{\mathfrak{b}}(M) \longrightarrow H^{f-1}_{\mathfrak{b}}(\overline{M})$$

and Remark 5.9(vi), it is now enough for us to show that $\mathcal{S}(H^{f-1}_{\mathfrak{b}}(\overline{M}))$ is contained in a $\mathcal{P}(\mathbf{m})$-domain in $\mathbb{Z}^r$.

As $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H^j_{\mathfrak{b}}(\Gamma_{R_{\mathbf{m}}R}(M)))}$ for all $j \in \mathbb{N}_0$, the exact sequence

$$H^i_{\mathfrak{b}}(M) \longrightarrow H^i_{\mathfrak{b}}(\overline{M}) \longrightarrow H^{i+1}_{\mathfrak{b}}\big(\Gamma_{R_{\mathbf{m}}R}(M)\big)$$

shows that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H^i_{\mathfrak{b}}(\overline{M}))}$ for all integers $i < f$. Set $\operatorname{Ass}_R(\overline{M}) =: \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. As $R_{\mathbf{m}}R$ does not consist entirely of zero-divisors on $\overline{M}$, we have $R_{\mathbf{m}} \not\subseteq \mathfrak{p}_i$ for each $i = 1, \dots, k$. Therefore, by Lemma 5.13, there exists $u' \in \mathbb{N}$ such that $R_{u'\mathbf{m}} \not\subseteq \bigcup_{i=1}^k \mathfrak{p}_i$, and hence there exists $y' \in R_{u'\mathbf{m}}$ which is not a zero-divisor on $\overline{M}$. We can now take a sufficiently high power y of y' to find $u \in \mathbb{N}$ and $y \in R_{u\mathbf{m}}$ such that $R_{u\mathbf{m}} H^{f-1}_{\mathfrak{b}}(\overline{M}) = 0$ and y is a non-zero-divisor on $\overline{M}$, so that there is a short exact sequence of $\mathbb{Z}^r$-graded R-modules

$$0 \longrightarrow \overline{M}(-u\mathbf{m}) \xrightarrow{\;y\;} \overline{M} \longrightarrow \overline{M}/y\overline{M} \longrightarrow 0.$$

It now follows from the long exact sequence of local cohomology modules induced from the above short exact sequence that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H^i_{\mathfrak{b}}(\overline{M}/y\overline{M}))}$ for all integers $i < f-1$. Therefore, by the inductive hypothesis, there is a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}''$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^{f-2}_{\mathfrak{b}}(\overline{M}/y\overline{M})) \subseteq \mathbb{X}''$. Let K be the kernel of the map $H^{f-1}_{\mathfrak{b}}(\overline{M}) \to H^{f-1}_{\mathfrak{b}}(\overline{M})(u\mathbf{m})$ provided by multiplication by y. The long exact sequence of local cohomology modules induced from the last-displayed short exact sequence now shows that $\mathcal{S}(K) \subseteq \mathbb{X}'' - u\mathbf{m}$.

We now apply Lemma 5.10(i) to $H^{f-1}_{\mathfrak{b}}(\overline{M})$, with $u\mathbf{m}$ playing the rôles of both $\mathbf{m}$ and $\mathbf{k}$: the conclusion is that there exists $v \in \mathbb{N}_0$ such that

$$\mathcal{S}\big(H^{f-1}_{\mathfrak{b}}(\overline{M})\big) \subseteq \bigcup_{j=0}^{v}\big(\mathcal{S}(K) - ju\mathbf{m}\big) \subseteq \bigcup_{j=0}^{v}(\mathbb{X}'' - u\mathbf{m} - ju\mathbf{m}).$$

We can now use Remarks 5.9(iii),(vi) to deduce the existence of a $\mathcal{P}(\mathbf{m})$-domain $\mathbb{X}_{f-1}$ in $\mathbb{Z}^r$ such that $\mathcal{S}(H^{f-1}_{\mathfrak{b}}(\overline{M})) \subseteq \mathbb{X}_{f-1}$. With this, the proof is complete. $\square$

We now connect the concept of $\mathcal{Q}$-finiteness dimension of M with respect to $\mathfrak{b}$, introduced in Definition 5.17, with the concept of $\mathfrak{a}$-finiteness dimension of M relative to $\mathfrak{b}$ (where $\mathfrak{a}$ is a second ideal of R), studied by Faltings in [5]. (See also [4, Chapter 9].)

5.19. Reminder. Assume that M is finitely generated, and let $\mathfrak{a}, \mathfrak{d}$ be ideals of R (not necessarily graded).

The $\mathfrak{a}$-*finiteness dimension* $f_{\mathfrak{d}}^{\mathfrak{a}}(M)$ *of* M *relative to* $\mathfrak{d}$ is defined by

$$f_{\mathfrak{d}}^{\mathfrak{a}}(M) = \inf\left\{i \in \mathbb{N}_0\colon\ \mathfrak{a} \not\subseteq \sqrt{\left(0 : H_{\mathfrak{d}}^i(M)\right)}\right\}$$

and the $\mathfrak{a}$-*minimum* $\mathfrak{d}$-*adjusted depth* $\lambda_{\mathfrak{d}}^{\mathfrak{a}}(M)$ *of* M is defined by

$$\lambda_{\mathfrak{d}}^{\mathfrak{a}}(M) := \inf\left\{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{d}+\mathfrak{p})/\mathfrak{p}\colon\ \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\right\}.$$

(Here, $\operatorname{Var}(\mathfrak{a})$ denotes the *variety* $\{\mathfrak{p} \in \operatorname{Spec}(R)\colon\ \mathfrak{p} \supseteq \mathfrak{a}\}$ *of* $\mathfrak{a}$.) It is always the case that $f_{\mathfrak{d}}^{\mathfrak{a}}(M) \leqslant \lambda_{\mathfrak{d}}^{\mathfrak{a}}(M)$; Faltings' (Extended) Annihilator Theorem [5] states that if R admits a dualizing complex or is a homomorphic image of a regular ring, then $f_{\mathfrak{d}}^{\mathfrak{a}}(M) = \lambda_{\mathfrak{d}}^{\mathfrak{a}}(M)$. (See [3, Corollary 3.8] for an account of the extended version of Faltings' Annihilator Theorem.)

5.20. Remark. Let the situation be as in Reminder 5.19, let $K \subseteq R$, and let $(K_j)_{j \in J}$ be a family of subsets of R.

(i) It is easy to deduce from the definition that $f_{\mathfrak{d}}^{KR}(M) = \inf\{f_{\mathfrak{d}}^{aR}(M)\colon\ a \in K\}$.
(ii) We can then deduce from part (i) that $f_{\mathfrak{d}}^{(\bigcup_{j\in J} K_j)R}(M) = \inf\left\{f_{\mathfrak{d}}^{K_jR}(M)\colon\ j \in J\right\}$.
(iii) Similarly, it is easy to deduce from the definition that $\lambda_{\mathfrak{d}}^{KR}(M) = \inf\{\lambda_{\mathfrak{d}}^{aR}(M)\colon\ a \in K\}$.
(iv) We can then deduce from part (iii) that $\lambda_{\mathfrak{d}}^{(\bigcup_{j\in J} K_j)R}(M) = \inf\left\{\lambda_{\mathfrak{d}}^{K_jR}(M)\colon\ j \in J\right\}$.

5.21. Theorem. *Assume that M is finitely generated, and let* $\emptyset \neq \mathcal{T} \subseteq \mathbb{N}_0^r$.

(i) *We have*

$$\begin{aligned}
&\sup\Big\{k \in \mathbb{N}_0\colon \textit{ for all } i < k \textit{ and all } \mathbf{m} \in \mathcal{T},\ \textit{there exists a } \mathcal{P}(\mathbf{m})\textit{-domain } \mathbb{X}_i^{(\mathbf{m})} \textit{ in } \mathbb{Z}^r \\
&\qquad\qquad \textit{such that } \mathcal{S}\big(H_{\mathfrak{b}}^i(M)\big) \subseteq \mathbb{X}_i^{(\mathbf{m})}\Big\} \\
&\quad = \inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)\colon\ \mathbf{m} \in \mathcal{T}\right\} \\
&\quad = f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) \leqslant \lambda_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M).
\end{aligned}$$

(ii) *If R admits a dualizing complex or is a homomorphic image of a regular ring, then we can replace the inequality in part* (i) *by equality.*

Proof. Apply Remark 5.17(iv) to the family $(\mathcal{P}(\mathbf{m}))_{\mathbf{m}\in\mathcal{T}}$ of subsets of $\{1,\dots,r\}$ to conclude that

$$\begin{aligned}
&\sup\Big\{k \in \mathbb{N}_0\colon \text{ for all } i < k \text{ and all } \mathbf{m} \in \mathcal{T},\ \text{there exists a } \mathcal{P}(\mathbf{m})\text{-domain } \mathbb{X}_i^{(\mathbf{m})} \text{ in } \mathbb{Z}^r \\
&\qquad\qquad \text{such that } \mathcal{S}\big(H_{\mathfrak{b}}^i(M)\big) \subseteq \mathbb{X}_i^{(\mathbf{m})}\Big\} \\
&\quad = \inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)\colon\ \mathbf{m} \in \mathcal{T}\right\}.
\end{aligned}$$

By Proposition 5.18, we have $g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) = f_{\mathfrak{b}}^{R_{\mathbf{m}}R}(M)$ for all $\mathbf{m} \in \mathcal{T}$. Therefore, on use of Remark 5.20(ii), we deduce that

$$\inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)\colon\ \mathbf{m} \in \mathcal{T}\right\} = \inf\left\{f_{\mathfrak{b}}^{R_{\mathbf{m}}R}(M)\colon\ \mathbf{m} \in \mathcal{T}\right\} = f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M).$$

We can now use Faltings' (Extended) Annihilator Theorem [5] (see Reminder 5.19) to complete the proof of part (i) and to obtain the statement in part (ii). $\square$

5.22. Corollary. *Assume that M is finitely generated.*

(i) *For each non-empty set $\mathcal{T} \subseteq \mathbf{1} + \mathbb{N}_0^r$, we have*

$$f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) = g_{\mathfrak{b}}^{\{1,\dots,r\}}(M).$$

(ii) *For each set $\mathcal{T} \subseteq \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ such that $\mathbb{N}\mathbf{e}_i \cap \mathcal{T} \neq \emptyset$ for all $i \in \{1,\dots,r\}$, we have*

$$\begin{aligned} f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) &= \sup\bigl\{k \in \mathbb{N}_0\colon\ \mathcal{S}\bigl(H_{\mathfrak{b}}^i(M)\bigr) \text{ is finite for all } i < k\bigr\} \\ &= \sup\bigl\{k \in \mathbb{N}_0\colon\ H_{\mathfrak{b}}^i(M) \text{ is finitely graded for all } i < k\bigr\}. \end{aligned}$$

(iii) *If $M \neq \mathfrak{b}M$, then $f_{\mathfrak{b}}^R(M) = g_{\mathfrak{b}}^{\emptyset}(M) = \operatorname{grade}_M \mathfrak{b}$.*

Note. If, in the case where $r = 1$, we take $\mathcal{T} = \mathbb{N}$, so that $\sum_{m\in\mathcal{T}} R_m R = R_+$, then the statement in part (ii) becomes

$$f_{\mathfrak{b}}^{R_+}(M) = \sup\bigl\{k \in \mathbb{N}_0\colon\ H_{\mathfrak{b}}^i(M) \text{ is finitely graded for all } i < k\bigr\},$$

a result proved by Marley in [14, Proposition 2.3].

Proof. (i) By Theorem 5.21(i), we have $f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) = \inf\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)\colon\ \mathbf{m} \in \mathcal{T}\}$. But $\mathcal{P}(\mathbf{m}) = \{1,\dots,r\}$ for all $\mathbf{m} \in \mathbf{1} + \mathbb{N}_0^r$.

(ii) By Theorem 5.21(i), we have

$$f_{\mathfrak{b}}^{\sum_{\mathbf{m}\in\mathcal{T}} R_{\mathbf{m}}R}(M) = \inf\bigl\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)\colon\ \mathbf{m} \in \mathcal{T}\bigr\}.$$

By the hypothesis, for each $i \in \{1,\dots,r\}$, there exists $\mathbf{m}_i \in \mathcal{T}$ with $\mathcal{P}(\mathbf{m}_i) = \{i\}$. It therefore follows from Remark 5.17(ii) that $\inf\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M)\colon\ \mathbf{m} \in \mathcal{T}\} = \min\{g_{\mathfrak{b}}^{\{1\}}(M),\dots,g_{\mathfrak{b}}^{\{r\}}(M)\}$. However, we noted in Remark 5.17(v) that

$$\min\bigl\{g_{\mathfrak{b}}^{\{1\}}(M),\dots,g_{\mathfrak{b}}^{\{r\}}(M)\bigr\} = \sup\bigl\{k \in \mathbb{N}_0\colon\ \mathcal{S}\bigl(H_{\mathfrak{b}}^i(M)\bigr) \text{ is finite for all } i < k\bigr\}.$$

(iii) Since $R = R_{\mathbf{0}}R$, we can deduce from Theorem 5.21(i) and Remark 5.17(i) that

$$f_{\mathfrak{b}}^R(M) = f_{\mathfrak{b}}^{R_{\mathbf{0}}R}(M) = g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{0})}(M) = g_{\mathfrak{b}}^{\emptyset}(M) = \sup\bigl\{k \in \mathbb{N}_0\colon\ H_{\mathfrak{b}}^i(M) = 0 \text{ for all } i < k\bigr\} = \operatorname{grade}_M \mathfrak{b}. \qquad \square$$

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