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A classification of nilpotent orbits in infinitesimal symmetric spaces

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ABSTRACT

Let G be a semisimple algebraic group defined over an algebraically closed field k whose characteristic is very good for G and not equal to 2. Suppose θ is an involution on G. We also denote the induced involution on $\mathfrak{g}$ by θ. Let $K = \{g \in G\colon\ \theta(g) = g\}$ and let $\mathfrak{p}$ be the -1-eigenspace of θ in $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ induces an action of K on $\mathfrak{p}$ and on the variety $\mathcal{N}(\mathfrak{p})$, which consists of the nilpotent elements in $\mathfrak{p}$. In this paper, we give a classification of the K-orbits in $\mathcal{N}(\mathfrak{p})$. To do so, we use the theory of associated cocharacters developed by Pommerening.

1. Introduction

Suppose G is a reductive linear algebraic group defined over an algebraically closed field k with $\mathfrak{g} = \mathrm{Lie}(G)$. The *nullcone* of $\mathfrak{g}$ is the algebraic variety consisting of the nilpotent elements of $\mathfrak{g}$ and is denoted by $\mathcal{N}(\mathfrak{g})$. The adjoint action of G on $\mathfrak{g}$ (denoted here by $g \cdot x$ for $g \in G$ and $x \in \mathfrak{g}$) induces an action of G on $\mathcal{N}(\mathfrak{g})$, and the G-orbits in $\mathcal{N}(\mathfrak{g})$ are called *nilpotent orbits*. Reductive groups always have only finitely many nilpotent orbits, regardless of the value of $\mathrm{char}(k)$. Nilpotent orbits have many applications in representation theory and have been extensively studied (cf. [9] and [5]).

1.1. Background

When $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \gg 0$, the Jacobson–Morozov Theorem says that for each $e \in \mathcal{N}(\mathfrak{g})$, there exist a semisimple element $h \in \mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$ such that (h, e, f) forms a basis for a copy of $\mathfrak{sl}_2(k)$ in $\mathfrak{g}$. Such triples are called *standard triples* and are crucial to the classification of the orbit set $\mathcal{N}(\mathfrak{g})/G$ given by Bala and Carter in [1,2]. Given a nilpotent orbit $\mathcal{O}$ with $e \in \mathcal{O}$, the Jacobson–Morozov Theorem gives a standard triple (h, e, f) as noted above. From this, we can con-

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struct a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ defined by $\mathfrak{q} = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{x \in \mathfrak{g}\colon\ [h, x] = ix\}$. Bala and Carter define the notions of *distinguished* elements and parabolic subalgebras. These will be analogous to the *featured* elements and parabolic subalgebras defined in Section 3. The Bala–Carter classification shows that there is a one-to-one correspondence from the set $\mathcal{N}(\mathfrak{g})/G$ to the set of G-conjugacy classes of pairs $(\mathfrak{r}, \mathfrak{q})$, where $\mathfrak{r}$ is a Levi subalgebra and $\mathfrak{q}$ is a distinguished parabolic subalgebra of $\mathfrak{r}$, constructed as above using a standard triple.

In this paper we consider groups defined over fields of *good* or *very good* characteristics, defined as follows. Suppose G is quasisimple. A characteristic of 0 is considered good and very good for G. The following primes, which depend on the Cartan type of G, are also good: all primes are good if G is of type A_n, all primes greater than 2 are good for types B_n, C_n, and D_n, and all primes greater than 3 are good for the exceptional types, except for type E_8, for which primes greater than 5 are good. A good prime p is very good for G if G is not of type A_n or if G is of type A_n and p does not divide $n + 1$. For an arbitrary reductive group G, which is an almost direct product of quasisimple groups G_i and a torus T, $\mathrm{char}(k)$ is good (resp., very good) for G if it is good (resp., very good) for each G_i.

In [15,16], Pommerening extended the Bala–Carter classification to reductive groups defined over fields of good characteristic. However, instead of using standard triples, which are not always available when $\mathrm{char}(k)$ is good, Pommerening used objects called *associated cocharacters*, which are defined in Section 2. He was able to show that the classification of $\mathcal{N}(\mathfrak{g})/G$ given by Bala and Carter remains the same when $\mathrm{char}(k)$ is good.

Now suppose G is equipped with an involution θ, i.e., an automorphism of order 2. Let $K = \{g \in G\colon\ \theta(g) = g\}$, and for the sake of convenience let θ also denote the differential of θ, which is an involution of $\mathfrak{g}$. Let $\mathfrak{k}$ be the $+1$-eigenspace of θ in $\mathfrak{g}$, and let $\mathfrak{p}$ be the -1-eigenspace. The adjoint action of G on $\mathfrak{g}$ induces an action of K on $\mathfrak{p}$. The subgroup K also acts on the nullcone $\mathcal{N}(\mathfrak{p})$ of $\mathfrak{p}$, which is defined to be the variety $\mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{p}$. The space $\mathfrak{p}$ is called an *infinitesimal symmetric space*, terminology inspired by the role that -1-eigenspaces of Cartan involutions play in the theory of Riemannian symmetric spaces.

Kostant and Rallis gave an extensive study of the action of K on $\mathfrak{p}$ in [11] when G is a complex reductive group. Recently, Levy extended many of Kostant and Rallis' results to fields of good characteristic (see [12]). In particular, he shows that each irreducible component of $\mathcal{N}(\mathfrak{p})$ has a dense open orbit and gives a new proof of the number of such components. He also gives results related to $k[\mathfrak{p}]^{\mathfrak{k}}$, the ring of $\mathfrak{k}$-invariant polynomials on $\mathfrak{p}$.

The following proposition (which is also true for a larger class of reductive groups, including $G = GL_n(k)$ when $\mathrm{char}(k) \neq 2$) will be important in Section 3.

Proposition 1. *When G is a semisimple group defined over an algebraically closed field with a characteristic which is very good for G and not 2, there are only finitely many K-orbits in $\mathcal{N}(\mathfrak{p})$.*

This statement follows immediately from a result of Richardson [18, Proposition 7.4] which says that there are only finitely many K-orbits in $\mathcal{U}(P)$ ($\mathcal{U}(P)$ being the set of unipotent elements in P) and a result of Bardsley and Richardson [3, Proposition 10.1] which says $\mathcal{U}(P)$ and $\mathcal{N}(\mathfrak{p})$ are isomorphic as K-varieties. This isomorphism follows from Springer's result in [19] that $\mathcal{U}(G)$ and $\mathcal{N}(\mathfrak{g})$ are isomorphic as G-varieties when $\mathrm{char}(k)$ is good for G.

If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \gg 0$, then for any $e \in \mathcal{N}(\mathfrak{p})$ one can use the Jacobson–Morozov Theorem to obtain a standard triple (h, e, f) in $\mathfrak{g}$ with the additional properties that $h \in \mathfrak{k}$ and $f \in \mathfrak{p}$. A standard triple with these properties is called a *normal triple*. Normal triples were used by Noël in [14] to give a classification of the orbit set $\mathcal{N}(\mathfrak{p})/K$ when $\mathrm{char}(k) = 0$.

When $\mathrm{char}(k) = 0$, normal triples play a part in the study of $\mathcal{N}(\mathfrak{p})/K$ analogous to the part played by standard triples in the study of $\mathcal{N}(\mathfrak{g})/G$. A goal of this paper is to construct cocharacters that will replace normal triples when $\mathrm{char}(k)$ is very good and not 2, similar to the way in which Pommerening's associated cocharacters replaced standard triples. These cocharacters are constructed in Section 2, and in Section 3, we will then use them to obtain a classification of $\mathcal{N}(\mathfrak{p})/K$ that will hold whenever $\mathrm{char}(k)$ is very good and not 2.

1.2. Assumptions and conventions

Throughout, we will make the following assumptions. The group G is a semisimple linear algebraic group defined over an algebraically closed field k whose characteristic is very good for G and not 2. Actually, the results contained herein hold even when $G = GL_n(k)$ and $\mathrm{char}(k) \neq 2$. Note that in this case when $\mathrm{char}(k)$ divides n, G is isomorphic to a Levi subgroup of $SL_{n+1}(k)$, for which $\mathrm{char}(k)$ is very good.

Any references to topological concepts refer to the Zariski topology on G and $\mathfrak{g} = \mathrm{Lie}(G)$. The group of units of k will be denoted k^*, and the identity component of G will be denoted by G°. The derived subgroup of an algebraic group H will be denoted $\mathcal{D}H$.

There is a bijection from the set of nilpotent G-orbits in $\mathfrak{g}$ to the set of nilpotent $G/Z(G)$-orbits in $\mathrm{Lie}(G/Z(G))$ (see [9, Proposition 2.7a]). Since $G/Z(G)$ is of adjoint type, we may thus assume that G also has this property when studying nilpotent orbits.

The assumptions on G imply that $\mathrm{Lie}(C_G(x)) = \mathfrak{g}^x$ for all $x \in \mathfrak{g}$, where $C_G(x)$ (resp., $\mathfrak{g}^x$) denotes the centralizer of x in G (resp., $\mathfrak{g}$) relative to the adjoint action. Thus, the tangent space $T_x(G \cdot x)$ at x to the orbit $G \cdot x$ is equal to $[\mathfrak{g}, x]$ (see [9, Section 2.2]).

Since involutions are semisimple automorphisms when $\mathrm{char}(k) \neq 2$, G contains a θ-stable Borel subgroup B which in turn contains a θ-stable torus T (see [20, Theorem 7.5]). The Lie algebra $\mathfrak{h}$ of T is a θ-stable Cartan subalgebra of $\mathfrak{g}$.

2. A replacement for normal triples

In this section, we first state some basic facts about characters and cocharacters. Then we develop a special cocharacter that will serve as a replacement for normal triples when we classify $\mathcal{N}(\mathfrak{p})/K$ in the next section.

2.1. Characters and cocharacters

Let $X^*(T)$ denote the group of characters of T, which consists of all algebraic group morphisms from T to k^*, and let $X_*(T)$ denote the group of cocharacters of T, which consists of all algebraic group morphisms from k^* to T. There is a perfect pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$, $(\alpha, \lambda) \mapsto \langle \alpha, \lambda \rangle$, defined by $\alpha(\lambda(t)) = t^{\langle \alpha, \lambda \rangle}$ for all $t \in k^*$.

Since T is θ-stable, the subgroup $\langle \theta \rangle$ of $\mathrm{Aut}(G)$ acts on $X^*(T)$ with an action defined by $\theta\alpha = \alpha \circ \theta$ for $\alpha \in X^*(T)$, and on $X_*(T)$ with an action defined by $\theta\lambda = \theta \circ \lambda$ for $\lambda \in X_*(T)$. Let Φ denote the root system of G defined by T, Φ^+ the positive roots defined by B, and Δ the simple roots in Φ. Since B and T are θ-stable, Φ, Φ^+, and Δ are θ-stable subsets of $X^*(T)$. The following lemma will be useful in Section 3.

Lemma 2. *The pairing $\langle\, , \rangle$ between $X^*(T)$ and $X_*(T)$ is θ-equivariant.*

Proof. For all $t \in k^*$, $\alpha \in X^*(T)$, and $\lambda \in X_*(T)$,

$$t^{\langle \theta\alpha, \theta\lambda \rangle} = \theta\alpha\big(\theta\lambda(t)\big) = \alpha\big(\theta\big(\theta\big(\lambda(t)\big)\big)\big) = \alpha\big(\lambda(t)\big) = t^{\langle \alpha, \lambda \rangle}.$$

Thus, $\langle \theta\alpha, \theta\lambda \rangle = \langle \alpha, \lambda \rangle$. $\square$

Cocharacters can be used to construct certain subalgebras of $\mathfrak{g}$. Given a cocharacter $\lambda \in X_*(G)$, we get a grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\lambda, i)$$

of $\mathfrak{g}$, where $\mathfrak{g}(\lambda,0)=\mathfrak{h}\oplus\bigoplus_{\langle\alpha,\lambda\rangle=0}\mathfrak{g}_\alpha$ and $\mathfrak{g}(\lambda,i)=\bigoplus_{\langle\alpha,\lambda\rangle=i}\mathfrak{g}_\alpha$ for $i\neq 0$. Since $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\colon\ g\cdot x=\alpha(g)x$ for all $g\in T\}$, $\mathfrak{g}(\lambda,i)=\{x\in\mathfrak{g}\colon\ \lambda(t)\cdot x=t^i x$ for all $t\in k^*\}$. Define

$$\mathfrak{q}=\mathfrak{q}(\lambda)=\bigoplus_{i\geqslant 0}\mathfrak{g}(\lambda,i).$$

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$. If we let $\mathfrak{l}=\mathfrak{l}(\lambda)=\mathfrak{g}(\lambda,0)$ and $\mathfrak{u}=\mathfrak{u}(\lambda)=\bigoplus_{i>0}\mathfrak{g}(\lambda,i)$, then $\mathfrak{l}\oplus\mathfrak{u}$ is a Levi decomposition of $\mathfrak{q}$. Furthermore, $L=L(\lambda)=C_G(\lambda)$ is a Levi subgroup of G such that $\mathrm{Lie}(L)=\mathfrak{l}$, where $C_G(\lambda)=\{g\in G\colon\ g\lambda(t)g^{-1}=\lambda(t)$ for all $t\in k^*\}$.

For $g\in L(\lambda)$, $x\in\mathfrak{g}(\lambda,i)$, and $t\in k^*$,

$$\lambda(t)\cdot(g\cdot x)=g\cdot\left(t^i x\right)=t^i(g\cdot x),$$

which shows that $\mathfrak{g}(\lambda,i)$ is $L(\lambda)$-stable for all i.

2.2. Associated cocharacters of K

In [15,16], Pommerening extended the Bala–Carter classification of the orbit set $\mathcal{N}(\mathfrak{g})/G$ to groups defined over fields of good characteristic. He found that the classification remained the same as the one given by Bala and Carter for $k=\mathbb{C}$. One obstacle in extending the Bala–Carter classification was the unavailability of the Jacobson–Morozov Theorem, which only holds when $\mathrm{char}(k)=0$ or $\mathrm{char}(k)\gg 0$. Given $e\in\mathcal{N}(\mathfrak{g})$, Jantzen in [9] formulated Pommerening's solution to this problem using a cocharacter $\lambda: k^*\to G$ which satisfies the following properties relative to e:

- $\lambda(t)\cdot e=t^2 e$ for all $t\in k^*$.
- $\lambda(k^*)$ is contained in the derived subgroup of a Levi subgroup R of G such that $\mathrm{Lie}(R)$ is a minimal Levi subalgebra containing e.

Such a cocharacter is defined to be *associated* with e. Associated cocharacters provide a partial substitute for standard triples, which were utilized by Bala and Carter in their classification.

Pommerening's proof was computational in nature and ultimately relied on case-checking by root system type. In [17], Premet gave a fairly short, conceptual proof of Pommerening's theorem using the theory of *optimal cocharacters*, as introduced by Kempf and Rousseau (see [17] for a short exposition of Kempf–Rousseau theory). The relationship between the optimal cocharacters used by Premet and the associated cocharacters used by Pommerening was stated precisely by McNinch in [13, Theorem 21]. He defines a cocharacter $\lambda\in X_*(G)$ to be *primitive* if there is no $\phi\in X_*(G)$ such that $\lambda=n\phi$ for some integer $n\geqslant 2$. He then showed that if a cocharacter λ is associated with $e\in\mathcal{N}(\mathfrak{g})$, then λ is optimal for e. Conversely, if λ is primitive and optimal for e, then either λ or 2λ is associated with e.

Let e be an element in $\mathcal{N}(\mathfrak{g})$, and let $N(e)=\{g\in G\colon\ g\cdot e\in ke\}$, a closed subgroup of G. Any cocharacter of G associated with e is in $X_*(N(e))$. The following lemma is the main result needed to construct our desired cocharacter.

Lemma 3. *(See [13, Lemma 25].) Let $e\in\mathcal{N}(\mathfrak{g})$, and let S be any maximal torus of $N(e)$. Then there is a unique cocharacter λ in $X_*(S)$ associated with e.*

Suppose $e\in\mathfrak{p}$. Then $\theta(e)=-e$, which means θ leaves $N(e)$ invariant. Hence by [20], $N(e)$ has a maximal torus which is θ-stable.

Theorem 4. *Let $e\in\mathcal{N}(\mathfrak{p})$, and let S be a θ-stable maximal torus of $N(e)$. There is a unique cocharacter λ in $X_*(S\cap K)$ associated with e.*

Proof. By Lemma 3, e has a unique cocharacter λ in $X_*(S)$ associated with it. Let R be the Levi subgroup which by the definition of an associated cocharacter contains the image of λ. Because S is θ-stable, $\theta \circ \lambda \in X_*(S)$. Since θ is a semisimple automorphism, θ is actually conjugation by a semisimple element s in a linear algebraic group $\overline{G}$ containing G. Thus $(\theta \circ \lambda)(t) \cdot e = (s\lambda(t)s^{-1}) \cdot e = t^2 e$ for all $t \in k^*$. Also $(\theta \circ \lambda)(k^*) \subset \mathcal{D}(sRs^{-1})$, and $\mathrm{Lie}(sRs^{-1}) = s \cdot \mathrm{Lie}(R)$ is a minimal Levi subalgebra containing $\theta(e) = -e$, and hence e itself. Thus $\theta \circ \lambda$ is associated with e. The uniqueness of λ now implies that $\theta \circ \lambda = \lambda$, which means λ is also a cocharacter of K. $\square$

Given $e \in \mathcal{N}(\mathfrak{p})$, Theorem 4 gives an associated cocharacter $\lambda \in X_*(S \cap K)$. Then there exists an element $g \in K$ such that $g\lambda g^{-1} \in X_*(T \cap K)$, and $g\lambda g^{-1}$ is associated with $g \cdot e$. Replacing e by $g \cdot e$ does not affect the orbit $K \cdot e$, so we may assume that $\lambda \in X_*(T \cap K)$. We will denote the unique cocharacter in $X_*(T \cap K)$ associated with $e \in \mathcal{N}(\mathfrak{p})$ by λ_e. The cocharacter λ_e, given that its image is in K, can be seen as an analogue to a normal triple, just as Pommerening's associated cocharacters are analogous to standard triples.

Since θ is a semisimple automorphism, it has the important property that $\mathrm{Lie}(G^\theta) = \mathfrak{g}^\theta$, where G^θ (resp., $\mathfrak{g}^\theta$) is the subgroup of θ-fixed points in G (resp., $\mathfrak{g}$). This fact will be needed in the proof of the following lemma and elsewhere below.

Lemma 5. *Let $e \in \mathcal{N}(\mathfrak{p})$, and let $\lambda = \lambda_e$. Let $L = C_G(\lambda)$, and let $\mathfrak{l} = \mathrm{Lie}(L)$.*

(a) $\mathrm{Lie}(L \cap K) = \mathfrak{l} \cap \mathfrak{k}$.
(b) $(C_{L\cap K}(e))^\circ$ *is reductive.*
(c) $\mathrm{Lie}(C_{L\cap K}(e)) = (\mathfrak{l} \cap \mathfrak{k})^e$.
(d) $\mathrm{Lie}(C_K(e)) = \mathfrak{k}^e$.

Proof. (a) Let $g \in L$. Then for all $t \in k^*$, $\theta(g)\lambda(t)\theta(g)^{-1} = \theta(g\lambda(t)g^{-1})$ since $\lambda(t) \in K$. But this equals $\theta(\lambda(t))$ since $g \in L$, and this in turn is just $\lambda(t)$. Thus $\theta(g) \in L$, and we have that θ restricts to a semisimple automorphism of L. Thus $\mathrm{Lie}(L \cap K) = \mathrm{Lie}(L^\theta) = \mathrm{Lie}(L)^\theta = \mathfrak{l}^\theta = \mathfrak{l} \cap \mathfrak{k}$.

(b) Since $e \in \mathcal{N}(\mathfrak{g})$, the centralizer $C_L(e)$ is reductive (see [9, Proposition 5.11]). It thus follows from [21] that $(C_L(e)^\theta)^\circ = (C_{L\cap K}(e))^\circ$ is also reductive.

(c) By [9, Proposition 5.10], $\mathrm{Lie}(C_L(e)) = \mathfrak{l}^e$. Using this fact, and the fact that θ restricts to a semisimple automorphism of $C_L(e)$, an argument similar to the one in part (a) gives the desired result.

(d) Since θ is semisimple, $\mathrm{Lie}(C_K(e)) = \mathrm{Lie}(C_G(e))^\theta = (\mathfrak{g}^e)^\theta = \mathfrak{k}^e$. $\square$

3. A classification of $\mathcal{N}(\mathfrak{p})/K$

In this section, we give a classification of the K-orbits in $\mathcal{N}(\mathfrak{p})$ which is similar in spirit to the Bala–Carter–Pommerening classification of $\mathcal{N}(\mathfrak{g})/G$ which holds whenever $\mathrm{char}(k)$ is good and to Noël's classification of $\mathcal{N}(\mathfrak{p})/K$ which holds when $\mathrm{char}(k) = 0$. In [10], Kawanaka gave a classification of $\mathcal{N}(\mathfrak{p})/K$ using objects similar to weighted Dynkin diagrams which holds when $\mathrm{char}(k)$ is good, but the one given here is significantly different. Also, a classification of $\mathcal{N}(\mathfrak{p})/K$ in good characteristic is given in [7] under the assumption that G is a classical group. The present classification makes no such assumption.

3.1. Featured elements and featured pairs

Let S be a torus in G. We call $C_G(S)$ a *special Levi subgroup* if $S \subset K$. The Lie algebra of a special Levi subgroup will be called a *special Levi subalgebra*. By definition, a special Levi subalgebra will have the form $\mathfrak{g}^{\mathfrak{s}}$ for some subset $\mathfrak{s}$ of $\mathfrak{k}$ consisting of semisimple elements. We call an element $e \in \mathcal{N}(\mathfrak{p})$ *featured* in $\mathfrak{g}$ if the only special Levi subalgebra of $\mathfrak{g}$ containing e is $\mathfrak{g}$ itself. Featured elements are analogous to Noël's noticed elements and Bala and Carter's distinguished elements.

Suppose $e \in \mathcal{N}(\mathfrak{p})$ is featured and $g \in K$. If $\mathfrak{r}$ is a special Levi subalgebra containing $g \cdot e$, then $g^{-1} \cdot \mathfrak{r}$ is a special Levi subalgebra containing e. Since e is featured, this implies $g^{-1} \cdot \mathfrak{r} = \mathfrak{g}$, and hence that $\mathfrak{r} = \mathfrak{g}$. This shows that if e is featured, then so is every element in the orbit $K \cdot e$. We call orbits consisting of featured nilpotent elements *featured nilpotent orbits*.

The first step is to classify the featured K-orbits in $\mathcal{N}(\mathfrak{p})$. The fact that every nilpotent element is featured in a minimal special Levi subalgebra containing it will allow us to then extend the classification to all orbits. The following characterizations of featured elements will be quite useful.

Proposition 6. *Let e be an element in $\mathcal{N}(\mathfrak{p})$, and let $\lambda = \lambda_e$. The following are equivalent*:

(a) *e is featured in $\mathfrak{g}$.*
(b) *$\mathfrak{k}^e$ contains no nonzero semisimple elements.*
(c) $\dim(\mathfrak{g}(\lambda, 0) \cap \mathfrak{k}) = \dim(\mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$.

Proof. ((a) $\Leftrightarrow$ (b)) By definition, if e is featured then all semisimple elements of $\mathfrak{k}$, and hence of $\mathfrak{k}^e$, are contained in the center of $\mathfrak{g}$. However, $\mathfrak{g}$ is semisimple (by our assumptions on G), so its center is trivial. (Recall that since $\operatorname{char}(k)$ is very good, $\mathfrak{g}$ has no factor isomorphic to $\mathfrak{sl}_n(k)$ where $\operatorname{char}(k)$ divides n, which would have a nonzero center.) Therefore $\mathfrak{k}^e$ contains no nonzero semisimple elements. Conversely, suppose $\mathfrak{k}^e$ contains no nonzero semisimple elements, and let $\mathfrak{r} = \mathfrak{g}^{\mathfrak{s}}$ be a special Levi subalgebra containing e. Then $\mathfrak{s} \subset \mathfrak{k}^e$, so $\mathfrak{s} = \{0\}$. Thus, $\mathfrak{r} = \mathfrak{g}$, and hence e is featured.

((b) $\Leftrightarrow$ (c)) Let $\mathfrak{q} = \mathfrak{q}(\lambda)$. Recall that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l} = \mathfrak{l}(\lambda) = \mathfrak{g}(\lambda, 0)$ and $\mathfrak{u} = \mathfrak{u}(\lambda)$ as defined in Section 2. Also recall the definition of the subgroup $L = L(\lambda)$. By [9, Lemma 5.7, Proposition 5.8] the map $(\operatorname{ad} e) : \mathfrak{l} \to \mathfrak{g}(\lambda, 2)$ is onto since $e \in \mathfrak{g}(\lambda, 2)$. Since e is also in $\mathfrak{p}$, $(\operatorname{ad} e) : \mathfrak{l} \cap \mathfrak{k} \to \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$ is onto as well, which means $\dim(\mathfrak{l} \cap \mathfrak{k}) = \dim(\mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$ if and only if $(\operatorname{ad} e)$ is one-to-one. The kernel of $(\operatorname{ad} e)$ is $(\mathfrak{l} \cap \mathfrak{k})^e$. Since $(\mathfrak{l} \cap \mathfrak{k})^e$ is the Lie algebra of the reductive group $(C_{L \cap K}(e))^\circ$ (by Lemma 5(b) and (c)), it will be nonzero if and only if it contains nonzero semisimple elements. Thus, we have so far that $\dim(\mathfrak{l} \cap \mathfrak{k}) = \dim(\mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$ if and only if $(\mathfrak{l} \cap \mathfrak{k})^e$ contains no nonzero semisimple elements. Now since λ is associated to e, by [9, Eq. (6), p. 55] $\mathfrak{g}^e = \mathfrak{q}^e$, which implies $\mathfrak{k}^e = (\mathfrak{l} \oplus \mathfrak{u})^e \cap \mathfrak{k}$. Thus $(\mathfrak{l} \cap \mathfrak{k})^e$ contains no nonzero semisimple elements if and only if $\mathfrak{k}^e$ contains no nonzero semisimple elements. $\square$

Remark 7. To see that Proposition 6 also holds when $G = GL_n(k)$ and $\operatorname{char}(k) \neq 2$, note that up to conjugacy by an inner automorphism, the only involutions on G are $g \mapsto (g^{\mathsf{T}})^{-1}$, $g \mapsto J_m^{-1} (g^{\mathsf{T}})^{-1} J_m$ (where $n = 2m$ and $J_m = \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right)$), and $g \mapsto J_{\alpha,\beta}\, g\, J_{\alpha,\beta}$ (where (α, β) is a partition of n and $J_{\alpha,\beta} = \left(\begin{smallmatrix} I_\alpha & 0 \\ 0 & -I_\beta \end{smallmatrix}\right)$) (see [8]). From this, one can compute the possible subalgebras $\mathfrak{k}$ of $\mathfrak{g}$ and conclude that they all have a trivial intersection with the center of $\mathfrak{g}$, which consists of the scalar matrices in $\mathfrak{g}$. Thus, we can still conclude that $\mathfrak{k}^e$ contains no nonzero semisimple elements.

In the next subsection, we will associate each featured orbit to an object called a featured pair, which is defined as follows. Let $\mathfrak{q}$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{l} \oplus \mathfrak{u}$. Let L be the connected subgroup of G such that $\operatorname{Lie}(L) = \mathfrak{l}$. Then $\mathfrak{q}$ is defined to be a *featured parabolic subalgebra* if $\mathfrak{q}$ is θ-stable and K-conjugate to a standard parabolic subalgebra.

Lemma 8. *Suppose $\mathfrak{q}$ is a standard parabolic subalgebra with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Let $\mathfrak{u}^1 = \mathfrak{u}$ and $\mathfrak{u}^i = [\mathfrak{u}, \mathfrak{u}^{i-1}]$ for $i \geqslant 2$, and let Δ_0 be the subset of Δ which defines $\mathfrak{q}$. Then $\mathfrak{u}$ has the following properties*:

(a) *For $i \geqslant 1$, $\mathfrak{u}^i$ is the sum of its one-dimensional root spaces.*
(b) *A root α of $\mathfrak{u}$ is a root of $\mathfrak{u}^i$ if and only if α is the sum of i roots of $\mathfrak{u}$.*
(c) *A root of $\mathfrak{u}^i$ which is not a root of $\mathfrak{u}^{i+1}$ is the sum of i roots in $\Delta \setminus \Delta_0$ plus various roots in Δ_0.*

Proof. Parts (a) and (b) follow easily from the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all α and β in Φ.

(c) Let α be a root of $\mathfrak{u}^i$ which is not a root of $\mathfrak{u}^{i+1}$. By part (b), α is the sum of i roots in $\mathfrak{u}$. The fact that α is not a root of $\mathfrak{u}^{i+1}$ implies that none of the roots of $\mathfrak{u}$ which are summands of α can be roots of $\mathfrak{u}^2$. By [6, Proposition 8.27(iii)], a root of $\mathfrak{u}$ which is not a root of $\mathfrak{u}^2$ is the sum of exactly one root in $\Delta \setminus \Delta_0$ and various roots in Δ_0. Thus α is the sum of exactly i roots in $\Delta \setminus \Delta_0$ and various roots in Δ_0. □

Suppose $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is featured and satisfies the hypotheses of Lemma 8. (We lose no generality in assuming that $\mathfrak{q}$ is standard since we will only be working up to K-conjugacy in what follows.) Denote by $\mathfrak{q}(2)$ the direct sum of all the root spaces $\mathfrak{g}_\alpha$ with the property that α is the sum of 2 roots in Δ and various roots in Δ_0. We define a *featured pair* $(\mathfrak{q}, \mathfrak{m})$ to be a pair consisting of a featured parabolic subalgebra $\mathfrak{q}$ and the closure $\mathfrak{m}$ of an $(L \cap K)^\circ$-orbit in $\mathfrak{q}(2) \cap \mathfrak{p}$ with the property that $\dim \mathfrak{m} = \dim(\mathfrak{l} \cap \mathfrak{k})$.

The set $\mathfrak{m}$ is an irreducible subset of $\mathfrak{p}$ since $(L \cap K)^\circ$ is irreducible. Because, by Proposition 1, there are only finitely many K-orbits in $\mathcal{N}(\mathfrak{p})$, there must be a unique K-orbit $\mathcal{O}_{(\mathfrak{q},\mathfrak{m})}$ such that $\mathcal{O}_{(\mathfrak{q},\mathfrak{m})} \cap \mathfrak{m}$ is open and dense in $\mathfrak{m}$. Notice that $\mathcal{O}_{(g\cdot\mathfrak{q}, g\cdot\mathfrak{m})} = \mathcal{O}_{(\mathfrak{q},\mathfrak{m})}$ for all $g \in K$. This shows that there is a well-defined map from the set of K-conjugacy classes of featured pairs to the set $\mathcal{N}(\mathfrak{p})/K$. If $e \in \mathcal{O}_{(\mathfrak{q},\mathfrak{m})} \cap \mathfrak{m}$, we call e a *Richardson element* associated with $(\mathfrak{q}, \mathfrak{m})$. The following proposition gives an example of a featured pair together with a Richardson element.

Proposition 9. *Let e be a featured element in $\mathcal{N}(\mathfrak{p})$, and let $\lambda = \lambda_e$. Let $\mathfrak{q} = \mathfrak{q}(\lambda)$ and $\mathfrak{m} = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$. Then $(\mathfrak{q}, \mathfrak{m})$ is a featured pair, and e is a Richardson element associated with $(\mathfrak{q}, \mathfrak{m})$.*

Proof. As above, we have $\mathfrak{q} = \mathfrak{q}(\lambda)$, $L = L(\lambda)$, $\mathfrak{l} = \mathfrak{l}(\lambda)$, and $\mathfrak{u} = \mathfrak{u}(\lambda)$. First, notice that $\mathfrak{q}$ is the standard parabolic subalgebra of $\mathfrak{g}$ defined by the subset $\{\alpha \in \Delta\colon \langle \alpha, \lambda \rangle = 0\}$, hence $\mathfrak{q}$ is trivially K-conjugate to a standard parabolic. Since $\lambda \in X_*(K)$, for $t \in k^*$ and $x \in \mathfrak{g}(\lambda, i)$, $\lambda(t) \cdot \theta(x) = \theta(\lambda(t) \cdot x) = t^i \theta(x)$. Thus $\mathfrak{g}(\lambda, i)$ is θ-stable for all i, which proves that $\mathfrak{q}$ is θ-stable. Therefore, $\mathfrak{q}$ is a featured parabolic.

By the proof of Proposition 6, $[\mathfrak{l} \cap \mathfrak{k}, e] = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$, which implies that $(L \cap K)^\circ \cdot e$ is dense in $\mathfrak{g}(\lambda, 2) \cap \mathfrak{p} = \mathfrak{m}$. Further, we have by Lemma 8 that $\mathfrak{u}^i = \bigoplus_{\langle \alpha, \lambda \rangle \geqslant i} \mathfrak{g}_\alpha$, so $\mathfrak{g}(\lambda, 2) = \mathfrak{q}(2)$. Thus, $\mathfrak{m}$ is the closure of an $(L \cap K)^\circ$-orbit in $\mathfrak{q}(2) \cap \mathfrak{p}$. Finally, since e is featured,

$$\dim \mathfrak{l} \cap \mathfrak{k} = \dim \mathfrak{g}(\lambda, 0) \cap \mathfrak{k} = \dim \mathfrak{g}(\lambda, 2) \cap \mathfrak{p} = \dim \mathfrak{m}.$$

Thus, $(\mathfrak{q}, \mathfrak{m}) = (\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$ is a featured pair.

The orbit $\mathcal{O}_{(\mathfrak{q},\mathfrak{m})}$ associated to this pair is the one containing $(L \cap K)^\circ \cdot e$. Since $e \in \mathcal{O}_{(\mathfrak{q},\mathfrak{m})} \cap \mathfrak{m}$, e is a Richardson element for the featured pair $(\mathfrak{q}, \mathfrak{m})$. □

3.2. Classification

The first step in obtaining a classification of $\mathcal{N}(\mathfrak{p})/K$ is to show that there is a one-to-one correspondence between the set of K-conjugacy classes of featured pairs and the set of featured K-orbits in $\mathcal{N}(\mathfrak{p})$. We can then obtain the classification of all of $\mathcal{N}(\mathfrak{p})/K$ by replacing $\mathfrak{g}$ with special Levi subalgebras.

Given an arbitrary featured pair $(\mathfrak{q}, \mathfrak{m})$, we first develop a way to explicitly describe $\mathfrak{q}$ and $\mathfrak{m}$ in terms of a cocharacter $\tau \in X_*(K)$. Since we are dealing with K-conjugacy classes of featured pairs, there is no loss of generality in assuming that $\mathfrak{q}$ is a standard parabolic in what follows.

Proposition 10. *Given a featured pair $(\mathfrak{q}, \mathfrak{m})$ with $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ a standard parabolic subalgebra, there exists a cocharacter $\tau \in X_*(K)$ such that $\mathfrak{q} = \mathfrak{q}(\tau)$ and $\mathfrak{m} \subset \mathfrak{g}(\tau, 2) \cap \mathfrak{p}$. Furthermore, the subgroup $L(\tau) = C_G(\tau)$ is a θ-stable Levi subgroup such that $\mathrm{Lie}(L(\tau)) = \mathfrak{l}$.*

Proof. Suppose $\mathfrak{q}$ is defined by the subset Δ_0 of Δ and has Levi decomposition $\mathfrak{l} \oplus \mathfrak{u}$.

Define a function $f : \Delta \to \mathbb{Z}$ by

$$f(\alpha) = \begin{cases} 0, & \alpha \in \Delta_0, \\ 1, & \text{otherwise} \end{cases}$$

and extend f linearly to $\mathbb{Z}\Phi$. Since G is of adjoint type, $\mathbb{Z}\Phi = X^*(T)$. Thus, f has the property that $f(X^*(T)) \subset \mathbb{Z}$. The perfect pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ therefore produces a cocharacter $\tau \in X_*(T)$ such that $f(\alpha) = \langle \alpha, \tau \rangle$ for all $\alpha \in \Phi \cup \{0\}$.

Because $\mathfrak{q}$ is θ-stable, so is $\mathfrak{l}$. Since $\mathfrak{h}$ and Δ are θ-stable, it can be shown that $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\theta\alpha}$ for all $\alpha \in \Delta$. Now

$$\mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta_0 \rangle} \mathfrak{g}_\alpha = \mathfrak{l} = \theta \mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta_0 \rangle} \mathfrak{g}_{\theta\alpha},$$

which implies that Δ_0 is θ-stable. This means that $f(\alpha) = f(\theta\alpha)$ for all $\alpha \in \Delta$. This, together with the fact stated in Lemma 2 that the pairing $\langle\, , \rangle$ is θ-equivariant, gives

$$\langle \alpha, \tau \rangle = f(\alpha) = f(\theta\alpha) = \langle \theta\alpha, \tau \rangle = \langle \alpha, \theta\tau \rangle$$

for all $\alpha \in \Delta$. Thus $\theta\tau = \tau$, and hence, $\tau \in X_*(K)$.

The fact that $\mathfrak{q}$ is defined by Δ_0 means

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta_0 \rangle} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+ \setminus \langle \Delta_0 \rangle^+} \mathfrak{g}_\alpha,$$

which says that $\mathfrak{q} = \mathfrak{q}(\tau)$. Also, by Lemma 8 and the definition of the function f, $\mathfrak{q}(2) \subset \mathfrak{g}(\tau, 2)$. We thus have $\mathfrak{m} \subset \mathfrak{g}(\tau, 2) \cap \mathfrak{p}$. (It will be shown later that we actually have $\mathfrak{m} = \mathfrak{g}(\tau, 2) \cap \mathfrak{p}$.)

Finally, $L(\tau)$ is θ-stable since $\tau(k^*) \subset K$. The equality $\mathrm{Lie}(L(\tau)) = \mathfrak{l}$ follows from the fact that $\mathfrak{l} = \mathfrak{g}(\tau, 0)$. □

Lemma 11. *Let* $(\mathfrak{q}, \mathfrak{m})$ *be a featured pair, and let e be a Richardson element associated with* $(\mathfrak{q}, \mathfrak{m})$. *Also, let* $L = L(\tau)$ *and* $\mathfrak{l} = \mathfrak{l}(\tau)$. *Then*

(a) $\mathfrak{k}^e \subset \mathfrak{q} \cap \mathfrak{k}$, *and*
(b) $\mathrm{Lie}(C_{L\cap K}(e)) = (\mathfrak{l} \cap \mathfrak{k})^e$.

Proof. (a) Let $\mathfrak{u} = \mathfrak{u}(\tau)$, and let Q be the connected subgroup of G such that $\mathrm{Lie}(Q) = \mathfrak{q}$. Suppose D is the unique nilpotent G-orbit in $\mathfrak{g}$ which meets $\mathfrak{u}$ in a dense set. (We know D exists because there are finitely many G-orbits in $\mathfrak{g}$.) Since $\mathcal{O} = \mathcal{O}_{(\mathfrak{q},\mathfrak{m})}$ is the unique nilpotent K-orbit in $\mathfrak{p}$ meeting $\mathfrak{m}$ in a dense set, $\mathcal{O} \subset D$. Thus, $e \in D \cap \mathfrak{u}$. Then by a theorem of Richardson (see [4, Corollary 5.2.4], which only requires that $\mathrm{char}(k)$ be good), $C_G(e)^\circ \subset Q$. Now, it is easy to show that $C_G(e)^\circ$ is θ-stable, and Q is θ-stable since $\mathfrak{q}$ is. Thus, taking the Lie algebras of the θ-fixed point subgroups of both sides, we get $\mathrm{Lie}(C_G(e)^\circ)^\theta \subset \mathrm{Lie}(Q^\theta)$, which simplifies to $\mathfrak{k}^e \subset \mathfrak{q} \cap \mathfrak{k}$.

(b) An argument identical to the one used to prove Lemma 5(c) proves this result. □

Proposition 12. *If* $(\mathfrak{q}, \mathfrak{m})$ *is a featured pair, then* $\mathcal{O}_{(\mathfrak{q},\mathfrak{m})}$ *is a featured orbit.*

Proof. Let $L = L(\tau)$ and $\mathfrak{l} = \mathfrak{l}(\tau)$. Suppose e is a Richardson element associated with $(\mathfrak{q}, \mathfrak{m})$. It suffices to show that e is featured. Since $\mathrm{Lie}(C_{L\cap K}(e)) = (\mathfrak{l} \cap \mathfrak{k})^e$, $T_e((L \cap K)^\circ \cdot e) = [\mathfrak{l} \cap \mathfrak{k}, e]$ (see [9, Section 2.2]). Then $\dim[\mathfrak{l} \cap \mathfrak{k}, e] = \dim(L \cap K)^\circ \cdot e = \dim \mathfrak{m} = \dim \mathfrak{l} \cap \mathfrak{k}$. This implies $(\mathfrak{l} \cap \mathfrak{k})^e = \{0\}$. Since $\mathfrak{k}^e = (\mathfrak{q} \cap \mathfrak{k})^e$ by Lemma 11(a), $\mathfrak{k}^e = (\mathfrak{u} \cap \mathfrak{k})^e$, which means $\mathfrak{k}^e$ contains no nonzero semisimple elements. Thus e is featured. □

This proposition gives a well-defined map ϕ from the set of K-conjugacy classes of featured pairs to the set of featured K-orbits in $\mathcal{N}(\mathfrak{p})$. It remains to show that ϕ is a bijection. The following restatement of Proposition 9 shows that ϕ is onto.

Proposition 13. *Let $e \in \mathcal{N}(\mathfrak{p})$ be featured, and let $\lambda = \lambda_e$. Then*

(a) $(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$ *is a featured pair, and*
(b) $e \in \mathcal{O}_{(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda,2)\cap\mathfrak{p})}$.

We now show that ϕ is one-to-one.

Proposition 14. *Let $e \in \mathcal{N}(\mathfrak{p})$ be a Richardson element associated to a featured pair $(\mathfrak{q}, \mathfrak{m})$, and let $\lambda = \lambda_e$. Then $(\mathfrak{q}, \mathfrak{m})$ is K-conjugate to $(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$.*

Proof. By definition, we may assume that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a standard parabolic subalgebra defined by $\Delta_0 \subset \Delta$. Then by Proposition 10, there exists a cocharacter τ such that $\mathfrak{q} = \mathfrak{q}(\tau)$ and $\mathfrak{l} = \mathfrak{l}(\tau)$. We will show that $\mathfrak{g}(\tau, i) = \mathfrak{g}(\lambda, i)$ for all $i \in \mathbb{Z}$, which will imply that $\mathfrak{q} = \mathfrak{q}(\lambda)$.

Choose $h_1 \in \mathfrak{h}$ such that $\alpha(h_1) = \langle \alpha, \tau \rangle$ for all $\alpha \in \Delta$. Because Δ_0 and $\Delta \setminus \Delta_0$ are θ-stable, h_1 is actually in $\mathfrak{h} \cap \mathfrak{k}$. Notice that $[h_1, e] = \alpha(h_1)e = \langle \alpha, \tau \rangle e = 2e$ since $e \in \mathfrak{m} \subset \mathfrak{g}(\tau, 2)$.

Since $\lambda \in X_*(K)$, the subsets $\{\alpha \in \Delta\colon \langle \alpha, \lambda \rangle = i\}$ of Δ are θ-stable for each $i \in \mathbb{Z}$. This allows us to choose $h_2 \in \mathfrak{h} \cap \mathfrak{k}$ such that $\alpha(h_2) = \langle \alpha, \lambda \rangle$ for all $\alpha \in \Delta$. Since $e \in \mathfrak{g}(\lambda, 2)$, $[h_2, e] = 2e$. Thus $[h_1 - h_2, e] = 0$, which means $h_1 - h_2$ is a semisimple element of $\mathfrak{k}^e$. But e is featured, so $h_1 = h_2$. Therefore, $\langle \alpha, \tau \rangle = \langle \alpha, \lambda \rangle$ for all $\alpha \in \Phi$, and hence, $\mathfrak{g}(\tau, i) = \mathfrak{g}(\lambda, i)$ for all $i \in \mathbb{Z}$. Therefore $\mathfrak{q} = \mathfrak{q}(\lambda)$.

It remains to show that $\mathfrak{m} = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$. We know that $\mathfrak{m} \subset \mathfrak{g}(\tau, 2) \cap \mathfrak{p}$ by Proposition 10. Since $(\mathfrak{q}, \mathfrak{m})$ is a featured pair,

$$\begin{aligned}
\dim \mathfrak{m} &= \dim \mathfrak{l} \cap \mathfrak{k} \\
&= \dim \mathfrak{g}(\tau, 0) \cap \mathfrak{k} \\
&= \dim \mathfrak{g}(\lambda, 0) \cap \mathfrak{k} \\
&= \dim \mathfrak{g}(\lambda, 2) \cap \mathfrak{p} \\
&= \dim \mathfrak{g}(\tau, 2) \cap \mathfrak{p}.
\end{aligned}$$

Then $\mathfrak{m} = \mathfrak{g}(\tau, 2) \cap \mathfrak{p} = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$ since $\mathfrak{g}(\tau, 2) \cap \mathfrak{p}$ is irreducible and $\mathfrak{m}$ is closed. □

We have proved the following:

Theorem 15. *There is a one-to-one correspondence between featured K-orbits in $\mathcal{N}(\mathfrak{p})$ and K-conjugacy classes of featured pairs. The K-orbit corresponding to a featured pair $(\mathfrak{q}, \mathfrak{m})$ is the unique one which intersects $\mathfrak{m}$ in a dense subset of $\mathfrak{m}$.*

Before stating the general classification, we need a couple of lemmas.

Lemma 16. *All minimal special Levi subalgebras of $\mathfrak{g}$ containing a fixed element $e \in \mathcal{N}(\mathfrak{p})$ are $C_K(e)$-conjugate.*

Proof. Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be special Levi subalgebras containing e. Then there exist tori S_1 and S_2 in K such that $\mathfrak{r}_1 = \operatorname{Lie}(C_G(S_1))$ and $\mathfrak{r}_2 = \operatorname{Lie}(C_G(S_2))$. Since $\mathfrak{r}_1$ and $\mathfrak{r}_2$ both contain e, S_1 and S_2 both fix e and are thus contained in $C_K(e)$. Further, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ will be minimal precisely when S_1 and S_2 are maximal in $C_K(e)$. Being maximal tori in $C_K(e)$, S_1 and S_2 are $C_K(e)$-conjugate, and thus $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are $C_K(e)$-conjugate. □

Lemma 17. *Let $\mathfrak{r}$ be a minimal special Levi subalgebra of $\mathfrak{g}$ containing a nilpotent element e. Then $e \in \mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}]$, and e is a featured element of $\mathfrak{r}'$.*

Proof. Since all elements of the center of $\mathfrak{r}$ are semisimple, we have $e \in \mathfrak{r}'$. The fact that e is featured in $\mathfrak{r}'$ follows from Proposition 6. $\square$

We can now give the classification of K-orbits in $\mathcal{N}(\mathfrak{p})$. Since λ_e is an associated cocharacter for each $e \in \mathcal{N}(\mathfrak{p})$, it has the property that $\lambda_e(k^*) \subset \mathcal{D}L$, where $L = C_G(M)$ for a maximal torus M in $C_G(e)$ (see [13]). Let $\overline{M} = M \cap K$, a maximal torus in $C_K(e)$, and let $R = C_G(\overline{M})$, a Levi subgroup of G. Since $\mathrm{Lie}(C_K(e)) = \mathfrak{k}^e$ and $\mathrm{Lie}(R) = \mathfrak{g}^{\mathrm{Lie}(\overline{M})}$, e is featured in $\mathrm{Lie}(R)$ by definition, and $\lambda_e(k^*) \subset \mathcal{D}R$.

Theorem 18. *There is a one-to-one correspondence between K-orbits in $\mathcal{N}(\mathfrak{p})$ and K-conjugacy classes of triples $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$, where $\mathfrak{r}$ is a special Levi subalgebra of $\mathfrak{g}$ and $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ is a featured pair of the semisimple part $\mathfrak{r}'$ of $\mathfrak{r}$. The K-orbit corresponding to a given triple $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ contains the $(R \cap K)$-orbit which intersects $\mathfrak{m}$ in a dense subset of $\mathfrak{m}$.*

Proof. We again have a well-defined map ψ from K-conjugacy classes of triples $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ to nilpotent K-orbits on $\mathfrak{p}$ which sends the triple $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ to the K-orbit containing the $(R \cap K)$-orbit which intersects $\mathfrak{m}$ in a dense subset of $\mathfrak{m}$.

Let $e \in \mathcal{N}(\mathfrak{p})$. We have that the image of $\lambda = \lambda_e$ is in $\mathcal{D}R$. This allows us to replace G by $\mathcal{D}R$ in the preceding arguments. By Lemma 17, e is a featured element in the semisimple subalgebra $\mathfrak{r}'$. Thus by Proposition 13, there is a featured pair $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ of $\mathfrak{r}'$ such that e is a Richardson element associated with $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$. This shows that ψ is onto.

We say that an element $e \in \mathcal{N}(\mathfrak{p})$ is a Richardson element associated with a triple $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ if $\mathfrak{r}$ is a minimal special Levi subalgebra containing e and e is a Richardson element associated with the pair $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$. Suppose $e \in \mathcal{N}(\mathfrak{p})$ is a Richardson element associated with the triples $(\mathfrak{r}_1, \mathfrak{q}_{\mathfrak{r}'_1}, \mathfrak{m}_1)$ and $(\mathfrak{r}_2, \mathfrak{q}_{\mathfrak{r}'_2}, \mathfrak{m}_2)$. By Lemma 16, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are conjugate by an element in $C_K(e)$, and thus by Proposition 14, $(\mathfrak{q}_{\mathfrak{r}'_1}, \mathfrak{m}_1)$ and $(\mathfrak{q}_{\mathfrak{r}'_2}, \mathfrak{m}_2)$ are conjugate by an element in $\mathcal{D}R_1 \cap K$ (or equivalently, by an element in $\mathcal{D}R_2 \cap K$). Thus, the triples $(\mathfrak{r}_1, \mathfrak{q}_{\mathfrak{r}'_1}, \mathfrak{m}_1)$ and $(\mathfrak{r}_2, \mathfrak{q}_{\mathfrak{r}'_2}, \mathfrak{m}_2)$ are in the same K-conjugacy class. This shows that ψ is also one-to-one. $\square$

Example 19. This example is adapted from [14]. Let $G = SL_3(k)$, and let θ be the involution on G defined by $\theta(g) = (g^{-1})^{\mathsf{T}}$. Then $K = SO_3(k)$, $\mathfrak{g} = \mathfrak{sl}_3(k)$, and the induced involution θ on $\mathfrak{g}$ is defined by $\theta(x) = -x^{\mathsf{T}}$ for all $x \in \mathfrak{g}$. Then $\mathfrak{p}$ is the subspace of symmetric matrices in $\mathfrak{g}$. By [7], we know that the nilpotent K-orbits in $\mathfrak{p}$ correspond to partitions of 3. Thus, there are two nonzero orbits, which correspond to the partitions $(2, 1)$ and (3).

Now, turning back to our classification scheme, let $i = \sqrt{-1}$ in k. Noël showed that up to K-conjugacy, the only θ-stable parabolic subalgebra of $\mathfrak{g}$ is

$$\mathfrak{q} = kH_1 \oplus kH_2 \oplus kE_1 \oplus kE_2 \oplus kE_3,$$

where

$$H_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{pmatrix}.$$

We have that $\mathfrak{q}$ is K-conjugate to the set of upper triangular matrices in $\mathfrak{sl}_3(k)$, which of course is a standard parabolic. A Levi decomposition for $\mathfrak{q}$ is $\mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l} = kH_1 \oplus kH_2$ and $\mathfrak{u} = kE_1 \oplus kE_2 \oplus kE_3$. Let $\mathfrak{m}_1 = kE_1$ and let $\mathfrak{m}_2 = kE_2$. Both of these contain dense $(L \cap K)^{\circ}$-orbits since $\mathfrak{l} \cap \mathfrak{k} = kH_1$ and $[H_1, E_1] = 2E_1$ and $[H_1, E_2] = E_2$. Also,

$$\dim(\mathfrak{l} \cap \mathfrak{k}) = 1 = \dim \mathfrak{m}_1 = \dim \mathfrak{m}_2.$$

Thus, $(\mathfrak{q}, \mathfrak{m}_1)$ is a featured pair corresponding to the orbit in $\mathcal{N}(\mathfrak{p})/K$ containing E_1 (which corresponds to the partition $(2,1)$), and $(\mathfrak{q}, \mathfrak{m}_2)$ is a featured pair corresponding to the orbit containing E_2 (which corresponds to the partition (3)). Since E_1 and E_2 are both featured nilpotent elements, the nonzero nilpotent K-orbits in $\mathcal{N}(\mathfrak{p})$ are completely classified by the K-conjugacy classes of the triples $(\mathfrak{g}, \mathfrak{q}, \mathfrak{m}_1)$ and $(\mathfrak{g}, \mathfrak{q}, \mathfrak{m}_2)$.

The classification of $\mathcal{N}(\mathfrak{p})/K$ given here is inspired by the one given by Noël in [14] under the assumption that $k = \mathbb{C}$. In this case, the well-known Kostant–Sekiguchi bijection states that the K-orbits in $\mathcal{N}(\mathfrak{p})$ correspond bijectively to the $G_{\mathbb{R}}$-orbits in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$, where $G_{\mathbb{R}}$ is a real Lie group with complexified maximal compact subgroup K and $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$. Thus, Noël's classification simultaneously classifies $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})/G_{\mathbb{R}}$. The classification given here only requires that $\mathrm{char}(k)$ be very good for the group G and not 2; in particular, it holds when $\mathrm{char}(k) = 0$. We thus get a streamlined classification of $\mathcal{N}(\mathfrak{p})/K$ when $\mathrm{char}(k) = 0$, and hence, by the Kostant–Sekiguchi correspondence, a new classification of $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})/G_{\mathbb{R}}$.

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