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F-coherent rings with applications to tight closure theory

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ABSTRACT

The aim of this paper is to introduce a new class of Noetherian rings of prime characteristic via perfect closure and study their basic properties. If the perfect closure of a Noetherian ring is coherent, we call it an F-coherent ring. Some applications are given to tight closure theory. In particular, we discuss some relationship between F-coherent rings and F-pure, F-regular, and F-injective rings. As a main tool, we use techniques from valuation theory. The final section discusses how the coherent property effects the behavior of tight closure of finitely generated ideals on general perfect rings.

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1. Introduction

In the present paper, all rings are assumed to be commutative and unitary. Recall that a ring is *coherent* if every finitely generated ideal is finitely presented. This condition is automatic for all Noetherian rings. We shall consider the following problem in connection with tight closure theory.

Problem 1. Let R be a Noetherian ring of prime characteristic and let R^{∞} be its perfect closure. Then what is characteristic of R, if R^{∞} is a coherent ring?

Tight closure theory was created in the mid 80's by Hochster and Huneke as a powerful tool with applications to many outstanding questions regarding rings of prime characteristic. In fact, a modification of the above problem has a connection with the so-called "localization problem", asking whether tight closure commutes with localization, or not. An attempt made by Aberbach

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and Hochster to establish the localization problem was to compare tight closure with plus closure. More precisely, if it were true that R^+ (this notation will be explained later) is coherent, then it would follow that tight closure commutes with localization. Unfortunately, it turns out that the localization problem is false, as was shown by the recent work of Brenner and Monsky [3].

Except the case when R is a field, the perfect ring R^∞ is almost never Noetherian, but R^∞ can be better-behaved than R itself. As we shall see below, perfect coherent rings of certain type force their finitely generated ideals to be tightly closed. This result relies on the flatness of the Frobenius map on perfect rings. Also, there are non-trivial cases when the perfect closures of rings, much smaller than R^+, are not coherent. In this respect, it would be adequate to call a Noetherian ring of prime characteristic *F-coherent* if its perfect closure is coherent. For example, every regular ring of prime characteristic is F-coherent. Especially, a subclass of affine semigroup rings forms an interesting class of F-coherent rings. We are led to the following question. Let $(R, \mathfrak{m})$ be a local Noetherian ring of prime characteristic. Then under what condition on R is a system of parameters of R a regular sequence on the perfect closure R^∞?

Recall that an algebra T over a local ring $(R, \mathfrak{m})$ is a *big Cohen–Macaulay R-algebra* if some system of parameters of R is a regular sequence on T. We prove by using a valuation, that if R is F-coherent, then every system of parameters of R is a regular sequence on R^∞. This result is related to the recent work by Roberts, Singh, and Srinivas. Quite generally, if R is any complete local domain of prime characteristic, then they proved that R^∞ is almost Cohen–Macaulay ([14] for this terminology). Note, however, that even if R^∞ is a big Cohen–Macaulay R-algebra, R itself is not necessarily Cohen–Macaulay. Finally, we mention that certain homological aspects of perfect closures, as well as absolute integral closures of Noetherian domains are studied by Asgharzadeh [2].

2. Preliminaries on tight closure

Throughout this article, assume that R is a commutative ring of prime characteristic $p > 0$, that is, R contains a field of characteristic p. We shall simply say that "R is of characteristic $p > 0$". Let $F_R^e : R \to R^{(e)}$ be the e-th iterated Frobenius map in which $R^{(e)} = R$ as a left R-module, while the right R-module structure of $R^{(e)}$ is provided by $a \cdot r = aF^e(r)$. The *Frobenius* (*Peskine–Szpiro*) *functor* $\mathbf{F}_R^e(-)$ is defined by $\mathbf{F}_R^e(M) = R^{(e)} \otimes_R M$ for any R-module M.

Let us denote by R^0 the complement of the set of all minimal primes of R and let I be an ideal. Then we define $I^{[q]}$ as an ideal generated by $q = p^e$-powers of elements of I. Then the *tight closure* I^* of I is the set of $x \in R$ such that there exists $c \in R^0$ for which $cx^q \in I^{[q]}$ for $q = p^e \gg 0$. It is easy to see that I^* is an ideal containing I. An ideal I is *tightly closed* if $I = I^*$. A Noetherian ring R is called *weakly F-regular* if every ideal of R is tightly closed. R is called *F-regular* if every localization of R is weakly F-regular.

The *perfect closure* of a ring R is defined as the limit of the direct system:

$$R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots$$

defined by the Frobenius map on R and denote it by R^∞. Since the iteration of the Frobenius map kills nilpotent elements, R^∞ is reduced and exists uniquely. Alternatively, R^∞ is an overring of R_{red} by adjoining to R_{red} all p-power roots of elements of R_{red}. So R^∞ is an overring of R_{red}. We also use the following notation. For a ring R, we let $R^{1/q}$ denote $(R_{\mathrm{red}})^{1/q}$. The *Frobenius closure* of I is defined as $I^F = IR^\infty \cap R$, where $IR^\infty \cap R$ is the inverse image of IR^∞ under the map $R \to R^\infty$. An alternative definition is that $x \in I^F$ if and only if $x^q \in I^{[q]}$ for all $q = p^e \gg 0$. It is obviously true that $I^F \subseteq I^*$. A Noetherian ring R is called *cyclically F-pure* if $I = I^F$ for every ideal I of R. R is called *F-pure* if R is a pure extension of R via the Frobenius map. These notions coincide when R is approximately Gorenstein (due to Hochster). If $(R, \mathfrak{m})$ is a local Noetherian ring of characteristic $p > 0$, then the Frobenius map on R naturally induces a map on the Čech complex of R, hence it gives rise to the Frobenius action on the local cohomology modules $H^k_{\mathfrak{m}}(R) \to H^k_{\mathfrak{m}}(R^{(1)})$. $(R, \mathfrak{m})$ is *F-injective* if the map $H^k_{\mathfrak{m}}(R) \to H^k_{\mathfrak{m}}(R^{(1)})$ is injective for all $k \geqslant 0$. If $(R, \mathfrak{m})$ is Cohen–Macaulay, then

R is F-injective if and only if $I = I^F$ for any parameter ideal I of R. Here I is a *parameter ideal* if I is generated by a system of parameters. Then F-purity implies F-injectivity.

The *absolute integral closure* of an integral domain R is the integral closure of R in an algebraic closure of the field of fractions of R and denote it by R^+. Then clearly $R^\infty \subseteq R^+$.

The most part of tight closure theory is concerned with Noetherian rings, while the definition of tight closure itself, at least, makes sense for any ring of characteristic $p > 0$. We will use only the definition of tight closure to derive some properties of coherence on certain perfect rings.

3. F-coherent rings

A finitely generated module M over a commutative ring R is *coherent* if every finitely generated submodule of M is finitely presented. A ring R is *coherent* if it is coherent as an R-module. We often use the following fundamental fact on coherent rings [8, Theorem 2.3.2].

Proposition 3.1. *Let R be a commutative ring. Then the following are equivalent*:

(1) *R is coherent.*
(2) *Every finitely presented R-module is a coherent module.*
(3) *Every finitely generated submodule of a free R-module is finitely presented.*
(4) *$(0 : a)$ and $I \cap J$ are finitely generated ideals for any $a \in R$ and any finitely generated ideals I, $J \subseteq R$.*
(5) *$(I : a)$ is a finitely generated ideal for any $a \in R$ and any finitely generated ideal $I \subseteq R$.*
(6) *Every finitely generated submodule of every finitely presented module over R is finitely presented over R.*

For more on coherent rings and modules, we refer the reader to [8]. Let us begin with the definition of F-coherent rings and establish some basic properties.

Definition 3.2. Let R be a Noetherian ring of characteristic $p > 0$. Then we say that R is *F-coherent* if the perfect closure R^∞ is coherent.

A simple observation shows that R is F-coherent if and only if R_{red} is F-coherent. In particular, we may assume that R is always reduced. In the following, when we say a *flat colimit* of commutative rings, it means the direct limit of a direct system $\{R_\alpha\}_{\alpha\in\Lambda}$ such that, whenever $\alpha < \beta$, the transition map $R_\alpha \to R_\beta$ is flat in the usual sense.

Proposition 3.3. *Let R be a Noetherian ring of characteristic $p > 0$.*

(1) *Any regular ring is F-coherent.*
(2) *Let S be a multiplicative subset of an F-coherent ring R. Then the localization $S^{-1}R$ is F-coherent as well.*
(3) *If $U = \{\mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p}$ is F-coherent$\}$ is constructible, then U is a Zariski open subset. Furthermore U is non-empty.*
(4) *Let $R \subseteq S$ be a faithfully flat extension. Then if S is F-coherent, so is R.*

Proof. For (1), note that $R \to R^{1/p} \to \cdots \to R^{1/p^e} \to \cdots$ is a direct system with flat transition maps by applying a theorem of Kunz [12], so that R^∞ is coherent, since the flat colimit of coherent rings is coherent.

For (2), let S be a multiplicative subset of R. We claim that there is an isomorphism: $(S^{-1}R)^\infty \simeq S^{-1}(R^\infty)$. Since the problem depends only on R_{red}, we may assume that R is reduced. Then the natural map $R \to R^\infty$ extends to an injection $S^{-1}R \to S^{-1}(R^\infty)$. Since this map is purely inseparable and the localization of a perfect ring is again perfect (an easy exercise), the required isomorphism follows. On the other hand, the property of coherence is stable under localization, so the stated isomorphism implies that $(S^{-1}R)^\infty$ is coherent.

For (3), we pick $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q} \in U$. Then since the localization of an F-coherent ring is F-coherent, it follows that $\mathfrak{p} \in U$, which implies that U is stable under generalization.

Hence U is open by assumption. Next let us show that U is non-empty. Again we may assume that R is reduced. Let $\mathfrak{p} \subseteq R$ be any minimal prime ideal. Then $R_{\mathfrak{p}}$ is a field, which is obviously F-coherent. Thus U is non-empty.

For (4), note that $R^{\infty} \subseteq S^{\infty}$ is also faithfully flat. We prove the following fact: Let M be an R^{∞}-module. Then M is finitely generated over R^{∞} if and only if $M \otimes_{R^{\infty}} S^{\infty}$ is finitely generated over S^{∞}. The "only if" part is obvious. Conversely, assume that the finite set of elements $s_1, \ldots, s_n$ of M generates the S^{∞}-module $M \otimes_{R^{\infty}} S^{\infty}$ under the inclusion $M \to M \otimes_{R^{\infty}} S^{\infty}$. Let N be an R^{∞}-submodule of M that is generated by $s_1, \ldots, s_n$. Then the short exact sequence: $0 \to N \to M \to M/N \to 0$ gives that $(M/N) \otimes_{R^{\infty}} S^{\infty} = 0$, hence $M/N = 0$ by faithful flatness of $R^{\infty} \to S^{\infty}$ and this implies that M is generated by $s_1, \ldots, s_n$. We may then apply this fact together with [13, Theorem 7.4] to conclude that $(0 :_{R^{\infty}} a)$ and $I \cap J$ are both finitely generated for any $a \in R^{\infty}$ and any finitely generated ideals I, J of R^{∞}. Thus if S is F-coherent, so is R. □

It is not clear as to what to expect on the topology of the F-coherent locus in $\operatorname{Spec} R$. For example, if R is an excellent domain, then the F-coherent locus contains an open subset, since the regular locus is open.

Remark 3.4. As a natural extension of the argument used to show that a regular ring is F-coherent, it can be shown that the flat colimit of F-coherent rings is F-coherent, if the colimit is Noetherian. In fact, this follows easily from the fact that the perfect closure of a ring R of characteristic $p > 0$ is obtained as the colimit:

$$\varinjlim\{R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots\},$$

where the map is the Frobenius map, together with the fact that if the map from one direct system to another system is flat, its colimit is also flat.

Although the next corollary is simple, it serves as a useful way to produce sufficiently many F-coherent rings.

Corollary 3.5. *Let $R \to S$ be a purely inseparable extension of Noetherian rings. Then R is F-coherent if and only if S is so. In particular, if R is a purely inseparable extension (or subextension) of a polynomial algebra over a field of characteristic $p > 0$, it is F-coherent.*

Proof. The proof of this corollary is immediate from the definition. □

Example 3.6. Affine semigroup rings provide non-trivial (non-regular) examples of F-coherent rings. They even include rings which are neither Cohen–Macaulay, nor normal. Let k be any field of characteristic 2 and let $R = k[x^4, x^3y, xy^3, y^4]$. Then $(x^3y)^2/x^4 = x^2y^2$, which is integral over R, but is not in R. The failure of Cohen–Macaulay property is obvious. On the other hand, we have $k[x^4, y^4] \subseteq R \subseteq k[x, y]$, which is a tower of purely inseparable extensions, hence R is F-coherent. Later on, we prove an easy-to-use criterion which implies that R is not F-pure in terms of the failure of Cohen–Macaulay property on R.

Discussion 3.7. We shall use a valuative method for deriving various properties for F-coherent rings. Let R be any Noetherian domain and let P be its prime ideal. Then there exists a discrete valuation ring (V, tV) such that $R \subseteq V \subseteq K$ and $P = R \cap tV$, where K is the field of fractions of R. If $R \subseteq S$ is an integral extension domain, then the valuation v extends to S, where $\mathbb{Z}$ or $\mathbb{Q}$ is its value group. In particular, for some non-zero element $a \in R$, we have $v(a^{1/r}) = \frac{1}{r} \cdot v(a)$. In what follows, we shall say that (V, tV) is *attached* to the pair (R, P) for the brevity of notation.

A question which naturally arises is, of course, to ask how F-coherent rings are related with those rings that are studied in tight closure theory. We occasionally use some basic facts on coherent modules, for which we refer to [8].

Theorem 3.8. *Let R be a reduced Noetherian ring of characteristic $p > 0$ with module-finite normalization. If R is F-coherent, then R^∞ is a normal ring. In particular, for any such F-coherent ring, the normalization $R \to \overline{R}$ is purely inseparable.*

Proof. Let $R \to \overline{R}$ be the normalization map. Since this map is module-finite, there exists a non-zero divisor $r \in R$ such that $r \cdot \overline{R} \subseteq R$. By iterating the Frobenius map, we get the commutative diagram

$$\begin{array}{ccccccc} R & \longrightarrow & R^{1/p} & \longrightarrow & R^{1/p^2} & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ r\overline{R} & \longrightarrow & r^{1/p}(\overline{R})^{1/p} & \longrightarrow & r^{1/p^2}(\overline{R})^{1/p^2} & \longrightarrow & \cdots \end{array}$$

and taking the direct limit, we have $r^{1/q}(\overline{R})^\infty \subseteq R^\infty$ for any $q = p^e$. What we need to show is $(\overline{R})^\infty = R^\infty$. Let $R^\infty \subseteq T \subseteq (\overline{R})^\infty$ be any module-finite extension. Since both $(\overline{R})^\infty$ and R^∞ have the same total ring of fractions, and T is finitely generated over R^∞, there exists a $t \in R^\infty$ such that t is T-regular and $t \cdot T \subseteq R^\infty$. That is, T is regarded as a submodule of R^∞, implying that T is a coherent R^∞-module by Proposition 3.1. Then

$$0 \longrightarrow R^\infty \longrightarrow T \longrightarrow T/R^\infty \longrightarrow 0$$

is a short exact sequence of coherent R^∞-modules [8, Theorem 2.2.1]. Next, let $N := T/R^\infty = R^\infty \cdot u_1 + \cdots + R^\infty \cdot u_k$ and define the R^∞-module map $\phi : R^\infty \to N^{\oplus k}$ by $\phi(a) = (au_1, \ldots, au_k)$. Then we have $\operatorname{Ker}\phi = (0 :_{R^\infty} N)$ and $r^{1/q} \in \operatorname{Ker}\phi$ for all $q = p^e$. But since ϕ is a map of coherent R^∞-modules, $\operatorname{Ker}\phi$ is finitely generated in R^∞ by [8, Corollary 2.2.2].

Assume $\operatorname{Ker}\phi \neq R^\infty$. Choosing a minimal prime P of R^∞ which is contained in a maximal ideal containing $\operatorname{Ker}\phi$, we consider the natural map $\Psi : R^\infty \to R^\infty/P$. Then $J := \Psi(\operatorname{Ker}\phi)$ is a finitely generated proper ideal, and for $r \in R$ as above, we find that $\bar{r} = \Psi(r)$ is non-zero because $r \notin P$. $A' := R/(R \cap P) \to A := R^\infty/P$ is an integral extension and A' is Noetherian. Extend A' to its module-finite extension B such that $B \subseteq A$ and $J = (a_1, \ldots, a_m)A$ with $a_i \in B$ for all i. We denote again by J the ideal of B generated by $a_1, \ldots, a_m$. Let (V, tV) be a discrete valuation ring attached to (B, Q) for $Q \in \operatorname{Spec} B$ with $J \subseteq Q$. Let v be the extended valuation to A. But then we get $v(\bar{r}^{1/q}) = \frac{1}{q} \cdot v(\bar{r}) \to 0$ $(q \to \infty)$ and $\inf\{v(a) \mid a \in J\} \geqslant v(t) = 1$, which are not compatible with each other, because $\bar{r}^{1/q} \cdot A \subseteq J$. Hence $R^\infty = T$, which implies that $R^\infty = (\overline{R})^\infty$. □

Corollary 3.9. *Let R be a one-dimensional reduced excellent ring of characteristic $p > 0$. Then R is F-coherent if and only if the normalization of R is purely inseparable over R.*

As we will see later, this result gives a geometric interpretation of one-dimensional singularities of "F-coherent type" in characteristic zero. Roughly speaking, if the normalization separates a single singular point of a variety into at least two smooth points, then it is not of F-coherent type.

Example 3.10. Let $R = k[x^4, x^2y, xy^2, y^4]$ for a field k of characteristic $p > 2$. Then the element $(xy^2)^2/y^4 = x^2$ is contained in the normalization of R. Now assume that R is F-coherent. In view of Theorem 3.8, x^2 is purely inseparable over R. But then $(x^4)^n = (x^2)^{p^k}$ for some $k > 0$ and thus $2n = p^k$, which cannot happen, because $p > 2$. Hence R is not F-coherent. However, if k has characteristic 2, then R is F-coherent. Finally, the normalization of R is just $k[x, y]$ for any field k, and so $k[x, y]$ is obviously F-coherent.

Theorem 3.11. *Let $(R, \mathfrak{m})$ be a reduced local ring which is a residue class ring of a Gorenstein local ring. Then R^∞ is almost Cohen–Macaulay in the sense that, for some non-zero element $c \in R$,*

$$c^{1/q} \cdot \big((x_1, \ldots, x_i) :_{R^\infty} x_{i+1}\big) \subseteq (x_1, \ldots, x_i)R^\infty$$

for any system of parameters $x_1, \ldots, x_d$ of R and any $q = p^e$. More generally, assume that $R \subseteq R^\infty \subseteq T$ is an integral extension such that T is reduced, quasilocal, and almost Cohen–Macaulay over R in the sense described for R^∞ as above. If T is not a big Cohen–Macaulay R-algebra, then T is not coherent.

Proof. The first assertion was already proved by Roberts, Singh and Srinivas [14, Proposition 1.4] when R is a complete local domain, but we give a proof for completeness. Let $d = \dim R$ and let $x_1, \ldots, x_d$ be a system of parameters. Then by [4, Corollary 8.1.4], there is a non-zero element $c \in R$ for which

$$c \cdot \big((x_1^{p^e}, \ldots, x_i^{p^e}) :_R x_{i+1}^{p^e}\big) \subseteq \big(x_1^{p^e}, \ldots, x_i^{p^e}\big)$$

for $0 \leqslant i \leqslant d-1$ and any integer $e > 0$. Since $R^{1/p^e} \simeq R$ under the e-th Frobenius map, we have $c^{1/p^e} \cdot ((x_1, \ldots, x_i) :_{R^{1/p^e}} x_{i+1}) \subseteq (x_1, \ldots, x_i)R^{1/p^e}$. Since $e > 0$ is arbitrary, we have

$$c^{1/q} \cdot \big((x_1, \ldots, x_i) :_{R^\infty} x_{i+1}\big) \subseteq (x_1, \ldots, x_i)R^\infty$$

for all $q = p^e$ and this is the desired claim.

We prove the second assertion. Then there exists a non-zero element $c \in R$ for which $c^{1/q} \cdot ((x_1, \ldots, x_i) :_T x_{i+1}) \subseteq (x_1, \ldots, x_i)T$. Assume that T is coherent and derive a contradiction. Consider the multiplication map:

$$T/(x_1, \ldots, x_i)T \xrightarrow{x_{i+1}} T/(x_1, \ldots, x_i)T$$

and assume that for some $i > 0$, the kernel of the above map contains a non-zero cyclic T-submodule N. Since $T/(x_1, \ldots, x_i)T$ is finitely presented over T, it is coherent by [8, Theorem 2.3.2], and thus we find that N is a finitely presented cyclic T-module. So we may write it as T/J for some finitely generated ideal J. Note that J is not a unit ideal.

Since T is reduced quasilocal and c is non-zero, we can find a minimal prime P of T such that $c \notin P$ and $(J + P)$ is a proper ideal. $R/P \cap R \to T/P$ is an integral extension of quasilocal domains. Then a valuation on R/P whose center is the unique maximal ideal of $R/P \cap R$ naturally extends as a valuation on T/P whose center is the unique maximal ideal of T/P. Taking the image of $c^{1/q} \cdot T \subseteq J$ under the map $T \to T/P$, we will get a contradiction as argued previously. Hence x_{i+1} must be a non-zero divisor of $T/(x_1, \ldots, x_i)T$. As our initial choice of a system of parameters was arbitrary, we complete the proof. □

For example, if R is a complete local domain and F-pure, but not Cohen–Macaulay, then in view of Theorem 3.11, a system of parameters of R is not a regular sequence on R^∞. Hence R is not F-coherent. Rings of this type abound. Example 3.6 shows that even if R^∞ is Cohen–Macaulay, R is not so in general. We will explore this issue later again. The following proposition is taken from [1, Proposition 2.3] with a slight modification.

Proposition 3.12 *(Criterion for F-purity and F-regularity). Let R be a reduced Noetherian ring of characteristic $p > 0$. Suppose that R is F-coherent and $I \subseteq R$ is any ideal. Then we have $I^F = I^*$. In particular, if R is F-coherent, cyclic F-purity is equivalent to F-regularity.*

Proof. We first recall that $I^F \subseteq I^*$. For a contradiction, let $u \in R$ be such that $u \in I^*$, but $u \notin I^F$. Then since R is F-coherent, it follows that $J := IR^\infty :_{R^\infty} u$ is a finitely generated non-unit ideal, and we have $cu^q \in I^{[q]}$ for $q = p^e \gg 0$ and $c \in R^0$ by our assumption. Hence $c^{1/q}u \in IR^{1/q} \subseteq IR^\infty$, or equivalently $c^{1/q} \in J$ for $q = p^e \gg 0$. Since $c \in R^0$, there is a minimal prime P of R^∞ such that $(J + P)$ is a proper ideal and thus $c \notin P$. Arguing as previously, it is now easy to see that $I^F = I^*$.

Next, suppose that R is F-coherent and cyclically F-pure. Then the above discussion gives that $I = I^F = I^*$ for any ideal I, which implies that every ideal of R is tightly closed. In order to show that every ideal in every localization of R is tightly closed, it suffices to recall that F-coherence is stable under localization, as shown previously. This completes the proof of the proposition. □

As a consequence of the above proposition, tight closure commutes with localization in any F-coherent ring. Namely, we have the following.

Corollary 3.13. *Let R be any reduced F-coherent ring of characteristic $p > 0$ and let $I \subseteq R$ be any ideal. Then $S^{-1}(I^*) = (S^{-1}I)^*$ for any multiplicative subset $S \subseteq R$.*

By the main results of [5], a positive affine semigroup ring $k[C]$ is F-pure if and only if it is normal (see also [10, Theorem 5.33]). However, there is an example of a normal semigroup ring which is not F-coherent. The following example is due to T. Shibuta and K. Watanabe.

Example 3.14. Let $R = k[x^2, xy, y^2]$ for a field k of characteristic $p > 2$. Then

$$\left(\left(x^2, y^2\right) :_{R^{1/q}} xy\right) = \left(x^{1+1/q}, xy^{1/q}, y^{1+1/q}, yx^{1/q}\right)$$

for all $q = p^e$, which implies that $((x^2, y^2) :_{R^\infty} xy)$ is not finitely generated. Note that since $q > 2$ is odd, we find that $(x^{1+1/q}, xy^{1/q}, y^{1+1/q}, yx^{1/q}) \subseteq R^{1/q}$ and $x^2, y^2 \in (x^{1+1/q}, xy^{1/q}, y^{1+1/q}, yx^{1/q})$. To see the above equality, it suffices to show that

$$\left(\left(x^{2q}, y^{2q}\right) :_R x^q y^q\right) = \left(x^{q+1}, x^q y, y^{q+1}, xy^q\right)$$

by taking the q-th Frobenius power. Let $x^q y^q (Ax^2 + Bxy + Cy^2) = Ax^{q+2}y^q + Bx^{q+1}y^{q+1} + Cx^q y^{q+2} \in (x^{2q}, y^{2q})$. By considering only those values of A, B, and C such that no monomial appearing in one of $Ax^{q+2}y^q$, $Bx^{q+1}y^{q+1}$, or $Cx^q y^{q+2}$ does not appear as a monomial in the other two, we have

$$Ax^{q+2}y^q \in \left(x^{2q}, y^{2q}\right), \qquad Bx^{q+1}y^{q+1} \in \left(x^{2q}, y^{2q}\right), \quad \text{and} \quad Cx^q y^{q+2} \in \left(x^{2q}, y^{2q}\right).$$

The rest of the calculation is easy. Since R is a direct summand of a regular ring $k[x, y]$, it follows that a pure subring of a regular ring is not necessarily F-coherent.

We state some consequences on the Cohen–Macaulay property for F-coherent rings under various conditions.

Corollary 3.15 *(Criterion for Cohen–Macaulayness I). Suppose that R is a reduced F-coherent ring of characteristic $p > 0$ and satisfies one of the following conditions*:

(1) *R is a residue class ring of a Cohen–Macaulay ring.*
(2) *R is excellent.*

Then if R is cyclically F-pure, it is Cohen–Macaulay.

Proof. By the previous proposition, we find that if R is cyclically F-pure, then it is F-regular, so that our assumptions imply that it is Cohen–Macaulay due to [11, Theorem 4.2 and Proposition 6.27]. □

The following corollary is very similar to the above one, but there are minor differences in their assumptions.

Corollary 3.16 *(Criterion for Cohen–Macaulayness II). Let $(R, \mathfrak{m})$ be a reduced local ring which is a residue class ring of a Gorenstein local ring. If R is F-coherent and F-injective, then R is Cohen–Macaulay and F-rational.*

Proof. We need only show that $H^k_{\mathfrak{m}}(R) = 0$ for all $k < \dim R$. As was already shown, every system of parameters of R is a regular sequence on R^∞, so $H^k_{\mathfrak{m}}(R^\infty) = 0$ and the natural map $H^k_{\mathfrak{m}}(R) \to H^k_{\mathfrak{m}}(R^\infty)$ is injective for all $k < \dim R$ since R is F-injective, which clearly yields $H^k_{\mathfrak{m}}(R) = 0$ for $k < \dim R$, and thus R is Cohen–Macaulay. Finally, it remains to prove that R is F-rational. Since tight closure coincides with Frobenius closure and R is Cohen–Macaulay F-injective, a simple computation of the top local cohomology module shows that it is F-rational. □

Definition 3.17. Let $R \to S$ be a ring extension. Then we say that R is an *algebra retract of S* if there is a ring homomorphism $\phi : S \to R$ such that the restriction of ϕ to R is the identity map on R.

Theorem 3.18. *Let $R \to S$ be a ring extension of reduced Noetherian rings of characteristic $p > 0$ and suppose that R is an algebra retract of S and S is F-coherent. Then R is F-coherent.*

Proof. Let $\phi : S \to R$ be a retraction map. Then we can extend ϕ to a ring homomorphism $\phi_\infty : S^\infty \to R^\infty$ such that the restriction $\phi_\infty|_{R^\infty}$ is the identity map as follows. The composition of ring homomorphisms:

$$S^{1/q} \xrightarrow{\simeq} S \xrightarrow{\phi} R \xrightarrow{\simeq} R^{1/q},$$

where the first and the third maps are given by the Frobenius maps, yields a compatible sequence of retraction maps ϕ_e with $\phi_0 = \phi$ for every $q = p^e$ and taking its direct limit, we find that $\varinjlim_e \phi_e$ is the desired map. Now we apply [8, Theorem 4.1.5] to conclude that R^∞ is coherent, as desired. □

The situation in the above theorem is found in the case of affine semigroup rings. Let $k[C]$ be an affine semigroup ring and let F be a face of the cone $\mathbb{R}_+C$. Then $k[F \cap C]$ is an algebra retract of $k[C]$. This follows from the fact that $C \setminus F$ is a (prime) ideal. Let P_F be an ideal generated by the monomials X^c for $c \in C \setminus F$. Then $k[C] = k[C \cap F] + P_F$ and clearly $k[C \cap F] \cap P_F = 0$. Hence $k[C \cap F]$ is an algebra retract of $k[C]$. We also remark that an algebra retract of a regular ring is regular, as was shown by Costa [6].

By the next proposition, we can always reduce to the henselian local rings to study F-coherent local rings.

Proposition 3.19. *Let $(R, \mathfrak{m})$ be a reduced local ring of characteristic $p > 0$. Then R is F-coherent if and only if the henselization R^{h} is so.*

Proof. Since $R \to R^{\mathrm{h}}$ is faithfully flat, R is F-coherent if R^{h} is so. Conversely, assume that R is F-coherent. Then we show that F-coherence passes to any finite étale extension $R \to S$. Since S is étale over R, the natural ring homomorphism $R^{1/q} \otimes_R S \to S^{1/q}$ is an isomorphism for all $q > 0$. Hence we have $R^\infty \otimes_R S \simeq S^\infty$. Since $R^\infty \otimes_R S$ is a module-finite free extension of R^∞, it is coherent as well. Recall that the localization of an F-coherent ring is F-coherent. Therefore, any standard étale extension (which is the same as the localization of a module-finite étale extension of a ring) of R is

F-coherent. Quite generally, if $R \to S \to T$ is a composition of ring homomorphisms such that $R \to T$ is étale and $R \to S$ is unramified, then $S \to T$ is étale. In particular, it is flat.

Finally, since the flat colimit of F-coherent rings is F-coherent if the colimit is Noetherian, and the henselization of R can be constructed as a colimit by taking all standard étale R-algebras, we conclude that R^{h} is F-coherent. □

Question 1. Let $R \to S$ be a faithfully flat map of Noetherian rings such that R is F-coherent. Then what conditions on the fibers of $R \to S$ are required in order that S is F-coherent?

We do not know if this question is true for the case where S is smooth over R, because we do not even know whether the polynomial ring $R[x]$ is F-coherent or not, if we assume that R is F-coherent (see [7] for a related result).

Question 2. Let R be a local ring and let $t \in R$ be a non-zero divisor. If R/tR is F-coherent, then is R also F-coherent?

This question is not obvious, since the property of coherence is not necessarily inherited from the quotient of a ring by a non-zero divisor. As there is an example of hypersurface ring (a quotient of a regular domain modulo a non-zero element) which is F-pure, but not F-regular, the converse of the above question is not true.

Question 3. Let R be an F-coherent local ring. Then is the Hilbert–Kunz multiplicity of R rational?

The Hilbert–Kunz multiplicity of a local ring having finite F-representation type is known to be rational, due to Seibert [15]. Although the author does not have any intuition on the connection between F-coherent rings and rings having finite F-representation type, the above question seems natural. As our examples of F-coherent rings always came from the perfect closure of regular rings, it is quite important to answer the following question:

Question 4. Can one find an example of an F-coherent ring whose perfect closure does not coincide with the perfect closure of a regular ring?

Question 5. Suppose R is a Noetherian ring of characteristic $p > 0$. Then can one find any practical characterization of F-coherent rings other than the definition itself?

Remark 3.20.

(1) Let us make some observations on singularities of "F-coherent type" in characteristic zero by considering the arithmetic deformation over $\operatorname{Spec}\mathbb{Z}$. First, we consider $\mathbb{Z}[t^2, t^3] \simeq \mathbb{Z}[x, y]/(x^3 - y^2)$. Then for a prime $p > 0$, the ring $(\mathbb{Z}/p\mathbb{Z})[t^2, t^3]$ is F-coherent, due to Corollary 3.5. Hence the generic fiber of $\operatorname{Spec}\mathbb{Z}[x, y]/(x^3 - y^2) \to \operatorname{Spec}\mathbb{Z}$ is supposed to be of F-coherent type. On the other hand, let us take $\mathbb{Z}[x, y]/(y^2 - x^3 - x^2)$, an ordinary double point. Then $\mathbb{Z}[x, y]/(y^2 - x^3 - x^2) \simeq \mathbb{Z}[t, t\sqrt{t+1}]$, so we have that $(\mathbb{Z}/p\mathbb{Z})[t, t\sqrt{t+1}] \to (\mathbb{Z}/p\mathbb{Z})[t, \sqrt{t+1}]$ is the normalization map and purely inseparable only when $p = 2$. In view of Corollary 3.9, the property that closed fibers are F-coherent can be isolated in the arithmetic deformation.
(2) The example $R = (\mathbb{Z}/p\mathbb{Z})[t, t\sqrt{t+1}]$ discussed above shows that the perfect closure R^{∞} is not normal, when $p \neq 2$.
(3) Other than coherence, there are various notions (e.g. stably coherent, flat dimension, etc.) defined over non-Noetherian rings. And thus, it may be interesting to discuss them on the perfect closure of Noetherian rings, as is done in this article.

4. Tight closure in perfect rings and coherence

In this section, we show that Brenner and Monsky's recent example [3] that tight closure does not commute with localization is related to the coherence of certain perfect rings. While the failure of coherence for the absolute integral closure was already found by Aberbach and Hochster [1], the main idea discussed in this section has the advantage that it can throw some light on the nature of general perfect rings.

We first establish some basic properties of tight closure of finitely generated ideals on certain perfect rings. Let A be a perfect ring, that is, the Frobenius map is bijective on A. Then the Frobenius functor $\mathbf{F}_A$ is obviously flat.

Lemma 4.1. *Let A be a perfect coherent ring of characteristic $p > 0$. Suppose that A is I-adically separated for every finitely generated ideal I of A; that is, $\bigcap_{n>0} I^n = 0$. Then every finitely generated ideal of A is tightly closed.*

Proof. Let I be any finitely generated ideal and let $z \in I^*$. Then there exists $c \in A^0$ such that $cz^{[q]} \in I^{[q]}$ for $q = p^e \gg 0$. If we assume $z \notin I$, then $c \in (I^{[q]} :_A z^{[q]})$ is a non-zero proper ideal of A. By the flatness of $\mathbf{F}_A$, we have

$$(I :_A z)^{[q]} = \left(I^{[q]} :_A z^{[q]}\right),$$

which is finitely generated, due to [8, Theorem 2.3.2]. Hence

$$c \in \bigcap_{q>0} (I :_A z)^{[q]} \subseteq \bigcap_{q>0} (I :_A z)^q = 0.$$

But this is a contradiction. □

The following lemma is due to Hochster [9] for a complete local domain.

Lemma 4.2. *Let R be any Noetherian domain and let I be its proper ideal. Then R^+ is I-adically separated.*

Proof. To get a contradiction, assume there is an element $0 \neq c \in \bigcap_{n>0} I^n R^+$. Extend R to its module-finite extension domain S so that $c \in IS$, and we may choose a prime ideal P of S such that $IS \subseteq P$ together with a discrete valuation ring (V, tV) attached to (S, P). Then we have $c \in \bigcap_{n>0} I^n R^+ \cap V \subseteq \bigcap_{n>0} t^n V^+ \cap V = \bigcap_{n>0} t^n V = 0$, which is a contradiction. □

The following proposition looks similar to Proposition 3.12. However, the difference is that Proposition 3.12 addresses consequences on the tight closure of ideals on Noetherian rings, while the following does so directly on perfect rings.

Proposition 4.3. *Let R be a Noetherian domain of characteristic $p > 0$. Suppose that S is any perfect domain that is integral over R. If there is a finitely generated ideal of S that is not tightly closed, then S is not coherent.*

Proof. Note first that $R \subseteq S \subseteq R^+$. Let I be any finitely generated ideal of S. Then extend R to its module-finite extension domain R' such that the generators of I are contained in the Noetherian domain R'. Then applying Lemma 4.2, we obtain $\bigcap_{n>0} I^n S \subseteq \bigcap_{n>0} I^n R^+ = 0$. The proposition now follows from Lemma 4.1. □

We recall the following example and prove the failure of coherence in the absolute integral closure.

Example 4.4 *(Brenner–Monsky).* Recently, an example of a Noetherian domain R of characteristic 2 together with an ideal I such that $I^+ \neq I^*$ has been constructed. Here $I^+ = IR^+ \cap R$ is the plus closure of I. Let $L = \overline{(\mathbb{Z}/2\mathbb{Z})}$, the algebraic closure of $\mathbb{Z}/2\mathbb{Z}$, and

$$R := R_t = L[x, y, z, t]/(g_t),$$

where $g_t = z^4 + xyz^2 + x^3z + y^3z + tx^2y^2$. Let also $S = L[t] - \{0\}$. The ring R is a three-dimensional normal hypersurface ring. Then they show that for an ideal $I = (x^4, y^4, z^4)$, $y^3z^3 \in (S^{-1}I)^*$, but y^3z^3 is not in $S^{-1}(I^*)$.

Proposition 4.5. *Let R be as in the above example. Then R^+ is not coherent.*

Proof. Quite generally, it is known that $I^+ \subseteq I^*$ [4, Remarks 10.1.6]. By definition, plus closure commutes with localization, so that $I^+ \neq I^*$. Their example, in fact, shows that $I^+ \subsetneq I^*$ and thus R is not F-coherent. Assume R^+ is coherent. Then we have by Lemma 4.1 that $IR^+ = (IR^+)^*$. Therefore:

$$I^* \subseteq I^*R^+ \cap R \subseteq \left(IR^+\right)^* \cap R \subseteq IR^+ \cap R = I^+,$$

which yields a contradiction that $I^* \subseteq I^*R^+ \cap R = I^+$. Here $I^*R^+ \subseteq (IR^+)^*$ can be verified easily from the definition of tight closure. Hence this shows that R^+ is not coherent. Of course, this fact is quite immediate from [1, Proposition 2.3], while our argument makes a direct use of tight closure on R^+. □

Let R be a Dedekind domain. Then both R^∞ and R^+ are coherent, since they are both obtained as the flat colimit of Dedekind domains due to a theorem of Krull and Akizuki. In higher dimension, R^+ is almost never coherent, while the coherence of R^∞ depends on R.

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