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Good filtrations and strong F-regularity of the ring of U_P-invariants

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ABSTRACT

Let k be an algebraically closed field of positive characteristic, G a reductive group over k, and V a finite dimensional G-module. Let P be a parabolic subgroup of G, and U_P its unipotent radical. We prove that if $S = \operatorname{Sym} V$ has a good filtration, then S^{U_P} is strongly F-regular.

1. Introduction

Throughout this paper, k denotes an algebraically closed field, and G a reductive group over k. We fix a maximal torus T and a Borel subgroup B which contains T. We fix a base Δ of the root system Σ of G so that B is negative. For any weight $\lambda \in X(T)$, we denote the induced module $\operatorname{ind}_B^G(\lambda)$ by $\nabla_G(\lambda)$. We denote the set of dominant weights by X^+. For $\lambda \in X^+$, we call $\nabla_G(\lambda)$ the *dual Weyl module* of highest weight λ. Note that for $\lambda \in X(T)$, $\operatorname{ind}_B^G(\lambda) \neq 0$ if and only if $\lambda \in X^+$ [Jan, (II.2.6)], and if this is the case, $\nabla_G(\lambda) = \operatorname{ind}_B^G(\lambda)$ is finite dimensional [Jan, (II.2.1)]. We denote $\nabla_G(-w_0\lambda)^*$ by $\Delta_G(\lambda)$, and call it the *Weyl module* of highest weight λ, where w_0 is the longest element of the Weyl group of G.

We say that a G-module W is *good* if $\operatorname{Ext}_G^1(\Delta_G(\lambda), W) = 0$ for any $\lambda \in X^+$. A filtration $0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_r$ or $0 = W_0 \subset W_1 \subset W_2 \subset \cdots$ of W is called a *good filtration* of W if $\bigcup_i W_i = W$, and for any $i \geqslant 1$, $W_i/W_{i-1} \cong \nabla_G(\lambda(i))$ for some $\lambda(i) \in X^+$. A G-module W has a good filtration if and only if W is good and of countable dimension [Don1]. See also [Fr] and [Has1, (III.1.3.2)].

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Let V be a finite dimensional G-module. Let P be a parabolic subgroup of G containing B, and U_P its unipotent radical. The objective of this paper is to prove the following.

Corollary 5.5. *Let k be of positive characteristic. Let V be a finite dimensional G-module, and assume that $S = \operatorname{Sym} V$ is good as a G-module. Then S^{U_P} is a finitely generated strongly F-regular Gorenstein UFD.*

A Noetherian ring R of characteristic p is said to be strongly F-regular if any R-submodule of any R-module is tightly closed, see (2.3). Note that if R is F-finite, then it is strongly F-regular if and only if for any nonzerodivisor a of R, there exists some $r > 0$ such that the $R^{(r)}$-linear map $aF^r : R^{(r)} \to R$ ($x^{(r)} \mapsto ax^{p^r}$) is $R^{(r)}$-pure (or equivalently, an $R^{(r)}$-split mono, as R is F-finite. This condition, defined in [HH1] for F-finite rings and simply called the strong F-regularity, is called the 'very strongly F-regular' property in this paper in order to avoid confusion, see Lemma 2.4). See (2.1) for the notation. A strongly F-regular Noetherian F-finite $\mathbb{F}_p$-algebra is F-regular in the sense of Hochster and Huneke [HH2], and hence it is Cohen–Macaulay normal ([HH3, (4.2)], [Kun], and [Vel, (0.10)]).

Under the same assumption as in Corollary 5.5, it has been known that S^G is strongly F-regular [Has2]. This old result is a corollary to our Corollary 5.5, since T is linearly reductive and $S^G = S^B = (S^U)^T$ is a direct summand subring of S^U. Under the same assumption as in Corollary 5.5, it has been proved that S^U is F-pure [Has6]. An F-finite Noetherian ring R of characteristic p is said to be F-pure if the Frobenius map $F : R^{(1)} \to R$ is pure (or equivalently, a split mono, as R is F-finite) as an $R^{(1)}$-linear map. So an F-finite strongly F-regular ring is F-pure, and hence Corollary 5.5 (or Corollary 4.14) yields this old result, too.

Popov [Pop3] proved that if the characteristic of k is zero, G is a reductive group over k, and A is a finitely generated G-algebra, then A has rational singularities if and only if A^U does so. Corollary 5.5 (or Corollary 4.14) can be seen as a weak characteristic p version of one direction of this result. For a characteristic p result related to the other direction, see Corollary 3.9.

Section 2 is preliminaries. We review the Frobenius twisting of rings, modules, and representations. We also review the basics of F-singularities such as F-rationality and F-regularity.

In Section 3, we study the ring theoretic properties of the invariant subring $k[G]^U$ of the coordinate ring $k[G]$. The main results of this section are Lemma 3.8 and Corollary 3.9.

In Section 4, we state and prove our main result for $P = B$. In order to do so, we introduce the notion of G-strong F-regularity and G-F-purity. These notions have already appeared in [Has2] essentially. Our main theorem in the most general form can be stated using these words (Theorem 4.12). As in [Has2], Steinberg modules play important roles.

In Section 5, we generalize the main results in Section 4 to the case of general P. Donkin's results on U_P-invariants of good G-modules play an important role here.

In Section 6, we give some examples. The first one is the action associated with a finite quiver. The second one is a special case of the first, and is a determinantal variety studied by De Concini and Procesi [DP]. The third one is also an example of the first. It gives some new understandings on the study of Goto, Hayasaka, Kurano and Nakamura [GHKN]. It also has some relationships with Miyazaki's study [Miy].

In Section 7, we prove the following.

Theorem 7.11. *Let S be a scheme, G a reductive S-group acting trivially on a Noetherian S-scheme X. Let M be a locally free coherent $(G, \mathcal{O}_X)$-module. Then*

$$\operatorname{Good}(\operatorname{Sym} M) = \bigl\{x \in X \bigm| \operatorname{Sym}\bigl(\kappa(x) \otimes_{\mathcal{O}_{X,x}} M_x\bigr) \textit{ is a good } \bigl(\operatorname{Spec} \kappa(x) \times_S G\bigr)\textit{-module}\bigr\},$$

and $\operatorname{Good}(\operatorname{Sym} M)$ *is Zariski open in X.*

Here, for a quasi-coherent $(G, \mathcal{O}_X)$-module N,

$$\operatorname{Good}(N) = \bigl\{x \in X \bigm| N_x \text{ is a good } (\operatorname{Spec} \mathcal{O}_{X,x} \times_S G)\text{-module}\bigr\}.$$

For a reductive group G over a field which is not linearly reductive, there is a finite dimensional G-module V such that $(\operatorname{Sym} V)^G$ is not Cohen–Macaulay [Kem]. On the other hand, in characteristic zero, a reductive group G is linearly reductive, and Hochster and Roberts [HR] proved that $(\operatorname{Sym} V)^G$ is Cohen–Macaulay for any finite dimensional G-module V. Later, Boutot proved that $(\operatorname{Sym} V)^G$ has rational singularities [Bt]. In view of Corollary 5.5 and Theorem 7.11, it seems that the condition $\operatorname{Sym} V$ being good is an appropriate condition to ensure that the good results in characteristic zero still holds.

2. Preliminaries

(2.1) Throughout this paper, p denotes a prime number. Let K be a perfect field of characteristic p.

For a K-space V and $e \in \mathbb{Z}$, we denote the abelian group V with the new K-space structure $\alpha \cdot v = \alpha^{p^{-e}} v$ by $V^{(e)}$, where the product of $\alpha^{p^{-e}}$ and v in the right hand side is given by the original K-space structure of V. An element of V, viewed as an element of $V^{(e)}$ is sometimes denoted by $v^{(e)}$ to avoid confusion. Thus we have $v^{(e)} + w^{(e)} = (v+w)^{(e)}$ and $\alpha v^{(e)} = (\alpha^{p^{-e}} v)^{(e)}$. If $f : V \to W$ is a K-linear map, then $f^{(e)} : V^{(e)} \to W^{(e)}$ given by $f^{(e)}(v^{(e)}) = (fv)^{(e)}$ is a K-linear map again. Note that $(?)^{(e)}$ is an autoequivalence of the category of K-vector spaces.

If A is a K-algebra, then $A^{(e)}$ with the multiplicative structure of A is a K-algebra. So $a^{(e)}b^{(e)} = (ab)^{(e)}$ for $a, b \in A$. If M is an A-module, then $M^{(e)}$ is an $A^{(e)}$-module by $a^{(e)}m^{(e)} = (am)^{(e)}$. For a K-algebra A and $r \geqslant 0$, the rth Frobenius map $F^r = F^r_A : A \to A$ is defined by $F^r(a) = a^{p^r}$. Then $F^r : A^{(r+e)} \to A^{(e)}$ is a K-algebra map for $e \in \mathbb{Z}$. Note that $F^r(a^{(r+e)}) = (a^{p^r})^{(e)}$. $F^r : A^{(r+e)} \to A^{(e)}$ is also written as $(F^r)^{(e)}$.

In commutative algebra, $A^{(e)}$ is sometimes denoted by ${}^{-e}A$, A^{p^e}, or $A^{(p^{-e})}$.

(2.2) For a K-scheme X, the scheme X with the new K-scheme structure $X \xrightarrow{f} \operatorname{Spec} K \xrightarrow{{}^a(F_K^{-e})} \operatorname{Spec} K$ is denoted by $X^{(e)}$, where f is the original structure map of X as a K-scheme. So for a K-algebra A, $\operatorname{Spec} A^{(e)}$ is identified with $(\operatorname{Spec} A)^{(e)}$. The Frobenius map $F^r : X \to X^{(r)}$ is a K-morphism. Note that $(?)^{(e)}$ is an autoequivalence of the category of K-schemes with the quasi-inverse $(?)^{(-e)}$, and it preserves the product. So the canonical map $(X \times Y)^{(e)} \to X^{(e)} \times Y^{(e)}$ is an isomorphism. If G is a K-group scheme, then with the product $G^{(e)} \times G^{(e)} \cong (G \times G)^{(e)} \xrightarrow{\mu^{(e)}} G^{(e)}$, $G^{(e)}$ is a K-group scheme, and $F^r : G^{(e)} \to G^{(e+r)}$ is a homomorphism of K-group schemes. If V is a G-module, then $V^{(e)}$ is a $G^{(e)}$-module in a natural way. Thus $V^{(r)}$ is a G-module again for $r \geqslant 0$ via $F^r : G \to G^{(r)}$. If V has a basis $v_1, \dots, v_n$, $g \in G(K)$, and $gv_j = \sum_i c_{ij} v_i$, then $gv_j^{(r)} = \sum_i c_{ij}^{p^r} v_i^{(r)}$. If A is a G-algebra, then $A^{(r)}$ is a G-algebra again. If M is a (G, A)-module, then $M^{(r)}$ is a $(G, A^{(r)})$-module. See [Has2].

(2.3) Let A be an $\mathbb{F}_p$-algebra. We say that A is *F-finite* if A is a finite $A^{(1)}$-module. An F-finite Noetherian K-algebra is excellent [Kun].

Let A be Noetherian. We denote by A° the set $A \setminus \bigcup_{P \in \operatorname{Min} A} P$, where $\operatorname{Min} A$ denotes the set of minimal primes of A. Let M be an A-module and N a submodule. We define

$$\operatorname{Cl}_A(N, M) = N^*_M := \{ x \in M \mid \exists c \in A^\circ \ \exists e_0 \geqslant 1 \ \forall e \geqslant e_0 \ x \otimes c^{(-e)} \in M/N \otimes_A A^{(-e)} \text{ is zero} \},$$

and call it the *tight closure* of N in M. Note that $\operatorname{Cl}_A(N, M)$ is an R-submodule of M containing N [HH2, Section 8]. We say that N is *tightly closed* in M if $\operatorname{Cl}_A(N, M) = N$. For an ideal I of A, $\operatorname{Cl}_A(I, A)$ is simply denoted by I^*. If $I^* = I$, then we say that I is tightly closed.

We say that A is *very strongly F-regular* if for any $a \in A^\circ$, there exists some $r \geqslant 1$ such that the $A^{(r)}$-linear map $aF^r_A : A^{(r)} \to A$ is pure as an $A^{(r)}$-linear map. That is, for any $A^{(r)}$-module M, the map $aF^r \otimes 1_M : A^{(r)} \otimes_{A^{(r)}} M \to A \otimes_{A^{(r)}} M$ is injective. We say that A is *strongly F-regular* if $\operatorname{Cl}_A(N, M) = N$ for any A-module M and any submodule N of it [Hoc, p. 166]. We say that A is *weakly F-regular* if $I = I^*$ for any ideal I of A [HH2]. We say that A is *F-regular* if for any prime ideal P of A, A_P is weakly F-regular [HH2]. We say that A is *F-rational* if $I = I^*$ for any ideal I generated by $\operatorname{ht} I$ elements, where $\operatorname{ht} I$ denotes the height of I.

Lemma 2.4. *Let A be a Noetherian $\mathbb{F}_p$-algebra.*

- **(i)** *If A is very strongly F-regular, then it is strongly F-regular. The converse is true, if A is either local, F-finite, or essentially of finite type over an excellent local ring.*
- **(ii)** *If A is strongly F-regular, then it is F-regular. An F-regular ring is weakly F-regular. A weakly F-regular ring is F-rational.*
- **(iii)** *A pure subring of a strongly F-regular ring is strongly F-regular.*
- **(iv)** *An F-rational ring is normal.*
- **(v)** *An F-rational ring which is a homomorphic image of a Cohen–Macaulay ring is Cohen–Macaulay.*
- **(vi)** *A locally excellent F-rational ring is Cohen–Macaulay.*
- **(vii)** *If $A = \bigoplus_{i \geqslant 0} A_i$ is graded and A_0 is a field, and if A is weakly F-regular, then A is very strongly F-regular.*
- **(viii)** *A Gorenstein F-rational ring is strongly F-regular.*

Proof. **(i)** is [Has5, (3.6), (3.9), (3.35)]. **(ii)** is [Has5, (3.7)], [HH2, (4.15)], and [HH3, (4.2)]. **(iii)** is [Has5, (3.17)]. **(iv)** and **(v)** are [HH3, (4.2)]. **(vi)** is [Vel, (0.10)].

(vii) is [LS, (4.3)], if the field A_0 is F-finite. We prove the general case. By [HH2, (4.15)], $A_{\mathfrak{m}}$ is weakly F-regular, where $\mathfrak{m} = \bigoplus_{i>0} A_i$ is the irrelevant ideal. Let K be the perfect closure (the largest purely inseparable extension) of A_0, and set $B := K \otimes_{A_0} A$. Then B is purely inseparable over A. It is easy to see that $B_{\mathfrak{m}} := B \otimes_A A_{\mathfrak{m}}$ is a local ring whose maximal ideal is $\mathfrak{m}B_{\mathfrak{m}}$. By [HH3, (6.17)], $B_{\mathfrak{m}}$ is weakly F-regular. By the proof of [LS, (4.3)], $B_{\mathfrak{m}}$ and B are strongly F-regular. By [Has5, (3.17)], A is strongly F-regular. As A is finitely generated over the field A_0, A is very strongly F-regular by **(i)**.

(viii) Let A be a Gorenstein F-rational ring. By [HH3, (4.2)], $A_{\mathfrak{m}}$ is Gorenstein F-rational for any maximal ideal $\mathfrak{m}$ of A. If $A_{\mathfrak{m}}$ is strongly F-regular for any maximal ideal $\mathfrak{m}$ of A, then A is strongly F-regular by [Has5, (3.6)]. Thus we may assume that $(A, \mathfrak{m})$ is local. Let $(x_1, \dots, x_d)$ be a system of parameters of A. Then an element of $H^d_{\mathfrak{m}}(A)$ as the dth cohomology group of the modified Čech complex [BH, (3.5)] is of the form $a/(x_1 \cdots x_d)^t$ for some $t \geqslant 0$ and $a \in A$. This element is zero if and only if $a \in (x_1^t, \dots, x_d^t)$, by [BH, (10.3.20)]. So this element is in the tight closure $(0)^*_{H^d_{\mathfrak{m}}(A)}$ of 0 if and only if $a \in (x_1^t, \dots, x_d^t)^* = (x_1^t, \dots, x_d^t)$, and hence $(0)^*_{H^d_{\mathfrak{m}}(A)} = 0$. As A is Gorenstein, $H^d_{\mathfrak{m}}(A)$ is isomorphic to the injective hull $E_A(A/\mathfrak{m})$ of the residue field $A/\mathfrak{m}$. By [Has5, (3.6)], A is strongly F-regular. $\square$

(2.5) Let K be a field of characteristic zero, and A a K-algebra of finite type. We say that A is of *strongly F-regular type* if there is a finitely generated $\mathbb{Z}$-subalgebra R of A and a finitely generated flat R-algebra A_R such that $A \cong K \otimes_R A_R$, and for any maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m} \otimes_R A_R$ is strongly F-regular. See [Har, (2.5.1)].

3. The invariant subring $k[G]^U$

(3.1) Let the notation be as in Introduction. Let Λ be an abelian group. We say that $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ is a *Λ-graded G-algebra* if A is both a G-algebra and a Λ-graded k-algebra, and each A_λ is a G-submodule of A for $\lambda \in \Lambda$. This is the same as to say that A is a $G \times \operatorname{Spec} k\Lambda$-algebra, where $k\Lambda$ is the group algebra of Λ over k. It is a commutative cocommutative Hopf algebra with each $\lambda \in \Lambda$ group-like.

We say that a $\mathbb{Z}$-graded k-algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is *positively graded* if $A_i = 0$ for $i < 0$ and $k \cong A_0$.

(3.2) Let the notation be as in Introduction.

We need to review Popov–Grosshans filtration [Pop2,Grs2].

Let us fix (until the end of this section) a function $h : X(T) \to \mathbb{Z}$ such that (i) $h(X^+) \subset \mathbb{N} = \{0, 1, \dots\}$; (ii) $h(\lambda) > h(\mu)$ whenever $\lambda > \mu$; (iii) $h(\chi) = 0$ for $\chi \in X(G)$. Such a function h exists [Grs2, Lemma 6].

Let V be a G-module. For a poset ideal π of X^+, we define $O_\pi(V)$ to be the sum of all the G-submodules W of V such that W belongs to π, that is, if $\lambda \in X^+$ and $W_\lambda \neq 0$, then $\lambda \in \pi$. $O_\pi(V)$

is the biggest G-submodule of V belonging to π. We set $\pi(n) := h^{-1}(\{0, 1, \dots, n\})$ for $n \geqslant 0$ and $\pi(n) = \emptyset$ for $n < 0$. We also define $V\langle n\rangle := O_{\pi(n)}(V)$.

For a G-algebra A, $(A\langle n\rangle)$ is a filtration of A. That is, $1 \in A\langle 0\rangle \subset A\langle 1\rangle \subset \cdots$, $\bigcup_n A\langle n\rangle = A$, and $A\langle n\rangle \cdot A\langle m\rangle \subset A\langle n+m\rangle$. The Rees ring $\mathcal{R}(A)$ of A is the subring $\bigoplus_n A\langle n\rangle t^n$ of $A[t]$. Letting G act on t trivially, $A[t]$ is a G-algebra, and $\mathcal{R}(A)$ is a G-subalgebra of $A[t]$. So the associated graded ring $\mathcal{G}(A) := \mathcal{R}(A)/t\mathcal{R}(A)$ is also a G-algebra.

We denote the opposite of U by U^+.

Theorem 3.3. *(See Grosshans [Grs1, Theorem 16].) Let A be a G-algebra which is good as a G-module. There is a G-algebra isomorphism $\Phi : \mathcal{G}(A) \to (A^{U^+} \otimes k[G]^U)^T$, where U acts right regularly on $k[G]$, T acts right regularly on $k[G]^U$ (because T normalizes U), and G acts left regularly on $k[G]^U$ and trivially on A^{U^+}.*

The direct product $G \times G$ acts on the coordinate ring $k[G]$ by

$$\big((g_1, g_2)f\big)(g) = f\big(g_1^{-1} g g_2\big) \quad \big(f \in k[G],\ g, g_1, g_2 \in G(k)\big).$$

In particular, $k[G]$ is a $G \times B$-algebra. Taking the invariant subring by the subgroup $U = \{e\} \times U \subset G \times B$, $k[G]^U$ is a $G \times T$-algebra, since T normalizes U. Thus $k[G]^U = \bigoplus_{\lambda \in X(T)} k[G]^U_\lambda$ is an $X(T)$-graded G-algebra. As a G-module,

$$k[G]^U_\lambda = \big\{f \in k[G] \bigm| f(gb) = \lambda(b)f(g)\big\} \cong \big((-\lambda) \otimes k[G]\big)^B = \operatorname{ind}_B^G(-\lambda)$$

for $\lambda \in X(B) = X(T)$ by the definition of induction, see [Jan, (I.3.3)]. Thus we have:

Lemma 3.4. $k[G]^U \cong \bigoplus_{\lambda \in X^+} \nabla_G(\lambda) \boxtimes (-\lambda)$ *as a $G \times T$-module. It is an integral domain.*

The converse is also true.

Lemma 3.5. *Let A be a $G \times T$-algebra such that $A \cong \bigoplus_{\lambda \in X^+} \nabla_G(\lambda) \boxtimes (-\lambda)$ as a $G \times T$-module and that A^{U^+} is a domain, where $U^+ = U^+ \times \{e\} \subset G \times T$. Then $A \cong k[G]^U$ as a $G \times T$-algebra.*

Proof. Let $\varphi : X^+ \to X(T) \times X(T)$ be the semigroup homomorphism given by $\varphi(\lambda) = (\lambda, -\lambda)$. A^{U^+} is a $\varphi(X^+)$-graded domain, and each homogeneous component $A^{U^+}_{\varphi\lambda} = \nabla_G(\lambda)^{U^+} \boxtimes (-\lambda)$ is one-dimensional. So by [Has3, Lemma 5.5], $A^{U^+} \cong k\varphi(X^+)$ as an $X(T) \times X(T)$-graded k-algebra.

Set $G' = G \times T$, and $T' = T \times T$. Define $h' : X(T') \cong X(T) \times X(T) \to \mathbb{Z}$ by $h'(\lambda, \mu) = h(\lambda)$. For a G'-algebra A', we have a filtration of A' from h' as in (3.2). We denote the associated graded algebra by $\mathcal{G}'(A')$.

It is easy to see that $A \cong \mathcal{G}'(A)$, and this is isomorphic to $C := (k\varphi(X^+) \otimes (k[G]^U \boxtimes k[T]))^{T'}$ by Theorem 3.3 (applied to G'). We define $\psi : k[G]^U \to C$ by

$$\psi(a \otimes t_{-\lambda}) = (t_\lambda \otimes t_{-\lambda}) \otimes (a \otimes t_{-\lambda}) \otimes t_\lambda,$$

where $a \in \nabla_G(\lambda)$ and for $\mu \in X(T)$, t_μ is the element μ considered as a basis element of $kX(T) = k[T]$. We consider that $\nabla_G(\lambda) \boxtimes (-\lambda) \subset k[G]^U$. With respect to the left regular action, t_λ is of weight $-\lambda$. So ψ is a $G \times T$-algebra isomorphism. Thus $A \cong k[G]^U$ as a $G \times T$-algebra, as desired. $\square$

Assume that G is semisimple simply connected. Then by [Pop2], $X(B) \to \operatorname{Pic}(G/B)$ $(\lambda \mapsto \mathcal{L}(\lambda))$ is an isomorphism, where $\mathcal{L}(\lambda) = \mathcal{L}_{G/B}(\lambda)$ is the G-linearized invertible sheaf on the flag variety G/B, associated to the B-module λ, see [Jan, (I.5.8)]. Thus we have

Lemma 3.6. *If G is semisimple and simply connected, then the Cox ring (the total coordinate ring, see [Cox,EKW]) $\mathrm{Cox}(G/B)$ is isomorphic to $\bigoplus_{\lambda\in X^+}\nabla_G(\lambda)$, as an $X(T)$-graded G-module (that is, $G\times T$-module), where both $H^0(G/B,\mathcal{L}(\lambda))\subset\mathrm{Cox}(G/B)$ and $\nabla_G(\lambda)$ are assigned degree $-\lambda$. The Cox ring $\mathrm{Cox}(G/B)$ is also an integral domain, and hence isomorphic to $k[G]^U$ as an $X(T)$-graded G-algebra.*

Proof. The first assertion follows from the fact that $H^0(G/B,\mathcal{L}(\lambda))=\nabla_G(\lambda)$ for $\lambda\in X^+$ and $H^0(G/B,\mathcal{L}(\lambda))=0$ for $\lambda\in X(B)\setminus X^+$ [Jan, (II.2.6)].

Consider the $\mathcal{O}_{G/B}$-algebra $\mathcal{S}:=\mathrm{Sym}(\mathcal{L}(\lambda_1)\oplus\cdots\oplus\mathcal{L}(\lambda_l))$, where $\lambda_1,\dots,\lambda_l$ are the fundamental dominant weights. Being a vector bundle over G/B, $\underline{\mathrm{Spec}}\,\mathcal{S}$ is integral. Hence $\mathrm{Cox}(G/B)\cong\Gamma(G/B,\mathcal{S})$ is a domain.

The last assertion follows from Lemma 3.5. $\square$

Lemma 3.7. *If G is semisimple and simply connected, then $k[G]^U$ is a UFD.*

Proof. This is a consequence of Lemma 3.6 and [EKW, Corollary 1.2].

There is another proof. Popov [Pop2] proved that $k[G]$ is a UFD. Moreover, G does not have a nontrivial character, since $G=[G,G]$, see [Hum, (29.5)]. It follows easily that $k[G]^\times=k^\times$ by [Ros, Theorem 3]. As U is unipotent, U does not have a nontrivial character. The lemma follows from Remark 3 after Proposition 2 of [Pop1]. See also [Has7, (4.31)]. $\square$

By [Grs1, (2.1)], $k[G]^U$ is finitely generated. See also [RR] and [Grs2, Theorem 9].

By [Has3, Lemma 5.6] and Lemma 3.4, $k[G]^U$ is strongly F-regular in positive characteristic, and strongly F-regular type in characteristic zero. In any characteristic, $k[G]^U$ is Cohen–Macaulay normal.

In any characteristic, if G is semisimple simply connected, being a finitely generated Cohen–Macaulay UFD, $k[G]^U$ is Gorenstein [Mur].

Combining the observations above, we have:

Lemma 3.8. *$k[G]^U$ is finitely generated. It is strongly F-regular in positive characteristic, and strongly F-regular type in characteristic zero. If G is semisimple and simply connected, then $k[G]^U$ is a Gorenstein UFD.*

Corollary 3.9. *Let k be of positive characteristic, and A be a G-algebra which is good as a G-module. If A^U is finitely generated and strongly F-regular, then A is finitely generated and F-rational.*

Proof. This is proved similarly to [Pop3, Proposition 10] and [Grs2, Theorem 17].

As U and U^+ are conjugate, $A^{U^+}\cong A^U$, and it is finitely generated and strongly F-regular by assumption. Note that A is finitely generated [Grs2, Theorem 9].

Note that $k[G]^U$ is finitely generated and strongly F-regular by Lemma 3.8. So the tensor product $A^{U^+}\otimes k[G]^U$ is finitely generated and strongly F-regular by [Has3, (5.2)]. Thus its direct summand subring $(A^{U^+}\otimes k[G]^U)^T$ is also finitely generated and strongly F-regular [HH1, (3.1)]. By Theorem 3.3, $\mathcal{G}(A)$ is finitely generated and strongly F-regular, hence is F-rational ([HH1, (3.1)] and [HH3, (4.2)]). By [HM, (7.14)], A is F-rational. $\square$

4. The main result

Let the notation be as in Introduction. In this section, the characteristic of k is $p>0$.

(4.1) For a G-module W and $r\geqslant 0$, $W^{(r)}$ denotes the rth Frobenius twist of W, see Section 2 and [Jan, (I.9.10)].

Let ρ denote the half-sum of positive roots. For $r\geqslant 0$, let St_r denote the rth Steinberg module $\nabla_G((p^r-1)\rho)$, if $(p^r-1)\rho$ is a weight of G. Note that $(p^r-1)\rho$ is a weight of G if p is odd or $[G,G]$ is simply connected.

The following lemma, which is the dual assertion of [Has2, Theorem 3], follows immediately from [Jan, (II.10.6)].

Lemma 4.2. *Let $(p^r-1)\rho$ be a weight of G for any $r \geqslant 0$. Let V be a finite dimensional G-module. Then there exists some $r_0 \geqslant 1$ such that for any $r \geqslant r_0$ and any subquotient W of V, any nonzero (or equivalently, surjective) G-linear map $\varphi : St_r \otimes W \to St_r$ admits a G-linear map $\psi : St_r \to St_r \otimes W$ such that $\varphi\psi = \mathrm{id}_{St_r}$.*

We set $\tilde{G} = \operatorname{rad} G \times \Gamma$, where $\operatorname{rad} G$ is the radical of G, and $\Gamma \to [G,G]$ is the universal covering of the derived subgroup $[G,G]$ of G. Note that there is a canonical surjective map $\tilde{G} \to G$, and hence any G-module (resp. G-algebra) is a $\tilde{G}$-module (resp. $\tilde{G}$-algebra) in a natural way. The restriction functor $\mathrm{res}^G_{\tilde{G}}$ is full and faithful.

Let $S = \bigoplus_{i \geqslant 0} S_i$ be a positively graded finitely generated G-algebra which is an integral domain.

Assume first that $(p^r-1)\rho$ is a weight of G for $r \geqslant 0$. We say that S is *G-strongly F-regular* if for any nonzero homogeneous element a of S^G, there exists some $r \geqslant 1$ such that the $(G, S^{(r)})$-linear map

$$\mathrm{id} \otimes aF^r : St_r \otimes S^{(r)} \to St_r \otimes S \quad \left(x \otimes s^{(r)} \mapsto x \otimes as^{p^r}\right)$$

is a split mono. In general, we say that S is G-strongly F-regular if it is so as a $\tilde{G}$-algebra.

The following is essentially proved in [Has2]. We give a proof for completeness.

Lemma 4.3. *If S is a G-strongly F-regular positively graded finitely generated G-algebra domain, then S^G is strongly F-regular.*

Proof. We may assume that $G = \tilde{G}$. Let $A := S^G$.

As we assume that S is a finitely generated positively graded domain, A is a finitely generated positively graded domain, see [MFK, Appendix A to Chapter 1]. Let a be a nonzero homogeneous element of A such that $A[1/a]$ is regular. Take $r \geqslant 1$ so that $\mathrm{id} \otimes aF^r : St_r \otimes S^{(r)} \to St_r \otimes S$ is a split mono. Let $\Phi : St_r \otimes S \to St_r \otimes S^{(r)}$ be a $(G, S^{(r)})$-linear map such that $\Phi \circ (\mathrm{id} \otimes aF^r) = \mathrm{id}$. Then consider the commutative diagram of $(G, A^{(r)})$-modules

$$\begin{tikzcd}
St_r \otimes A^{(r)} \arrow[r, hook] \arrow[d, "aF^r"] & St_r \otimes S^{(r)} \arrow[r, "\mathrm{id}"] \arrow[d, "aF^r"] & St_r \otimes S^{(r)} \\
St_r \otimes A \arrow[r, hook] & St_r \otimes S. \arrow[ur, "\Phi"] &
\end{tikzcd}$$

Then applying the functor $\mathrm{Hom}_G(St_r, ?)$ to this diagram, we get the commutative diagram of $A^{(r)}$-modules

$$\begin{tikzcd}
A^{(r)} \arrow[r, "\mathrm{id}"] \arrow[d, "aF^r"] & A^{(r)} \arrow[r, "\mathrm{id}"] \arrow[d] & A^{(r)} \\
A \arrow[r] & \mathrm{Hom}_G(St_r, St_r \otimes S), \arrow[ur] &
\end{tikzcd}$$

see [Has2, Proposition 1, 5]. This shows that the $A^{(r)}$-linear map $aF^r : A^{(r)} \to A$ splits. By [HH1, (3.3)], A is strongly F-regular. $\square$

The following is also proved in [Has2] (see the proof of [Has2, Theorem 6]).

Theorem 4.4. *Let V be a finite dimensional G-module. If $S = \operatorname{Sym} V$ has a good filtration (see Introduction for definition), then S is G-strongly F-regular.*

Lemma 4.5. *Let S be a G-strongly F-regular positively graded finitely generated G-algebra domain, and assume that there exists some $a \in S^G \setminus \{0\}$ such that $S[1/a]$ is strongly F-regular. Then S is strongly F-regular.*

Proof. We may assume that $G = \tilde{G}$. Let I be the radical ideal of S which defines the non-strongly F-regular locus of S. Such an ideal exists, see [HH1, (3.3)]. Then I is $G \times \mathbb{G}_m$-stable, and hence $I \cap S^G$ is $\mathbb{G}_m$-stable. In other words, $I \cap S^G$ is a homogeneous ideal of S^G. By assumption, $0 \neq a \in I \cap S^G$. So $I \cap S^G$ contains a nonzero homogeneous element b. Take $r \geqslant 1$ so that $1 \otimes bF^r : St_r \otimes S^{(r)} \to St_r \otimes S$ has a spitting. Let x be any nonzero element of St_r. Then $x \otimes \mathrm{id} : S^{(r)} \cong k \otimes S^{(r)} \to St_r \otimes S^{(r)}$ given by $s^{(r)} \mapsto x \otimes s^{(r)}$ is a split mono as an $S^{(r)}$-linear map. Thus $(x \otimes \mathrm{id}_S)(bF^r) = (\mathrm{id} \otimes bF^r)(x \otimes \mathrm{id}_{S^{(r)}})$ is a split mono as an $S^{(r)}$-linear map, and hence so is $bF^r : S^{(r)} \to S$. By [HH1, (3.3)], S is strongly F-regular. $\square$

Let S be a finitely generated G-algebra. We say that S is *G-F-pure* if there exists some $r \geqslant 1$ such that $\mathrm{id} \otimes F^r : St_r \otimes S^{(r)} \to St_r \otimes S$ splits as a $(G, S^{(r)})$-linear map. Obviously, a G-strongly F-regular finitely generated positively graded G-algebra domain is G-F-pure. The following is essentially proved in [Has6].

Lemma 4.6. *Let S be a G-F-pure finitely generated G-algebra. Then S^G is F-pure.*

Proof. This is proved similarly to Lemma 4.3. See also [Has6]. $\square$

Lemma 4.7. *Let S and S' be a G-F-pure finitely generated G-algebras. Then the tensor product $S \otimes S'$ is G-F-pure.*

Proof. This is easy, and we omit the proof. $\square$

Lemma 4.8. *Let S be a G-F-pure finitely generated G-algebra, and assume that the $(G, S^{(r)})$-linear map*

$$\mathrm{id} \otimes F^r : St_r \otimes S^{(r)} \to St_r \otimes S$$

splits. Then the $(G, S^{(nr)})$-linear map

$$\mathrm{id} \otimes F^{nr} : St_{nr} \otimes S^{(nr)} \to St_{nr} \otimes S$$

splits for any $n \geqslant 0$.

Proof. Induction on n. The case that $n = 0$ is trivial. (Note that $S^{(0)} = S$, and St_0 should be understood to be the trivial representation k.) Assume that $n > 0$. Note that $St_{nr} \cong St_r \otimes St^{(r)}_{(n-1)r}$. So

$$\mathrm{id} \otimes \left(F^{(n-1)r}\right)^{(r)} : St_{nr} \otimes S^{(nr)} \to St_{nr} \otimes S^{(r)}$$

is identified with the map

$$\mathrm{id} \otimes \left(\mathrm{id} \otimes F^{(n-1)r}\right)^{(r)} : St_r \otimes \left(St_{(n-1)r} \otimes S^{((n-1)r)}\right)^{(r)} \to St_r \otimes \left(St_{(n-1)r} \otimes S\right)^{(r)},$$

and it has a $(G, S^{(nr)})$-linear splitting by the induction assumption. On the other hand, $\mathrm{id} \otimes F^r : St_{nr} \otimes S^{(r)} \to St_{nr} \otimes S$ splits by assumption, as $St_{nr} \cong St_r \otimes St^{(r)}_{(n-1)r}$. Thus the composite

$$St_{nr} \otimes S^{(nr)} \xrightarrow{\mathrm{id} \otimes (F^{(n-1)r})^{(r)}} St_{nr} \otimes S^{(r)} \xrightarrow{\mathrm{id} \otimes F^r} St_{nr} \otimes S,$$

which agrees with $\mathrm{id} \otimes F^{nr}$, has a splitting, as desired. $\square$

Lemma 4.9. *Let $S = \bigoplus_{n \geqslant 0} S_n$ be a finitely generated positively graded G-algebra which is an integral domain. Then the following are equivalent.*

1 *S is G-strongly F-regular.*
2 *S is $[G, G]$-strongly F-regular.*
3 *S is Γ-strongly F-regular, where $\Gamma \to [G, G]$ is the universal covering.*

Proof. The implications $\mathbf{1} \Rightarrow \mathbf{2} \Leftrightarrow \mathbf{3}$ are trivial. We prove the direction $\mathbf{3} \Rightarrow \mathbf{1}$. Replacing G by $\tilde{G}$ if necessary, we may assume that $G = R \times \Gamma$, where R is a torus, and Γ is a semisimple and simply connected algebraic group. Let $a \in S^G$ be any nonzero homogeneous element. Then by assumption, the $(R, (S^{(r)})^\Gamma)$-linear map

$$\left(aF^r\right)^* : \operatorname{Hom}_{\Gamma, S^{(r)}}\left(St_r \otimes S, St_r \otimes S^{(r)}\right) \to \operatorname{Hom}_{\Gamma, S^{(r)}}\left(St_r \otimes S^{(r)}, St_r \otimes S^{(r)}\right)$$

is surjective. Taking the R-invariant,

$$\left(aF^r\right)^* : \operatorname{Hom}_{G, S^{(r)}}\left(St_r \otimes S, St_r \otimes S^{(r)}\right) \to \operatorname{Hom}_{G, S^{(r)}}\left(St_r \otimes S^{(r)}, St_r \otimes S^{(r)}\right)$$

is still surjective, since R is linearly reductive. This is what we wanted to prove. □

The following is proved similarly.

Lemma 4.10. *Let $S = \bigoplus_{n \geqslant 0} S_n$ be a finitely generated G-algebra. Then the following are equivalent.*

1 *S is G-F-pure.*
2 *S is $[G, G]$-F-pure.*
3 *S is Γ-F-pure, where $\Gamma \to [G, G]$ is the universal covering.*

Lemma 4.11. *Let G be semisimple and simply connected. Then $k[G]^U$ is G-F-pure.*

Proof. This is [Has6, Lemma 3]. □

The following is the main theorem of this paper.

Theorem 4.12. *Let $S = \bigoplus_{n \geqslant 0} S_n$ be a finitely generated positively graded G-algebra. Assume that*

1 *S is F-rational and Gorenstein.*
2 *S is G-F-pure.*

Then S is a G-strongly F-regular integral domain.

Proof. Note that S is normal [HH3, (4.2)]. As S is positively graded, S is an integral domain.

Replacing G by Γ, where $\Gamma \to [G, G]$ is the universal covering, we may assume that G is semisimple and simply connected, by Lemma 4.9 and Lemma 4.10.

As S is G-F-pure, there exists some $l \geqslant 1$ such that $\mathrm{id} \otimes F^l : St_l \otimes S^{(l)} \to St_l \otimes S$ has a $(G, S^{(l)})$-linear splitting $\psi : St_l \otimes S \to St_l \otimes S^{(l)}$.

Note that any graded (G, S)-module which is rank one free as an S-module is of the form $S(n)$, where $S(n)$ is S as a (G, S)-module, but the grading is given by $S(n)_i = S_{n+i}$. In fact, let $-n$ be the generating degree of the rank one free graded (G, S)-module, say M, then $M_{-n} \otimes S \to M$ is a (G, S)-isomorphism. As M_{-n} is trivial as a G-module (since G is semisimple), $M_{-n} \cong k(n)$ as a graded G-module. Thus $M \cong k(n) \otimes S \cong S(n)$.

Let a be the a-invariant of the Gorenstein positively graded ring S. Namely, $\omega_S \cong S(a)$ (as a graded (G, S)-module, see the last paragraph). Then by the G-equivariant duality for finite G-morphisms (see [Has4, Theorem 25.2, Theorem 27.8]),

$$\operatorname{Hom}_{S^{(r)}}\big(S, S^{(r)}\big) \cong \operatorname{Hom}_{S^{(r)}}\big(S, (\omega_S)^{(r)}\big(-p^r a\big)\big) \cong \omega_S\big(-p^r a\big) \cong S\big(\big(1-p^r\big)a\big)$$

for $r \geqslant 0$.

Let σ be any nonzero element of $\operatorname{Hom}_{S^{(1)}}(S, S^{(1)})_{(p-1)a} \cong S_0$. As $S_0 = k$ is G-trivial, $\sigma : S \to S^{(1)}$ is $(G, S^{(1)})$-linear of degree $(p-1)a$.

For $r \geqslant 0$, let σ_r be the composite

$$S \xrightarrow{\sigma} S^{(1)} \xrightarrow{\sigma^{(1)}} S^{(2)} \xrightarrow{\sigma^{(2)}} \cdots \xrightarrow{\sigma^{(r-1)}} S^{(r)}.$$

It is $(G, S^{(r)})$-linear of degree $(p^r-1)a$. Note that $\sigma_u = \sigma_{u-r}^{(r)}\sigma_r$ for $u \geqslant r$.

Hence by the composite map

$$\begin{aligned} Q_{r,u} : \operatorname{Hom}_{S^{(r)}}\big(S, S^{(r)}\big) &\xrightarrow{\sigma_{u-r}^{(r)}} \operatorname{Hom}_{S^{(r)}}\big(S, \operatorname{Hom}_{S^{(u)}}\big(S^{(r)}, S^{(u)}\big)\big) \\ &\cong \operatorname{Hom}_{S^{(u)}}\big(S^{(r)} \otimes_{S^{(r)}} S, S^{(u)}\big) \cong \operatorname{Hom}_{S^{(u)}}\big(S, S^{(u)}\big), \end{aligned} \tag{1}$$

the element σ_r is mapped to σ_u, where the first map $\sigma_{u-r}^{(r)}$ maps $f \in \operatorname{Hom}_{S^{(r)}}(S, S^{(r)})$ to the map $x \mapsto f(x) \cdot \sigma_{u-r}^{(r)}$. More precisely, we have $Q_{r,u}(f) = \sigma_{u-r}^{(r)} \circ f$.

By the induction on u, it is easy to see that $Q_{1,u}$ is an isomorphism, and σ_u is a generator of the rank one S-free module $\operatorname{Hom}_{S^{(u)}}(S, S^{(u)})$. It follows that $Q_{r,u}$ is an isomorphism for any $u \geqslant r$.

We continue the proof of Theorem 4.12. Take a nonzero homogeneous element b of $A = S^G$. It suffices to show that there exists some $u \geqslant 1$ such that $\mathrm{id} \otimes bF^u : St_u \otimes S^{(u)} \to St_u \otimes S$ splits as a $(G, S^{(u)})$-linear map.

As S is F-rational Gorenstein, it is strongly F-regular by Lemma 2.4**(viii)**. So there exists some $r \geqslant 1$ such that

$$\big(bF_S^r\big)^* : \operatorname{Hom}_{S^{(r)}}\big(S, S^{(r)}\big) \to \operatorname{Hom}_{S^{(r)}}\big(S^{(r)}, S^{(r)}\big)$$

given by $(bF_S^r)^*(\varphi) = \varphi bF_S^r$ is surjective.

Let V be the degree $-(p^r-1)a-d$ component of S, where d is the degree of b. Note that $V \cong \operatorname{Hom}_{S^{(r)}}(S, S^{(r)})_{-d}$ is mapped onto $k \cong \operatorname{Hom}_{S^{(r)}}(S^{(r)}, S^{(r)})_0$ by $(bF_S^r)^*$. In particular, $-(p^r-1)a-d \geqslant 0$. So $a \leqslant 0$. If $S \neq k$, then it is easy to see that $a < 0$.

By Lemma 4.2, there exists some $u_0 \geqslant 1$ such that for any $u \geqslant u_0$, for any subquotient W of V, and any G-linear nonzero map $f : St_u \otimes W \to St_u$, there exists some G-linear map $g : St_u \to St_u \otimes W$ such that $fg = \mathrm{id}$. Take $u \geqslant u_0$ such that $u-r$ is divisible by l.

Now the diagram

$$\begin{tikzcd} \operatorname{Hom}_{S^{(r)}}(S, S^{(r)}) \arrow[r, two heads, "(bF^r)^*"] \arrow[d, "\cong"', "Q_{r,u}"] & \operatorname{Hom}_{S^{(r)}}(S^{(r)}, S^{(r)}) \arrow[d, "\cong"', "Q_{0,u-r}^{(r)}"] \\ \operatorname{Hom}_{S^{(u)}}(S, S^{(u)}) \arrow[r, "(bF^r)^*"] & \operatorname{Hom}_{S^{(u)}}(S^{(r)}, S^{(u)}) \end{tikzcd}$$

is commutative. So the bottom $(bF^r)^*$ is surjective. Let us consider the surjection

$$\big(bF^r\big)^* : W = \big(\big(bF^r\big)^*\big)^{-1}\big(k \cdot \sigma_{u-r}^{(r)}\big) \cap \operatorname{Hom}_{S^{(u)}}\big(S, S^{(u)}\big)_{ap^r(p^{u-r}-1)-d} \to k \cdot \sigma_{u-r}^{(r)}.$$

By definition, W is contained in the degree $ap^r(p^{u-r}-1)-d$ component of $\mathrm{Hom}_{S^{(u)}}(S, S^{(u)}) = S(-a(p^u-1))$, which is isomorphic to V as a G-module. So W is isomorphic to a G-submodule of V.

Let $E := \mathrm{Hom}(St_u, St_u)$. Then by the choice of u_0 and u, there exists some G-linear map $g_1 : E \to E \otimes W$ such that the composite

$$E \xrightarrow{g_1} E \otimes W \xrightarrow{1\otimes(bF^r)^*} E \otimes \bigl(k \cdot \sigma^{(r)}_{u-r}\bigr)$$

maps φ to $\varphi \otimes \sigma^{(r)}_{u-r}$.

We identify $E \otimes \mathrm{Hom}_{S^{(u)}}(S, S^{(u)})$ with $\mathrm{Hom}_{S^{(u)}}(St_u \otimes S, St_u \otimes S^{(u)})$ in a natural way. Similarly, $E \otimes \mathrm{Hom}_{S^{(r)}}(S^{(r)}, S)$ is identified with $\mathrm{Hom}_{S^{(r)}}(St_u \otimes S^{(r)}, St_u \otimes S)$, and so on.

Then letting $\nu := g_1(\mathrm{id}_{St_u})$, the composite

$$St_u \otimes S^{(r)} \xrightarrow{\mathrm{id}\otimes bF^r} St_u \otimes S \xrightarrow{\nu} St_u \otimes S^{(u)}$$

agrees with $\mathrm{id} \otimes \sigma^{(r)}_{u-r}$.

Since S is G-F-pure, $u-r$ is a multiple of l, and $St_u \cong St_r \otimes St^{(r)}_{u-r}$, there exists some $(G, S^{(u)})$-linear map $\Phi : St_u \otimes S^{(r)} \to St_u \otimes S^{(u)}$ such that $\Phi \circ (\mathrm{id}_{St_u} \otimes F_S^{u-r}) = \mathrm{id}$ by Lemma 4.8.

Viewing Φ as an element of $E \otimes \mathrm{Hom}_{S^{(u)}}(S^{(r)}, S^{(u)})$, let

$$\beta \in E \otimes \mathrm{Hom}_{S^{(r)}}\bigl(S^{(r)}, S^{(r)}\bigr)$$

be the element $(\mathrm{id}_E \otimes (Q^{(r)}_{r,u})^{-1})(\Phi)$. In other words, $\beta : St_u \otimes S^{(r)} \to St_u \otimes S^{(r)}$ is the unique map such that the composite

$$St_u \otimes S^{(r)} \xrightarrow{\beta} St_u \otimes S^{(r)} \xrightarrow{\mathrm{id}\otimes\sigma^{(r)}_{u-r}} St_u \otimes S^{(u)}$$

is Φ.

Write $\beta = \sum_i \varphi_i \otimes a_i^{(r)}$, where $\varphi_i \in E$ and $a_i \in S$. Define $\beta' \in E \otimes \mathrm{Hom}_S(S, S)$ by $\beta' = \sum_i \varphi_i \otimes a_i^{p^r}$. Then it is easy to check that $(\mathrm{id} \otimes bF^r) \circ \beta = \beta' \circ (\mathrm{id} \otimes bF^r)$ as maps $St_u \otimes S^{(r)} \to St_u \otimes S$.

Combining the observations above, the whole diagram of $(G, S^{(u)})$-modules

$$\begin{tikzcd}
St_u \otimes S^{(u)} \arrow[r, "\mathrm{id}\otimes F^{u-r}"] \arrow[d, "\mathrm{id}"] & St_u \otimes S^{(r)} \arrow[dl, "\Phi"'] \arrow[d, "\beta"] \arrow[dr, "\mathrm{id}\otimes bF^r"] & \\
St_u \otimes S^{(u)} & St_u \otimes S^{(r)} \arrow[l, "\mathrm{id}\otimes\sigma^{(r)}_{u-r}"'] \arrow[d, "\mathrm{id}\otimes bF^r"] & St_u \otimes S \arrow[dl, "\beta'"] \\
& St_u \otimes S \arrow[ul, "\nu"] &
\end{tikzcd}$$

is commutative. So $\mathrm{id} \otimes bF^u : St_u \otimes S^{(u)} \to St_u \otimes S$ has a $(G, S^{(u)})$-linear splitting $\nu\beta'$. □

Corollary 4.13. *Let S be as in Theorem 4.12. Then S^U is finitely generated and strongly F-regular.*

Proof. Finite generation is by [Grs2, Theorem 9].

We prove the strong F-regularity. We may assume that G is semisimple and simply connected. Then $k[G]^U$ is a strongly F-regular Gorenstein domain by Lemma 3.8. Hence the tensor product

$S \otimes k[G]^U$ is also a strongly F-regular Gorenstein domain, see [Has3, Theorem 5.2]. As S is assumed to be G-F-pure and $k[G]^U$ is G-F-pure by Lemma 4.11, the tensor product $S \otimes k[G]^U$ is also G-F-pure by Lemma 4.7. Hence by the theorem, $S \otimes k[G]^U$ is G-strongly F-regular. It follows that $(S \otimes k[G]^U)^G$ is strongly F-regular by Lemma 4.3. As $S^U \cong (S \otimes k[G]^U)^G$ (see the proof of [Grs1, (1.2)]. See also [Dol, Lemma 4.1]), we are done. □

Corollary 4.14. *Let V be a finite dimensional G-module, and assume that $S = \operatorname{Sym} V$ has a good filtration as a G-module. Then S^U is finitely generated and strongly F-regular.*

Proof. It follows immediately from Corollary 4.13 and Theorem 4.4. □

5. The unipotent radicals of parabolic subgroups

Let the notation be as in Introduction. Let I be a subset of Δ. Let $L = L_I$ be the corresponding Levi subgroup $C_G(\bigcap_{\alpha\in I}(\operatorname{Ker}\alpha)^\circ)$, where $(?)^\circ$ denotes the identity component, and C_G denotes the centralizer. Let $P = P_I$ be the parabolic subgroup generated by B and L. Let U_P be the unipotent radical of P. Let $B_L := B \cap L$, and U_L be the unipotent radical of B_L.

Here are two theorems due to Donkin.

Theorem 5.1. *(See Donkin [Don3, (1.2)].) Let w_0 and w_L denote the longest elements of the Weyl groups of G and L, respectively. For $\lambda \in X^+$, we have $\nabla_G(\lambda)^{U_P} \cong \nabla_L(w_L w_0 \lambda)$ as L-modules.*

Theorem 5.2. *(See Donkin [Don3, (1.4)], [Don4, (3.9)].) Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of G-modules. If M_1 is good, then

$$0 \to M_1^{U_P} \to M_2^{U_P} \to M_3^{U_P} \to 0$$

is exact. In other words, if M is a good G-module, then $R^i(H^0(U_P, ?) \circ \operatorname{res}^G_{U_P})(M) = 0$ for $i > 0$.

From these two theorems, it follows immediately:

Lemma 5.3. *Let M be a good G-module. Then M^{U_P} is a good L-module.*

So we have:

Proposition 5.4. *Let k be of positive characteristic. Let S be a finitely generated positively graded G-algebra. Assume that S is Gorenstein F-rational, and G-F-pure. Then S^{U_P} is finitely generated and F-rational.*

Proof. By Lemma 5.3, S^{U_P} is good as an L-module. By Corollary 4.13, $(S^{U_P})^{U_L} \cong S^U$ is finitely generated and strongly F-regular. By Corollary 3.9, applied to the action of L on S^{U_P}, we have that S^{U_P} is finitely generated and F-rational. □

Corollary 5.5. *Let k be of positive characteristic. Let V be a finite dimensional G-module, and assume that $S = \operatorname{Sym} V$ is good as a G-module. Then S^{U_P} is a finitely generated strongly F-regular Gorenstein UFD.*

Proof. As U_P is unipotent, S^{U_P} is a UFD by Remark 3 after Proposition 2 of [Pop1].

On the other hand, S satisfies the assumption of Proposition 5.4 by Theorem 4.4. So by Proposition 5.4, S^{U_P} is finitely generated and F-rational. Being a finitely generated Cohen–Macaulay UFD, it is Gorenstein [Mur], and hence is strongly F-regular by Lemma 2.4**(viii)**. □

Remark 5.6. Let k be of characteristic zero. The characteristic-zero counterpart of Proposition 5.4 is stated as follows: If S is a finitely generated G-algebra with rational singularities, then S^{U_P} is finitely generated with rational singularities. This is proved in the same line as Proposition 5.4. Note that S^U is finitely generated with rational singularities by [Pop3, Corollary 4, Theorem 6]. Then applying [Pop3, Corollary 4, Theorem 6] again to the action of L on S^{U_P}, S^{U_P} is finitely generated and has rational singularities, since $(S^{U_P})^{U_L} \cong S^U$ is so.

The characteristic-zero counterpart of Corollary 5.5 is stated as follows: If S is a finitely generated G-algebra with rational singularities and is a UFD, then S^{U_P} is a Gorenstein finitely generated UFD which is of strongly F-regular type. As we have already seen, S^{U_P} is finitely generated with rational singularities. S^{U_P} is a UFD by Remark 3 after Proposition 2 of [Pop1]. As S^{U_P} is also Cohen–Macaulay [KKMS, p. 50, Proposition], S^{U_P} is Gorenstein [Mur]. A Gorenstein finitely generated algebra with rational singularities is of strongly F-regular type, see [Har, (1.1), (5.2)].

6. Applications

The following is pointed out in the proof of [SvdB, (5.2.3)].

Lemma 6.1. *Let K be a field of characteristic zero, H be an extension of a finite group scheme by a torus over K, and A a finitely generated H-algebra of strongly F-regular type. Then A^H is of strongly F-regular type.*

Proof. Set $B = A^H$. Let $\bar{K}$ be the algebraic closure of K. As can be seen easily, if $\bar{K} \otimes_K B$ is of strongly F-regular type, then so is B. Since $\bar{K} \otimes_K B \cong (\bar{K} \otimes_K A)^{\bar{K} \otimes_K H}$, replacing K by $\bar{K}$, we may assume that K is algebraically closed. Then H is an extension of a finite group Γ by a split torus $\mathbb{G}_m^r$ for some r. As $A^H \cong (A^{\mathbb{G}_m^r})^{\Gamma}$, we may assume that H is either a split torus $\mathbb{G}_m^r$ or a finite group Γ.

Now we can take a finitely generated $\mathbb{Z}$-subalgebra R of K and a finitely generated flat R-algebra A_R such that $K \otimes_R A_R \cong A$, and for any closed point x of $\operatorname{Spec} R$, $\kappa(x) \otimes_R A_R$ is strongly F-regular. Extending R if necessary, we have an action of H_R on A_R which is extended to the action of H on A, where $H_R = (\mathbb{G}_m)_R^r$ or $H_R = \Gamma$. Extending R if necessary, we may assume that $n \in R^\times$, where n is the order of Γ, if $H = \Gamma$.

Now set $B_R := A_R^{H_R}$.

If $H = (\mathbb{G}_m)_R^r$, then B_R is the degree zero component of the $\mathbb{Z}^r$-graded finitely generated R-algebra A_R, and it is finitely generated, and is a direct summand subring of A_R.

If $H = \Gamma$, then $B_R \to A_R$ is an integral extension and B_R is finitely generated by [AM, (7.8)]. As $\rho : A_R \to B_R$ given by $\rho(a) = (1/n)\sum_{\gamma \in \Gamma} \gamma a$ is a splitting, B_R is a direct summand subring of A_R.

In either case, B_R is finitely generated over R, so extending R if necessary, we may assume that B_R is R-flat. Note that $B \cong K \otimes_R B_R$, since K is R-flat, and the invariance is compatible with a flat base change. Note also that $\kappa(x) \otimes_R B_R$ is a direct summand subring of $\kappa(x) \otimes_R A_R$, and $\kappa(x) \otimes_R A_R$ is strongly F-regular. Hence $\kappa(x) \otimes_R B_R$ is strongly F-regular by Lemma 2.4**(iii)**. This shows that $A^H = B$ is of strongly F-regular type. $\square$

The following is a refinement of [SvdB, (5.2.3)].

Corollary 6.2. *Let K be a field of characteristic zero, H an affine algebraic group scheme over K such that H° is reductive. Let S be a finitely generated H-algebra which has rational singularities and is a UFD. Then S^H is of strongly F-regular type.*

Proof. Let $H' := [H^\circ, H^\circ]$. Then $\bar{K} \otimes_K H'$ is semisimple, and does not have a nontrivial character. Thus $S^{H'}$ has rational singularities by Boutot's theorem [Bt] and is a UFD by [Has7, (4.28)]. So it is of strongly F-regular type by Hara [Har, (1.1), (5.2)]. As $(H/H')^\circ$ is a torus, $S^H = (S^{H'})^{H/H'}$ is of strongly F-regular type by Lemma 6.1. $\square$

Theorem 6.3. *Let k be an algebraically closed field, and $Q = (Q_0, Q_1, s, t)$ a finite quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows, and s and t are the source and the target maps $Q_1 \to Q_0$, respectively. Let*

$d : Q_0 \to \mathbb{N}$ be a map. For $i \in Q_0$, set $M_i := k^{d(i)}$, and let H_i be a closed subgroup scheme of $GL(M_i)$ of one of the following forms:

(1) *$GL(M_i)$, $SL(M_i)$;*
(2) *$Sp_{d(i)}$ (in this case, $d(i)$ is required to be even);*
(3) *$SO_{d(i)}$ (in this case, the characteristic of k must not be two);*
(4) *Levi subgroup of any of* **(1)**–**(3)**;
(5) *Derived subgroup of any of* **(1)**–**(4)**;
(6) *Unipotent radical of a parabolic subgroup of any of* **(1)**–**(5)**;
(7) *Any subgroup H_i of $GL(M_i)$ with a closed normal subgroup N_i of H_i such that N_i is any of* **(1)**–**(6)***, and H_i/N_i is a linearly reductive group scheme. In characteristic zero, we require that $(H_i/N_i)^\circ$ is a torus.*

Set $H := \prod_{i\in Q_0} H_i$ and $M := \prod_{\alpha\in Q_1} \operatorname{Hom}(M_{s(\alpha)}, M_{t(\alpha)})$. Then $(\operatorname{Sym} M^*)^H$ is finitely generated, and strongly F-regular if the characteristic of k is positive, and strongly F-regular type if the characteristic of k is zero.

Proof. If H_i satisfies **(7)** and the corresponding N_i satisfies $(\mathbf{x})$, where $1 \leqslant \mathbf{x} \leqslant 6$, then we say that H_i is of type $(\mathbf{7}, \mathbf{x})$.

Note that $\operatorname{Sym} M^* \cong \bigotimes_{\alpha\in Q_1} \operatorname{Sym}(M_{s(\alpha)} \otimes M^*_{t(\alpha)})$.

First we prove that $\operatorname{Sym} M^*$ has a good filtration as an H-module if each H_i is as in **(1)**–**(5)**. To verify this, we only have to show that $\operatorname{Sym}(M_{s(\alpha)} \otimes M^*_{t(\alpha)})$ is a good H-module for each α, by Mathieu's tensor product theorem [Mat]. This module is trivial as an H_i-module if $i \neq s(\alpha), t(\alpha)$. Thus it suffices to show that this is good as an $H_{s(\alpha)} \times H_{t(\alpha)}$-module if $s(\alpha) \neq t(\alpha)$, and as an $H_{s(\alpha)}$-module if $s(\alpha) = t(\alpha)$, see [Has2, Lemma 4]. By [Has2, Lemma 3, **3**, **5**, **6**] and [Don2, (3.2.7), (3.4.3)], the assertion is true for $H_{s(\alpha)}$, $H_{t(\alpha)}$ of **(1)**–**(3)**. By Mathieu's theorem [Mat, Theorem 1], the groups of type **(4)** is also allowed. By [Don2, (3.2.7)] again, the groups of type **(5)** is also allowed. By [Has2, Theorem 6], the conclusion of the theorem holds this case.

We consider the general case. If H_i is of the form **(1)**–**(5)**, then considering $N_i = U_i \subset B_i \subset H_i$, where B_i is a Borel subgroup of H_i and U_i its unipotent radical, B_i is a group of the form **(7)**, and as the H_i-invariant and the B_i-invariant are the same thing for an H_i-module, we may replace H_i by B_i without changing the invariant subring. Hence in this case, we may assume that H_i is of the form **(7, 6)**. Clearly, a group of the form **(6)** is also of the form **(7, 6)**, letting $N_i = H_i$. So we may assume that each H_i is of type **(7)**. If $(\operatorname{Sym} M^*)^N$ is strongly F-regular (type), where $N = \prod_{i\in Q_0} N_i$, then $(\operatorname{Sym} M^*)^H \cong ((\operatorname{Sym} M^*)^N)^{H/N}$ is also strongly F-regular in positive characteristic, since $H/N \cong \prod_{i\in Q_0} H_i/N_i$ is linearly reductive and $(\operatorname{Sym} M^*)^H$ is a direct summand subring of $(\operatorname{Sym} M^*)^N$. In characteristic zero, $(H/N)^\circ$ is a torus, and we can invoke Lemma 6.1. Thus we may assume that each H_i is of the form **(1)**–**(6)**. Then again by the argument above, we may assume that each H_i is of the form **(7, 6)**. Again by the argument above, we may assume that each H_i is of the form **(6)**. Now suppose that $H_i \subset G_i \subset GL(M_i)$, and each G_i if of the form **(1)**–**(5)**, and H_i is the unipotent radical of the parabolic subgroup P_i of G_i. Then letting $G := \prod G_i$ and $P := \prod P_i$, $H = \prod H_i$ is the unipotent radical of the parabolic subgroup P of G. As $\operatorname{Sym} M^*$ has a good filtration as a G-module by the first paragraph, $(\operatorname{Sym} M^*)^H$ is finitely generated and strongly F-regular (type) by Corollary 5.5 and Remark 5.6. $\square$

This covers Example 1 and Example 2 of [Has2], except that we do not consider the case $p = 2$ here, if O_n or SO_n is involved. For example,

Example 6.4. Let $Q = 1 \to 2 \to 3$, $(d(1), d(2), d(3)) = (m, t, n)$, $H_1 = H_3 = \{e\}$, and $H_2 = GL_t$. Then $M = \operatorname{Hom}(M_2, M_3) \times \operatorname{Hom}(M_1, M_2)$, and $M \to M /\!\!/ H$ is identified with

$$\pi : M \to Y_t = \{ f \in \operatorname{Hom}(M_1, M_3) \mid \operatorname{rank} f \leqslant t \},$$

where $\pi(\varphi, \psi) = \varphi\psi$ (De Concini and Procesi [DP]). Thus (the coordinate ring of) Y_t is strongly F-regular (type), as was proved by Hochster and Huneke [HH4, (7.14)] (F-regularity and strong F-regularity are equivalent for positively graded rings, see Lemma 2.4).

Next we consider an example which really requires a group of type **(7)** in Theorem 6.3.

Let K be a field, and $M = K^m$, $N = K^n$. Let $1 \leqslant s \leqslant n$, and $\underline{a} = (0 = a_0 < a_1 < \cdots < a_s = n)$ be an increasing sequence of integers. Let $\mathfrak{G}$, $\mathfrak{S}$, and $\mathfrak{T}$ be disjoint subsets of $\{1, \ldots, s\}$ such that $\mathfrak{G} \coprod \mathfrak{S} \coprod \mathfrak{T} = \{1, \ldots, s\}$. Let

$$H = H(\underline{a}; \mathfrak{G}, \mathfrak{S}, \mathfrak{T}) := \begin{pmatrix} \boxed{H_1} & & & * \\ & \boxed{H_2} & & \\ & & \ddots & \\ 0 & & & \boxed{H_s} \end{pmatrix} \subset GL_m(K) \cong GL(M),$$

where H_l is $GL_{a_l - a_{l-1}}$ if $l \in \mathfrak{G}$, $SL_{a_l - a_{l-1}}$ if $l \in \mathfrak{S}$, and $\{E_{a_l - a_{l-1}}\}$ if $l \in \mathfrak{T}$. Here $E_{a_l - a_{l-1}}$ is the identity matrix of size $a_l - a_{l-1}$.

Let us consider the symmetric algebra $S = \mathrm{Sym}(M \otimes N)$. It is a graded polynomial algebra over K with each variable degree one. Let $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ be the standard bases of $M = K^m$ and $N = K^n$, respectively. For sequences $1 \leqslant c_1, \ldots, c_u \leqslant m$ and $1 \leqslant d_1, \ldots, d_u \leqslant n$, we define $[c_1, \ldots, c_u \mid d_1, \ldots, d_u]$ to be the determinant $\det(e_{c_i} \otimes f_{d_j})_{1 \leqslant i, j \leqslant u}$. It is a minor of the matrix $(e_i \otimes f_j)$ up to sign, or zero. Let Σ be the set of minors

$$\bigl\{[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \bigm| 1 \leqslant u \leqslant \min(m, n),\ 1 \leqslant c_1 < \cdots < c_u \leqslant m,\ 1 \leqslant d_1 < \cdots < d_u \leqslant n\bigr\}.$$

We say that $[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \leqslant [c'_1, \ldots, c'_v \mid d'_1, \ldots, d'_v]$ if $u \geqslant v$, and $c'_i \geqslant c_i$ and $d'_i \geqslant d_i$ for $1 \leqslant i \leqslant v$. It is easy to see that Σ is a distributive lattice.

Set $\epsilon := \min \mathfrak{G}$. For $1 \leqslant l < \epsilon$, set

$$\Gamma_l := \bigl\{[1, \ldots, a_l \mid d_1, \ldots, d_{a_l}] \bigm| 1 \leqslant d_1 < \cdots < d_{a_l} \leqslant n\bigr\}$$

if $l \in \mathfrak{S}$, and

$$\begin{aligned} \Gamma_l := \bigl\{&[c_1, \ldots, c_u \mid d_1, \ldots, d_u] \bigm| a_{l-1} < u \leqslant a_l, \\ &1 \leqslant c_1 < \cdots < c_u \leqslant a_l,\ c_t = t\ (t \leqslant a_{l-1}),\ 1 \leqslant d_1 < \cdots < d_u \leqslant n\bigr\} \end{aligned}$$

if $l \in \mathfrak{T}$. Set $\Gamma = \bigcup_{l < \epsilon} \Gamma_l$. Note that Γ is a sublattice of Σ.

It is well-known that S is an ASL on Σ over K [BH, (7.2.7)]. For the definition of ASL, see [BH, (7.1)].

Lemma 6.5. *Let B be a graded ASL on a poset Ω over a field K. Let Ξ be a subset of Ω such that for any two incomparable elements $\xi, \eta \in \Xi$,*

$$\xi\eta = \sum c_i m_i \tag{2}$$

in S with each m_i in the right hand side being a monomial of Ξ divisible by an element ξ_i in Ξ smaller than both ξ and η. Then the subalgebra $K[\Xi]$ of B is a graded ASL on Ξ.

Proof. We may assume that m_i in the right hand side of (2) has the same degree as that of $\xi\eta$ for each ξ, η, and m_i. For a monomial $m = \prod_{\omega \in \Omega} \omega^{c(\omega)}$, the weight $w(m)$ of m is defined to be $\sum_\omega c(\omega) 3^{\mathrm{coht}(\omega)}$, where $\mathrm{coht}(\omega)$ is the maximum of the lengths of chains $\omega = \omega_0 < \omega_1 < \cdots$ in Ω. Then $w(mm') = w(m) + w(m')$, and for each i, $w(m_i) > w(\xi\eta)$ in (2). So each time we use (2) to rewrite a monomial, the weight goes up. On the other hand, there are only finitely many monomials of a given degree, this rewriting procedure will stop eventually, and we get a linear combination of standard monomials in Ξ. Now (H_2) condition in [BH, (7.1)] is clear, while (H_0) and (H_1) are trivial. $\square$

We call $K[\Xi]$ a *subASL* of B generated by Ξ if the assumption of the lemma is satisfied.

Theorem 6.6. *Let the notation be as above. Let H act on S via $h(m \otimes n) = h(m) \otimes n$. Set $A := S^H$. Then*

(1) *$A = K[\Gamma]$.*
(2) *$K[\Gamma]$ is a subASL of $S = K[\Sigma]$ generated by Γ.*
(3) *A is a Gorenstein UFD. It is strongly F-regular if the characteristic of K is positive, and is of strongly F-regular type if the characteristic of K is zero.*

Proof. First we prove that A is strongly F-regular (type). To do so, we may assume that $K = k$ is algebraically closed. Let B^+ be the subgroup of upper triangular matrices in GL_m, and set $B^+_H := B^+ \cap H$. Then it is easy to see that $A = S^{B^+_H}$.

Now let Q be the quiver $1 \to 2$, $d = (d(1), d(2)) = (m, n)$, $G_1 = B^+_H \subset GL_m$, and $G_2 = \{e\}$. Let U^+_H be the unipotent radical of B^+_H. Then U^+_H is the unipotent radical of an appropriate parabolic subgroup of GL_m, U^+_H is normal in B^+_H, and B^+_H/U^+_H is a torus. Thus the assumption **(7)** of Theorem 6.3 is satisfied, and thus $A = S^{B^+_H}$ is strongly F-regular (type).

The assertion **(2)** is a consequence of the straightening relation of the ASL S. See [ABW] for details.

Assume that **(1)** is proved. Then by the definition of Γ, letting M' be the subspace of M spanned by $e_1, \dots, e_{a_{\epsilon-1}}$, $A = \operatorname{Sym}(M' \otimes N)^{H'}$, where $H' = H \cap GL(M')$, by **(1)** again ($GL(M')$ is viewed as a subgroup of $GL(M)$ via $g'(e_i) = e_i$ for $i > a_{\epsilon-1}$). As H' is connected and $\bar{K} \otimes_K H'$ does not have a non-trivial character, A is a UFD by [Has7, (4.28)], where $\bar{K}$ is the algebraic closure of K. So assuming **(1)**, the assertion **(3)** is proved.

It remains to prove **(1)**. It is easy to see that $\Gamma \subset A$. So it suffices to prove that $\dim_K A_d = \dim_K K[\Gamma]_d$ for each degree $d \geqslant 0$. To do so, we may assume that K is algebraically closed.

Let P^+ be the parabolic subgroup $H(\underline{a}; \{1, \dots, s\}, \emptyset, \emptyset)$ of GL_m, and U_{P^+} the unipotent radical of P^+. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of good $GL(M) \times GL(N)$-modules, then

$$0 \to M_1^{U_{P^+}} \to M_2^{U_{P^+}} \to M_3^{U_{P^+}} \to 0 \tag{3}$$

is an exact sequence of good P^+/U_{P^+}-modules by Lemma 5.3 and Theorem 5.2. Note that P^+/U_{P^+} is identified with $\prod_{l=1}^{s} GL_{a_l - a_{l-1}}$, and H/U_{P^+} is identified with its subgroup $\prod_{l=1}^{s} H_l$. As each H_l is either $GL_{a_l - a_{l-1}}$, $SL_{a_l - a_{l-1}}$, or trivial, it follows that a good P^+/U_{P^+}-module is also good as an H/U_{P^+}-module. Applying the invariance functor $(?)^{H/U_{P^+}}$ to (3),

$$0 \to M_1^H \to M_2^H \to M_3^H \to 0$$

is exact.

Now we employ the standard convention for $GL(M)$. Let T be the set of diagonal matrices in $G := GL(M) = GL_m$, and we identify $X(T)$ with $\mathbb{Z}^m$ by the isomorphism

$$\mathbb{Z}^m \ni (\lambda_1, \lambda_2, \dots, \lambda_m) \mapsto \left(\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_m \end{pmatrix} \mapsto t^\lambda = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_m^{\lambda_m} \right) \in X(T).$$

We fix the base of the root system of $GL(M)$ so that the set of lower triangular matrices in $GL(M)$ is negative. Then the set of dominant weights $X^+_{GL(M)}$ is the set

$$\{\lambda = (\lambda_1, \dots, \lambda_m) \in X(T) \mid \lambda_1 \geqslant \cdots \geqslant \lambda_m\}.$$

We use a similar convention for $GL(N)$. See [Jan, (II.1.21)] for more information on this convention.

For $\lambda \in X^+_{GL(M)}$, $\nabla_{GL(M)}(\lambda)^{U_{P^+}}$ is a single dual Weyl module by Theorem 5.1. But obviously, the highest weight of $\nabla_{GL(M)}(\lambda)^{U_{P^+}}$ is λ. Thus $\nabla_{GL(M)}(\lambda)^{U_{P^+}} \cong \nabla_{P^+/U_{P^+}}(\lambda)$. Now the following is easy to verify:

Lemma 6.7. *For $\lambda = (\lambda_1, \dots, \lambda_m) \in X^+_{GL(M)}$,*

$$\nabla_{GL(M)}(\lambda)^H \cong \begin{cases} \nabla_{GL_{a_1}}(\lambda(1)) \otimes \cdots \otimes \nabla_{GL_{a_s - a_{s-1}}}(\lambda(s)) & (\lambda \in \Theta), \\ 0 & (\mathit{otherwise}) \end{cases}$$

as P^+/H-modules, where $\lambda(l) := (\lambda_{a_{l-1}+1}, \dots, \lambda_{a_l})$ for each l, and Θ is the subset of $X^+_{GL(M)}$ consisting of sequences $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $\lambda(l) = (0, 0, \dots, 0)$ for each $l \in \mathfrak{G}$, and $\lambda(l) = (t, t, \dots, t)$ for some $t \in \mathbb{Z}$ for each $l \in \mathfrak{S}$.

Let $r := \min(m, n)$, and set

$$\mathcal{P}(d) = \{\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0,\ |\lambda| = d\},$$

where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. We consider that

$$(\lambda_1, \dots, \lambda_r) = (\lambda_1, \dots, \lambda_r, 0, \dots, 0),$$

and $\mathcal{P}(d) \subset X^+_{GL(M)}$. Similarly, we also consider that $\mathcal{P}(d) \subset X^+_{GL(N)}$. By the Cauchy formula [ABW, (III.1.4)], S_d has a good filtration as a $GL(M) \times GL(N)$-module whose associated graded object is

$$\bigoplus_{\lambda \in \mathcal{P}(d)} \nabla_{GL(M)}(\lambda) \boxtimes \nabla_{GL(N)}(\lambda).$$

Note that $\nabla_{GL(M)}(\lambda)$ is isomorphic to the Schur module $L_{\tilde{\lambda}} M$ in [ABW], where $\tilde{\lambda}$ is the transpose of λ. That is, $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ is given by $\tilde{\lambda}_i = \#\{j \geqslant 1 \mid \lambda_j \geqslant i\}$.

By Lemma 6.7, S_d^H has a filtration whose associated graded object is

$$\bigoplus_{\lambda \in \mathcal{P}(d) \cap \Theta} \nabla_{GL_{a_1}}\bigl(\lambda(1)\bigr) \otimes \cdots \otimes \nabla_{GL_{a_s - a_{s-1}}}\bigl(\lambda(s)\bigr) \boxtimes \nabla_{GL(N)}(\lambda).$$

In particular,

$$\dim S_d^H = \sum_{\lambda \in \mathcal{P}(d) \cap \Theta} \dim \nabla_{GL(N)}(\lambda) \prod_l \dim \nabla_{GL_{a_l - a_{l-1}}}\bigl(\lambda(l)\bigr). \tag{4}$$

Next we count the dimension of $K[\Gamma]_d$. This is the number of standard monomials of degree d in $K[\Gamma]$. For a standard monomial

$$v = \prod_{b=1}^{\alpha} [c_{b,1}, \dots, c_{b,\mu_b} \mid d_{b,1}, \dots, d_{b,\mu_b}]$$

(where $[c_{b,1}, \dots, c_{b,\mu_b} \mid d_{b,1}, \dots, d_{b,\mu_b}]$ increases when b increases) in Σ, we define $\mu(v) = (\mu_1, \dots, \mu_\alpha)$, and $\lambda(v)$ its transpose. Such a standard monomial v of Γ of degree d exists if and only if $\lambda(v) \in \Theta \cap \mathcal{P}(d)$.

For a standard monomial v of Σ such that $\lambda(v) = \lambda \in \mathcal{P}(d) \cap \Theta$, v is a monomial of Γ if and only if the following condition holds. For each $1 \leqslant b \leqslant \lambda_1$, $1 \leqslant l \leqslant s$, and each $a_{l-1} < i \leqslant a_l$, it holds $a_{l-1} < c_{s,i} \leqslant a_l$. The number of such monomials agrees with $\dim \nabla_{GL(N)}(\lambda) \prod_l \dim \nabla_{GL_{a_l - a_{l-1}}}(\lambda(l))$, as can be seen easily from the standard basis theorem [ABW, (II.2.16)]. So $\dim_K K[\Gamma]$ agrees with the right hand side of (4), and we have $\dim_K A_d = \dim_K S^H_d = \dim_K K[\Gamma]_d$, as desired. $\square$

Remark 6.8. The case that $s = 2$, $a_1 = l$, $\mathfrak{G} = \emptyset$, $\mathfrak{S} = \{2\}$, and $\mathfrak{T} = \{1\}$ is studied by Goto, Hayasaka, Kurano and Nakamura [GHKN]. Gorenstein property and factoriality are proved there for this case. The case that $s = m$, $a_l = l$ $(l = 1, \dots, m)$, $\mathfrak{G} = \mathfrak{S} = \emptyset$, and $\mathfrak{T} = \{1, \dots, m\}$ is a very special case of the study of Miyazaki [Miy].

7. Openness of good locus

(7.1) Let R be a Noetherian commutative ring, and G a split reductive group over R. We fix a split maximal torus T of G whose embedding into G is defined over $\mathbb{Z}$. We fix a base Δ of the root system, and let B be the negative Borel subgroup. For a dominant weight λ, the dual Weyl module $\nabla_G(\lambda)$ is defined to be $\operatorname{ind}_B^G(\lambda)$, and the Weyl module $\Delta_G(\lambda)$ is defined to be $\nabla_G(-w_0\lambda)^*$.

A G-module M is said to be *good* if $\operatorname{Ext}^1_G(\Delta_G(\lambda), M) = 0$ for any $\lambda \in X^+$, where X^+ is the set of dominant weights, see [Has1, (III.2.3.8)].

Lemma 7.2. *The notion of goodness of a G-module M is independent of the choice of T or Δ, and depends only on M.*

Proof. Let T' and Δ' be another choice of a split maximal torus defined over $\mathbb{Z}$ and a base of the root system (with respect to T'). Let B' be the corresponding negative Borel subgroup.

Assume that R is an algebraically closed field. Then there exists some $g \in G(R)$ such that $gBg^{-1} = B'$. So $\operatorname{ind}_B^G \lambda \cong \operatorname{ind}_{B'}^G(\lambda')$ for any $\lambda \in X(B)$, where λ' is the composite

$$B' \xrightarrow{b' \mapsto g^{-1}b'g} B \xrightarrow{\lambda} \mathbb{G}_m.$$

So this case is clear.

When R is a field, then a G-module M is good if and only if $\bar{R} \otimes_R M$ is so as an $\bar{R} \otimes_R G$-module, and this notion is independent of the choice of B, where $\bar{R}$ is the algebraic closure of R.

Now consider the general case. If M is R-finite R-projective, then the assertion follows from [Has1, (III.4.1.8)] and the discussion above. If M is general, then M is good if and only if there exists some filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots$$

of M such that $\bigcup_i M_i = M$, and for each $i \geqslant 1$, $M_i/M_{i-1} \cong N_i \otimes V_i$ for some R-finite R-projective good G-module N_i and an R-module V_i. Indeed, the only if part is [Has1, (III.2.3.8)], while the if part is a consequence of the goodness of $N_i \otimes V_i$, see [Has1, (III.4.1.8)]. This notion is independent of the choice of T or Δ, and we are done. $\square$

Note that if $R \to R'$ is a Noetherian R-algebra, then an $R' \otimes_R G$-module M' is good if and only if it is so as a G-module. This comes from the isomorphism

$$\operatorname{Ext}^i_G\bigl(\Delta_G(\lambda), M'\bigr) \cong \operatorname{Ext}^i_{R' \otimes G}\bigl(\Delta_{R' \otimes G}(\lambda), M'\bigr).$$

If M is a good G-module, and R' is R-flat or M is R-finite R-projective, then $R' \otimes_R M$ is a good $R' \otimes_R G$-module by [Has1, (I.3.6.20)] and [Has1, (III.1.4.8)], see [Has1, (III.2.3.15)]. If M is good and V is a flat R-module, then $M \otimes V$ is good. This follows from the canonical isomorphism

$$\operatorname{Ext}^i_G\bigl(\Delta_G(\lambda), M \otimes V\bigr) \cong \operatorname{Ext}^i_G\bigl(\Delta_G(\lambda), M\bigr) \otimes V,$$

see [Has1, (I.3.6.16)].

If R' is faithfully flat over R and $R' \otimes_R M$ is good, then M is good by [Has1, (I.3.6.20)].

(7.3) Let S be a scheme, and G a reductive group scheme over S, and X a Noetherian S-scheme on which G acts trivially. Let M be a quasi-coherent $(G, \mathcal{O}_X)$-module. For $(G, \mathcal{O}_X)$-modules, see [Has4, Chapter 29]. Almost by definition, a $(G, \mathcal{O}_X)$-module and a $(G \times_S X, \mathcal{O}_X)$-module (note that $G \times_S X$ is an X-group scheme) are the same thing.

We say that M is good if there is a Noetherian commutative ring R and a faithfully flat morphism of finite type $f : \operatorname{Spec} R \to X$ such that $G_R := \operatorname{Spec} R \times_S G$ is a split reductive group scheme over R, and $\Gamma(\operatorname{Spec} R, f^*M)$ is a good G_R-module. This notion is independent of the choice of f such that G_R is split reductive. When $X = \operatorname{Spec} B$ is affine, then we also say that $\Gamma(X, M)$ is good, if M is good. If $g : X' \to X$ is a flat morphism of Noetherian schemes and M is a good quasi-coherent $(G, \mathcal{O}_X)$-module, then g^*M is good. If M is a quasi-coherent $(G, \mathcal{O}_X)$-module, g is faithfully flat, and g^*M is good, then M is good.

For a quasi-coherent $(G, \mathcal{O}_X)$-module M, we define the good locus of M to be

$$\operatorname{Good}(M) = \bigl\{x \in X \bigm| M_x \text{ is a good } (\operatorname{Spec} \mathcal{O}_{X,x} \times_S G)\text{-module}\bigr\}.$$

If $g : X' \to X$ is a flat morphism of Noetherian schemes, then $g^{-1}(\operatorname{Good}(M)) = \operatorname{Good}(g^*M)$. If $X = \operatorname{Spec} R$ is affine, then for a (G, R)-module N, $\operatorname{Good}(N)$ stands for $\operatorname{Good}(\tilde{N})$, where $\tilde{N}$ is the sheaf associated with N.

(7.4) Let the notation be as in (7.1).

For a poset ideal π of X^+ and a G-module M, we say that M belongs to π if $M_\lambda = 0$ for $\lambda \in X^+ \setminus \pi$.

Proposition 7.5. *Let π be a poset ideal of X^+ and M a G-module. Then the following are equivalent.*

(1) *M belongs to π.*
(2) *For any R-finite subquotient N of M and any R-algebra K that is a field, $K \otimes_R N$ belongs to π.*
(3) *For any R-finite subquotient N of M, any R-algebra K that is a field, and $\lambda \in X^+ \setminus \pi$, $\operatorname{Hom}_G(\Delta_G(\lambda), K \otimes_R N) = 0$.*
(4) *For any $\lambda \in X^+ \setminus \pi$, $\operatorname{Hom}_G(\Delta_G(\lambda), M) = 0$.*
(5) *M is a C_π-comodule, where C_π is the Donkin subcoalgebra of C with respect to π, see [Has1, (III.2.3.13)].*

Proof. **(1) ⇒ (2)** is obvious.

(2) ⇒ (3) We may assume that $R = K$ and $N = M$. Then

$$\begin{aligned}\operatorname{Hom}_G\bigl(\Delta_G(\lambda), M\bigr) &\cong \operatorname{Hom}_G\bigl(M^*, \operatorname{ind}_B^G(-w_0\lambda)\bigr)\\ &\cong \operatorname{Hom}_B\bigl(M^*, -w_0\lambda\bigr) \cong \operatorname{Hom}_B(w_0\lambda, M) \subset M_{w_0\lambda} = 0.\end{aligned}$$

(3) ⇒ (4) As M is the inductive limit of R-finite G-submodules of M, we may assume that M is R-finite. We use the Noetherian induction, and we may assume that the implication is true for R/I for any nonzero ideal I of R. If R is not a domain, then there is a nonzero ideal I of R such that the annihilator $0 : I$ of R is also nonzero. As $\operatorname{Hom}_G(\Delta_G(\lambda), M/IM) = 0$ and $\operatorname{Hom}_G(\Delta_G(\lambda), IM) = 0$, we have that $\operatorname{Hom}_G(\Delta(\lambda), M) = 0$. So we may assume that R is a domain. Let N be the torsion part of M. Note that

$$0 \to N \to M \to K \otimes_R M$$

is exact, where K is the field of fractions of R. Hence N is a G-submodule of M. The annihilator of N is nontrivial, and hence $\operatorname{Hom}_G(\Delta_G(\lambda), N) = 0$. On the other hand, by assumption, $\operatorname{Hom}_G(\Delta_G(\lambda), K \otimes_R M) = 0$. So $\operatorname{Hom}_G(\Delta_G(\lambda), M) = 0$, and we are done.

(4) $\Rightarrow$ **(5)** is [Has1, (III.2.3.5)].

(5) $\Rightarrow$ **(1)** As the coaction $\omega_M : M \to M' \otimes_R C_\pi$ is injective, it suffices to show that $M' \otimes_R C_\pi$ belongs to π, where M' is the R-module M with the trivial G-action. For this, it suffices to show that C_π belongs to π. This is proved easily by induction on the number of elements of π, if π is finite, almost by the definition of the Donkin system [Has1, (III.2.2)], and the fact that $\nabla_G(\lambda)$ belongs to π. Then the general case follows easily from the definition of C_π, see [Has1, (III.2.3.13)]. □

Corollary 7.6. *Let M be a G-module, and π a poset ideal of the set of dominant weights X^+. If M belongs to π, then $\operatorname{Ext}^i_G(\Delta_G(\lambda), M) = 0$ for $i \geqslant 0$ and $\lambda \in X^+ \setminus \pi$.*

Proof. We use the induction on i. The case $i = 0$ is already proved in Proposition 7.5.

Let $i > 0$. Let C_π denote the Donkin subcoalgebra of $k[G]$. Consider the exact sequence

$$0 \to M \xrightarrow{\omega_M} M' \otimes_R C_\pi \to N \to 0.$$

Then N belongs to π, and $\operatorname{Ext}^{i-1}_G(\Delta_G(\lambda), N) = 0$ by induction assumption. On the other hand, as C_π is good and R-finite R-projective by construction, $M' \otimes_R C_\pi$ is also good by [Has1, (III.4.1.8)]. Hence $\operatorname{Ext}^i_G(\Delta_G(\lambda), M' \otimes_R C_\pi) = 0$. By the long exact sequence of the Ext-modules, we have that $\operatorname{Ext}^i_G(\Delta_G(\lambda), M) = 0$. □

Lemma 7.7. *Let the notation be as in (7.3). Let M be a coherent $(G, \mathcal{O}_X)$-module. Then* $\operatorname{Good}(M)$ *is Zariski open in X.*

Proof. Let $f : \operatorname{Spec} R \to X$ be a faithfully flat morphism of finite type such that G_R is split reductive. Let $M_R := \Gamma(\operatorname{Spec} R, f^*M)$. Then $\operatorname{Good}(M_R) = f^{-1}(\operatorname{Good}(M))$. As f is a surjective open map, it suffices to show that $\operatorname{Good}(M_R)$ is open in $\operatorname{Spec} R$. So we may assume that $S = X = \operatorname{Spec} R$ is affine and G is split, and we are to prove that $\operatorname{Good}(N)$ is open for an R-finite G-module N.

As N is R-finite, there exists some finite poset ideal π of X^+ to which N belongs. Then $\operatorname{Ext}^i_G(\Delta_G(\lambda), N) = 0$ for $\lambda \in X^+ \setminus \pi$ and $i \geqslant 0$. Set $L := \bigoplus_{\lambda \in \pi} \Delta_G(\lambda)$. Then $\operatorname{Good}(N)$ is nothing but the complement of the support of the R-module $\operatorname{Ext}^1_G(L, N)$ by [Has1, (III.2.3.8)]. As $\operatorname{Ext}^1_G(L, N)$ is R-finite by [Has1, (III.2.3.19)], the support of $\operatorname{Ext}^1_G(L, N)$ is closed, and we are done. □

(7.8) Let the notation be as in (7.3). For a quasi-coherent $(G, \mathcal{O}_X)$-module M, the good dimension $\operatorname{GD}(M)$ is defined to be $-\infty$ if $M = 0$. If $M \neq 0$ and there is an exact sequence

$$0 \to M \to N_0 \to \cdots \to N_s \to 0 \tag{5}$$

such that each N_i is good, then $\operatorname{GD}(M)$ is defined to be the smallest s such that such an exact sequence exists. If there is no such an exact sequence, $\operatorname{GD}(M)$ is defined to be ∞.

(7.9) Assume that $X = \operatorname{Spec} R$ is affine and G is split reductive. For a G-module M,

$$\operatorname{GD}(M) = \sup\left\{ i \;\middle|\; \bigoplus_{\lambda \in X^+} \operatorname{Ext}^i_G\bigl(\Delta_G(\lambda), M\bigr) \neq 0 \right\}.$$

Note that M is good if and only if $\mathrm{GD}(M) \leqslant 0$. If $r \geqslant 0$, $s \geqslant -1$, and

$$0 \to M \to M_s \to \cdots \to M_0 \to N \to 0$$

is an exact sequence of G-modules with $\mathrm{GD}(M_i) \leqslant i + r$, then $\mathrm{GD}(M) \leqslant s + r + 1$ if and only if $\mathrm{GD}(N) \leqslant r$.

If M and N are good and M is R-finite R-projective, then $M \otimes N$ is good, see [Has1, (III.4.5.10)]. Moreover, if M is R-finite R-projective with $\mathrm{GD}(M) \leqslant s$, then M has an exact sequence of the form (5) such that each N_i is R-finite R-projective and good. Indeed, M belongs to some finite poset ideal π of X^+, and when we truncate the cobar resolution of M as a C_π-comodule, then we obtain such a sequence.

It follows that for an R-finite R-projective G-module M, $\mathrm{GD}(M) \leqslant s$ if and only if $\mathrm{GD}(\kappa(\mathfrak{m}) \otimes_R M) \leqslant s$ for any maximal ideal $\mathfrak{m}$ of R by [Has1, (III.4.1.8)].

It also follows that if $\mathrm{GD}(M) \leqslant s$ and $\mathrm{GD}(N) \leqslant t$ with M being R-finite R-projective, then $\mathrm{GD}(M \otimes N) \leqslant s + t$.

Lemma 7.10. *Let V be an R-finite R-projective G-module with $\operatorname{rank} V \leqslant n < \infty$. Then the following are equivalent.*

(1) $\operatorname{Sym} V$ *is good.*
(2) $\bigoplus_{i=1}^{n-1} \operatorname{Sym}_i V$ *is good.*
(3) *For $i = 1, \ldots, n-1$, $\mathrm{GD}(\bigwedge^i V) \leqslant i - 1$.*
(4) *For $i \geqslant 1$, $\mathrm{GD}(\bigwedge^i V) \leqslant i - 1$.*

Proof. We may assume that R is a field.

(1) $\Rightarrow$ **(2)** is trivial.

(2) $\Rightarrow$ **(3)** We use the induction on i.

By assumption and the induction assumption, $\mathrm{GD}(\operatorname{Sym}_{i-j} V \otimes \bigwedge^j V) \leqslant j - 1$ for $j = 1, \ldots, i-1$. On the other hand, $\operatorname{Sym}_i V$ is good. So by the exact sequence

$$0 \to \bigwedge^i V \to \operatorname{Sym}_1 V \otimes \bigwedge^{i-1} V \to \cdots \to \operatorname{Sym}_{i-1} V \otimes \bigwedge^1 V \to \operatorname{Sym}_i V \to 0, \tag{6}$$

$\mathrm{GD}(\bigwedge^i V) \leqslant i - 1$.

(3) $\Rightarrow$ **(4)** is trivial, as $\dim \bigwedge^i V \leqslant 1$ for $i \geqslant n$.

(4) $\Rightarrow$ **(1)** Note that $\operatorname{Sym}_0 V = R$ is good. Now use induction on $i \geqslant 1$ to prove that $\operatorname{Sym}_i V$ is good (use the exact sequence (6) again). $\square$

Theorem 7.11. *Let S be a scheme, G a reductive S-group acting trivially on a Noetherian S-scheme X. Let M be a locally free coherent $(G, \mathcal{O}_X)$-module. Then*

$$\operatorname{Good}(\operatorname{Sym} M) = \bigl\{x \in X \bigm| \operatorname{Sym}\bigl(\kappa(x) \otimes_{\mathcal{O}_{X,x}} M_x\bigr) \text{ is a good } \bigl(\operatorname{Spec} \kappa(x) \times_S G\bigr)\text{-module}\bigr\}, \tag{7}$$

and $\operatorname{Good}(\operatorname{Sym} M)$ *is Zariski open in X.*

Proof. Take a faithfully flat morphism of finite type $f : \operatorname{Spec} R \to X$ such that $\operatorname{Spec} R \times_S G$ is split reductive. Note that f is a surjective open map, and $f^{-1}(\operatorname{Good}(\operatorname{Sym} M)) = \operatorname{Good}(\operatorname{Sym} f^* M)$.

First we prove that $\operatorname{Good}(\operatorname{Sym} M)$ is open. We may assume that $S = X = \operatorname{Spec} R$ is affine, and G is split reductive.

Then by Lemma 7.10 and Lemma 7.7,

$$\mathrm{Good}(\mathrm{Sym}\, M) = \mathrm{Good}\left(\bigoplus_{i=1}^{n} \mathrm{Sym}_i\, M\right)$$

is open, where the rank of M is less than or equal to n.

Next we prove that the equality (7) holds. Let $P \in \mathrm{Spec}\, R$, and $x = f(P)$. Then $\mathrm{Sym}(\kappa(x) \otimes_{\mathcal{O}_{X,x}} M_x)$ is good if and only if $\mathrm{Sym}(\kappa(P) \otimes_{R_P} \Gamma(\mathrm{Spec}\, R, f^*M)_P)$ is good. So we may assume that $S = X = \mathrm{Spec}\, R$ is affine, and G is split reductive. Let N be an R-finite R-projective G-module of rank at most n. Then $(\mathrm{Sym}\, N)_P$ is good if and only if $(\bigoplus_{i=1}^n \mathrm{Sym}_i\, N)_P$ is good by Lemma 7.10. By [Has1, (III.4.1.8)], $(\bigoplus_{i=1}^n \mathrm{Sym}_i\, N)_P$ is good if and only if $\kappa(P) \otimes_{R_P} (\bigoplus_{i=1}^n \mathrm{Sym}_i\, N)_P$ is good. By Lemma 7.10 again, it is good if and only if $\kappa(P) \otimes_{R_P} (\mathrm{Sym}\, N)_P$ is so. Thus the equality (7) was proved. □

Corollary 7.12. *Let R be a Noetherian domain of characteristic zero, and G a reductive group over R. If M is an R-finite R-projective G-module, then $\{P \in \mathrm{Spec}\, R \mid \mathrm{Sym}(\kappa(P) \otimes_R M)$ is good$\}$ is a dense open subset of* $\mathrm{Spec}\, R$.

Proof. By Theorem 7.11, it suffices to show that $\mathrm{Good}(\mathrm{Sym}\, M)$ is non-empty. But the generic point η of $\mathrm{Spec}\, R$ is in $\mathrm{Good}(\mathrm{Sym}\, M)$. Indeed, $\kappa(\eta)$ is a field of characteristic zero, and any $\kappa(\eta) \otimes_R G$-module is good. □

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References

[ABW] K. Akin, D.A. Buchsbaum, J. Weyman, Schur functors and Schur complexes, Adv. Math. 44 (1982) 207–278.
[AM] M.F. Atiyah, I.G. MacDonald, Introduction to Commutative Algebra, Addison–Wesley, 1969.
[Bt] J.-F. Boutot, Singularités rationnelles et quotients par les groupes réductifs, Invent. Math. 88 (1987) 65–68.
[BH] W. Bruns, J. Herzog, Cohen–Macaulay Rings, first paper back ed., Cambridge Univ. Press, Cambridge, 1998.
[Cox] D.A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995) 17–50.
[DP] C. De Concini, C. Procesi, A characteristic free approach to invariant theory, Adv. Math. 21 (1976) 330–354.
[Dol] I. Dolgachev, Lectures on Invariant Theory, London Math. Soc. Lecture Note Ser., vol. 296, Cambridge Univ. Press, Cambridge, 2003.
[Don1] S. Donkin, A filtration for rational modules, Math. Z. 177 (1981) 1–8.
[Don2] S. Donkin, Rational Representations of Algebraic Groups, Lecture Notes in Math., vol. 1140, Springer, 1985.
[Don3] S. Donkin, Invariants of unipotent radicals, Math. Z. 198 (1988) 117–125.
[Don4] S. Donkin, On tilting modules and invariants for algebraic groups, in: Finite Dimensional Algebras and Related Topics, Ottawa, ON, 1992, Kluwer, 1994, pp. 59–77.
[EKW] E.J. Elizondo, K. Kurano, K.-i. Watanabe, The total coordinate ring of a normal projective variety, J. Algebra 276 (2004) 625–637.
[Fr] E.M. Friedlander, A canonical filtration for certain rational modules, Math. Z. 188 (1985) 433–438.
[GHKN] S. Goto, F. Hayasaka, K. Kurano, Y. Nakamura, Rees algebras of the second syzygy module of the residue field of a regular local ring, in: Commutative Algebra and Algebraic Geometry, in: Contemp. Math., vol. 390, Amer. Math. Soc., 2005, pp. 97–108.
[Grs1] F.D. Grosshans, The invariants of unipotent radicals of parabolic subgroups, Invent. Math. 73 (1983) 1–9.
[Grs2] F.D. Grosshans, Contractions of the actions of reductive algebraic groups in arbitrary characteristic, Invent. Math. 107 (1992) 127–133.
[Har] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, Amer. J. Math. 120 (1998) 981–996.
[Has1] M. Hashimoto, Auslander–Buchweitz Approximations of Equivariant Modules, London Math. Soc. Lecture Note Ser., vol. 282, Cambridge Univ. Press, Cambridge, 2000.
[Has2] M. Hashimoto, Good filtrations of symmetric algebras and strong F-regularity of invariant subrings, Math. Z. 236 (2001) 605–623.

[Has3] M. Hashimoto, Surjectivity of multiplication and F-regularity of multigraded rings, in: Commutative Algebra, Grenoble/Lyon, 2001, in: Contemp. Math., vol. 331, Amer. Math. Soc., 2003, pp. 153–170.
[Has4] M. Hashimoto, Equivariant twisted inverses, in: J. Lipman, M. Hashimoto (Eds.), Foundations of Grothendieck Duality for Diagrams of Schemes, in: Lecture Notes in Math., vol. 1960, Springer, 2009, pp. 261–478.
[Has5] M. Hashimoto, F-pure homomorphisms, strong F-regularity, and F-injectivity, Comm. Algebra 38 (2010) 4569–4596.
[Has6] M. Hashimoto, Good filtrations and F-purity of invariant subrings, J. Math. Soc. Japan 63 (2011) 815–818.
[Has7] M. Hashimoto, Equivariant total ring of fractions and factoriality of rings generated by semiinvariants, arXiv:1009.5152.
[HM] M. Hashimoto, M. Miyazaki, G-prime and G-primary G-ideals on G-schemes, Comm. Algebra (2013), in press, http://dx.doi.org/10.1080/00927872.2012.656335.
[Hoc] M. Hochster, Foundations of tight closure theory, available at http://www.math.lsa.umich.edu/~hochster/mse.html.
[HH1] M. Hochster, C. Huneke, Tight closure and strong F-regularity, Mém. Soc. Math. France 38 (1989) 119–133.
[HH2] M. Hochster, C. Huneke, Tight closure, invariant theory, and the Briançon–Skoda theorem, J. Amer. Math. Soc. 3 (1990) 31–116.
[HH3] M. Hochster, C. Huneke, F-regularity, test elements, and smooth base change, Trans. Amer. Math. Soc. 346 (1994) 1–62.
[HH4] M. Hochster, C. Huneke, Tight closure of parameter ideals and splitting in module-finite extensions, J. Algebraic Geom. 3 (1994) 599–670.
[HR] M. Hochster, J. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay, Adv. Math. 13 (1974) 115–175.
[Hum] J.E. Humphreys, Linear Algebraic Groups, third printing, Grad. Texts in Math., vol. 21, 1987.
[Jan] J.C. Jantzen, Representations of Algebraic Groups, 2nd ed., Amer. Math. Soc., 2003.
[Kem] G. Kemper, A characterization of linearly reductive groups by their invariants, Transform. Groups 5 (2000) 1–8.
[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, Toroidal Embeddings I, Lecture Notes in Math., vol. 339, Springer, 1973.
[Kun] E. Kunz, On Noetherian rings of characteristic p, Amer. J. Math. 98 (1976) 999–1013.
[LS] G. Lyubeznik, K.E. Smith, Strong and weak F-regularity are equivalent for graded rings, Amer. J. Math. 121 (1999) 1279–1290.
[Mat] O. Mathieu, Filtrations of G-modules, Ann. Sci. École Norm. Sup. (4) 23 (1990) 625–644.
[Miy] M. Miyazaki, Invariants of the unipotent radical of a Borel subgroup, in: The 29th Symposium on Commutative Algebra in Japan, Nagoya, 2007, pp. 43–50, http://www.math.nagoya-u.ac.jp/~hasimoto/commalg-proc.html.
[MFK] D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory, third edition, Springer, 1994.
[Mur] M.P. Murthy, A note on factorial rings, Arch. Math. 15 (1964) 418–420.
[Pop1] V.L. Popov, On the stability of the action of an algebraic group on an algebraic variety, Math. USSR-Izv. 6 (1972) 367–379.
[Pop2] V.L. Popov, The Picard group of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles, Math. USSR-Izv. 8 (1975) 301–327.
[Pop3] V.L. Popov, Contraction of the actions of reductive algebraic groups, Math. USSR-Sb. 58 (1987) 311–335.
[RR] S. Ramanan, A. Ramanathan, Projective normality of flag varieties and Schubert varieties, Invent. Math. 79 (1985) 217–224.
[Ros] M. Rosenlicht, Toroidal algebraic groups, Proc. Amer. Math. Soc. 12 (1961) 984–988.
[SvdB] K.E. Smith, M. Van den Bergh, Simplicity of rings of differential operators in prime characteristic, Proc. London Math. Soc. 75 (1997) 32–62.
[Vel] J.D. Vélez, Openness of the F-rational locus and smooth base change, J. Algebra 172 (1995) 425–453.