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The spectrum of the Burnside Tambara functor on a finite cyclic p-group ☆

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ABSTRACT

For a finite group G, a Tambara functor on G is regarded as a G-bivariant analog of a commutative ring. In this analogy, previously we have defined an ideal of a Tambara functor. In this article, we calculate the prime spectrum of the Burnside Tambara functor, when G is a finite cyclic p-group for a prime integer p.

1. Introduction and preliminaries

Tambara functors on a finite group G were firstly defined by Tambara [13] under the name 'TNR-functors', to treat the multiplicative transfers of Green functors. The terminology *Tambara functor* firstly appeared in Brun's paper, when he used it to describe the structure of Witt–Burnside rings [2]. It consists of a triplet $T = (T^*, T_+, T_\bullet)$, where the *additive part* (T^*, T_+) forms a Mackey functor, whereas the *multiplicative part* $(T^*, T_\bullet)$ forms a semi-Mackey functor. This is just like a commutative ring consists of an additive abelian group structure and a multiplicative commutative semi-group structure. In fact a Tambara functor is nothing but a commutative ring when G is trivial. In this sense, this notion is regarded as a Mackey-functor-theoretic analog (or, 'G-bivariant analog') of a commutative ring [16].

In this analogy, some algebraic notions in commutative ring theory find their analogs in Tambara functor theory. We have ideals [7], fractions [6], and polynomials [8] of Tambara functors. These are mutually related, as they should be, and moreover in connection with the celebrated Dress

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construction [12,9]. The most typical Tambara functor is the *Burnside Tambara functor* Ω_G (Example 1.4), which plays a role just like $\mathbb{Z}$ in the ordinary commutative ring theory.

Especially, in the G-bivariant analog of the ideal theory, the prime spectrum *Spec* T has been defined for any Tambara functor T. In our previous short note [10], we have demonstrated a calculation of the prime spectrum of Ω_G, when G is a group of prime order p. In this article, extending the method in [10], we calculate *Spec* Ω_G when G is a *cyclic p-group* for a prime integer p. To determine the prime ideals of Ω_G, the key observation is that the set of subgroups of G is totally ordered

$$e = H_0 < H_1 < \cdots < H_r = G,$$

and thus Ω_G can be regarded as a sequence of commutative rings equipped with adjacent structure morphisms

$$R_0 \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} R_1 \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} R_2 \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} \cdots \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} R_k \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} \cdots \underset{\text{jnd}}{\overset{\text{ind}}{\rightleftarrows}} R_r,$$

(each pair of adjacent rings is connected by ind and jnd to the right and res to the left)

where $R_k = \Omega_G(G/H_k)$. With this identification, an ideal of Ω_G can be regarded as a sequence $[I_0, \dots, I_r]$ of ideals $I_k \subseteq R_k$. A sequence $[I_0, \dots, I_r]$ forms an ideal of Ω_G if and only if the condition

$\boxed{\mathcal{I}(k)}$ $\mathrm{ind}_{k-1}^k(I_{k-1}) \subseteq I_k$, $\mathrm{res}_{k-1}^k(I_k) \subseteq I_{k-1}$, $\mathrm{jnd}_{k-1}^k(I_{k-1}) \subseteq I_k$.

is satisfied for each $1 \leqslant k \leqslant r$ (Corollary 3.3). The restriction of an ideal $\mathscr{I} = [I_0, \dots, I_k]$ of Ω_{H_k} onto H_i is given by $\mathscr{I}|_{H_i} = [I_0, \dots, I_i]$ for any $0 \leqslant i \leqslant k$. On the contrary, we can consider an extension of $\mathscr{I}$ onto H_{k+1}, namely, an ideal $\mathscr{I}'$ of $\Omega_{H_{k+1}}$ satisfying $\mathscr{I}'|_{H_k} = \mathscr{I}$. In particular, the largest and the smallest among such $\mathscr{I}'$ are explicitly given by $\mathcal{L}\mathscr{I}$ and $\mathcal{S}\mathscr{I}$ in Definition 3.5. This allows us an inductive construction of ideals in Ω_G.

Whether an ideal $\mathscr{I} = [I_0, \dots, I_r]$ is prime or not can be also checked inductively on k. In fact, it is prime if and only if the condition

$\boxed{\mathcal{P}(k)}$ For any $0 \leqslant i \leqslant k$, $a \in (\mathrm{res}_{k-1}^k)^{-1}(I_{k-1})$ and $b \in (\mathrm{res}_{i-1}^i)^{-1}(I_{i-1}) \setminus I_i$,

$$a \cdot \mathrm{jnd}_i^k(b) \in I_k \quad \Rightarrow \quad a \in I_k.$$

is satisfied for each $0 \leqslant k \leqslant r$ (Proposition 4.4). Consequently, any restriction $\mathscr{I}|_{H_i}$ of a prime ideal $\mathscr{I} = [I_0, \dots, I_k] \subseteq \Omega_{H_k}$ is again prime. Especially I_0 should be a prime ideal in $R_0 \cong \mathbb{Z}$, and thus equal to 0, (p), or (q) for some prime integer $q \neq p$. By determining prime ideals over 0, (p), (q), namely, prime ideals $\mathscr{I} = [I_0, \dots, I_r]$ satisfying $I_0 = 0$, (p), (q) respectively, we obtain the following result. In particular, the dimension of Ω_G is calculated as $\dim \Omega_G = r + 1$ (Corollary 6.13).

Theorem 1.1. *Let G be a cyclic p-group of order p^r. The prime ideals of $\Omega_G \in \mathrm{Ob}(Tam(G))$ are as follows.*

(i) *Over $(q) \subseteq R_0$, there are $r + 1$ prime ideals*

$$\mathcal{S}^r(q) \subsetneq \mathcal{L}\mathcal{S}^{r-1}(q) \subsetneq \cdots \subsetneq \mathcal{L}^r(q).$$

(ii) *Over $(p) \subseteq R_0$, there is only one prime ideal $\mathcal{L}^r(p)$.*

(iii) *Over $0 \subseteq R_0$, there are $r + 1$ prime ideals*

$$0 \subsetneq \mathcal{L}(0) \subsetneq \cdots \subsetneq \mathcal{L}^r(0).$$

Thus we have

$$\begin{aligned}\mathit{Spec}\,\Omega_G = \{\mathcal{L}^r(p)\} \cup \{\mathcal{L}^i(0) \mid 0 \leqslant i \leqslant r\} \\ \cup \{\mathcal{L}^i\mathcal{S}^{r-i}(q) \mid 0 \leqslant i \leqslant r,\ q \text{ is a prime different from } p\}.\end{aligned}$$

Throughout this article, the unit of a finite group G will be denoted by e. Abbreviately we denote the trivial subgroup of G by e, instead of $\{e\}$. The notation $H \leqslant G$ means H is a subgroup of G. The symbol ${}_G\mathit{set}$ denotes the category of finite G-sets and G-equivariant maps. If $H \leqslant G$ and $g \in G$, then ${}^gH$ denotes the conjugate ${}^gH = gHg^{-1}$. Similarly, $H^g = g^{-1}Hg$. A monoid is always assumed to be unitary and commutative. Similarly a ring is assumed to be commutative, with an additive unit 0 and a multiplicative unit 1. We denote the category of monoids by *Mon*, the category of rings by *Ring*. A monoid homomorphism preserves units, and a ring homomorphism preserves 1. For any category $\mathscr{C}$ and any pair of objects X and Y in $\mathscr{C}$, the set of morphisms from X to Y in $\mathscr{C}$ is denoted by $\mathscr{C}(X,Y)$.

We briefly recall the definition of a Tambara functor.

Definition 1.2. (See [13].) A *Tambara functor T on G* is a triplet $T = (T^*, T_+, T_\bullet)$ of two covariant functors

$$T_+ : {}_G\mathit{set} \to \mathit{Set}, \qquad T_\bullet : {}_G\mathit{set} \to \mathit{Set}$$

and one contravariant functor

$$T^* : {}_G\mathit{set} \to \mathit{Set}$$

which satisfies the following. Here *Set* is the category of sets.

(1) $T^\alpha = (T^*, T_+)$ is a Mackey functor on G.
(2) $T^\mu = (T^*, T_\bullet)$ is a semi-Mackey functor on G.
Since T^α, T^μ are semi-Mackey functors, we have $T^*(X) = T_+(X) = T_\bullet(X)$ for each $X \in \mathrm{Ob}({}_G\mathit{set})$. We denote this by $T(X)$.
(3) (Distributive law) If we are given an exponential diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] \arrow[d, phantom, "exp"] & Z \arrow[l, "\lambda"'] \arrow[d, "\rho"] \\ Y & & B \arrow[ll, "q"] \end{tikzcd}$$

in ${}_G\mathit{set}$, then

$$\begin{tikzcd} T(X) \arrow[d, "T_\bullet(f)"'] & T(A) \arrow[l, "T_+(p)"'] \arrow[r, "T^*(\lambda)"] \arrow[d, phantom, "\circlearrowright"] & T(Z) \arrow[d, "T_\bullet(\rho)"] \\ T(Y) & & T(B) \arrow[ll, "T_+(q)"] \end{tikzcd}$$

is commutative. For the definition and basic properties of exponential diagrams, see [13].

If $T = (T^*, T_+, T_\bullet)$ is a Tambara functor, then $T(X)$ becomes a ring for each $X \in \mathrm{Ob}({}_G\mathit{set})$ [13]. Since T^α is a Mackey functor, by definition T is '*additive*', in the sense that for any $X_1, X_2 \in \mathrm{Ob}({}_G\mathit{set})$, the inclusions $\iota_1 : X_1 \hookrightarrow X_1 \amalg X_2$ and $\iota_2 : X_2 \hookrightarrow X_1 \amalg X_2$ induce a natural isomorphism of rings $(T^*(\iota_1), T^*(\iota_2)) : T(X_1 \amalg X_2) \xrightarrow{\cong} T(X_1) \times T(X_2)$. For each $f \in {}_G\mathit{set}(X, Y)$,

- $T^*(f) : T(Y) \to T(X)$ is a ring homomorphism, called the *restriction* along f,
- $T_+(f) : T(X) \to T(Y)$ is an additive homomorphism, called the *additive transfer* along f,
- $T_\bullet(f) : T(X) \to T(Y)$ is a multiplicative homomorphism, called the *multiplicative transfer* along f.

$T^*(f)$, $T_+(f)$, $T_\bullet(f)$ are often abbreviated to f^*, f_+, $f_\bullet$.

Remark 1.3. If f is the natural projection $\mathrm{pr}_K^H : G/K \to G/H$ for some $K \leqslant H \leqslant G$, then f^*, f_+, $f_\bullet$ is written as

$$\mathrm{res}_K^H = \left(\mathrm{pr}_K^H\right)^*,$$
$$\mathrm{ind}_K^H = \left(\mathrm{pr}_K^H\right)_+,$$
$$\mathrm{jnd}_K^H = \left(\mathrm{pr}_K^H\right)_\bullet.$$

For a conjugate map $c_g : G/H^g \to G/H$, we define $c_{g,H} : T(G/H) \to T(G/H^g)$ (or simply $c_g : T(G/H) \to T(G/H^g)$) by

$$c_{g,H} = T^*(c_g).$$

If g belongs to the normalizer $N_G(H)$ of H in G, then this gives an automorphism $c_{g,H} : T(G/H) \to T(G/H)$. With this $N_G(H)$-action, every $T(G/H)$ becomes an $N_G(H)/H$-ring.

Since any G-map is a union of compositions of natural projections and conjugate maps, the structure morphisms of a Tambara functor are completely determined by res_K^H, ind_K^H, jnd_K^H, $c_{g,H}$ for $K \leqslant H \leqslant G$, $g \in G$, by virtue of the additivity.

Example 1.4. If we define Ω_G by

$$\Omega_G(X) = K_0({}_G\mathit{set}/X)$$

for each $X \in \mathrm{Ob}({}_G\mathit{set})$, where the right-hand side is the Grothendieck ring of the category of finite G-sets over X, then Ω_G becomes a Tambara functor on G [13]. This is called the *Burnside Tambara functor*. For each $f \in {}_G\mathit{set}(X, Y)$,

$$f_\bullet : \Omega_G(X) \to \Omega_G(Y)$$

is the one determined by

$$f_\bullet(A \xrightarrow{p} X) = \left(\Pi_f(A) \xrightarrow{\varpi} Y\right) \quad \left({}^\forall(A \xrightarrow{p} X) \in \mathrm{Ob}({}_G\mathit{set}/X)\right),$$

where $\Pi_f(A)$ and ϖ are

$$\Pi_f(A) = \left\{ (y, \sigma) \;\middle|\; \begin{array}{l} y \in Y, \\ \sigma : f^{-1}(y) \to A \text{ is a map of sets}, \\ p \circ \sigma = \mathrm{id}_{f^{-1}(y)} \end{array} \right\},$$
$$\varpi(y, \sigma) = y.$$

G acts on $\Pi_f(A)$ by $g \cdot (y, \sigma) = (gy, {}^g\sigma)$, where ${}^g\sigma$ is the map defined by

$$ {}^g\sigma(x) = g\sigma\left(g^{-1}x\right) \quad \left({}^\forall x \in f^{-1}(gy)\right). $$

$f_+ : \Omega_G(X) \to \Omega_G(Y)$ is an additive homomorphism satisfying

$$ f_+(A \xrightarrow{p} X) = (A \xrightarrow{f \circ p} Y) \quad \left({}^\forall (A \xrightarrow{p} X) \in \mathrm{Ob}({}_G set/X)\right). $$

f^* is defined by using a fiber product [13]. Namely, the Mackey-functor structure on the additive part of Ω is the usual one as in [1].

Remark 1.5. For any $K \leqslant H \leqslant G$, we have a natural isomorphism (cf. [1]) $\Omega_G(G/K) \cong \Omega_H(H/K)$, and we will identify them through this isomorphism.

We often abbreviate Ω_G to Ω, if the base group is obvious from the context.

2. The Burnside Tambara functor on a cyclic p-group

Throughout this article, we fix a prime number p. Let G be a cyclic p-group of order p^r, and let $H_k \leqslant G$ be its subgroup of order p^k for each $0 \leqslant k \leqslant r$. In the following argument, without loss of generality we may assume

$$ G = \mathbb{Z}/p^r\mathbb{Z}, \qquad H_k = p^{r-k}\mathbb{Z}/p^r\mathbb{Z}, $$
$$ e = H_0 < H_1 < \cdots < H_k < \cdots < H_r = G. $$

Then the G-set G/H_k is canonically isomorphic to $\mathbb{Z}/p^{r-k}\mathbb{Z}$, and the natural projection

$$ G/H_k \xrightarrow{\mathrm{pr}_k^\ell} G/H_\ell \quad (k \leqslant \ell) $$

is identified with a map given by

$$ \mathbb{Z}/p^{r-k}\mathbb{Z} \to \mathbb{Z}/p^{r-\ell}\mathbb{Z}; \quad a \bmod p^{r-k}\mathbb{Z} \mapsto a \bmod p^{r-\ell}\mathbb{Z} $$

for any $a \in \mathbb{Z}$.

Since G is commutative, each $\Omega(G/H_k)$ has a trivial G-action, and admits a natural $\mathbb{Z}$-basis

$$ \left\{ (G/H_i \xrightarrow{\mathrm{pr}_i^k} G/H_k) \;\middle|\; 0 \leqslant i \leqslant k \right\}. $$

Thus if we denote $(G/H_i \xrightarrow{\mathrm{pr}_i^k} G/H_k)$ by $X_{k,i}$, then it is a free $\mathbb{Z}$-module

$$ \Omega(G/H_k) = \bigoplus_{0 \leqslant i \leqslant k} \mathbb{Z}X_{k,i} $$

with a trivial G-action. Therefore, if we put $R_k = \bigoplus_{0 \leqslant i \leqslant k} \mathbb{Z}X_{k,i}$, then the Tambara functor Ω_G is regarded just as a sequence of commutative rings R_k and structure morphisms

$$\begin{tikzcd}[column sep=large] R_0 \arrow[r, shift left=3, "\mathrm{ind}_0^1"] \arrow[r, shift right=3, "\mathrm{jnd}_0^1"'] & R_1 \arrow[l, "\mathrm{res}_0^1" description] \arrow[r, shift left=3, "\mathrm{ind}_1^2"] \arrow[r, shift right=3, "\mathrm{jnd}_1^2"'] & R_2 \arrow[l, "\mathrm{res}_1^2" description] \arrow[r, shift left=3, "\mathrm{ind}_2^3"] \arrow[r, shift right=3, "\mathrm{jnd}_2^3"'] & \cdots \arrow[l, "\mathrm{res}_2^3" description] \arrow[r, shift left=3, "\mathrm{ind}_{k-1}^k"] \arrow[r, shift right=3, "\mathrm{jnd}_{k-1}^k"'] & R_k \arrow[l, "\mathrm{res}_{k-1}^k" description] \arrow[r, shift left=3, "\mathrm{ind}_k^{k+1}"] \arrow[r, shift right=3, "\mathrm{jnd}_k^{k+1}"'] & \cdots \arrow[l, "\mathrm{res}_k^{k+1}" description] \arrow[r, shift left=3, "\mathrm{ind}_{r-1}^r"] \arrow[r, shift right=3, "\mathrm{jnd}_{r-1}^r"'] & R_r \arrow[l, "\mathrm{res}_{r-1}^r" description] \end{tikzcd}$$

satisfying conditions in Definition 1.2. Here, ind_{k-1}^k, res_{k-1}^k, jnd_{k-1}^k are the abbreviations of $\mathrm{ind}_{H_{k-1}}^{H_k}$, $\mathrm{res}_{H_{k-1}}^{H_k}$, $\mathrm{jnd}_{H_{k-1}}^{H_k}$. We use similar abbreviations in the rest. Remark that any structure morphism of Ω_G can be realized as a composition of these morphisms.

Remark 2.1. By virtue of Remark 1.5, the first k-terms

$$\begin{tikzcd}[column sep=large] R_0 \arrow[r, shift left=3, "\mathrm{ind}_0^1"] \arrow[r, shift right=3, "\mathrm{jnd}_0^1"'] & R_1 \arrow[l, "\mathrm{res}_0^1" description] \arrow[r, shift left=3, "\mathrm{ind}_1^2"] \arrow[r, shift right=3, "\mathrm{jnd}_1^2"'] & R_2 \arrow[l, "\mathrm{res}_1^2" description] \arrow[r, shift left=3, "\mathrm{ind}_2^3"] \arrow[r, shift right=3, "\mathrm{jnd}_2^3"'] & \cdots \arrow[l, "\mathrm{res}_2^3" description] \arrow[r, shift left=3, "\mathrm{ind}_{k-1}^k"] \arrow[r, shift right=3, "\mathrm{jnd}_{k-1}^k"'] & R_k \arrow[l, "\mathrm{res}_{k-1}^k" description] \end{tikzcd} \tag{2.1}$$

can be regarded as a sequence representing the Tambara functor Ω_{H_k}. We always work under this identification in this paper. With this identification, forgetting the entire group G, we can regard the Burnside Tambara functor Ω_{H_k} on a cyclic p-group H_k of order p^k, simply as a length k sequence of rings (2.1) obtained inductively by adding R_k, ind_{k-1}^k, res_{k-1}^k, jnd_{k-1}^k to the length $k-1$ sequence

$$\begin{tikzcd}[column sep=large] R_0 \arrow[r, shift left=3, "\mathrm{ind}_0^1"] \arrow[r, shift right=3, "\mathrm{jnd}_0^1"'] & R_1 \arrow[l, "\mathrm{res}_0^1" description] \arrow[r, shift left=3, "\mathrm{ind}_1^2"] \arrow[r, shift right=3, "\mathrm{jnd}_1^2"'] & \cdots \arrow[l, "\mathrm{res}_1^2" description] \arrow[r, shift left=3, "\mathrm{ind}_{k-2}^{k-1}"] \arrow[r, shift right=3, "\mathrm{jnd}_{k-2}^{k-1}"'] & R_{k-1} \arrow[l, "\mathrm{res}_{k-2}^{k-1}" description] \end{tikzcd}$$

corresponding to $\Omega_{H_{k-1}}$. This observation enables us an inductive construction of ideals of in Ω.

We go on to describe R_k and the structure morphisms. As above, we have $R_k = \bigoplus_{0\leqslant i\leqslant k} \mathbb{Z}X_{k,i}$ as a module.

Remark 2.2. Additive transfer $\mathrm{ind}_k^\ell : R_k \to R_\ell$ is given by

$$\mathrm{ind}_k^\ell(X_{k,i}) = X_{\ell,i} \quad (0 \leqslant i \leqslant k)$$

for each $k \leqslant \ell$.

Proof. This is obvious. $\square$

If we take a fiber product of $\mathrm{pr}_i^k : G/H_i \to G/H_k$ and $\mathrm{pr}_j^k : G/H_j \to G/H_k$

$$\begin{tikzcd} G/H_i \times_{G/H_k} G/H_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G/H_j \arrow[d, "\mathrm{pr}_j^k"] \\ G/H_i \arrow[r, "\mathrm{pr}_i^k"'] & G/H_k \end{tikzcd}$$

where $0 \leqslant i, j \leqslant k$, then by the Mackey decomposition formula, we have

$$G/H_i \underset{G/H_k}{\times} G/H_j = \begin{cases} \coprod_{p^{k-j}} G/H_i & (i \leqslant j), \\ \coprod_{p^{k-i}} G/H_j & (j \leqslant i). \end{cases} \tag{2.2}$$

In any case, the cardinality of this G-set is

$$|G/H_i \underset{G/H_k}{\times} G/H_j| = p^{r+k-(i+j)}.$$

Eq. (2.2) is also written as

$$G/H_i \underset{G/H_k}{\times} G/H_j = \coprod_{p^{k-\max(i,j)}} G/H_{\min(i,j)}.$$

As a corollary, we obtain the following.

Corollary 2.3. *For any $0 \leqslant k \leqslant r$, the following holds.*

(1) *For any $i, j \leqslant k$,*

$$X_{k,i} \cdot X_{k,j} = \begin{cases} p^{k-j} X_{k,i} & (i \leqslant j), \\ p^{k-i} X_{k,j} & (j \leqslant i) \end{cases}$$

in R_k. In particular, $X_{k,k} = 1$ is the unit of R_k. In fact, R_k is written as the residue ring of the polynomial ring $\mathbb{Z}[X_{k,i} \mid 0 \leqslant i \leqslant k]$ over indeterminates $X_{k,i}$ $(0 \leqslant i \leqslant k)$, by the ideal generated by

$$\{X_{k,k} - 1\} \cup \bigl\{X_{k,i} X_{k,j} - p^{k-\max(i,j)} X_{k,\min(i,j)} \bigm| 0 \leqslant i, j \leqslant k\bigr\}.$$

(2) *For any $\ell \geqslant k$, $\mathrm{res}^\ell_k : R_\ell \to R_k$ is given by*

$$\mathrm{res}^\ell_k(X_{\ell,i}) = \begin{cases} p^{\ell-k} X_{k,i} & (i \leqslant k), \\ p^{\ell-i} X_{k,k} = p^{\ell-i} & (i \geqslant k). \end{cases}$$

As for multiplicative transfers, we need some calculation. For the detail, see Appendix A.

Corollary 2.4. *For any $\ell \geqslant k$, $\mathrm{jnd}^\ell_k : R_k \to R_\ell$ is given by*

$$\mathrm{jnd}^\ell_k\Bigl(\sum_{0 \leqslant i \leqslant k} m_i X_{k,i}\Bigr) = m_k X_{\ell,\ell} + \sum_{k \leqslant i < \ell} \frac{(m_k)^{p^{\ell-i}} - (m_k)^{p^{\ell-i-1}}}{p^{\ell-i}} X_{\ell,i}$$
$$+ \sum_{0 \leqslant i < k} \frac{(\sum_{s=i}^k m_s p^{k-s})^{p^{\ell-k}} - (\sum_{s=i+1}^k m_s p^{k-s})^{p^{\ell-k}}}{p^{\ell-i}} X_{\ell,i}$$

for any $m_0, \ldots, m_k \in \mathbb{Z}$.

3. Ideals of Ω as a sequence

An ideal of a Tambara functor is defined in [7] as follows.

Definition 3.1. Let T be a Tambara functor. An *ideal* $\mathscr{I}$ of T is a family of ideals $\{\mathscr{I}(X) \subseteq T(X)\}_{X \in \mathrm{Ob}({}_G set)}$ satisfying

(i) $f^*(\mathscr{I}(Y)) \subseteq \mathscr{I}(X)$,
(ii) $f_+(\mathscr{I}(X)) \subseteq \mathscr{I}(Y)$,
(iii) $f_\bullet(\mathscr{I}(X)) \subseteq f_\bullet(0) + \mathscr{I}(Y)$

for any $f \in {}_G set(X, Y)$.

These conditions also imply

$$\mathscr{I}(X_1 \amalg X_2) \cong \mathscr{I}(X_1) \times \mathscr{I}(X_2)$$

for any $X_1, X_2 \in \mathrm{Ob}({}_G set)$. From this, an ideal of T is determined by a family

$$\{I_H = \mathscr{I}(G/H)\}_{H \leqslant G}$$

of ideals $I_H \subseteq T(G/H)$, indexed by the set of subgroups of G, as follows.

Proposition 3.2. *Let T be a Tambara functor on G. To give an ideal $\mathscr{I}$ of T is equivalent to give a family $\{I_H\}_{H \leqslant G}$ of ideals $I_H \subseteq T(G/H)$ satisfying the conditions*

(i) $\mathrm{res}^H_K(I_H) \subseteq I_K$,
(ii) $\mathrm{ind}^H_K(I_K) \subseteq I_H$,
(iii) $\mathrm{jnd}^H_K(I_K) \subseteq I_H$,
(iv) $c_{g,H}(I_H) \subseteq I_{H^g}$

for any $K \leqslant H \leqslant G$ and $g \in G$.

Proof. This is straightforward (cf. Corollary 2.2 in [10]). $\square$

In our particular case, an ideal of Ω_G is written as follows.

Corollary 3.3. *Let G be a cyclic p-group of order p^r. An ideal $\mathscr{I}$ of Ω_G is given by a sequence*

$$\mathscr{I} = [I_0, \dots, I_r]$$

of ideals $I_k \subseteq R_k$, satisfying the following condition $\mathcal{I}(k)$ for each $1 \leqslant k \leqslant r$.

$\boxed{\mathcal{I}(k)}$ $\mathrm{ind}^k_{k-1}(I_{k-1}) \subseteq I_k$, $\mathrm{res}^k_{k-1}(I_k) \subseteq I_{k-1}$, $\mathrm{jnd}^k_{k-1}(I_{k-1}) \subseteq I_k$.

Remark that for ideals $\mathscr{I} = [I_0, \dots, I_r]$ and $\mathscr{J} = [J_0, \dots, J_r]$, we have $\mathscr{I} \subseteq \mathscr{J}$ if and only if $I_k \subseteq J_k$ holds for any $0 \leqslant k \leqslant r$.

In the following, we will simply say “$[I_0, \dots, I_k]$ is an ideal of Ω_{H_k}” to mean that $[I_0, \dots, I_k]$ satisfies $\mathcal{I}(i)$ for any $1 \leqslant i \leqslant k$. This makes sense by virtue of Remark 2.1. The restriction of an ideal $\mathscr{I} = [I_0, \dots, I_k]$ of Ω_{H_k} onto H_i is given by $[I_0, \dots, I_i]$, for each $0 \leqslant i \leqslant k$. We denote this by $\mathscr{I}|_{H_i}$. Obviously, restriction preserves inclusions of ideals.

On the contrary, Corollary 3.3 also enables us to extend an ideal $[I_0,\dots,I_{k-1}]$ of $\Omega_{H_{k-1}}$ to an ideal $[I_0,\dots,I_k]$ of Ω_{H_k}, by adding an ideal $I_k \subseteq R_k$ satisfying $\mathcal{I}(k)$. Among the possible extensions of $[I_0,\dots,I_{k-1}]$, the largest and the smallest are given as follows. If an ideal $\mathscr{I}' \subseteq \Omega_{H_k}$ satisfies $\mathscr{I}'|_{H_i} = \mathscr{I}$ for a given ideal $\mathscr{I} \subseteq \Omega_{H_i}$, then we say that *$\mathscr{I}'$ is an ideal over $\mathscr{I}$*.

Proposition 3.4. *For a fixed $1 \leqslant k \leqslant r$, suppose we are given an ideal $\mathscr{I} = [I_0,\dots,I_{k-1}]$ of $\Omega_{H_{k-1}}$.*

(1) *If we define $L_k(I_{k-1}) = L(I_{k-1}) \subseteq R_k$ by*

$$L(I_{k-1}) = \left(\mathrm{res}^k_{k-1}\right)^{-1}(I_{k-1}),$$

then $[I_0,\dots,I_{k-1},L(I_{k-1})]$ is the largest ideal of Ω_{H_k} over $[I_0,\dots,I_{k-1}]$.

(2) *If we define $S_k(I_{k-1}) = S(I_{k-1}) \subseteq R_k$ to be the ideal of R_k generated by*

$$\left\{\mathrm{ind}^k_{k-1}(\alpha) \,\middle|\, \alpha \in I_{k-1}\right\} \quad \textit{and} \quad \left\{\mathrm{jnd}^k_{k-1}(\alpha) \,\middle|\, \alpha \in I_{k-1}\right\},$$

shortly,

$$S(I_{k-1}) = \left(\mathrm{ind}^k_{k-1}(I_{k-1})\right) + \left(\mathrm{jnd}^k_{k-1}(I_{k-1})\right),$$

then $[I_0,\dots,I_{k-1},S(I_{k-1})]$ is the smallest ideal of Ω_{H_k} over $[I_0,\dots,I_{k-1}]$.

Definition 3.5. For an ideal $\mathscr{I} = [I_0,\dots,I_{k-1}]$ of $\Omega_{H_{k-1}}$, define ideals $\mathcal{L}_k(\mathscr{I}) = \mathcal{L}\mathscr{I}$ and $\mathcal{S}_k(\mathscr{I}) = \mathcal{S}\mathscr{I}$ of Ω_{H_k} by

$$\mathcal{L}\mathscr{I} = \left[I_0,\dots,I_{k-1},L(I_{k-1})\right],$$
$$\mathcal{S}\mathscr{I} = \left[I_0,\dots,I_{k-1},S(I_{k-1})\right],$$

where $L(I_{k-1})$ and $S(I_{k-1})$ are those in Proposition 3.4. Additionally, we denote the n-times iterations of L, S, $\mathcal{L}$, $\mathcal{S}$ by L^n, S^n and $\mathcal{L}^n$, $\mathcal{S}^n$. For example, an ideal $\mathscr{I} = [I_0,\dots,I_k] \subseteq \Omega_{H_k}$ yields $\mathcal{L}^n\mathscr{I} = [I_0,\dots,I_k,L(I_k),\dots,L^n(I_k)] \subseteq \Omega_{H_{k+n}}$.

Proof of Proposition 3.4. (1) To show $\mathcal{L}\mathscr{I}$ is an ideal of Ω_{H_k}, it suffices to confirm $\mathcal{I}(k)$ is satisfied. By definition, $\mathrm{res}^k_{k-1}(L(I_{k-1})) \subseteq I_{k-1}$ is obvious. In addition, by the existence of a pullback diagram

$$\begin{tikzcd}
\amalg_p G/H_{k-1} \arrow[r, "\nabla"] \arrow[d] \arrow[dr, phantom, "\square"] & G/H_{k-1} \arrow[d, "\mathrm{pr}^k_{k-1}"] \\
G/H_{k-1} \arrow[r, "\mathrm{pr}^k_{k-1}"'] & G/H_k
\end{tikzcd}$$

where $\nabla : \amalg_p G/H_{k-1} \to G/H_{k-1}$ is the folding map, we have

$$\mathrm{res}^k_{k-1} \circ \mathrm{ind}^k_{k-1}(\alpha) = p\alpha,$$
$$\mathrm{res}^k_{k-1} \circ \mathrm{jnd}^k_{k-1}(\alpha) = \alpha^p$$

for any $\alpha \in R_k$. Thus if $\alpha \in I_{k-1}$, then

$$\mathrm{ind}^k_{k-1}(\alpha) \in \left(\mathrm{res}^k_{k-1}\right)^{-1}(I_{k-1}),$$
$$\mathrm{jnd}^k_{k-1}(\alpha) \in \left(\mathrm{res}^k_{k-1}\right)^{-1}(I_{k-1})$$

hold. This shows $\mathrm{ind}^k_{k-1}(I_{k-1}) \subseteq L(I_{k-1})$ and $\mathrm{jnd}^k_{k-1}(I_{k-1}) \subseteq L(I_{k-1})$, and thus $\mathcal{L}\mathscr{I}$ is an ideal of Ω_{H_k}. Moreover, since any ideal $[I_0, \dots, I_{k-1}, J_k]$ of Ω_{H_k} should satisfy $\mathrm{res}^k_{k-1}(J_k) \subseteq I_{k-1}$, obviously $\mathcal{L}\mathscr{I}$ is the largest.

(2) It suffices to confirm $\mathcal{S}\mathscr{I}$ satisfies $\mathcal{I}(k)$. Since res^k_{k-1} is a ring homomorphism, $\mathrm{res}^k_{k-1}(S(I_{k-1})) \subseteq I_{k-1}$ follows from the fact that any $\alpha \in I_{k-1}$ satisfies

$$\mathrm{res}^k_{k-1} \circ \mathrm{ind}^k_{k-1}(\alpha) = p\alpha \in I_{k-1},$$
$$\mathrm{res}^k_{k-1} \circ \mathrm{jnd}^k_{k-1}(\alpha) = \alpha^p \in I_{k-1}.$$

The other conditions

$$\mathrm{ind}^k_{k-1}(I_{k-1}) \subseteq S(I_{k-1}),$$
$$\mathrm{jnd}^k_{k-1}(I_{k-1}) \subseteq S(I_{k-1})$$

are obviously satisfied by the definition of $S(I_{k-1})$. Thus $\mathcal{S}\mathscr{I}$ is an ideal of Ω_{H_k}. Moreover, since any ideal $[I_0, \dots, I_{k-1}, J_k]$ of Ω_{H_k} should satisfy $\mathrm{ind}^k_{k-1}(I_{k-1}) \subseteq J_k$ and $\mathrm{jnd}^k_{k-1}(I_{k-1}) \subseteq J_k$, obviously $\mathcal{S}\mathscr{I}$ is the smallest. $\square$

4. Inductive criterion of primality

In [7], a prime ideal of a Tambara functor is defined as follows.

Definition 4.1. Let G be an arbitrary finite group, and let T be a Tambara functor on G. An ideal $\mathscr{I} \subsetneq T$ is prime if and only if the following two conditions become equivalent, for any transitive $X, Y \in \mathrm{Ob}({}_G\mathit{set})$ and any $a \in T(X)$, $b \in T(Y)$.

(1) For any $C \in \mathrm{Ob}({}_G\mathit{set})$ and for any pair of diagrams

$$C \xleftarrow{v} D \xrightarrow{w} X, \qquad C \xleftarrow{v'} D' \xrightarrow{w'} Y$$

in ${}_G\mathit{set}$, $\mathscr{I}(C)$ satisfies

$$\left(v_\bullet w^*(a) - v_\bullet(0)\right) \cdot \left(v'_\bullet w'^*(b) - v'_\bullet(0)\right) \in \mathscr{I}(C).$$

(2) $a \in \mathscr{I}(X)$ or $b \in \mathscr{I}(Y)$.

Remark that (2) always implies (1).

By a straightforward argument, we may assume C, D, D' are transitive, and this condition can be also written as follows.

Proposition 4.2. *Let G be an arbitrary finite group, and let $\mathscr{I} = \{I_H\}_{H \leqslant G}$ be an ideal of T. Then $\mathscr{I}$ is prime if and only if the following condition is satisfied for any $H, H' \leqslant G$ and any $a \in T(G/H)$, $b \in T(G/H')$.*

(∗) *If* $(\mathrm{jnd}^L_{K^g} \circ c_{g,K} \circ \mathrm{res}^H_K(a)) \cdot (\mathrm{jnd}^L_{K'^{g'}} \circ c_{g',K'} \circ \mathrm{res}^{H'}_{K'}(a)) \in I_L$ *is satisfied for any* $L, K, K' \leqslant G$ *and* $g, g' \in G$ *satisfying* $L \geqslant K^g$, $K \leqslant H$, $L \geqslant K'^{g'}$, $K' \leqslant H'$, *then*

$$a \in I_H \quad \text{or} \quad b \in I_{H'}$$

holds.

In our case, this can be reduced to the following condition.

Corollary 4.3. *Let* G *be a cyclic* p*-group of order* $p^r \geqslant 1$. *An ideal* $\mathscr{I} = [I_0, \dots, I_r]$ *of* Ω_G *is prime if and only if it satisfies the following condition* $\mathcal{P}(k, \ell)$ *for each* $0 \leqslant \ell \leqslant k \leqslant r$.

$\boxed{\mathcal{P}(k,\ell)}$ *For any* $a \in R_k$ *and* $b \in R_\ell$,

$$\bigl(\mathrm{jnd}^m_i \circ \mathrm{res}^k_i(a)\bigr) \cdot \bigl(\mathrm{jnd}^m_j \circ \mathrm{res}^\ell_j(b)\bigr) \in I_m \quad \bigl(0 \leqslant {}^\forall i \leqslant k,\ 0 \leqslant {}^\forall j \leqslant \ell,\ m = \max(i, j)\bigr)$$

implies

$$a \in I_k \quad \text{or} \quad b \in I_\ell.$$

Proof. For any $0 \leqslant k, \ell \leqslant r$ and any $a \in R_k$, $b \in R_\ell$, the condition (∗) in Proposition 4.2 is equivalent to the following.

(∗) If

$$\bigl(\mathrm{jnd}^m_i \circ c_g \circ \mathrm{res}^k_i(a)\bigr) \cdot \bigl(\mathrm{jnd}^m_j \circ c_{g'} \circ \mathrm{res}^\ell_j(b)\bigr) \in I_m \tag{4.1}$$

is satisfied for any $g, g' \in G$ and any $m \geqslant i \leqslant k$, $m \geqslant j \leqslant \ell$, then

$$a \in I_k \quad \text{or} \quad b \in I_\ell$$

holds.

Remark that we have $c_g = \mathrm{id}$, $c_{g'} = \mathrm{id}$. Besides, by (iii) of Proposition 3.2, assumption (4.1) is only have to be confirmed for $m = \max(i, j)$. Moreover, by the symmetry in k and ℓ, we may assume $\ell \leqslant k$. □

Furthermore, this condition can be checked on each k-th step, as follows.

Proposition 4.4. *Let* G *be as above. An ideal* $\mathscr{I} = [I_0, \dots, I_r]$ *of* Ω_G *is prime if and only if it satisfies the following condition* $\mathcal{P}(k)$ *for each* $0 \leqslant k \leqslant r$.

$\boxed{\mathcal{P}(k)}$ *For any* $0 \leqslant i \leqslant k$ *and any* $a \in L_k(I_{k-1})$, $b \in L_i(I_{i-1}) \setminus I_i$,

$$a \cdot \mathrm{jnd}^k_i(b) \in I_k \quad \Rightarrow \quad a \in I_k \tag{4.2}$$

holds.

Here, when $k = 0$, *we define* $L_0(I_{-1})$ *to be* R_0. *Namely,* $\mathcal{P}(0)$ *is as follows.*

$\boxed{\mathcal{P}(0)}$ *For any $a \in R_0$ and $b \in R_0 \setminus I_0$,*

$$ab \in I_0 \quad \Rightarrow \quad a \in I_0$$

holds. (This is saying $I_0 \subseteq R_0$ is prime in the ordinary ring-theoretic meaning.)

Proof. For each $0 \leqslant k \leqslant r$, we define condition $\mathcal{Q}(k)$ as follows.

$\boxed{\mathcal{Q}(k)}$ $\mathcal{P}(k, \ell)$ holds for all $0 \leqslant \ell \leqslant k$.

It suffices to show that $\mathcal{Q}(k)$ holds for any $0 \leqslant k \leqslant r$ if and only if $\mathcal{P}(k)$ holds for any $0 \leqslant k \leqslant r$. This follows from:

Claim 4.5. *For any $0 \leqslant k \leqslant r$, the following holds.*

(1) *$\mathcal{Q}(k)$ implies $\mathcal{P}(k)$.*
(2) *If $k \geqslant 1$, then $\mathcal{Q}(k-1)$ and $\mathcal{P}(k)$ imply $\mathcal{Q}(k)$.*
(3) *$\mathcal{Q}(0)$ is equivalent to $\mathcal{P}(0)$.*

In fact if this is shown, then by an induction on k, we can easily show that the following are equivalent for each $0 \leqslant k \leqslant r$.

- $\mathcal{Q}(k')$ holds for any $0 \leqslant k' \leqslant k$.
- $\mathcal{P}(k')$ holds for any $0 \leqslant k' \leqslant k$.

This proves Proposition 4.4. Thus it remains to show Claim 4.5.

Proof of Claim 4.5. (3) When $k = 0$, then the condition

$\boxed{\mathcal{Q}(0)}$ $\mathcal{P}(0, 0)$ holds. Namely, for any $a, b \in R_0$,

$$ab \in I_0 \quad \Rightarrow \quad a \in I_0 \quad \text{or} \quad b \in I_0$$

holds.

is obviously equivalent to $\mathcal{P}(0)$.

(1) Fix k, suppose we are given $0 \leqslant \ell \leqslant k$ and $a \in L_k(I_{k-1}), b \in L_\ell(I_{\ell-1}) \setminus I_\ell$ satisfying

$$a \cdot \mathrm{jnd}_\ell^k(b) \in I_k. \tag{4.3}$$

It suffices to show $a \in I_k$. Since $\mathcal{Q}(k)$ (in particular $\mathcal{P}(k, \ell)$) is assumed, it is enough to confirm that

$$\left(\mathrm{jnd}_i^m \circ \mathrm{res}_i^k(a)\right) \cdot \left(\mathrm{jnd}_j^m \circ \mathrm{res}_j^\ell(b)\right) \in I_m \tag{4.4}$$

is satisfied for any $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant \ell$ and $m = \max(i, j)$. However, when $i < k$ or $j < \ell$, (4.4) follows from $\mathrm{res}_i^k(a) \in I_i$ and $\mathrm{res}_j^\ell(b) \in I_j$, since we have $a \in L_k(I_{k-1})$ and $b \in L_\ell(I_{\ell-1})$. In the remaining case where $i = k$ and $j = \ell$, (4.4) is also satisfied by the assumption (4.3).

(2) Fix $k \geqslant 1$, and assume $\mathcal{Q}(k-1)$. Under this assumption, we show $\mathcal{P}(k)$ implies $\mathcal{P}(k, \ell)$ for any $0 \leqslant \ell \leqslant k$. By an induction on ℓ, this is reduced to the following.

Claim 4.6. *For any* $1 \leqslant k \leqslant r$ *and* $0 \leqslant \ell \leqslant k$*, we have*

$$\mathcal{Q}(k-1), \mathcal{P}(k), \mathcal{P}(k, \ell-1) \quad \Rightarrow \quad \mathcal{P}(k, \ell).$$

(*Here, for* $\ell = 0$*,* $\mathcal{P}(k, -1)$ *is regarded as an empty condition.*)

We only have to show this claim in the rest. Suppose $a \in R_k$ and $b \in R_\ell$ satisfy

$$\big(\mathrm{jnd}_i^m \circ \mathrm{res}_i^k(a)\big) \cdot \big(\mathrm{jnd}_j^m \circ \mathrm{res}_j^\ell(b)\big) \in I_m \tag{4.5}$$

for any $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant \ell$, $m = \max(i, j)$. Claim 4.6 will follow immediately, if the following are shown.

(A) If $a \in R_k \setminus I_k$, then $b \in L_\ell(I_{\ell-1})$. (This is trivial when $\ell = 0$, since we have defined as $L_0(I_{-1}) = R_0$.)
(B) If $b \in R_\ell \setminus I_{\ell-1}$, then $a \in L_k(I_{k-1})$.

In fact, if (A) and (B) are shown, then the above a and b will satisfy

(i) $a \in I_k$, or
(ii) $b \in I_\ell$, or
(iii) $a \in L_k(I_{k-1})$ and $b \in L_\ell(I_{\ell-1}) \setminus I_\ell$.

In the third case, since $a \cdot \mathrm{jnd}_\ell^k(b) \in I_k$ is satisfied by (4.5) for $i = k$ and $j = \ell$, it follows $a \in I_k$ by $\mathcal{P}(k)$.

Thus it remains to show (A) and (B).

(A) By applying $\mathcal{P}(k, \ell-1)$ to

$$a \in R_k \quad \text{and} \quad \mathrm{res}_{\ell-1}^\ell(b) \in R_{\ell-1},$$

we obtain $\mathrm{res}_{\ell-1}^\ell(b) \in I_{\ell-1}$, namely $b \in L_\ell(I_{\ell-1})$.

(B) When $\ell < k$, by applying $\mathcal{Q}(k-1)$, in particular $\mathcal{P}(k-1, \ell)$ to

$$\mathrm{res}_{k-1}^k(a) \in R_{k-1} \quad \text{and} \quad b \in R_\ell,$$

we obtain $\mathrm{res}_{k-1}^k(a) \in I_{k-1}$, namely $a \in L_k(I_{k-1})$.

When $\ell = k$, by applying $\mathcal{P}(k, k-1)$ to

$$b \in R_k \quad \text{and} \quad \mathrm{res}_{k-1}^k(a) \in R_{k-1},$$

we obtain $\mathrm{res}_{k-1}^k(a) \in I_{k-1}$, namely $a \in L_k(I_{k-1})$. $\square$

By Proposition 4.4, whether an ideal is prime or not can be checked inductively on k using $\mathcal{P}(k)$. This is applied to restrictions and extensions of prime ideals as follows.

Corollary 4.7. *For any* $0 \leqslant i \leqslant k \leqslant r$*, if an ideal* $\mathscr{I} = [I_0, \dots, I_k] \subseteq \Omega_{H_k}$ *is prime, then its restriction* $\mathscr{I}|_{H_i} = [I_0, \dots, I_i]$ *onto* H_i *is also prime.*

Proof. This immediately follows from Proposition 4.4. $\square$

Corollary 4.8. *For* $k \geqslant 1$*, let* $\mathscr{I} = [I_0, \dots, I_{k-1}]$ *be an ideal of* Ω_{H_k}*. If* $\mathscr{I}$ *is prime, then* $\mathcal{L}\mathscr{I} \subseteq \Omega_{H_k}$ *is also prime.*

Proof. It suffices to show $\mathcal{P}(k)$ is satisfied. However for $\mathcal{L}\mathscr{I} = [I_0, \dots, I_{k-1}, I_k = L(I_{k-1})]$, this condition becomes trivial as follows.

- For any $0 \leqslant i \leqslant k$, $a \in L_k(I_{k-1})$ and $b \in L_i(I_{i-1}) \setminus I_i$,

$$a \cdot \mathrm{jnd}_i^k(b) \in I_k \quad \Rightarrow \quad a \in I_k \tag{4.6}$$

  holds.

Of course this is satisfied, since a belongs to $I_k = L_k(I_{k-1})$ from the first. $\square$

5. Ideals $J_{\ell,k}(x) \subseteq R_\ell$

In this section, we introduce ideals $J_{\ell,k}(x)$ of R_ℓ, which will perform an essential role in determining the prime ideals of Ω.

First we prepare another $\mathbb{Z}$-basis for R_ℓ, which is more suitable for calculation.

Definition 5.1. For any $0 \leqslant i \leqslant \ell \leqslant r$, define $F_{\ell,i} \in R_\ell$ by

$$F_{\ell,i} = X_{\ell,i} - p^{\ell-i}.$$

Obviously, each R_ℓ admits a $\mathbb{Z}$-basis

$$\{1, F_{\ell,0}, F_{\ell,1}, \dots, F_{\ell,\ell-1}\}$$

for $0 \leqslant \ell \leqslant r$, and thus any element $\alpha \in R_\ell$ can be written as

$$\alpha = m_\ell + \sum_{i=0}^{\ell-1} m_i F_{\ell,i}$$

for some uniquely determined $m_0, \dots, m_\ell \in \mathbb{Z}$.

This basis behaves well with the multiplication and the structure morphisms as follows.

Proposition 5.2. *The following holds.*

(1) *For any $0 \leqslant i, j \leqslant \ell$, we have*

$$X_{\ell,j} \cdot F_{\ell,i} = \begin{cases} p^{\ell-j} F_{\ell,i} - p^{\ell-i} F_{\ell,j} & (i \leqslant j), \\ 0 & (j \leqslant i). \end{cases}$$

(2) *For any $0 \leqslant i, k \leqslant \ell$, we have*

$$\mathrm{res}_k^\ell(F_{\ell,i}) = \begin{cases} p^{\ell-k} F_{k,i} & (i \leqslant k), \\ 0 & (k \leqslant i). \end{cases}$$

In particular, we have

$$\mathrm{res}^{\ell}_{\ell-1}(\alpha)=m_\ell+\sum_{i=0}^{\ell-2}m_i pF_{\ell-1,i}$$

for any $\alpha=m_\ell+\sum_{i=0}^{\ell-1}m_iF_{\ell,i}\in R_\ell$.

(3) *When* $\ell\geqslant 1$, *for any*

$$\alpha=n_{\ell-1}+\sum_{i=0}^{\ell-2}m_iF_{\ell-1,i}\in R_{\ell-1},$$

we have

$$\mathrm{jnd}^{\ell}_{\ell-1}(\alpha)=(n_{\ell-1})^p+\sum_{i=0}^{\ell-1}u_iF_{\ell,i}$$

for some $u_0,\dots,u_{\ell-1}\in\mathbb{Z}$.

(4) *When* $\ell\geqslant 2$, *for any* $0\leqslant i\leqslant \ell-1$, *we have*

$$\mathrm{ind}^{\ell}_{\ell-1}(F_{\ell-1,i})=F_{\ell,i}-p^{\ell-i-1}F_{\ell,\ell-1}.$$

Moreover, $F_{\ell,\ell-1}$ *is calculated as*

$$F_{\ell,\ell-1}=\mathrm{jnd}^{\ell}_{\ell-1}(F_{\ell-1,\ell-2})+\frac{(-p)^p}{p^2}\,\mathrm{ind}^{\ell}_{\ell-1}(F_{\ell-1,\ell-2}).$$

Proof. (1) and (2) follow immediately from Corollary 2.3.

(3) For any $\alpha=n_{\ell-1}+\sum_{i=0}^{\ell-2}m_iF_{\ell-1,i}\in R_{\ell-1}$, if we put $m_{\ell-1}=n_{\ell-1}-\sum_{i=0}^{\ell-2}m_ip^{\ell-i-1}$, then we have

$$\alpha=m_{\ell-1}+\sum_{i=0}^{\ell-2}m_iX_{\ell-1,i}$$

and thus by Corollary 2.4, we obtain

$$\begin{aligned}\mathrm{jnd}^{\ell}_{\ell-1}(\alpha)&=m_{\ell-1}+\frac{(m_{\ell-1})^p-m_{\ell-1}}{p}X_{\ell,\ell-1}\\&\quad+\sum_{i=0}^{\ell-2}\frac{(\sum_{s=i}^{\ell-1}m_sp^{\ell-1-s})^p-(\sum_{s=i+1}^{\ell-1}m_sp^{\ell-1-s})^p}{p^i}X_{\ell,i}\\&=m_{\ell-1}+\left\{(m_{\ell-1})^p-m_{\ell-1}\right\}+\sum_{i=0}^{\ell-2}\left\{\left(\sum_{s=i}^{\ell-1}m_sp^{\ell-1-s}\right)^p-\left(\sum_{s=i+1}^{\ell-1}m_sp^{\ell-1-s}\right)^p\right\}\\&\quad+\frac{(m_{\ell-1})^p-m_{\ell-1}}{p}F_{\ell,\ell-1}+\sum_{i=0}^{\ell-2}\frac{(\sum_{s=i}^{\ell-1}m_sp^{\ell-1-s})^p-(\sum_{s=i+1}^{\ell-1}m_sp^{\ell-1-s})^p}{p^i}F_{\ell,i}\end{aligned}$$

$$= (n_{\ell-1})^p + \frac{(m_{\ell-1})^p - m_{\ell-1}}{p} F_{\ell,\ell-1}$$
$$+ \sum_{i=0}^{\ell-2} \frac{(n_{\ell-1} - \sum_{s=0}^{i-1} m_s p^{\ell-1-s})^p - (n_{\ell-1} - \sum_{s=0}^{i} m_s p^{\ell-1-s})^p}{p^i} F_{\ell,i}.$$

(4) By Remark 2.2, we have

$$\begin{aligned}\mathrm{ind}_{\ell-1}^{\ell}(F_{\ell-1,i}) + p^{\ell-i-1} F_{\ell,\ell-1} &= \left(X_{\ell,i} - p^{\ell-i-1} X_{\ell,\ell-1}\right) + \left(p^{\ell-i-1} X_{\ell,\ell-1} - p^{\ell-i}\right) \\ &= F_{\ell,i}.\end{aligned}$$

Moreover, by Remark 2.2 and Corollary 2.4, we have

$$\begin{aligned}&\mathrm{jnd}_{\ell-1}^{\ell}(F_{\ell-1,\ell-2}) + \frac{(-p)^p}{p^2} \mathrm{ind}_{\ell-1}^{\ell}(F_{\ell-1,\ell-2}) \\ &= \left(-p + \frac{(-p)^p + p}{p} X_{\ell,\ell-1} - \frac{(-p)^p}{p^2} X_{\ell,\ell-2}\right) + \frac{(-p)^p}{p^2}(X_{\ell,\ell-2} - pX_{\ell,\ell-1}) \\ &= X_{\ell,\ell-1} - p = F_{\ell,\ell-1}. \qquad \square\end{aligned}$$

Lemma 5.3. *For $1 \leqslant \ell \leqslant r$ and $n \in \mathbb{Z}$, we have the following.*

(1) *If n is not divisible by p, then we have*

$$n = \mathrm{jnd}_{\ell-1}^{\ell}(n) - \frac{n^{p-1} - 1}{p} \mathrm{ind}_{\ell-1}^{\ell}(n).$$

Remark that $\frac{n^{p-1}-1}{p}$ is an integer, by assumption.

(2) *If $n = pu$ for some $u \in \mathbb{Z}$, then we have*

$$n = \mathrm{ind}_{\ell-1}^{\ell}(u) - uF_{\ell,\ell-1}.$$

Proof. (1) By Remark 2.2 and Corollary 2.4, we have

$$\mathrm{jnd}_{k-1}^{k}(n) - \frac{n^{p-1} - 1}{p} \mathrm{ind}_{k-1}^{k}(n) = n + \frac{n^p - n}{p} X_{k,k-1} - \frac{n^{p-1} - 1}{p} nX_{k,k-1} = n.$$

(2) This follows from

$$\begin{aligned}\mathrm{ind}_{\ell-1}^{\ell}(u) - uF_{\ell,\ell-1} &= uX_{\ell,\ell-1} - u(X_{\ell,\ell-1} - p) \\ &= pu = n. \qquad \square\end{aligned}$$

Definition 5.4. For any $0 \leqslant \ell \leqslant r$, $0 \leqslant k \leqslant \ell$ and $x \in \mathbb{Z}$, we define an ideal $J_{\ell,k}(x) \subseteq R_\ell$ by

$$J_{\ell,k}(x) = \begin{cases} (x, F_{\ell,k}, F_{\ell,k+1}, \ldots, F_{\ell,\ell-1}) & (k \leqslant \ell - 1), \\ (x) & (k = \ell). \end{cases}$$

$J_{\ell,k}(x)$ can be calculated as follows.

Proposition 5.5. *For any $1 \leqslant \ell \leqslant r$, $0 \leqslant k \leqslant \ell - 1$ and $x \in \mathbb{Z}$, we have*

$$J_{\ell,k}(x) = \left\{ x \cdot \left(n_\ell + \sum_{i=0}^{k-1} n_i F_{\ell,i} \right) + \sum_{i=k}^{\ell-1} n_i F_{\ell,i} \;\middle|\; n_0, \dots, n_\ell \in \mathbb{Z} \right\}$$
$$= \left\{ m_\ell + \sum_{i=0}^{\ell-1} m_i F_{\ell,i} \;\middle|\; \begin{array}{l} m_0, \dots, m_\ell \in \mathbb{Z}, \\ m_\ell \in \mathbb{Z}x, \\ m_i \in \mathbb{Z}x \ (0 \leqslant i \leqslant k-1) \end{array} \right\}.$$

Proof. Obviously, $J_{\ell,k}(x)$ contains any element of the form

$$x \cdot \left(n_\ell + \sum_{i=0}^{k-1} n_i F_{\ell,i} \right) + \sum_{i=k}^{\ell-1} n_i F_{\ell,i}. \tag{5.1}$$

To show the converse, since any element in $J_{\ell,k}(x)$ can be written as an R_ℓ-coefficient sum of

$$x, F_{\ell,k}, \dots, F_{\ell,\ell-1},$$

it suffices to show that any element

(i) αx,
(ii) $\alpha F_{\ell,i}$ $(k \leqslant i \leqslant \ell - 1)$

can be written in the form of (5.1) for any $\alpha \in R_\ell$.

(i) For any $\alpha = m_\ell + \sum_{i=0}^{\ell-1} m_i F_{\ell,i}$ $(m_i \in \mathbb{Z})$, we have

$$\alpha x = x \cdot \left(m_\ell + \sum_{i=0}^{k-1} m_i F_{\ell,i} \right) + \sum_{i=k}^{\ell-1} x m_i F_{\ell,i}.$$

(ii) For any $\alpha = \sum_{j=0}^{\ell} m_j X_{\ell,j}$ $(m_j \in \mathbb{Z})$, we have

$$\begin{aligned} \alpha F_{\ell,i} &= \sum_{j=0}^{i} m_j X_{\ell,j} F_{\ell,i} + \sum_{j=i+1}^{\ell} m_j X_{\ell,j} F_{\ell,i} \\ &= \sum_{j=i+1}^{\ell} m_j \bigl(p^{\ell-j} F_{\ell,i} - p^{\ell-i} F_{\ell,j} \bigr) \\ &= \left(\sum_{j=i+1}^{\ell} m_j p^{\ell-j} \right) F_{\ell,i} - \sum_{j=i+1}^{\ell} m_j p^{\ell-i} F_{\ell,j} \end{aligned}$$

for any $k \leqslant i \leqslant \ell - 1$ by Proposition 5.2. $\square$

Proposition 5.6. *For any $1 \leqslant \ell \leqslant r$, $0 \leqslant k \leqslant \ell - 1$ and any $n \in \mathbb{Z}$, we have the following.*

(1) *If $n = 0$ or $n = q$ for some prime integer q different from p, then we have*
 (i) $L(J_{\ell-1,k}(n)) = J_{\ell,k}(n)$,

(ii) $S(J_{\ell-1,k}(n)) = \begin{cases} J_{\ell,k}(n) & (k \leqslant \ell-2), \\ J_{\ell,\ell}(n) = (n) & (k = \ell-1). \end{cases}$
Moreover, when $n = q$ is a prime different from p, then there is no ideal between $J_{\ell,k+1}(q) \subsetneq J_{\ell,k}(q)$.

(2) *For $k = 0$ and $n = p^{e+1}$ for some $e \in \mathbb{N}_{\geqslant 0}$, we have*
(i) $L(J_{\ell-1,0}(p^{e+1})) = J_{\ell,0}(p^{e+1})$,
(ii) $S(J_{\ell-1,0}(p^{e+1})) = J_{\ell,0}(p^{e+2})$.
Moreover, there is no ideal between $J_{\ell,0}(p^{e+2}) \subsetneq J_{\ell,0}(p^{e+1})$.

Proof. (1) (i) By definition,

$$L\big(J_{\ell-1,k}(n)\big) = \big\{\alpha \in R_\ell \bigm| \mathrm{res}^\ell_{\ell-1}(\alpha) \in J_{\ell-1,k}(n)\big\}.$$

For any $\alpha = m_\ell + \sum_{i=0}^{\ell-1} m_i F_{\ell,i}$ ($m_i \in \mathbb{Z}$), since we have

$$\begin{aligned} \mathrm{res}^\ell_{\ell-1}(\alpha) &= m_\ell + \sum_{i=0}^{\ell-1} m_i p F_{\ell-1,i} \\ &= m_\ell + \sum_{i=0}^{k-1} m_i p F_{\ell-1,i} + \sum_{i=k}^{\ell-1} m_i p F_{\ell-1,i}, \end{aligned}$$

this satisfies $\mathrm{res}^\ell_{\ell-1}(\alpha) \in J_{\ell-1,k}(n)$ if and only if

$$\begin{aligned} &m_\ell \in n\mathbb{Z}, \\ &m_i p \in n\mathbb{Z} \quad (0 \leqslant i \leqslant k-1), \end{aligned}$$

by Proposition 5.5. Since n is 0 or a prime different from p, this is equivalent to

$$m_\ell \in n\mathbb{Z} \quad \text{and} \quad m_i \in n\mathbb{Z} \quad (0 \leqslant i \leqslant k-1),$$

namely, to $\alpha \in J_{\ell,k}(n)$.

(ii) When $k \leqslant \ell-2$, it suffices to show

$$J_{\ell,k}(n) \subseteq S\big(J_{\ell-1,k}(n)\big).$$

In fact, this implies

$$J_{\ell,k}(n) \subseteq S\big(J_{\ell-1,k}(n)\big) \subseteq L\big(J_{\ell-1,k}(n)\big) = J_{\ell,k}(n),$$

and thus $S(J_{\ell-1,k}(n)) = L(J_{\ell-1,k}(n)) = J_{\ell,k}(n)$ follows.

Thus it remains to show $J_{\ell,k}(n) = (n, F_{\ell,k}, \dots, F_{\ell,\ell-1}) \subseteq S(J_{\ell-1,k}(n))$. However, this immediately follows from Proposition 5.2 and Lemma 5.3, since we have

$$\begin{aligned} n &= \mathrm{jnd}^\ell_{\ell-1}(n) - \frac{n^{p-1}-1}{p} \quad (\text{for } n \neq 0), \\ F_{\ell,\ell-1} &= \mathrm{jnd}^\ell_{\ell-1}(F_{\ell-1,\ell-2}) + \frac{(-p)^p}{p^2}\,\mathrm{ind}^\ell_{\ell-1}(F_{\ell-1,\ell-2}), \\ F_{\ell,i} &= \mathrm{ind}^\ell_{\ell-1}(F_{\ell-1,i}) + p^{\ell-i-1} F_{\ell,\ell-1} \quad (k \leqslant i \leqslant \ell-1). \end{aligned}$$

When $k=\ell-1$, remark that we have $J_{\ell-1,\ell-1}(q)=(q)\subseteq R_{\ell-1}$. If $n=0$, $(0)\subseteq S(J_{\ell-1,\ell-1}(0))$ is trivial. If $n=q$ is a prime different from p, then by Lemma 5.3 we have

$$q=\mathrm{jnd}^{\ell}_{\ell-1}(q)-\frac{q^{p-1}-1}{p}\mathrm{ind}^{\ell}_{\ell-1}(q)\in S\big((q)\big),$$

which means $(q)\subseteq S(J_{\ell-1,\ell-1}(q))$.

Conversely, if $n=q$ is a prime different from p, then for any $\alpha\in R_{\ell-1}$, we have

$$\begin{aligned}
\mathrm{jnd}^{\ell}_{\ell-1}(\alpha q)&=\mathrm{jnd}^{\ell}_{\ell-1}(\alpha)\cdot\mathrm{jnd}^{\ell}_{\ell-1}(q)\\
&=\mathrm{jnd}^{\ell}_{\ell-1}(\alpha)\cdot\left(q+\frac{q^p-q}{p}X_{\ell,\ell-1}\right)\\
&=\mathrm{jnd}^{\ell}_{\ell-1}(\alpha)\cdot\left(1+\frac{q^{p-1}-1}{p}X_{\ell,\ell-1}\right)\cdot q\in(q),\\
\mathrm{ind}^{\ell}_{\ell-1}(\alpha q)&=\mathrm{ind}^{\ell}_{\ell-1}(\alpha)\cdot q\in(q),
\end{aligned}$$

which imply $S(J_{\ell-1,\ell-1}(q))\subseteq(q)$. Similarly, $S(J_{\ell-1,\ell-1}(0))\subseteq(0)$ follows from $\mathrm{ind}^{\ell}_{\ell-1}(0)=\mathrm{jnd}^{\ell}_{\ell-1}(0)=0$.

Thus it remains to show there is no ideal between $J_{\ell,k+1}(q)\subsetneq J_{\ell,k}(q)$ for a prime $q\neq p$. Suppose there is an ideal

$$J_{\ell,k+1}(q)\subsetneq I\subseteq J_{\ell,k}(q).$$

By Proposition 5.5, I should contain an element

$$\alpha=q\beta+\sum_{i=k}^{\ell-1}n_iF_{\ell,i}$$

in $J_{\ell,k}(q)$, for some $\beta\in R_\ell$ and $n_i\in\mathbb{Z}$ $(k\leqslant i\leqslant\ell-1)$, which does not belong to $J_{\ell,k+1}(q)$. Then I should contain

$$n_kF_{\ell,k}=\alpha-\left\{q\beta+\sum_{i=k+1}^{\ell-1}n_iF_{\ell,i}\right\}.$$

Since α does not belong to $J_{\ell,k+1}(q)$, it follows that q does not divide n_k, and thus I contains an element of the form

$$n_kF_{\ell,k}\quad(n_k\in\mathbb{Z},\ \text{not divisible by }q).$$

On the other hand, $q\in J_{\ell,k+1}(q)\subseteq I$ implies

$$qF_{\ell,k}\in I.$$

Since q and n_k are coprime, it follows $F_{\ell,k}\in I$, which means $I=J_{\ell,k}(q)$.

(2) (i) For an element $\alpha = m_\ell + \sum_{i=0}^{\ell-1} m_i F_{\ell,i}$, since we have

$$\mathrm{res}^{\ell}_{\ell-1}(\alpha) = m_\ell + \sum_{i=0}^{\ell-2} m_i p F_{\ell-1,i}$$

by Proposition 5.2, it satisfies $\alpha \in L(J_{\ell-1,0}(p^{e+1}))$ if and only if $m_\ell \in p^{e+1}\mathbb{Z}$, namely $\alpha \in J_{\ell,0}(p^{e+1})$.

(ii) By Proposition 5.2 and Lemma 5.3, we have

$$\begin{aligned} F_{\ell,\ell-1} &= \mathrm{jnd}^{\ell}_{\ell-1}(F_{\ell-1,\ell-2}) + \frac{(-p)^p}{p^2}\,\mathrm{ind}^{\ell}_{\ell-1}(F_{\ell-1,\ell-2}), \\ F_{\ell,i} &= \mathrm{ind}^{\ell}_{\ell-1}(F_{\ell-1,i}) + p^{\ell-i-1}(F_{\ell,\ell-1}) \quad (0 \leqslant i \leqslant \ell-1), \\ p^{e+2} &= \mathrm{ind}^{\ell}_{\ell-1}\bigl(p^{e+1}\bigr) - p^{e+1}F_{\ell,\ell-1}, \end{aligned}$$

which imply $J_{\ell,0}(p^{e+2}) \subseteq S(L_{\ell-1,0}(p^{e+1}))$.

To show the converse, by Proposition 5.5, it suffices to show any element

$$\alpha = p^{e+1}n_{\ell-1} + \sum_{i=0}^{\ell-2} m_i F_{\ell-1,i} \quad \bigl(n_{\ell-1}, m_i \in \mathbb{Z}\ (0 \leqslant i \leqslant \ell-2)\bigr)$$

in $J_{\ell-1,0}(p^{e+1})$ satisfies $\mathrm{ind}^{\ell}_{\ell-1}(\alpha) \in J_{\ell,0}(p^{e+2})$ and $\mathrm{jnd}^{\ell}_{\ell-1}(\alpha) \in J_{\ell,0}(p^{e+2})$. However, these follow from

$$\begin{aligned} \mathrm{ind}^{\ell}_{\ell-1}(\alpha) &= p^{e+1}n_{\ell-1}X_{\ell,\ell-1} + \sum_{i=0}^{\ell-2} m_i\bigl(F_{\ell,i} - p^{\ell-i-1}F_{\ell,\ell-1}\bigr) \\ &= p^{e+2}n_{\ell-1} + \sum_{i=0}^{\ell-2} m_i F_i + \left(p^{e+1}n_{\ell-1} - \sum_{i=0}^{\ell-2} p^{\ell-i-1}m_i\right) F_{\ell,\ell-1} \end{aligned}$$

and

$$\mathrm{jnd}^{\ell}_{\ell-1}(\alpha) = \bigl(p^{e+1}n_{\ell-1}\bigr)^p + \sum_{i=0}^{\ell-1} u_i F_{\ell,i}$$

for some $u_0, \dots, u_{\ell-1} \in \mathbb{Z}$, by Proposition 5.2.

It remains to show there is no ideal between $J_{\ell,0}(p^{e+2}) \subsetneq J_{\ell,0}(p^{e+1})$. Suppose there is an ideal

$$J_{\ell,0}\bigl(p^{e+2}\bigr) \subsetneq I \subseteq J_{\ell,0}\bigl(p^{e+1}\bigr).$$

Then by Proposition 5.5, I should contain an element

$$\alpha = p^{e+1}n_\ell + \sum_{i=0}^{\ell-1} m_i F_{\ell,i}$$

in $J_{\ell,0}(p^{e+1})$, for some $n_\ell, m_i \in \mathbb{Z}$ $(0 \leqslant i \leqslant \ell - 1)$, which does not belong to $J_{\ell,0}(p^{e+2})$. Then I should contain

$$p^{e+1} n_\ell = \alpha - \sum_{i=0}^{\ell-1} m_i F_{\ell,i}.$$

Since α is not in $J_{\ell,0}(p^{e+2})$, it follows that p does not divide n_ℓ. On the other hand, we have

$$p^{e+2} \in J_{\ell,0}\big(p^{e+2}\big) \subseteq I.$$

These imply $p^{e+1} \in I$, which means $I = J_{\ell,0}(p^{e+1})$. $\square$

6. Structure of *Spec Ω*

Let $0 \leqslant \ell \leqslant r$ be any integer. For any ideal $\mathscr{I} = [I_0, \ldots, I_\ell] \subseteq \Omega_{H_\ell}$, define $F(\mathscr{I})$ to be the ideal $F(\mathscr{I}) = I_0$ of R_0. Since I_0 becomes prime if $\mathscr{I}$ is prime, this gives a map

$$F : \mathit{Spec}\,\Omega_G \to \mathit{Spec}\,R_0.$$

To determine $\mathit{Spec}\,\Omega$, it suffices to investigate each fiber of F over the points of $\mathit{Spec}\,R_0$. Since $\mathit{Spec}\,R_0 = \mathit{Spec}\,\mathbb{Z} = \{0\} \cup \{(q) \mid q \text{ is a prime integer}\}$, in the rest we will study the following three cases.

(i) Prime ideals over (q), where q is a prime integer different from p.
(ii) Prime ideals over (0).
(iii) Prime ideals over (p).

6.1. Prime ideals over $q \neq p$

Let q be any prime integer different from p.

Proposition 6.1. *We have the following.*

(1) *For any $k \geqslant 0$, we have*

$$S^k(q) = (q) \subseteq R_k,$$

and thus $\mathcal{S}^k(q) = [(q), (q), \ldots, (q)]$.

(2) *For any $1 \leqslant \ell \leqslant r$ and $0 \leqslant k \leqslant \ell$, we have*

$$L^{\ell-k} S^k(q) = J_{\ell,k}(q) \subseteq R_\ell.$$

(3) *For any $1 \leqslant \ell \leqslant r$ and $0 \leqslant k \leqslant \ell - 2$, we have*

$$S L^{\ell-k-1} S^k(q) = L^{\ell-k} S^k(q).$$

Proof. This follows from Proposition 5.6, by an induction on k and ℓ. $\square$

Proposition 6.2. *For any* $0 \leqslant k \leqslant r$,

$$\mathcal{S}^k(q) = \big[(q), \ldots, (q)\big] \subseteq \Omega_{H_k}$$

is a prime ideal.

Proof. We show this by an induction on k. If $k = 0$, then $(q) \subseteq \Omega_e \cong \mathbb{Z}$ is indeed a prime ideal.

When $k \geqslant 1$, suppose $\mathcal{S}^{k-1}(q) \subseteq \Omega_{H_{k-1}}$ is prime. It suffices to show $\mathcal{S}^k(q) = [(q), \ldots, (q)]$ satisfies $\mathcal{P}(k)$. Namely, we only have to show the following.

Claim 6.3. *For any* $0 \leqslant i \leqslant k$, *any* $a \in L_k(q)$ *and any* $b \in L_i(q) \setminus (q)$,

$$a \cdot \mathrm{jnd}_i^k(b) \in (q) \quad \Rightarrow \quad a \in (q)$$

holds.

Proof. Since $L_k(q) = J_{k,k-1}(q)$, a can be written as

$$a = q\beta + mF_{k,k-1} \quad (\beta \in R_k,\ m \in \mathbb{Z})$$

by Proposition 5.5.

Similarly by Proposition 5.5, $b \in L_i(q) = J_{i,i-1}(q)$ can be written in the form

$$\begin{aligned} b &= q\left(n_i + \sum_{j=0}^{i-2} n_j F_{i,j}\right) + n_{i-1}F_{i,i-1} \\ &= q\left(n_i - \sum_{j=0}^{i-2} n_j p^{i-j}\right) - pn_{i-1} + \sum_{j=0}^{i-2} n_j q X_{i,j} + n_{i-1}X_{i,i-1} \end{aligned}$$

for some $n_j \in \mathbb{Z}$ $(0 \leqslant j \leqslant i)$. If we put $n_i' = n_i - \sum_{j=0}^{i-2} n_j p^{i-j}$, then we have

$$\mathrm{jnd}_i^k(b) = \big(qn_i' - pn_{i-1}\big) + \sum_{t=0}^{k-1} u_t X_{k,t}$$

for some $u_t \in \mathbb{Z}$ $(0 \leqslant t < k)$, by Corollary 2.4.

Now we have

$$a \cdot \mathrm{jnd}_i^k(b) = q\big(\beta \cdot \mathrm{jnd}_i^k(b)\big) + \big(qn_i' - pn_{i-1}\big)mF_{k,k-1}.$$

This satisfies $a \cdot \mathrm{jnd}_i^k(b) \in (q)$ if and only if

$$pn_{i-1}mF_{k,k-1} \in (q),$$

which is equivalent to $pn_{i-1}m \in q\mathbb{Z}$ by Proposition 5.2. Since b is not in (q), n_{i-1} is not divisible by q. Thus it follows $m \in q\mathbb{Z}$, which means $a \in (q)$. $\square$

Proposition 6.4. *In Ω_{H_ℓ}, there are exactly $(\ell+1)$ ideals over $(q) \subseteq R_0$*

$$\mathcal{S}^\ell(q) \subsetneq \mathcal{L}\mathcal{S}^{\ell-1}(q) \subsetneq \cdots \subsetneq \mathcal{L}^{\ell-k}\mathcal{S}^k(q) \subsetneq \cdots \subsetneq \mathcal{L}^\ell(q), \tag{6.1}$$

all of which are prime.

Proof. By Proposition 6.2 and Corollary 4.8, these are prime.

We show the proposition by an induction on ℓ. Suppose we have done for $\ell-1$, and take any ideal $\mathscr{I} = [I_0, \dots, I_\ell] \subseteq \Omega_{H_\ell}$ over $(q) \subseteq R_0$. By the assumption of the induction, there exists some $0 \leqslant k \leqslant \ell-1$ satisfying

$$\mathscr{I}|_{H_{\ell-1}} = \mathcal{L}^{\ell-k-1}\mathcal{S}^k(q).$$

If $k \leqslant \ell-2$, then I_ℓ should satisfy

$$I_\ell = SL^{\ell-k-1}S^k(q) = L^{\ell-k}S^k(q) = J_{\ell,k}(q)$$

by Proposition 6.1, and thus $\mathscr{I} = \mathcal{L}^{\ell-k}\mathcal{S}^k(q)$.

If $k = \ell-1$, then I_ℓ should satisfy

$$S^\ell(q) \subseteq I_\ell \subseteq LS^{\ell-1}(q).$$

Since there is no ideal between $S^\ell(q) = (q)$ and $LS^{\ell-1}(q) = J_{\ell,\ell-1}(q)$ by Proposition 5.6, we have

$$I_\ell = S^\ell(q) \quad \text{or} \quad LS^{\ell-1}(q),$$

namely

$$\mathscr{I} = \mathcal{S}^\ell(q) \quad \text{or} \quad \mathcal{L}\mathcal{S}^{\ell-1}(q). \qquad \square$$

6.2. *Prime ideals over p*

Proposition 6.5. *Consider an ideal in R_ℓ, obtained from $(p) \subseteq R_0$ by an iterated application of L and S*

$$S^{a_1}L^{b_1}S^{a_2}L^{b_2}\cdots S^{a_s}L^{b_s}(p) \subseteq R_\ell$$

for $a_i, b_i \in \mathbb{N}_{\geqslant 0}$ satisfying $\sum_{i=1}^s a_i + \sum_{i=1}^s b_i = \ell$. Then this depends only on $k = \sum_{i=1}^s a_i$ $(0 \leqslant k \leqslant \ell)$, and is equal to $J_{\ell,0}(p^{k+1})$. Namely we have

$$\begin{aligned} S^{a_1}L^{b_1}S^{a_2}L^{b_2}\cdots S^{a_s}L^{b_s}(p) &= S^kL^{\ell-k}(p) \\ &= L^{\ell-k}S^k(p) = J_{\ell,0}\big(p^{k+1}\big). \end{aligned}$$

Proof. By an induction on ℓ, this immediately follows from Proposition 5.6. $\square$

Corollary 6.6. *For any $1 \leqslant \ell \leqslant r$, any ideal $\mathscr{I} = [I_0 = (p), I_1, \dots, I_\ell]$ of Ω_{H_ℓ} over (p) satisfies*

(1) *$I_i = S(I_{i-1})$ or $I_i = L(I_{i-1})$,*
(2) *$I_i = J_{i,0}(p^{k+1})$ for some $0 \leqslant k \leqslant i$*

for each $1 \leqslant i \leqslant \ell$.

Proof. We proceed by an induction on ℓ. Suppose we have shown for $\ell - 1$. Then, any ideal $\mathscr{I} = [(p), I_1, \dots, I_\ell]$ should satisfy

$$I_{\ell-1} = J_{\ell-1,0}\big(p^{k+1}\big)$$

for some $0 \leqslant k \leqslant \ell - 1$. Then by Proposition 5.6, I_ℓ should lie between $S(I_{\ell-1}) = J_{\ell,0}(p^{k+2})$ and $L(I_{\ell-1}) = J_{\ell,0}(p^{k+1})$, while there is no ideal between them. Thus I_ℓ should agree with $S(I_{\ell-1}) = J_{\ell,0}(p^{k+2})$ or $L(I_{\ell-1}) = J_{\ell,0}(p^{k+1})$. □

Proposition 6.7. *Among all the ideals of Ω_{H_ℓ} over (p) determined in Corollary 6.6,*

$$\mathcal{L}^\ell(p) = \big[(p), J_{1,0}(p), \dots, J_{\ell,0}(p)\big]$$

is the only one prime ideal.

Proof. $\mathcal{L}^\ell(p)$ is prime by Corollary 4.8. Remark that any other ideal $\mathscr{I} = [I_0 = (p), I_1, \dots, I_\ell]$ should satisfy $I_k = S(I_{k-1})$ for some $1 \leqslant k \leqslant \ell$. If we take the smallest such k, then it satisfies

$$I_{k-1} = J_{k-1,0}(p) \quad \text{and} \quad I_k = J_{k,0}\big(p^2\big).$$

Then $\mathcal{P}(k)$ fails for $\mathscr{I}$. In fact, for

$$a = b = p \in L(I_{k-1}) = J_{k,0}(p),$$

we have $ab = p^2 \in J_{k,0}(p^2)$, while neither a nor b belong to I_k. □

6.3. Prime ideals over 0

Proposition 6.8. *We have the following.*

(1) *For any $k \geqslant 0$, we have*

$$S^k(0) = (0).$$

(2) *For any $1 \leqslant \ell \leqslant r$ and $0 \leqslant k \leqslant \ell$, we have*

$$L^{\ell-k} S^k(0) = J_{\ell,k}(0) \subseteq R_\ell.$$

(3) *For any $1 \leqslant \ell \leqslant r$ and $0 \leqslant k \leqslant \ell - 2$, we have*

$$S L^{\ell-k-1} S^k(0) = L^{\ell-k} S^k(0).$$

Proof. Similarly to Proposition 6.1, this follows from Proposition 5.6 by an induction. □

Proposition 6.9. *For any $0 \leqslant k \leqslant r$,*

$$\mathcal{S}^k(0) = [0, \dots, 0] = 0 \subseteq \Omega_{H_k}$$

is prime.

Proof. This is shown in a similar way as in Proposition 6.2. More generally, for any finite group G, it was shown that the zero ideal $0 \subseteq \Omega_G$ is prime (Theorem 4.40 in [7]). $\square$

Lemma 6.10. *For any $1 \leqslant \ell \leqslant r$ and $0 \leqslant k \leqslant \ell - 1$, any ideal I satisfying*

$$J_{\ell,k+1}(0) \subseteq I \subseteq J_{\ell,k}(0)$$

is of the form

$$\begin{aligned} I &= J_{\ell,k+1}(0) + (nF_{\ell,k}) \\ &= (nF_{\ell,k}, F_{\ell,k+1}, F_{\ell,k+2}, \dots, F_{\ell,\ell-1}) \end{aligned}$$

for some $n \in \mathbb{Z}$.

Proof. By Proposition 5.5, we have

$$J_{\ell,k}(0) = \left\{ \sum_{i=k}^{\ell-1} n_i F_{\ell,i} \,\middle|\, n_i \in \mathbb{Z} \ (k \leqslant i \leqslant \ell - 1) \right\},$$
$$J_{\ell,k+1}(0) = \left\{ \sum_{i=k+1}^{\ell-1} n_i F_{\ell,i} \,\middle|\, n_i \in \mathbb{Z} \ (k+1 \leqslant i \leqslant \ell - 1) \right\}.$$

When $I \neq J_{\ell,k+1}(0)$, if we put

$$n = \min\{n \in \mathbb{N}_{>0} \mid nF_{\ell,k} \in I\},$$

then we can show easily

$$I = J_{\ell,k+1}(0) + (nF_{\ell,k}). \qquad \square$$

Proposition 6.11. *In Ω_{H_ℓ}, there are exactly $\ell + 1$ prime ideals*

$$0 \subsetneq \mathcal{L}(0) \subsetneq \mathcal{L}^2(0) \subsetneq \cdots \subsetneq \mathcal{L}^\ell(0)$$

over $(0) \subseteq R_0$.

Here, $\mathcal{L}^k(0)$ is the ideal obtained by a k-times application of $\mathcal{L}$ to $0 \subseteq \Omega_{H_{\ell-k}}$, for each $0 \leqslant k \leqslant \ell$.

Proof. By Corollary 4.8 and Proposition 6.9, each $\mathcal{L}^k(0)$ is prime.

We show the proposition by an induction on ℓ. Suppose we have done for $\ell - 1$, and take any prime ideal $\mathscr{I} = [I_0, \dots, I_\ell] \subseteq \Omega_{H_\ell}$ over $0 \subseteq R_0$. Since $\mathscr{I}|_{H_{\ell-1}}$ is prime by Corollary 4.7, we have $\mathscr{I}|_{H_{\ell-1}} = \mathcal{L}^k(0)$ on $H_{\ell-1}$ for some $0 \leqslant k \leqslant \ell - 1$, namely

$$[I_0, \dots, I_{\ell-1}] = \big[0, \dots, 0, L(0), \dots, L^k(0)\big].$$

If $k \geqslant 1$, then Proposition 6.8 shows

$$S(I_{\ell-1}) = SL^k(0) = L^{k+1}(0) = L(I_{\ell-1}),$$

and thus I_ℓ should be equal to $L^{k+1}(0)$. This means $\mathscr{I} = \mathcal{L}^{k+1}(0)$.

If $k=0$, then I_ℓ should satisfy

$$0 \subseteq I_\ell \subseteq L_\ell(0) = (F_{\ell,\ell-1}).$$

By Lemma 6.10, there exists $n \in \mathbb{N}_{\geqslant 0}$ satisfying $I_\ell = (nF_{\ell,\ell-1})$. Thus it suffices to show that $\mathscr{I} = [0,\dots,0,(nF_{\ell,\ell-1})] \subseteq \Omega_{H_\ell}$ is not prime unless $n=0$ or 1.

Suppose $\mathscr{I}$ is prime for some $n \neq 0$. Then for $a = F_{\ell,\ell-1}$ and $b = n \in R_0 \setminus (0)$, since we have

$$\begin{aligned} a \cdot \mathrm{jnd}_0^\ell(\ell) &= F_{\ell,\ell-1} \cdot \left(n + \sum_{i=0}^{\ell-1} m_i X_{\ell,i}\right) \quad (\text{for some } m_i \in \mathbb{Z}) \\ &= nF_{\ell,\ell-1} \in I_\ell, \end{aligned}$$

we obtain $F_{\ell,\ell-1} \in I_\ell$ by $\mathcal{P}(\ell)$. This implies $I_{\ell-1} = J_{\ell,\ell-1}(0)$ and $n=1$. □

6.4. Total picture

Putting Propositions 6.4, 6.7, 6.11 together, we obtain the following.

Theorem 6.12. *Let G be a cyclic p-group of order p^r. The prime ideals of $\Omega_G \in \mathrm{Ob}(Tam(G))$ are as follows.*

(i) *Over $(q) \subseteq R_0$, there are $r+1$ prime ideals*

$$\mathcal{S}^r(q) \subsetneq \mathcal{L}\mathcal{S}^{r-1}(q) \subsetneq \cdots \subsetneq \mathcal{L}^r(q).$$

(ii) *Over $(p) \subseteq R_0$, there is only one prime ideal $\mathcal{L}^r(p)$.*

(iii) *Over $0 \subseteq R_0$, there are $r+1$ prime ideals*

$$0 \subsetneq \mathcal{L}(0) \subsetneq \cdots \subsetneq \mathcal{L}^r(0).$$

Thus we have

$$\begin{aligned} \mathit{Spec}\,\Omega_G = {} & \left\{\mathcal{L}^r(p)\right\} \cup \left\{\mathcal{L}^i(0) \;\middle|\; 0 \leqslant i \leqslant r\right\} \\ & \cup \left\{\mathcal{L}^i\mathcal{S}^{r-i}(q) \;\middle|\; 0 \leqslant i \leqslant r,\ q \text{ is a prime different from } p\right\}. \end{aligned}$$

For each $0 \leqslant i \leqslant r$, there is an inclusion $\mathcal{L}^i(0) \subsetneq \mathcal{L}^i\mathcal{S}^{r-i}(q)$, while we have $\mathcal{L}^j(0) \nsubseteq \mathcal{L}^i\mathcal{S}^{r-i}(q)$ for any $j > i$. Moreover, we have

$$\mathcal{L}^r(0) \subseteq \mathcal{L}^r(p), \qquad \mathcal{L}^r(p) \nsubseteq \mathcal{L}^r(q), \qquad \mathcal{L}^r(q) \nsubseteq \mathcal{L}^r(p).$$

Corollary 6.13. *The longest sequence of prime ideals of Ω_G is of length $r+1$, such as*

$$0 \subsetneq \mathcal{S}^r(q) \subsetneq \mathcal{L}\mathcal{S}^{r-1}(q) \subsetneq \cdots \subsetneq \mathcal{L}^r(q),$$

and thus the dimension of Spec Ω_G, which we denote simply by $\dim \Omega_G$, is

$$\dim \Omega_G = r+1.$$

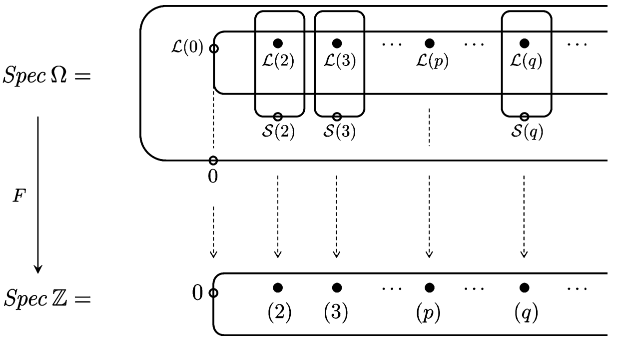

Fig. 1. $Spec\,\Omega$ for $G = \mathbb{Z}/p\mathbb{Z}$.

Let us draw some picture of $Spec\,\Omega_G$, by indicating the closures of points. If $r = 1$ and G is a group of prime order p, then the picture will become as follows (§3.7 in [10]).

Example 6.14. Let G be the group of order p. Then we have

$$\begin{aligned} Spec\,\Omega = {} & \{\mathcal{L}(p)\} \cup \{0\} \cup \{\mathcal{L}(0)\} \\ & \cup \{\mathcal{S}(q) \mid q \in \mathbb{Z} \text{ is prime},\ q \neq p\} \cup \{\mathcal{L}(q) \mid q \in \mathbb{Z} \text{ is prime},\ q \neq p\}. \end{aligned}$$

Inclusions are

$$\begin{array}{ccccc} (0) & \subsetneq & \mathcal{L}(0) & \subsetneq & \mathcal{L}(p) \\ \cup\!\!\!\!\neq & & \cup\!\!\!\!\neq & & \\ \mathcal{S}(q) & \subsetneq & \mathcal{L}(q) & & {\scriptstyle (q \neq p)}. \end{array}$$

Especially, the dimension is 2. $\mathcal{L}(p)$ and $\mathcal{L}(q)$'s are the closed points, and 0 is the generic point in $Spec\,\Omega$. If we represent the points in $Spec\,\Omega$ by their closures, $Spec\,\Omega$ with fibration F can be depicted as in Fig. 1.

A similar description is also possible for an arbitrary r. $\mathcal{L}^r(p)$ and $\mathcal{L}^r(q)$'s are the closed points, and 0 is the generic point in $Spec\,\Omega$. The picture becomes as in Fig. 2.

6.5. Topology on $Spec\,\Omega$

It is also possible to determine all the closed subsets of $Spec\,\Omega$. Throughout, let G be a cyclic p-group of order p^r.

First, we remark that the closed subsets in the closed fibers are determined by Theorem 6.12.

Remark 6.15. For any prime integer $q \neq 0$, any closed subset in the fiber $F^{-1}(q)$ is irreducible of the form $V(\mathscr{I})$ for some prime ideal $\mathscr{I} \in F^{-1}(q)$. Moreover for each q, these are totally ordered with respect to the inclusion.

Proof. If $q = p$, this is trivial. For $q \neq p$, by Theorem 6.12, the set of prime ideals in $F^{-1}(q)$ is totally ordered as

$$\mathcal{S}^r(q) \subsetneq \mathcal{L}\mathcal{S}^{r-1}(q) \subsetneq \cdots \subsetneq \mathcal{L}^r(q).$$

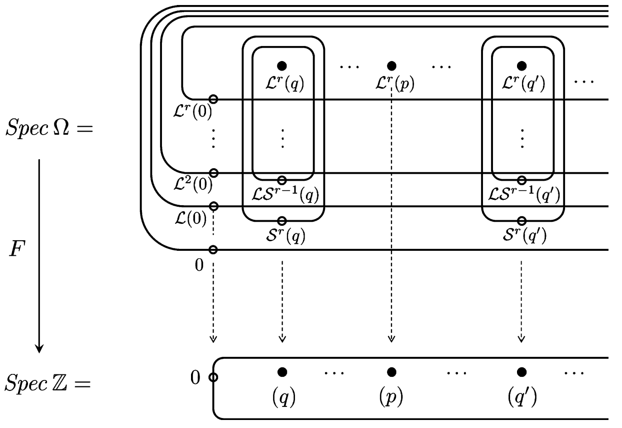

Fig. 2. *Spec* Ω for $G = \mathbb{Z}/p^r\mathbb{Z}$.

For any closed subset $V \subseteq F^{-1}(q)$, put

$$k_0 = \min\bigl\{k \bigm| \mathcal{L}^k\mathcal{S}^{r-k}(q) \in V\bigr\}.$$

Since V should contain the closure of the singleton $\{\mathcal{L}^{k_0}\mathcal{S}^{r-k_0}(q)\}$, which is

$$V\bigl(\mathcal{L}^{k_0}\mathcal{S}^{r-k_0}(q)\bigr) = \bigl\{\mathcal{L}^k\mathcal{S}^{r-k}(q) \bigm| k_0 \leqslant k \leqslant r\bigr\},$$

we have

$$V \subseteq V\bigl(\mathcal{L}^{k_0}\mathcal{S}^{r-k_0}(q)\bigr) \subseteq V,$$

and thus obtain $V = V(\mathcal{L}^{k_0}\mathcal{S}^{r-k_0}(q))$. This means that V is irreducible, and the closed sets in $F^{-1}(q)$ are totally ordered as

$$V\bigl(\mathcal{S}^r(q)\bigr) \supsetneq V\bigl(\mathcal{LS}^{r-1}(q)\bigr) \supsetneq \cdots \supsetneq V\bigl(\mathcal{L}^r(q)\bigr). \qquad \square$$

Remark that there also exist closed sets in *Spec* Ω 'transverse' to fibers.

Remark 6.16. For each $0 \leqslant t \leqslant r$, we have an irreducible closed subset

$$V\bigl(\mathcal{L}^{r-t}(0)\bigr) \subseteq Spec\,\Omega.$$

We show that any closed subset V of *Spec* Ω can be written as a union of closed sets given in Remarks 6.15 and 6.16. In fact, this gives the irreducible decomposition of V.

Remark that any closed subset in *Spec* Ω is, by definition, of the form $V(\mathscr{I})$ for some ideal $\mathscr{I} \subseteq \Omega$.

Proposition 6.17. *Let $\mathscr{I} = [I_0, \ldots, I_r]$ be any ideal in Ω. If we put*

$$t_0 = \min\{t \mid I_t \neq 0\} - 1$$

and

$$P_{\mathscr{I}} = \{q \in \mathbb{Z}_{>0},\ prime \mid (V(\mathscr{I}) - V(\mathcal{L}^{r-t_0}(0))) \cap F^{-1}(q) \neq \emptyset\},$$

then we have the following. (*When* $t_0 = -1$, *we put* $V(\mathcal{L}^{r+1_0}(0)) = \emptyset$.)

(1) $P_{\mathscr{I}}$ *is a finite set.*
(2) $V(\mathscr{I}) = V(\mathcal{L}^{r-t_0}(0)) \cup (\bigcup_{q \in P_{\mathscr{I}}} V_q(\mathscr{I}))$, *where* $V_q(\mathscr{I}) = V(\mathscr{I}) \cap F^{-1}(q)$. *This gives the irreducible decomposition of* $V(\mathscr{I})$.

Proof. Since (2) follows immediately from (1), we only show (1). For any prime $q \in \mathbb{Z}_{>0}$, remark that we have

$$F^{-1}(q) - V(\mathcal{L}^{r-t_0}(0)) = \{\mathcal{L}^{r-k}\mathcal{S}^k(q) \mid t_0 < k \leqslant r\}$$

and $\mathcal{L}^{r-t_0-1}\mathcal{S}^{t_0+1}(q)$ is the largest in this set.

Thus q satisfies

$$(V(\mathscr{I}) - V(\mathcal{L}^{r-t_0})) \cap F^{-1}(q) \neq \emptyset$$

if and only if it satisfies

$$V(\mathscr{I}) \ni \mathcal{L}^{r-t_0-1}\mathcal{S}^{t_0+1}(q),$$

which is equivalent to $\mathscr{I} \subseteq \mathcal{L}^{r-t_0-1}\mathcal{S}^{t_0+1}(q)$.

Since we have

$$\mathscr{I} = [\underbrace{0, \ldots, 0}_{t_0}, I_{t_0+1}, I_{t_0+2}, \ldots, I_r],$$
$$\mathcal{L}^{r-t_0-1}\mathcal{S}^{t_0+1}(q) = [\underbrace{(q), \ldots, (q), (q)}_{t_0+1}, L(q), \ldots, L^{r-t_0-1}(q)],$$

this means $I_{t_0+1} \subseteq (q)$. Thus we have

$$0 \neq I_{t_0+1} \subseteq \bigcap_{q \in P_{\mathscr{I}}} (q)$$

in R_{t_0+1}, which implies that $P_{\mathscr{I}}$ is finite. □

Remark 6.18. Since *Spec* Ω is a *spectral space* in the sense of [5], it should be homeomorphic to a prime spectrum of a commutative ring. However it will not be so easy to construct the ring which plays this role.

Nevertheless, we can find some relationship with the prime spectrum *Spec* $\Omega(G)$ of the Burnside ring $\Omega(G)$.

6.6. Relation to Spec $\Omega(G)$

A celebrated Dress' theorem [3] states that the prime spectrum of the Burnside ring $\Omega(G)$ detects the solvability of a finite group G. It would be a natural question whether $Spec\,\Omega(G)$ and our $Spec\,\Omega$ are related in some way. In the case of a cyclic p-group, we can construct an ad hoc continuous map $Spec\,\Omega(G) \to Spec\,\Omega$, which is bijective as a map of sets.

In fact, Dress' result gives the whole picture of $Spec\,\Omega(G)$ as follows.

Fact 6.19. Let G be a cyclic p-group of order p^r, and let

$$\rho = (\rho_{H_k})_{0\leqslant k\leqslant r} : \Omega(G) \to \prod_{0\leqslant k\leqslant r} \mathbb{Z}$$

be the mark morphism. Then any prime ideal in $\Omega(G)$ is of the form

$$\mathfrak{p}_{k,q} = \rho_{H_k}^{-1}(q),$$

where $q \in \mathbb{Z}$ is 0 or a prime, and k is an integer satisfying $0 \leqslant k \leqslant r$. Remark that ρ_{H_k} is given by

$$\rho_{H_k} = \left(\Omega(G) = R_r \xrightarrow{\mathrm{res}_k^r} R_k = \bigoplus_{0\leqslant i\leqslant k} \mathbb{Z}X_{k,i} \to \mathbb{Z}\right),$$

where the last map is given by $\sum_i m_i X_{k,i} \mapsto m_k$. Thus we have

$$\begin{aligned}\mathfrak{p}_{k,q} &= \left\{ \sum_{i=0}^{r} m_i X_{r,i} \in R_r \;\middle|\; \sum_{i=k}^{r} \left(m_i p^{r-i}\right) \in q\mathbb{Z} \right\} \\ &= \begin{cases} (q, F_{r,r-1}, \dots, F_{r,k}, X_{r,k-1}, \dots, X_{r,0}) & (q \neq p), \\ (p, F_{r,r-1}, \dots, F_{r,0}) & (q = p). \end{cases}\end{aligned}$$

In particular $\mathfrak{p}_{k,p}$ does not depend on k. We denote this simply by $\mathfrak{p}_p$. No other prime ideal equals to any other. The only inclusions among these ideals are

$$\mathfrak{p}_{k,0} \subseteq \mathfrak{p}_{k,q} \quad (0 \leqslant k \leqslant r,\ q \neq 0, p),$$
$$\mathfrak{p}_{k,0} \subseteq \mathfrak{p}_p \quad (0 \leqslant k \leqslant r).$$

Also remark that there is a canonical continuous map $E : Spec\,\Omega(G) \to Spec\,\mathbb{Z}$ induced from the ring homomorphism $\mathbb{Z} \to \Omega(G)$. The fibers of E are

$$E^{-1}(0) = \{\mathfrak{p}_{k,0} \mid 0 \leqslant k \leqslant r\},$$
$$E^{-1}(q) = \{\mathfrak{p}_{k,q} \mid 0 \leqslant k \leqslant r\} \quad (q \neq 0, p),$$
$$E^{-1}(p) = \{\mathfrak{p}_p\}.$$

All points in $E^{-1}(q)$ and $E^{-1}(p)$ are the closed points in $Spec\,\Omega(G)$.

Corollary 6.20. *If we define $f : \mathit{Spec}\,\Omega(G) \to \mathit{Spec}\,\Omega$ by*

$$\begin{aligned} f(\mathfrak{p}_{k,q}) &= \mathcal{L}^{r-k}\mathcal{S}^k(q) \quad (q \neq 0, p), \\ f(\mathfrak{p}_{k,0}) &= \mathcal{L}^{r-k}(0), \\ f(\mathfrak{p}_p) &= \mathcal{L}^r(p), \end{aligned}$$

then f is a continuous bijective (not homeomorphic) map, which makes the following diagram commutative.

$$\begin{tikzcd} \mathit{Spec}\,\Omega(G) \arrow[rr, "f"] \arrow[dr, "E"'] & \circlearrowleft & \mathit{Spec}\,\Omega \arrow[dl, "F"] \\ & \mathit{Spec}\,\mathbb{Z} & \end{tikzcd}$$

Proof. This immediately follows from Theorem 6.12, Proposition 6.17 and Fact 6.19. □

Appendix A. An inductive calculation of jnd, when G is abelian

We demonstrate how to calculate the multiplicative transfers of Ω inductively, when G is an abelian group. This will show Corollary 2.4. A detailed investigation of an exponential map will be found in [14]. A more clear argument using Möbius inversion will be found in [11].

Since $\mathrm{jnd}_H^L : \Omega_L(L/H) \to \Omega_L(L/L)$ and $\mathrm{jnd}_H^L : \Omega_G(G/H) \to \Omega_G(G/L)$ are essentially equal for $H \leqslant L \leqslant G$, we may assume $L = G$ from the first, and calculate $\mathrm{jnd}_H^G : \Omega_G(G/H) \to \Omega_G(G/G)$ for any $H \leqslant G$. Remark that any element in $\Omega_H(H/H)$ is of the form $\sum_{I \leqslant H} m_I H/I$ for some $m_I \in \mathbb{Z}$ $(I \leqslant H)$, and is identified with $\sum_{I \leqslant H} m_I(G/I \xrightarrow{\mathrm{pr}_I^H} G/H) \in \Omega_G(G/H)$. If each m_I satisfies $m_I \geqslant 0$, this is equal to $(\coprod_{I \leqslant H}(\amalg_{m_I} G/I) \xrightarrow{\mathrm{Pr}} G/H)$ in $\Omega_G(G/H)$, where Pr is the union of the projections $G/I \xrightarrow{\mathrm{pr}_I^H} G/H$.

Remark that $\mathrm{jnd}_H^G : \Omega_G(G/H) \to \Omega_G(G/G)$ is a polynomial map [15] as in the following. (As for polynomial maps, see [4].)

Fact A.1. There exist polynomials

$$P_K \in \mathbb{Z}[m_I \mid I \leqslant H] \quad (K \leqslant G)$$

which satisfy

$$\mathrm{jnd}_H^G\Big(\sum_{I \leqslant H} m_I\big(G/I \xrightarrow{\mathrm{pr}_I^H} G/H\big)\Big) = \sum_{K \leqslant G} P_K\big(\{m_I\}_{I \leqslant H}\big)G/K \tag{A.1}$$

for any $\{m_I\}_{I \leqslant H}$.

In particular if we obtain polynomials P_K which satisfy (A.1) whenever $\{m_I\}_I$ satisfies $m_I \geqslant 0$ $({}^{\forall}I \leqslant H)$, then (A.1) should hold for any $\{m_I\}_{I \leqslant H}$ with these P_K's. Now we calculate P_K, for each $K \leqslant G$. Let $\sum_{I \leqslant H} m_I H/I \in \Omega_H(H/H)$ be any element satisfying $m_I \geqslant 0$ $({}^{\forall}I \leqslant H)$, and denote the corresponding G-set $\coprod_{I \leqslant H}(\amalg_{m_I} G/I)$ by A. By the definition of $\mathrm{jnd}_H^G = \Omega_{\bullet}(\mathrm{pr}_H^G)$, the G-set $S = \mathrm{jnd}_H^G(A \xrightarrow{\mathrm{Pr}} G/H)$ is given by

$$\begin{aligned} S &= \left\{(y,s) \mid y \in G/G,\ s \in \mathrm{Map}\left(\left(\mathrm{pr}_H^G\right)^{-1}(y), A\right),\ \mathrm{Pr} \circ s = \mathrm{id}_{G/H}\right\} \\ &= \left\{s \in \mathrm{Map}(G/H, A) \mid \mathrm{Pr} \circ s = \mathrm{id}_{G/H}\right\} \\ &= \left\{s \in \mathrm{Map}(G/H, A) \mid s \text{ is a section of } \mathrm{Pr}\right\} \end{aligned}$$

(remark that G/G consists of only one element). The set S is equipped with a G-action

$${}^{g}s(x) = gs\left(g^{-1}x\right) \quad \left({}^{\forall}g \in G,\ {}^{\forall}s \in S,\ {}^{\forall}x \in G/H\right).$$

Definition A.2. Let G, H, A, S be as above. For each $K \leqslant G$, define $c(K) \in \mathbb{Z}$ by

$$c(K) = \sharp\{s \in S \mid G_s = K\}.$$

Then, since G is abelian, the number of G-orbits in S isomorphic to G/K should be equal to $\frac{c(K)}{|G:K|}$, and thus we have

$$S = \sum_{K \leqslant G} \frac{c(K)}{|G:K|} G/K,$$

namely, $P_K = \frac{c(K)}{|G:K|}$. Thus it remains to calculate $c(K)$.

Since G is abelian, we have the following.

Remark A.3. If we decompose G/K into K-orbits

$$G/H = X_1 \amalg \cdots \amalg X_{r(K)},$$

then $r(K) = |K\backslash G/H| = |G:KH|$, and each X_i is isomorphic to

$$KH/H \cong K/(K \cap H). \tag{A.2}$$

Fix an element $x_i \in X_i$ for each $1 \leqslant i \leqslant r(K)$. Then, we have the following.

Proposition A.4. *For any element $s \in S$, the following are equivalent.*

(1) $K \leqslant G_s$.
(2) *For any $1 \leqslant i \leqslant r(K)$ and any $k \in K$, we have*

$$ks(x_i) = s(kx_i). \tag{A.3}$$

Proof. By definition, $K \leqslant G_s$ holds if and only if for any $1 \leqslant i \leqslant r(K)$, any $x \in X_i$ and any $k \in K$,

$${}^{k}s(x) = s(x) \tag{A.4}$$

is satisfied. Since X_i is K-transitive, there is some $k' \in K$ satisfying $x = k'x_i$. Then (A.4) is written as

$$ks\left(k^{-1}k'x_i\right) = s\left(k'x_i\right).$$

This is easily shown to be equivalent to (2). $\square$

Corollary A.5. *Let G, H, A, S be as above. For each $K \leqslant G$, there is a bijection*

$$\{s \in S \mid G_s \geqslant K\} \cong \prod_{1 \leqslant i \leqslant r(K)} \left(\coprod_{I \leqslant H} \Bigl(\coprod_{m_I} \bigl\{ s_i \in G/I \bigm| \mathrm{pr}_I^H(s_i) = x_i,\ K_{s_i} = K_{x_i} \bigr\} \Bigr) \right).$$

Proof. By Proposition A.4, the set of $s \in S$ satisfying $G_s \geqslant K$ is determined by the values $s(x_i)$ ($1 \leqslant i \leqslant r(K)$). In fact, to a set of elements $s_1, \ldots, s_{r(K)} \in A$, we may associate a map $s : G/H \to A$ satisfying $s(x_i) = s_i$, by using (A.3). This map s becomes well-defined if and only if $K_{x_i} \leqslant K_{s_i}$ is satisfied for any i. Moreover, we see that the following are equivalent.

(i) s is a section of $\mathrm{Pr} : A \to G/H$, and satisfies $G_s \geqslant K$.
(ii) $\mathrm{Pr}(s_i) = x_i$ ($1 \leqslant {}^\forall i \leqslant r(K)$), and $K_{s_i} = K_{x_i}$.

Thus, to give an element in $\{s \in S \mid G_s \geqslant K\}$ is equivalent to give a set of elements $s_1, \ldots, s_{r(K)} \in A$ satisfying (ii). Namely, we have a bijection

$$\begin{aligned} \{s \in S \mid G_s \geqslant K\} &\cong \prod_{1 \leqslant i \leqslant r(K)} \bigl\{ s_i \in A \bigm| \mathrm{Pr}(s_i) = x_i,\ K_{s_i} = K_{x_i} \bigr\} \\ &= \prod_{1 \leqslant i \leqslant r(K)} \left(\coprod_{I \leqslant H} \Bigl(\coprod_{m_I} \bigl\{ s_i \in G/I \bigm| \mathrm{pr}_I^H(s_i) = x_i,\ K_{s_i} = K_{x_i} \bigr\} \Bigr) \right). \qquad \square \end{aligned}$$

Corollary A.6. *For any $K \leqslant G$, we have*

$$\sharp\{s \in S \mid G_s \geqslant K\} = \left(\sum_{(K \cap H) \leqslant I \leqslant H} m_I |H : I| \right)^{r(K)}.$$

Consequently, we can calculate $c(K)$ inductively by

$$c(K) = \left(\sum_{(K \cap H) \leqslant I \leqslant H} m_I |H : I| \right)^{r(K)} - \sum_{K < L \leqslant G} c(L).$$

Proof. Since G is abelian, we have $K_{x_i} = K \cap H$ by (A.2). Similarly, for each $s_i \in G/I$, we have $K_{s_i} = K \cap I$. Since $I \leqslant H$, we obtain

$$K_{x_i} = K_{s_i} \quad \Leftrightarrow \quad K \cap H = K \cap I \quad \Leftrightarrow \quad K \cap H \leqslant I.$$

As a consequence, the bijection in Corollary A.5 is reduced to

$$\{s \in S \mid G_s \geqslant K\} \cong \prod_{1 \leqslant i \leqslant r(K)} \left(\coprod_{(K \cap H) \leqslant I \leqslant H} \Bigl(\coprod_{m_I} \bigl\{ s_i \in G/I \bigm| \mathrm{pr}_I^H(s_i) = x_i \bigr\} \Bigr) \right).$$

Since the fiber $(\mathrm{pr}_I^H)^{-1}(x_i)$ has $|H : I|$ elements, it follows

$$\begin{aligned} \sharp\{s \in S \mid G_s \geqslant K\} &= \prod_{1 \leqslant i \leqslant r(K)} \left(\sum_{(K \cap H) \leqslant I \leqslant H} \bigl(m_I |H : I| \bigr) \right) \\ &= \left(\sum_{(K \cap H) \leqslant I \leqslant H} m_I |H : I| \right)^{r(K)}. \qquad \square \end{aligned}$$

Example A.7. When $G = \mathbb{Z}/p^\ell\mathbb{Z}$, let $H_k \leqslant G$ be the subgroup of order p^k;

$$e = H_0 < H_1 < \cdots < H_\ell = G.$$

For $H = H_k$, we can determine $c(H_j)$ $(0 \leqslant j \leqslant \ell)$ of $\mathrm{jnd}_H^G(\sum_{0\leqslant i\leqslant k} m_i H/H_i)$ as follows.

By Corollary A.6, for each $0 \leqslant j \leqslant \ell$ we have

$$c(H_j) = \begin{cases} m_k, & j = r, \\ (m_k)^{p^{\ell-j}} - (m_k)^{p^{\ell-j-1}}, & k \leqslant j < r, \\ (\sum_{s=j}^k m_s p^{k-s})^{p^{\ell-k}} - (\sum_{s=j+1}^k m_s p^{k-s})^{p^{\ell-k}}, & 0 \leqslant j < k, \end{cases}$$

which leads to Corollary 2.4.

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