Journal of Algebra 449 (2016) 446–459

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Canonical sections of the Hodge bundle over Ekedahl–Oort strata of Shimura varieties of Hodge type

Jean-Stefan Koskivirta

A R T I C L E   I N F O

Article history:
Received 2 December 2014
Available online 15 December 2015
Communicated by Steven Dale Cutkosky

Keywords:
Abelian varieties
Hasse invariant
Shimura varieties
Ekedahl–Oort stratification
Hodge bundle

A B S T R A C T

We construct canonical non-vanishing global sections of powers of the Hodge bundle on each Ekedahl–Oort stratum of a Hodge type Shimura variety. In particular we recover the quasi-affineness of the Ekedahl–Oort strata. In the projective case, this gives a very short proof of non-emptiness of Ekedahl–Oort strata. It follows that the Newton strata are also nonempty, by a result of S. Nie. From the canonicity of our construction, we deduce the fact that the μ-ordinary locus is determined by the Ekedahl–Oort strata of its image under any embedding of Shimura varieties.

Introduction

The Siegel modular variety $\mathcal{A}_{g,N}$ arises as a moduli space of g-dimensional principally polarized abelian varieties with level-N structure. More generally, Shimura varieties of PEL-type classify abelian varieties endowed with a polarization, an action of a semisimple algebra, and a level structure. These varieties satisfy nice properties due to their nature as moduli spaces. For example, the special fiber of a PEL-Shimura variety at a place of good reduction carries different stratifications (Ekedahl–Oort, Newton, p-rank, etc.).

E-mail address: jeanstefan.koskivirta@gmail.com.

http://dx.doi.org/10.1016/j.jalgebra.2015.11.030

Shimura varieties which can be embedded into a Siegel modular variety are called 'of Hodge type'. In general, they do not have an interpretation as a moduli space. In particular, all PEL-Shimura varieties are of Hodge type. For Hodge type varieties, one can still define the Ekedahl–Oort stratification using the stack of G-zips, introduced by Wedhorn, Moonen, Pink, Ziegler in [9,12] and [13]. More precisely, let $(\mathbf{G}, \mu)$ be a Hodge type Shimura datum and let S_K denote the special fiber of the associated Shimura variety at a place of good reduction. Zhang has constructed in [19] a G-zip over S_K and he has shown that the corresponding morphism

$$S_K \to G\text{-}\mathtt{Zip}^{\mu}$$

is smooth. By definition, the Ekedahl–Oort strata of S_K are the fibers of ζ. In particular, Zhang proves that the μ-ordinary Ekedahl–Oort stratum is open and dense. It is also possible to define group theoretically the Newton stratification on S_K. Wortmann has shown in [18] that the open Newton stratum coincides with the μ-ordinary locus.

Fix an embedding $\iota : S_K \to \mathcal{A}_{g,N}$ of a Hodge type Shimura variety S_K into a Siegel modular variety. Denote by $\mathscr{A} \to S_K$ the pull-back of the universal abelian scheme on $\mathcal{A}_{g,N}$ via ι. Let $e : S_K \to \mathscr{A}$ denote the identity section of $\mathscr{A}$. Then the Hodge bundle is by definition

$$\omega_{S_K} := \det(e^*\Omega_{\mathscr{A}/S_K}).$$

This line bundle is ample on S_K. In an upcoming paper by the author and T. Wedhorn, it is proved that this line bundle admits a canonical global section, a generalized Hasse invariant, which vanishes exactly outside the μ-ordinary locus [8, Theorem 4.12]. In this paper we construct sections of ω_S on each Ekedahl–Oort stratum. Here are our main results:

Theorem 1. *Let S^w be a nonempty Ekedahl–Oort stratum. There exists $N_w \geq 1$ such that for all $d \geq 1$, there exists a (canonical) non-vanishing section in the space*

$$H^0(S^w, \omega_{S_K}^{\otimes N_w d}).$$

This section is canonical, in the sense that it is a pull-back of a non-vanishing global section on a certain substack of the stack of G-zips. Therefore it only depends on the choice of the embedding into a Siegel Shimura variety. Our sections arise as pull-backs from a one-dimensional vector space (see Theorem 3.1). This ensures "functoriality" (up to a scalar) of the sections under any embedding of Shimura varieties (see paragraph 3.2). Such a non-vanishing section induces an isomorphism $\mathcal{O}_{S^w} \to \omega_{S_K}^{\otimes N_w d}$. In other words, Theorem 1 says that the line bundle ω_{S_K} is torsion on S^w. This implies that $\mathcal{O}_{S^w}$ is ample on S^w, so we deduce the following result:

Corollary 1. *The Ekedahl–Oort strata are quasi-affine.*

As pointed out to me by Ulrich Görtz and Chia-Fu Yu, this result can also be deduced immediately from the case of Siegel modular varieties. Let us also mention that by a result of Wedhorn and Yaroslav, the inclusion $S^w \to S$ is an affine morphism [17, Theorem 4.1]. From the quasi-affineness of Ekedahl–Oort strata, we deduce the following:

Corollary 2. *Let S_K be a Hodge type Shimura variety. Assume that S_K is projective. Then all Ekedahl–Oort and Newton strata are nonempty.*

For the definition of Newton strata, see paragraph 4.1. For a more detailed and thorough exposition, see [18, §5.2]. For PEL-Shimura varieties, all Ekedahl–Oort strata are known to be nonempty, by a result of Viehmann and Wedhorn in [16]. For more general Hodge type Shimura varieties, nonemptiness is expected to hold, even though no proof has been given so far.

Here is an idea of the proof of Corollary 2. The ampleness of the Hasse bundle and the existence of the Hasse invariant imply that an Ekedahl–Oort stratum of positive dimension cannot be projective. Using an inductive argument, we deduce that the superspecial locus (the Ekedahl–Oort stratum of dimension zero) is nonempty, and then the result is a consequence of the flatness of the map ζ. The rigidity of the construction in Theorem 1 has the following consequence:

Corollary 3. *Let S^μ be the μ-ordinary Ekedahl–Oort stratum, and let S_0^μ be the Ekedahl–Oort stratum of the Siegel modular variety $\mathcal{A}_{g,N}$ containing the image of S^μ by the embedding $\iota : S_K \to \mathcal{A}_{g,N}$. Then one has the equality:*

$$S^\mu = \iota^{-1}(S_0^\mu).$$

For example, assume S is a PEL-Shimura variety parametrizing tuples $(A, \lambda, \iota, \bar{\nu})$ where A is an abelian variety, λ a polarization, ι an action of a $\mathbb{Z}_{(p)}$-algebra, and $\bar{\nu}$ a level structure. Then the μ-ordinary locus of S is entirely determined by the isomorphism class of the p-torsion $A[p]$, forgetting the rest of the structure. This is a somewhat surprising result, even in the PEL-case. Of course it can happen that two Ekedahl–Oort strata are mapped to the same stratum via an embedding of Shimura varieties. For example, consider a Hilbert–Blumenthal Shimura variety S_K attached to a totally real field E which is a cyclic extension of $\mathbb{Q}$, and look at the reduction modulo an inert prime p. Then the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ acts non-trivially on the set of Ekedahl–Oort strata. Since the Galois action on the Ekedahl–Oort strata of a Siegel modular variety is trivial, it follows that two Galois-conjugate strata of S_K must be mapped to the same stratum under any embedding of Shimura varieties $S_K \to \mathcal{A}_{g,N}$ defined over $\mathbb{Q}$.

We now give an overview of the paper. In the first section we recall the parametrization of Ekedahl–Oort strata using the stack of G-zips and the map ζ. Then in the second part we state some general facts about the Picard group of a quotient stack and the space of global sections of a line bundle, which we apply to the stack of G-zips. In the third part we construct Hasse invariants on each Ekedahl–Oort stratum. Finally, we prove the corollaries of Theorem 1 in the last subsection.

1. Parametrization of Ekedahl–Oort strata

1.1. Shimura varieties of Hodge type

We follow the general setup of [8, (4.1)]. If (G_1, X_1) and (G_2, X_2) are two Shimura data (as in [2, 2.1.1]), a morphism of Shimura data $(G_1, X_1) \to (G_2, X_2)$ is a map of groups $G_1 \to G_2$, which induces a map $X_1 \to X_2$.

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type. We fix an embedding $\iota : (\mathbf{G}, \mathbf{X}) \to (\mathrm{GSp}(V), S^{\pm})$ of Shimura data, where $V = (V, \psi)$ is a symplectic space over $\mathbb{Q}$ and $S^{\pm}$ the double Siegel half space. Let $h \in X$ be a morphism $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ and let $[\mu]$ the $\mathbf{G}(\mathbb{C})$-conjugacy class of the component of $h_{\mathbb{C}} \colon \prod_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ corresponding to $\mathrm{id} \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let E be the reflex field, i.e. the field of definition of $[\mu]$.

For every sufficiently small open compact subgroup $K \subset G(\mathbb{A}_f)$ we obtain a Shimura variety $S^0_K(\mathbf{G}, \mathbf{X})$ defined over E. For every choice of compact open subsets $K \subset \mathbf{G}(\mathbb{A}_f)$ and $\tilde{K} \subset \mathrm{GSp}(V \otimes \mathbb{A}_f)$ such that $\iota(K) \subset \tilde{K}$, the embedding ι induces a map over E:

$$\iota : S^0_K(\mathbf{G}, \mathbf{X}) \longrightarrow S^0_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm}) \otimes_{\mathbb{Q}} E \tag{1.1}$$

where $S^0_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm})$ denotes the Siegel modular variety attached to the data $(\mathrm{GSp}(V), S^{\pm})$ at the level $\tilde{K}$.

We choose a prime number $p > 2$ and a place v of E over p and we denote by $\kappa := \kappa(v)$ the residue field of v. We now fix a pair $(K, \tilde{K})$ of compact subgroups as above such that the following assumptions are satisfied:

- (i) There exists a reductive $\mathbb{Z}_{(p)}$-model $\mathcal{G}$ of $\mathbf{G}$, such that $K_p = \mathcal{G}(\mathbb{Z}_p)$ (this is called a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$). In particular, $G_{\mathbb{Q}_p}$ is unramified, and hence quasi-split. It follows from the smoothness of the stack parametrizing Borel pairs in $\mathcal{G}$ that the group $\mathcal{G}$ is then also quasi-split (i.e. admits a Borel pair over $\mathbb{Z}_p$). We say that the Shimura variety $S^0_K(\mathbf{G}, \mathbf{X})$ has good reduction at p.
- (ii) There is a $\mathbb{Z}_{(p)}$-lattice Λ of V such that ι is induced by an embedding $\mathcal{G} \to \mathrm{GL}(\Lambda)$ (this is always possible by [5, Lemma (2.3.1)]).
- (iii) The bilinear form ψ induces a perfect $\mathbb{Z}_{(p)}$-pairing on Λ and one has $\tilde{K}_p = \mathrm{GSp}(\Lambda, \psi)(\mathbb{Z}_p)$. This implies that $S^0_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm})$ has good reduction at p.
- (iv) The compact subgroups K^p and $\tilde{K}^p$ are sufficiently small (see below for details).
- (v) The map (1.1) is a closed embedding. For fixed K and $\tilde{K}_p$ as above, it is possible to find $\tilde{K}^p$ such that this condition is satisfied (see [5, Lemma 2.1.2]).

Assumption (iii) may fail for our fixed ι, but it is possible to replace the symplectic space (V, ψ) such that it is satisfied (this is Zarhin's trick — see [6, (1.3.3)]). In assumption (iv), we choose K^p and $\tilde{K}^p$ such that the Shimura varieties $S^0_{\tilde{K}}(\mathrm{GSp}(V), S^{\pm})$ and

$S_K^0(\mathbf{G},\mathbf{X})$ have smooth integral models $\mathcal{S}_{\tilde{K}}(\mathrm{GSp}(V),S^{\pm})$ over $\mathbb{Z}_p$ and $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ over $\mathcal{O}_{E_v}$ respectively. By the main results of [5] and [14], the scheme $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ is defined as the normalization of the scheme-theoretic image of $S_K^0(\mathbf{G},\mathbf{X})$ inside $\mathcal{S}_{\tilde{K}}(\mathrm{GSp}(V),S^{\pm})$ by the embedding (1.1).

We denote by $S_K := \mathcal{S}_K(\mathbf{G},\mathbf{X}) \times_{\mathcal{O}_{E_v}} \kappa$ the special fiber of $\mathcal{S}_K(\mathbf{G},\mathbf{X})$. It is a smooth quasi-projective scheme over κ. By assumptions (ii) and (iii) we have an embedding

$$\iota\colon \mathcal{G} \hookrightarrow \mathrm{GSp}(\Lambda) \tag{1.2}$$

over $\mathbb{Z}_{(p)}$ and $\mathcal{G}$ is the scheme-theoretic stabilizer of a finite set s of tensors in $\Lambda^{\otimes}$ (see [5, Proposition 1.3.2]). By construction, there is a finite morphism:

$$\mathcal{S}_K(\mathbf{G},\mathbf{X}) \longrightarrow \mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda),S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E_v} \tag{1.3}$$

Let $\tilde{\mathscr{A}} \to \mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda),S^{\pm})$ be the universal abelian scheme of the integral Siegel modular variety, and let $\mathscr{A}$ be its pull-back to $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ by the map (1.3). We define a line bundle on $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ by

$$\omega := \det(e^*\Omega^1_{\mathscr{A}/\mathcal{S}})$$

where e is the identity section of $\mathscr{A}$, and we call it the *Hodge line bundle*. The line bundle ω is ample as the pull-back of an ample line bundle by a finite map. We denote by ω_{S_K} the pull-back of ω to the special fiber S_K.

1.2. The stack of G-zips

We denote by k an algebraic closure of κ and we write $G = \mathcal{G} \times_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ for the special fiber of $\mathcal{G}$.

The conjugacy class of cocharacters $[\mu^{-1}]$ extends to a conjugacy class over $\mathcal{O}_{E_v}$. As $\mathcal{G}$ is quasi-split, there exists a representative defined over $\mathcal{O}_{E_v}$. We denote by $\chi : \mathbb{G}_{m,\kappa} \to G_\kappa$ the reduction of this representative to the special fiber. Let $P_\pm = P_\pm(\chi)$ be the pair of opposite parabolic subgroups of G_κ attached to the cocharacter χ, with common Levi subgroup L (the centralizer of χ). Then $(G,P_+,\sigma(P_-),\varphi)$ is an algebraic zip datum in the sense of [12, 10.1], where $\sigma(-)$ denotes the pull-back under the absolute Frobenius $\sigma\colon x \mapsto x^p$ and where $\varphi\colon L \to \sigma(L)$ is the relative Frobenius. We set $P := P_+$, $Q := \sigma(P_-)$ and $M := \sigma(L)$, so that M is a Levi subgroup of Q.

We may assume (see [8, Lemma 4.2]), possibly after replacing χ by a conjugate cocharacter, that there is a Borel pair (T,B) defined over $\mathbb{F}_p$ such that $B_- \subset P$, and therefore $B \subset Q$. Let $(X,\Phi,X^\vee,\Phi^\vee,\Delta)$ be the based root datum of (G,B,T). Denote by $W = W(G,T) := N_G(T)/T$ the Weyl group and by $I \subset W$ the set of simple reflections defined by B. The subsets of I correspond bijectively to the parabolic subgroups of G_k containing B, which are called *standard*. For $J \subset I$, denote by Q_J the corresponding standard parabolic and by M_J the unique Levi subgroup of Q_J containing T. We have an inclusion $W_J := W(M_J, T) \hookrightarrow W(G, T)$ such that $J = W_J \cap I$. Every parabolic subgroup P' of G_k is conjugate to a unique standard parabolic subgroup Q_J and $J \subset I$ is called the *type of* P'.

We define the following set:

$$^JW := \{w \in W,\ \ell(sw) > \ell(w)\ \forall s \in J\}.$$

Any element $w \in {}^JW$ is the minimal element in the right coset $W_J w$. Hence the set ^JW is a set of representatives for the quotient $W_J \backslash W$.

For $x \in P$, we denote by $\bar{x}$ the image of x in $P/R_u(P) = L$, and similarly for the image of $y \in Q$ in $Q/R_u(Q) = M$. The associated *zip group* is defined by

$$E := \{\, (x, y) \in P \times Q\ ;\ \varphi(\bar{x}) = \bar{y} \,\}$$

and E acts on G_κ by $(x, y) \cdot g := xgy^{-1}$. Note that $\dim(E) = \dim(G)$. By [12, Proposition 7.3], there are finitely many E-orbits in G_k, which are parametrized as follows. Let J and K denote the types of P and Q, respectively. For every $w \in W$ we choose a representative $\dot{w} \in \mathrm{Norm}_G(T)$ such that $(w_1 w_2)^{\cdot} = \dot{w}_1 \dot{w}_2$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (this can be achieved by choosing representatives in $\mathrm{Norm}_G(T)$ attached to a Chevalley system [3, Exp. XXIII, §6]). Let $w_{0,J} \in W_J$ and $w_0 \in W$ be the longest elements and set $g_0 := (w_0 w_{0,J})^{\cdot}$. By [12, Theorem 5.12 and Theorem 7.5] we obtain a bijection

$$^JW \xrightarrow{\sim} \{E\text{-orbits in } G_k\}, \qquad w \mapsto O^w := E \cdot (g_0 \dot{w}) \tag{1.4}$$

such that $\dim O^w = \ell(w) + \dim(P)$.

1.3. Ekedahl–Oort strata

The algebraic quotient stack over κ

$$G\text{-}\mathtt{Zip}^\chi := [E \backslash G_\kappa] \tag{1.5}$$

is called the *stack of G-zips*. The underlying topological space of $G\text{-}\mathtt{Zip}^\chi$ is homeomorphic to ^JW endowed with the order topology with respect to a certain partial order $\preceq$ (see [12, Definition 6.1] for the definition of $\preceq$).

Zhang has constructed in [19] a G-zip of type χ over S_K and he has shown in [19] that the corresponding morphism $S_K \to G\text{-}\mathtt{Zip}^\chi$ is smooth. In this paper, we prefer to use the construction given by Wortmann in [18, §5] and we obtain again a smooth morphism

$$\zeta := \zeta_G \colon S_K \longrightarrow G\text{-}\mathtt{Zip}^\chi\,. \tag{1.6}$$

The Ekedahl–Oort strata of S_K are defined as the fibers of ζ. For $w \in {}^JW$, we denote by $S^w := \zeta^{-1}(w)$ the corresponding stratum endowed with the reduced scheme structure as a locally closed subset of S_K. Then S^w is smooth by [17] and if nonempty, has dimension $\ell(w)$. In the case of PEL-Shimura varieties, every Ekedahl–Oort stratum is nonempty [16, Theorem 10.1]. The map (1.6) restricts to a smooth map of stacks:

$$\zeta\colon S^w \longrightarrow [E\backslash O^w]. \tag{1.7}$$

2. Equivariant Picard group

2.1. G-linearizations

In this section we consider an arbitrary smooth algebraic group over k acting on a k-variety X. If $\pi : L \to X$ is a line bundle, a G-linearization of L is a map

$$G \times L \to L$$

defining an action of G on L, satisfying the conditions:

(i) The map π is G-equivariant.
(ii) The action of G on L is linear on the fibers.

We denote by $\mathrm{Pic}^G(X)$ the group of isomorphism classes of G-linearized line bundles on X. The image of the natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ is the subgroup of G-linearizable line bundles, and is denoted by $\mathrm{Pic}_G(X)$. The group $\mathrm{Pic}^G(X)$ can be identified with the Picard group of the quotient stack $[G\backslash X]$, defined as the group of isomorphism classes of line bundles on the stack $[G\backslash X]$.

We define $\mathcal{E}(X) := \mathbb{G}_m(X)/k^\times$. If X is an irreducible variety over k, it is a free abelian group of finite type.

Proposition 2.1. *Let G be a smooth algebraic group, and X an irreducible G-variety. Then there is an exact sequence:*

$$1 \to k^\times \to \mathbb{G}_m(X)^G \to \mathcal{E}(X) \to X^*(G) \to \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

Proof. See [7, Proposition 2.3 and Lemma 2.2]. The assumption that k is of characteristic 0 is not needed in the proof. $\square$

The map $X^*(G) \to \mathrm{Pic}^G(X)$, $\lambda \mapsto \mathscr{L}(\lambda)$ is defined as follows. A character $\lambda \in X^*(G)$ induces a G-linearization of the trivial line bundle $\mathbb{A}^1_k \times X$ on X given by $(g, x, s) \mapsto (g \cdot x, \lambda(g)s)$ for all $g \in G$, $x \in X$, $s \in \mathbb{A}^1_k$.

Proposition 2.2. *Let $H \subset G$ be algebraic groups. Then there is a natural isomorphism:*

$$\mathrm{Pic}^G(G/H) \simeq X^*(H).$$

Proof. One has $\mathrm{Pic}^G(G/H) \simeq \mathrm{Pic}([G\backslash(G/H)]) \simeq \mathrm{Pic}([1/H]) \simeq \mathrm{Pic}^H(1) \simeq X^*(H)$. □

2.2. The space of global sections

Proposition 2.3. *Let G be an algebraic group and let X be an irreducible G-variety containing an open G-orbit U. Denote by $\pi : X \to [X/G]$ the projection. Let $\mathscr{L}$ be a line bundle on the stack $[X/G]$ and write $L = \pi^*\mathscr{L}$. Then:*

(i) *$H^0([X/G], \mathscr{L})$ identifies with $H^0(X, L)^G$. In particular, for $\lambda \in X^*(G)$ one has:*

$$H^0([X/G], \mathscr{L}(\lambda)) = \{f : X \to k, f(g \cdot x) = \lambda(g)f(x), \ \forall g \in G, x \in X\}.$$

(ii) *The k-vector space $H^0([X/G], \mathscr{L})$ has dimension less than 1.*
(iii) *If $H^0([X/G], \mathscr{L}) \neq 0$ then $\mathscr{L}$ restricts to the trivial line bundle on $[U/G]$.*
(iv) *If $\mathscr{L}$ is trivial, $H^0([X/G], \mathscr{L}) = k$.*

Proof. See [8, Proposition 1.18]. □

3. Hasse invariants on Ekedahl–Oort strata

3.1. Construction

In this section we construct a canonical non-vanishing section of the Hodge bundle on each Ekedahl–Oort stratum of S_K. We expect that this section extends to the Zariski closure of the stratum, so we call it abusively a Hasse invariant of the stratum. In the particular case of the μ-ordinary stratum, it was proved in [8, Theorem 4.12], that this canonical section does extend to the whole Shimura variety (and even to its minimal compactification), and that the non-vanishing locus is exactly the μ-ordinary stratum. In [4, Theorem 3.1.1], Goldring and the author prove that H_w extends to the Zariski closure $\overline{S^w}$ under a certain weak condition on the prime number p. See also [1] for similar results in PEL-cases of type A and C.

Let G be a reductive group over $\mathbb{F}_p$, and $\mu : \mathbb{G}_{m,k} \to G_k$ a minuscule cocharacter. Denote by (G, P, Q, φ) the associated zip datum, and by E the attached zip group.

Theorem 3.1. *For all E-orbit $C \subset G_k$, the Picard group $\mathrm{Pic}^E(C)$ is finite. Denote by N_C its exponent. Then for all $d \geq 1$ and all $\lambda \in X^*(E)$, the space of global sections*

$$H^0\left([E\backslash C], \mathscr{L}(\lambda)^{\otimes N_C d}\right)$$

is one-dimensional.

Proof. We apply Proposition 2.1 to the E-variety C. Clearly $\mathbb{G}_m(C)^E = k^\times$. Hence we get an exact sequence:

$$1 \to \mathcal{E}(C) \to X^*(E) \to \mathrm{Pic}^E(C) \to \mathrm{Pic}(C) \tag{3.1}$$

Let x be an arbitrary element of C. The map $E \to C$, $e \mapsto e \cdot x$ identifies C with the quotient E/A_x, where A_x is the scheme-theoretic stabilizer of x in E. We have an exact sequence

$$1 \to A_{x,\mathrm{red}} \to A_x \to A_x/A_{x,\mathrm{red}} \to 1$$

where $A_x/A_{x,\mathrm{red}}$ is a finite group scheme. Hence we have an exact sequence

$$0 \to X^*(A_x/A_{x,\mathrm{red}}) \to X^*(A_x) \to X^*(A_{x,\mathrm{red}}).$$

By [12, Theorem 8.1], the group $A_{x,\mathrm{red}}$ has the form $U_x \rtimes H_x$ where U_x is unipotent and H_x finite. We deduce that $X^*(A_{x,\mathrm{red}})$ is finite, and hence so is $X^*(A_x)$. It follows from Proposition 2.2 that $\mathrm{Pic}^E(C)$ is finite. Let N_C be its exponent.

For all $d \geq 1$, the character $N_C d\lambda$ maps to zero in $\mathrm{Pic}^E(C)$. Therefore there exists a function $f \in \mathcal{E}(C)$ mapping to $N_C d\lambda$. By definition, this is a non-vanishing function on C satisfying the relation $f(e \cdot x) = \lambda(e)^{N_C d} f(x)$, $\forall e \in E, x \in C$, so it is a nonzero global section of $\mathscr{L}(\lambda)^{\otimes N_C d}$. This concludes the proof. $\square$

Remark 3.2. For a fixed character $\lambda \in X^*(E)$, let $N_C(\lambda)$ be the order of $\mathscr{L}(\lambda)$ in $\mathrm{Pic}^E(C)$. The set of integers r such that $H^0\left([E\backslash C], \mathscr{L}(\lambda)^{\otimes r}\right) \neq 0$ is the subgroup of $\mathbb{Z}$ generated by $N_C(\lambda)$.

The first projection $E \to P$ induces an isomorphism $X^*(E) = X^*(P)$. A character $\lambda \in X^*(E) = X^*(P)$ is said to be *ample* if the associated line bundle on G/P is ample, see Definition 3.2 in [8]. This defines a cone in $X^*(E)$. The following result is a reformulation of Theorem 3.8 in [8].

Theorem 3.3. *Let $U \subset G_k$ denote the open E-orbit of G_k. Let $\lambda \in X^*(E)$ be an ample character. Then one has*

$$H^0\left([E\backslash U], \mathscr{L}(\lambda)^{\otimes n}\right) = H^0\left([E\backslash G_k], \mathscr{L}(\lambda)^{\otimes n}\right)$$

for all $n \geq 1$. For N_U as in Theorem 3.1, $d \geq 1$, and $n = N_U d$, this space is one-dimensional and any nonzero element induces a function $G_k \to \mathbb{A}^1_k$ which vanishes exactly on the complement of U.

Proof. The natural pull-back map $H^0\left([E\backslash G_k], \mathscr{L}(\lambda)^{\otimes n}\right) \to H^0\left([E\backslash U], \mathscr{L}(\lambda)^{\otimes n}\right)$ is clearly injective. Since $\mathrm{Pic}^E(U)$ is finite, we have an isomorphism

$$\mathcal{E}(U)_{\mathbb{Q}} \simeq X^*(E)_{\mathbb{Q}}.$$

The space $H^0\left([E\backslash U], \mathscr{L}(\lambda)^{\otimes n}\right)$ is one-dimensional if and only if the function in $\mathcal{E}(U)_{\mathbb{Q}}$ corresponding to $n\lambda$ is in $\mathcal{E}(U) \subset \mathcal{E}(U)_{\mathbb{Q}}$. In this case the function extends to a regular function on G_k which vanishes exactly outside U, by [8, Theorem 3.8]. □

Now let us return to the notations of section 1. For an element $w \in {}^JW$, denote by N_w the integer associated with the E-orbit O^w as in Theorem 3.1. The map (1.7) induces by pull-back a map

$$H^0\left([E\backslash O^w], \mathscr{L}(\lambda)^{\otimes N_w d}\right) \to H^0\left(S^w, \zeta^*\mathscr{L}(\lambda)^{\otimes N_w d}\right)$$

As explained in [8, 4.6] and in the proof of Theorem 4.12, there is a character λ_ω of E such that

$$\zeta^*\mathscr{L}(\lambda_\omega) = \omega_{S_K}.$$

Thus we get a non-vanishing section H_w of $\omega_{S_K}^{N_w d}$ over S^w (well-defined up to a scalar), which proves Theorem 1. Note that this construction depends only on the choice of the Siegel embedding. Therefore we call H_w a canonical Hasse invariant for S^w.

We deduce the following corollary:

Corollary 3.4. *The Ekedahl–Oort strata are quasi-affine.*

Proof. Let S^w be an Ekedahl–Oort stratum. The non-vanishing section H_w induces an isomorphism $\omega_{S_K} \simeq \mathcal{O}_{S^w}$ on S^w. Hence the trivial bundle $\mathcal{O}_{S^w}$ is ample, so S^w is quasi-affine. □

As mentioned in the introduction, this result can also be deduced from the quasi-affineness of Ekedahl–Oort strata in the Siegel case (proved in [11, Theorem 1.2]). Each stratum S^w is in fact locally closed in the preimage of the corresponding Siegel Ekedahl–Oort stratum, and is therefore quasi-affine.

3.2. Functoriality

Let $f : G_1 \to G_2$ be a morphism of connected reductive groups over $\mathbb{F}_p$. Let $\mu_1 : \mathbb{G}_{m,k} \to G_{1,k}$ be a minuscule cocharacter, and set $\mu_2 := f \circ \mu_1$. For $i = 1, 2$, denote by (G_i, P_i, Q_i, φ) the zip datum attached to μ_i. Denote by E_1 and E_2 respectively the corresponding zip groups. The map f induces naturally a map $E_1 \to E_2$, which we denote again by f. We get a map of stacks:

$$[E_1\backslash G_{1,k}] \longrightarrow [E_2\backslash G_{2,k}].$$

Let C_1 be an E_1-orbit in $G_{1,k}$ and let C_2 be the E_2-orbit containing $f(C_1)$. Let $\lambda \in X^*(E_2)$ be a character of E_2 and denote by $N_1(\lambda \circ f)$ and $N_2(\lambda)$ the integers attached to the pairs $(C_1, \lambda \circ f)$ and (C_2, λ) as in Remark 3.2. We get a map:

$$\tilde{f} : H^0\left([E_2\backslash C_2], \mathscr{L}(\lambda)^{\otimes N_2(\lambda)}\right) \to H^0\left([E_1\backslash C_1], \mathscr{L}(\lambda \circ f)^{\otimes N_2(\lambda)}\right)$$

One sees readily that this map is injective. Since the space on the left has dimension one, we deduce that it is an isomorphism. In particular the integer $N_1(\lambda \circ f)$ divides $N_2(\lambda)$.

Assume now that f is an embedding and that C_1 is the open E_1-orbit in $G_{1,k}$. Define again C_2 to be the E_2-orbit containing $f(C_1)$. Let $\lambda \in X^*(E_2)$ be an ample character. Then $\lambda \circ f$ is again ample (Remark 3.5 in [11]). We deduce the following isomorphism:

$$H^0\left([E_2\backslash C_2], \mathscr{L}(\lambda)^{\otimes N_2(\lambda)}\right) \simeq H^0\left([E_1\backslash G_{1,k}], \mathscr{L}(\lambda \circ f)^{\otimes N_2(\lambda)}\right).$$

Any nonzero element H in this space induces a function $H : G_{1,k} \to \mathbb{A}^1_k$ which vanishes exactly outside C_1 by Theorem 3.3. But by definition it does not vanish on the preimage of C_2, so we get the following:

Proposition 3.5. *Assume that f is an embedding. Let C_1 denote the open E_1-orbit in $G_{1,k}$, and let C_2 be the E_2-orbit containing C_1. Then we have the following:*

$$C_1 = f^{-1}(C_2).$$

4. Consequences

Throughout this section, we use the same notations as in section 1.

4.1. Nonemptiness of Ekedahl–Oort and Newton strata in the projective case

We recall briefly the Newton stratification of the special fiber S_K. For more details, we refer to [18, §5.2]. Let k_0 be any algebraically closed field containing $\mathbb{F}_p$, and let $W := W(k_0)$ be its Witt ring endowed with the Frobenius σ, and let L be the fraction field of W. We denote by $B(\mathbf{G})$ the set of σ-conjugacy classes in $\mathbf{G}(L)$. This set is independent of the choice of the algebraically closed field k_0.

Let $x \in S$ be any point and we take k_0 to be an algebraic closure of the field of definition $\kappa(x)$ of x. To the point x one can attach a “Dieudonné module with $\mathbf{G}$-structure” $(\mathbb{D}_x, s_{\mathrm{cris}})$ (Construction 5.3 in [18]). It is a free W-module of finite type $\mathbb{D}_x$ endowed with a σ-linear map F, a σ^{-1}-linear map V such that $FV = VF = p$, and a family of tensors s_{cris} stabilized by F and V. Furthermore, there is an isomorphism $\mathbb{D}_x \simeq \Lambda \otimes_{\mathbb{Z}_{(p)}} W$ carrying the tensors s_{cris} to the tensors s (see (1.2)). After choosing any such isomorphism, the linearization of the Frobenius of $\mathbb{D}_x$ induces an element of $\mathbf{G}(L)$. Taking its class in $B(\mathbf{G})$ induces a well-defined map (between sets):

$$\mathrm{Newt} : S_K \longrightarrow B(\mathbf{G}). \tag{4.1}$$

The Newton strata of S_K are defined as the fibers of this map. One can be more precise and define a finite subset $B(\mathbf{G}, \mu) \subset B(\mathbf{G})$ such that the image of (4.1) is contained in $B(\mathbf{G}, \mu)$. See definition 5.6 in [18]. In particular, there are only finitely many Newton strata. One can also show that Newton strata are locally closed [15, 5.3.1.(ii)].

In this paragraph we will prove the following result:

Proposition 4.1. *Assume S_K is a projective variety. Then all Ekedahl–Oort strata and all Newton strata are nonempty.*

It follows from the parametrization (1.4) that there exists a unique Ekedahl–Oort stratum of dimension zero (corresponding to the element $1 \in {}^J W$), called the superspecial stratum. Since the map ζ is smooth, it is open. Hence the nonemptiness of the superspecial stratum implies that all Ekedahl–Oort strata are nonempty.

Lemma 4.2. *Assume that S is projective. Let S^w be an Ekedahl–Oort stratum of positive dimension. Then S^w is not closed.*

Proof. Since S^w is quasi-affine, it cannot be projective unless it is zero-dimensional. $\square$

We deduce that for any nonempty Ekedahl–Oort stratum of positive dimension $d \geq 1$, there is a nonempty Ekedahl–Oort stratum of dimension $< d$ in its closure. Using this argument recursively (beginning at the μ-ordinary stratum, which is nonempty), we deduce that there is a nonempty Ekedahl–Oort stratum of dimension 0, which must be the superspecial one. This proves the nonemptiness of Ekedahl–Oort strata.

Now Corollary 1.6 in [10] shows that every Newton stratum contains a fundamental Ekedahl–Oort stratum. Note that the assumption in [10] that the Shimura variety is of PEL-type is unnecessary, since the proof is entirely group theoretic. Hence it follows that all Newton strata are nonempty.

Remark 4.3. The proof of the nonemptiness of Ekedahl–Oort strata will work for any Shimura variety of Hodge type, provided that the closure of an Ekedahl–Oort stratum in the minimal compactification $S_K^{\min}$ intersects the boundary in a closed subset of codimension ≥ 2.

4.2. Embeddings of Shimura varieties

We set $\bar{\Lambda} := \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, endowed with the symplectic form induced by ψ. We denote by G_0 the group $\mathrm{GSp}(\bar{\Lambda})$. As explained in [8, 4.5], the embedding (1.2) induces a commutative diagram:

$$\begin{tikzcd} S_K \arrow[r, "\zeta"] \arrow[d] & G\text{-}\mathtt{Zip}^{\chi} \arrow[d, "\iota"] \\ S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^{\pm})_{\kappa} \arrow[r, "\zeta_0"] & G_0\text{-}\mathtt{Zip}^{\iota\circ\chi} \end{tikzcd}$$

where $S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^{\pm})$ denotes the special fiber of the Siegel modular variety $\mathcal{S}_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^{\pm})$ and ζ_0 the corresponding zip map. See diagram (4.9) in [8] for details. The cocharacter $\iota \circ \chi$ of G_0 gives rise to a zip datum (G_0, P_0, Q_0, φ) and we get a map between the quotient stacks:

$$[E\backslash G_{\kappa}] \longrightarrow [E_0\backslash G_{0,\kappa}].$$

Denote by U the open E-orbit in G_k and by U_0 the E_0-orbit containing $\iota(U)$. We deduce from Proposition 3.5 that $U = \iota^{-1}(U_0)$, and the following result follows:

Corollary 3. *Let S^{μ} be the μ-ordinary Ekedahl–Oort stratum, and let S_0^{μ} be the Ekedahl–Oort stratum of the Siegel modular variety $S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^{\pm})$ containing the image of S^{μ} by the embedding $\iota : S_K \to S_{\tilde{K}}(\mathrm{GSp}(\Lambda), S^{\pm})$. Then one has the equality:*

$$S^{\mu} = \iota^{-1}(S_0^{\mu}).$$

In particular the μ-ordinary locus S^{μ} is entirely determined in S by the isomorphism class of the p-divisible group $A[p^{\infty}]$ (or by the group scheme $A[p]$), where A is the abelian variety over k attached to a k-point in S_K.

Acknowledgments

I would like to thank Torsten Wedhorn for his very useful comments and fruitful conversations. I am also grateful to Ulrich Görtz and Wushi Goldring for some remarks on the paper.

References

[1] G. Boxer, Torsion in the coherent cohomology of Shimura varieties and Galois representations, preprint, arXiv:1507.05922, 2015.
[2] P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, in: A. Borel, W. Casselman (Eds.), Automorphic Forms, Representations, and L-Functions 2, in: Proc. Sympos. Pure Math., vol. 33, 1977, pp. 247–289.
[3] M. Demazure, A. Grothendieck, Schémas en groupes. III: Structure des schémas en groupes réductifs, in: Séminaire de Géométrie Algébrique du Bois Marie, 1962/64 (SGA 3), in: Lecture Notes in Math., vol. 153, Springer-Verlag, Berlin, 1962/1964.
[4] W. Goldring, J.-S. Koskivirta, Strata Hasse invariants, Hecke algebras and Galois representations, preprint, arXiv:1507.05032, 2015.
[5] M. Kisin, Integral models for Shimura varieties of abelian type, J. Amer. Math. Soc. 23 (2010) 967–1012.
[6] M. Kisin, Mod p points on Shimura varieties of abelian type, preprint, 2013.

[7] F. Knop, H. Kraft, T. Vust, The Picard group of a G-variety, in: Algebraic Transformation Groups and Invariant Theory, in: DMV-Seminar, vol. 13, Birkhäuser, 1989.
[8] J.-S. Koskivirta, T. Wedhorn, Generalized Hasse invariants for Shimura varieties of Hodge type, preprint, arXiv:1406.2178, 2014.
[9] B. Moonen, T. Wedhorn, Discrete invariants of varieties in positive characteristic, Int. Math. Res. Not. IMRN 72 (2004) 3855–3903.
[10] S. Nie, Fundamental elements of an affine Weyl group, preprint, arXiv:1310.2229, 2014.
[11] F. Oort, A stratification of a moduli space of polarized abelian varieties in positive characteristic, in: Moduli of Abelian Varieties, in: Progr. Math., vol. 195, Birkhäuser, 2001, pp. 345–416.
[12] R. Pink, T. Wedhorn, P. Ziegler, Algebraic zip data, Doc. Math. 16 (2011) 253–300.
[13] R. Pink, T. Wedhorn, P. Ziegler, F-zips with additional structure, preprint, Pacific J. Math. (2012), in press, arXiv:1208.3547.
[14] A. Vasiu, Integral canonical models of Shimura varieties of preabelian type, Asian J. Math. 3 (1999) 401–518.
[15] A. Vasiu, Manin problems for Shimura varieties of Hodge type, J. Ramanujan Math. Soc. 26 (1) (2011) 31–84.
[16] E. Viehmann, T. Wedhorn, Ekedahl–Oort and Newton strata for Shimura varieties of PEL type, Math. Ann. 356 (4) (2013) 1493–1550.
[17] T. Wedhorn, Y. Yaroslav, Purity of zip strata with applications to Ekedahl–Oort strata of Shimura varieties of Hodge type, preprint, arXiv:1404.0577, 2014.
[18] D. Wortmann, The μ-ordinary locus for Shimura varieties of Hodge type, preprint, arXiv:1310.6444, 2013.
[19] C. Zhang, Ekedahl–Oort strata for good reductions of Shimura varieties of Hodge type, preprint, arXiv:1312.4869, 2013.