



Representation theory of 0-Hecke–Clifford algebras



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ABSTRACT

The representation theory of 0-Hecke–Clifford algebras as a degenerate case is not semisimple and also with rich combinatorial meaning. Bergeron, Hivert and Thibon have proved that the Grothendieck ring of the category of finitely generated supermodules of 0-Hecke–Clifford algebras is isomorphic to the algebra of peak quasisymmetric functions defined by Stembridge. In this paper we further study the category of finitely generated projective supermodules and clarify the correspondence between it and the peak algebra of symmetric groups. In particular, two kinds of restriction rules for induced projective supermodules are obtained. After that, we consider the corresponding Heisenberg double and its Fock representation to prove that the ring of peak quasisymmetric functions is free over its subring of symmetric functions spanned by Schur's Q -functions.

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1. Introduction

As a q -deformation of the Sergeev algebra, the Hecke–Clifford algebra $HCl_n(q)$ was defined by G. Olshanski in [23] mixing the Hecke algebra $H_n(q)$ and the Clifford algebra Cl_n . When q is generic, it satisfies the Schur–Sergeev–Olshanski super-duality with the

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quantum enveloping algebra of queer Lie superalgebras $\mathfrak{q}_n$. Moreover, the Grothendieck group of the tower of Hecke–Clifford algebras is isomorphic to the subalgebra of symmetric functions spanned by Schur’s Q -functions, parallel to the classical case of Sergeev algebras ($q = 1$) [10, §3.3], [28]. For the root of unity case, Brundan and Kleshchev in [7] considered the (affine) Hecke–Clifford algebra and related its representation category with the positive part of the universal enveloping algebra of the affine Kac–Moody algebra $\mathfrak{g} = A_{2l}^{(2)}$. Recently, Mori in [22] extends the method of cellular algebras to the superalgebra setting, in order to uniformly study the cellular representation theory of Hecke–Clifford algebras only requiring the invertibility of q . In a word, the representation theory of (affine) Hecke–Clifford algebras has been widely studied; see also [16,31,32], etc.

There is still a degenerate case where $q = 0$, and it also has nice combinatorial aspect. The 0-Hecke–Clifford algebra $HCl_n(0)$ was first considered by Bergeron, Hivert and Thibon in [3], where they mainly constructed the simple supermodules of $HCl_n(0)$, and proved the Frobenius isomorphism between the Grothendieck group of the category of finitely generated supermodules of 0-Hecke–Clifford algebras and the Stembridge algebra of peak quasisymmetric functions. Since the cellular approach fails in the degenerate case, one needs different techniques to handle its representation theory. In this paper, we dually discuss the category of finitely generated projective supermodules. Note that the graded Hopf dual of the algebra of peak quasisymmetric functions is the peak algebra of symmetric groups [27]. Here we confirm the graded Hopf pairing between the Grothendieck groups of the above two supermodule categories and explicitly relate the projective ones to the peak algebra. In fact, there have already been many nice papers considering towers of algebras together with their corresponding Grothendieck groups, which provide a bunch of (combinatorial) Hopf algebras; see [7,9,12,19,26], etc.

Recently, Berg, Bergeron, Saliola, Serrano and Zabrocki in [1] construct a noncommutative lift of Schur functions, called the immaculate basis, and then in [2] find the corresponding objects of its dual basis in the category of finitely generated modules of 0-Hecke algebras under the Frobenius isomorphism defined in [19]. Inspired by their work, we construct a lift of Schur’s Q -functions to the peak algebra and thus extract a new basis from it in [17]. Dually we define a new basis, called the quasisymmetric Schur Q -functions, in the Stembridge algebra, whose expansion in the peak functions is expected to be positive based on concrete examples [17, Conjecture 4.15]. It implies the possibility for a representation theoretic meaning of such basis on the category of supermodules of 0-Hecke–Clifford algebras. For this purpose, we want to further study the representation theory of 0-Hecke–Clifford algebras.

As an application, we also use the corresponding Heisenberg double and its Fock representation from the tower of 0-Hecke–Clifford algebras to prove that the ring of peak quasisymmetric functions is free over its subring of symmetric functions spanned by Schur’s Q -functions. Such method was proposed by Savage and Yacobi in [26] to give a new proof for the freeness of the ring of quasisymmetric functions over its subring of

symmetric functions. For further discussion of twisted Heisenberg doubles, one can refer to [24,25].

The organization of this paper is as follows. In §2 we provide some notation, definitions for all combinatorial Hopf algebras discussed here. Some preliminaries on superalgebras and the terminology of towers of superalgebras are recalled. In §3 we focus on the representation theory of 0-Hecke–Clifford algebras. Their projective supermodules induced from those of 0-Hecke algebras are mainly considered. Finally we clarify a dual pair of graded Hopf algebra structures on two Grothendieck groups of finitely generated supermodules and projective supermodules of 0-Hecke–Clifford algebras, whose Frobenius superalgebra structures are also figured out. In §4 we further give the concrete relation between the Grothendieck groups defined in the previous section and the peak algebra of symmetric groups together with its dual, extending the results in [3]. In particular for those induced projective supermodules, two restriction rules and the indecomposable decomposition formula are given. Lastly, the freeness of the ring of peak quasisymmetric functions over its subring of symmetric functions spanned by Schur’s Q -functions is proved.

2. Preliminaries

2.1. Notation and definitions

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 for simplicity. Denote by $\mathbb{N}$ (resp. $\mathbb{N}_0$) the set of positive (resp. nonnegative) integers. Given any $m, n \in \mathbb{N}$, let $[m, n] := \{m, m+1, \dots, n\}$ if $m \leq n$ and $\emptyset$ otherwise. Also $[n] := [1, n]$ for short. Let $2^{[n]}$ be the set of subsets of $[n]$ and $\mathcal{C}(n)$ be the set of compositions of n , consisting of ordered tuples of positive integers summing up to n . We denote $\alpha \models n$ when $\alpha \in \mathcal{C}(n)$. Let $\mathcal{C} := \bigcup_{n \geq 1} \mathcal{C}(n)$. Given $\alpha = (\alpha_1, \dots, \alpha_r) \models n$, let $\ell(\alpha) = r$ be its length and define its associated *descent set* as

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{r-1}\} \subseteq [n-1].$$

Given a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$, we also define its associated *descent set* as

$$D(w) := \{i \in [n-1] : w_i > w_{i+1}\}$$

and *descent composition* $c(w) \models n$ such that $D(c(w)) = D(w)$. Note that $\mathcal{C}(n) \rightarrow 2^{[n-1]}$, $\alpha \mapsto D(\alpha)$ is a bijection. The refining order $\leq$ on $\mathcal{C}(n)$ is defined by

$$\alpha \leq \beta \text{ if and only if } D(\beta) \subseteq D(\alpha), \forall \alpha, \beta \models n.$$

In general, for $\alpha \in \mathbb{N}_0^n$, let $\ell(\alpha) = |\{i : \alpha_i > 0\}|$. Given $\alpha \models n$, the corresponding *descent class* of the symmetric group $\mathfrak{S}_n$ is defined by

$$\mathfrak{D}_\alpha := \{w \in \mathfrak{S}_n : D(w) = D(\alpha)\}. \quad (2.1)$$

For $\alpha = (\alpha_1, \dots, \alpha_r) \models n$, define its three counterparts:

- (1) the *reverse* of α , $\bar{\alpha} := (\alpha_r, \dots, \alpha_1)$, such that $D(\bar{\alpha}) = \{i \in [n-1] : n-i \in D(\alpha)\}$.
- (2) the *complement* of α , $\alpha^c \models n$ such that $D(\alpha^c) = [n-1] \setminus D(\alpha)$.
- (3) the *conjugation* of α , $\alpha^* := \bar{\alpha}^c$.

We also recall the concept of peaks. A subset $P \subseteq [n]$ is called a *peak set* in $[n]$ if $P \subseteq [2, n-1]$ and $i \in P \Rightarrow i-1 \notin P$. Denote by $\mathcal{P}_n$ the collection of peak sets in $[n]$, $\mathcal{P} := \bigcup_{n \geq 1} \mathcal{P}_n$, and $\emptyset_n$ the empty set $\emptyset$ in $\mathcal{P}_n$. Given $\alpha = (\alpha_1, \dots, \alpha_r) \models n$, let

$$P(\alpha) := \{x \in [2, n-1] : x-1 \notin D(\alpha), x \in D(\alpha)\}$$

be its associated peak set in $[n]$, while its associated *valley set* $V(\alpha) \subseteq [n]$ is defined by

$$V(\alpha) \cap [2, n] = \{x \in [2, n] : x-1 \in D(\alpha), x \notin D(\alpha)\}$$

and $1 \in V(\alpha) \Leftrightarrow 1 \notin D(\alpha)$. Note that $|V(\alpha)| = |P(\alpha)| + 1$. Given a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$, its *peak set*

$$P(w) := \{i \in [2, n-1] : w_{i-1} < w_i > w_{i+1}\} = P(c(w)).$$

Define the algebra of *noncommutative symmetric functions* as the free associative $\mathbb{K}$ -algebra generated by the symbols H_n ($n \in \mathbb{N}$) and denote it by NSym [13]. Define another set of generators E_n ($n \in \mathbb{N}$) in NSym by

$$\sum_{i=0}^n (-1)^i E_i H_{n-i} = \delta_{n,0},$$

where $H_0 = E_0 = 1$ for convenience. The algebra $\text{NSym} = \bigoplus_{n=0}^{\infty} \text{NSym}_n$ is a $\mathbb{Z}$ -graded algebra under the gradation given by $\deg(H_n) = n$, where NSym_n is the subspace of homogeneous elements of degree n . Let

$$H_\alpha := H_{\alpha_1} \cdots H_{\alpha_r}, E_\alpha := E_{\alpha_1} \cdots E_{\alpha_r}, \alpha = (\alpha_1, \dots, \alpha_r) \models n.$$

If we change the base ring from $\mathbb{K}$ to $\mathbb{Z}$, then both $\{H_\alpha\}_{\alpha \models n}$ and $\{E_\alpha\}_{\alpha \models n}$ are $\mathbb{Z}$ -bases of NSym_n , called the *noncommutative complete* and *elementary symmetric functions* respectively. There exists another important $\mathbb{Z}$ -basis $\{R_\alpha\}_{\alpha \models n}$ of NSym_n , called the *noncommutative ribbon Schur functions* and defined by

$$R_\alpha := \sum_{\beta \geq \alpha} (-1)^{l(\beta) - l(\alpha)} H_\beta.$$

With the coproduct defined by

$$\Delta(H_n) = \sum_{k=0}^n H_k \otimes H_{n-k}, \quad (2.2)$$

NSym becomes a graded and connected Hopf algebra. Moreover, the graded Hopf dual of NSym is the algebra of *quasisymmetric functions*, denoted by QSym [21]. It is a subring of the power series ring $\mathbb{K}[[x_1, x_2, \dots]]$ in the commuting variables $x_1, x_2, \dots$ and has a linear basis, the *monomial quasisymmetric functions*, defined by

$$M_\alpha := M_\alpha(x) = \sum_{i_1 < \dots < i_r} x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r},$$

where $\alpha = (\alpha_1, \dots, \alpha_r)$ varies over the set $\mathcal{C}$ of compositions. There is another important basis, the *fundamental quasisymmetric functions*, defined by

$$F_\alpha := F_\alpha(x) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ i_k < i_{k+1} \text{ if } k \in D(\alpha)}} x_{i_1} \cdots x_{i_n}, \quad \alpha \models n.$$

In other words, $F_\alpha = \sum_{\beta \leq \alpha} M_\beta$. Meanwhile, the canonical pairing $\langle \cdot, \cdot \rangle$ between NSym and QSym is defined by

$$\langle H_\alpha, M_\beta \rangle = \langle R_\alpha, F_\beta \rangle = \delta_{\alpha, \beta}$$

for any $\alpha, \beta \in \mathcal{C}$.

Let Λ be the graded ring of symmetric functions in the commuting variables $x_1, x_2, \dots$, with integer coefficients, and Ω be the subring of Λ generated by the symmetric functions q_n ($n \geq 1$). These functions are defined by

$$\sum_{n \geq 0} q_n z^n = \prod_{i \geq 1} \frac{1 + x_i z}{1 - x_i z}.$$

Note that $\Omega_{\mathbb{Q}} := \Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}[p_{2k-1} : k \in \mathbb{N}]$, where p_n 's are the power-sum symmetric functions. It has the following canonical inner product $[\cdot, \cdot]$ defined by

$$[p_\lambda, p_\mu] = z_\lambda 2^{-\ell(\lambda)} \delta_{\lambda, \mu}$$

for any partitions λ, μ only with odd parts, where $z_\lambda := \prod_{i \geq 1} m_i! i^{m_i}$ for $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ (see [20, Ch. III, §8]). Now introduce two Hopf algebra epimorphisms. One is

$$\theta : \Lambda \rightarrow \Omega, \quad h_n \mapsto q_n, \quad n \geq 1,$$

such that $\theta(p_n) = (1 - (-1)^n)p_n$, $n \geq 1$, and the other is

$$\pi : \text{NSym} \rightarrow \Lambda, H_n \mapsto h_n,$$

called the *forgetful map* and satisfying

$$\langle F, f \rangle = \langle \pi(F), f \rangle, F \in \text{NSym}, f \in \Lambda.$$

2.2. The peak subalgebra and its Hopf dual

Introduce

$$Q_0 := 1, Q_n := \sum_{k=0}^n E_k H_{n-k}, n \geq 1$$

and also $Q_n := 0$, $n < 0$ for convenience. Let $Q_\alpha := Q_{\alpha_1} \cdots Q_{\alpha_r}$, $\alpha = (\alpha_1, \dots, \alpha_r) \models n$. Then Q_n ($n \geq 1$) satisfy the following *Euler relations* (also called the *generalized Dehn–Sommerville relation*)

$$\sum_{r+s=n} (-1)^r Q_r Q_s = 0, \forall n \geq 1, \quad (2.3)$$

or equivalently,

$$\sum_{i=1}^{n-1} (-1)^{i-1} Q_i Q_{n-i} = \begin{cases} 2Q_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (2.4)$$

Let Peak be the Hopf subalgebra of NSym generated by Q_n ($n \geq 1$). Then $\text{Peak}_n := \text{Peak} \cap \text{NSym}_n$ is isomorphic to the *peak algebra* of the symmetric group $\mathfrak{S}_n$ when endowed with the internal product [3,27]. For any $P \in \mathcal{P}_n$, define

$$\Xi_P := \sum_{P(\alpha)=P} R_\alpha \in \text{NSym}. \quad (2.5)$$

Then $\{\Xi_P\}_{P \in \mathcal{P}_n}$ forms a linear basis of Peak_n [3, §2]. Note that by [3, Eq. (6)],

$$Q_n = 2\Xi_{\emptyset_n} = 2 \sum_{k=0}^{n-1} R_{1^k, n-k}, n \geq 1. \quad (2.6)$$

There exists a surjective Hopf algebra homomorphism

$$\Theta : \text{NSym} \rightarrow \text{Peak}, H_n \mapsto Q_n, n \geq 1,$$

called the *descent-to-peak transform* [18, §5], where $\text{Ker } \Theta$ is the Hopf ideal of NSym generated by $\sum_{r+s=n} (-1)^r H_r H_s$, $n \geq 1$ [6, Theorem 5.4].

Next we introduce the graded Hopf dual of Peak , the *Stembridge algebra* Peak^* of peak quasisymmetric functions, defined in [29]. This is a Hopf subalgebra of QSym , with a natural basis called the Stembridge's *peak functions*. They can be defined by [29, Prop. 3.5]

$$K_P := 2^{|P|+1} \sum_{\substack{\alpha \models n \\ P \subseteq D(\alpha) \triangle (D(\alpha)+1)}} F_\alpha, \quad P \in \mathcal{P}_n, \quad (2.7)$$

where $D \triangle (D+1) = D \setminus (D+1) \cup (D+1) \setminus D$ for any $D = \{D_1 < \dots < D_r\} \subseteq [n-1]$ and $D+1 := \{x+1 : x \in D\}$. By [29, Prop. 2.2], we also have

$$K_P = \sum_{\substack{\alpha \models n \\ P \subseteq D(\alpha) \cup (D(\alpha)+1)}} 2^{\ell(\alpha)} M_\alpha, \quad P \in \mathcal{P}_n$$

and in particular, by [29, (2.5)],

$$K_{\emptyset_n} = q_n = 2 \sum_{\alpha \models n} F_\alpha = \sum_{\alpha \models n} 2^{\ell(\alpha)} M_\alpha. \quad (2.8)$$

There also exists a surjective Hopf algebra homomorphism

$$\vartheta : \text{QSym} \rightarrow \text{Peak}^*, \quad F_\alpha \mapsto K_{P(\alpha)},$$

called the *descent-to-peak map*.

Note that Peak can be regarded as a noncommutative lift of Ω . The following commutative diagrams illustrate the situation.

$$\begin{array}{ccc} \text{NSym} & \xrightarrow{\Theta} & \text{Peak} \\ \pi \downarrow & & \downarrow \pi \\ \Lambda & \xrightarrow{\theta} & \Omega \end{array} \quad , \quad \begin{array}{ccc} \text{QSym} & \xrightarrow{\vartheta} & \text{Peak}^* \\ \uparrow & & \uparrow \\ \Lambda & \xrightarrow{\theta} & \Omega \end{array} \quad (2.9)$$

where the vertical maps in the second diagram are inclusions. Also, the graded Hopf dual pairing between Peak and Peak^* is defined by

$$[\cdot, \cdot] : \text{Peak} \times \text{Peak}^* \rightarrow \mathbb{K}, \quad [\Xi_P, K_Q] = \delta_{P,Q}, \quad P, Q \in \mathcal{P}, \quad (2.10)$$

which satisfies the following property [27, Cor. 5.6.],

$$\langle \Theta(F), f \rangle = \langle F, \vartheta(f) \rangle = [\Theta(F), \vartheta(f)], \quad F \in \text{NSym}, f \in \text{QSym}. \quad (2.11)$$

2.3. Preliminaries on superalgebras

We first recall some standard results about the representation theory of finite dimensional (associative) superalgebras, referring to [7,8]. A *superalgebra* A over $\mathbb{K}$ is a $\mathbb{Z}_2$ -graded $\mathbb{K}$ -vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ which is also an algebra such that $A_i A_j \subseteq A_{i+j}$, $i, j \in \mathbb{Z}_2$. Any superalgebra A considered here has the unit $1 = 1_A \in A_{\bar{0}}$. Given $a \in A_i$ ($i \in \mathbb{Z}_2$), let the *degree* of a be $|a| := i$. Homomorphisms between two superalgebras are usual algebra homomorphisms. For two $\mathbb{Z}_2$ -graded $\mathbb{K}$ -vector spaces V, W , $\text{Hom}_{\mathbb{K}}(V, W)$ is $\mathbb{Z}_2$ -graded such that $f \in \text{Hom}_{\mathbb{K}}(V, W)_i$ if $f(V_j) \subseteq W_{i+j}$, $i, j \in \mathbb{Z}_2$. The base field $\mathbb{K}$ serves as a one dimensional even space.

A left A -*supermodule* M is a $\mathbb{Z}_2$ -graded $\mathbb{K}$ -vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ which is also an A -module such that $A_i M_j \subseteq M_{i+j}$, $i, j \in \mathbb{Z}_2$. Given $m \in M_i$ ($i \in \mathbb{Z}_2$), also let the *degree* of m be $|m| := i$. A *morphism* f between two left A -supermodules M, N is a linear map such that $f(am) = (-1)^{|f||a|} af(m)$, $a \in A, m \in M$. We denote the $\mathbb{Z}_2$ -graded $\mathbb{K}$ -vector space of all such morphisms by $\text{Hom}_A(M, N)$. On the other hand, $M^* := \text{Hom}_{\mathbb{K}}(M, \mathbb{K})$ has the following natural left A -supermodule structure:

$$(af)(m) = (-1)^{|a|(|f|+|m|)} f(ma), \quad a \in A, f \in M^*, m \in M,$$

if M is a right A -supermodule, and the natural right A -supermodule structure: $(fa)(m) = f(am)$, if M is a left A -supermodule.

An A -supermodule is *irreducible* (or *simple*) if it is non-zero and has no non-zero proper super submodules. Then it is either irreducible as an ordinary A -module (called *of type M*) or else reducible as an A -module (called *of type Q*).

All the finitely generated A -supermodules together with their morphisms constitute a *superadditive* category, denoted by $A\text{-mod}$. Besides, all the finitely generated projective A -supermodules form a full subcategory of $A\text{-mod}$, denoted by $A\text{-pmod}$. There exists the *parity change functor* $\Pi : A\text{-mod} \rightarrow A\text{-mod}$ such that ΠM is the same underlying vector space but with $(\Pi M)_i = M_{i+1}$ ($i \in \mathbb{Z}_2$) and the following new action:

$$a.m := (-1)^{|a|} am, \quad a \in A, m \in M.$$

Note that one can identify $\text{Hom}_A(M, N)_i$ with $\text{Hom}_A(M, \Pi N)_{i+1}$ for any $i \in \mathbb{Z}_2$, and obviously the map $M \rightarrow \Pi M$, $m \mapsto (-1)^{|m|} m$ is a homomorphism of usual A -modules.

Suppose that ν is an automorphism of a superalgebra A . For any $M \in A\text{-mod}$, we can twist the left action on M with ν to define the following twisted left A -module, denoted by ${}^\nu M$:

$$a.{}^\nu m := \nu(a)m, \quad m \in M, a \in A.$$

If ν is an *unsigned* anti-automorphism of A , i.e. $\nu(ab) = \nu(b)\nu(a)$, $a, b \in A$, then we can twist the right action of A on M^* with ν to define the following twisted left A -module,

also denoted by ${}^\nu(M^*)$:

$$(a \cdot_\nu f)(m) := f(\nu(a)m), f \in M^*, m \in M, a \in A.$$

For any right A -supermodule M , we use the notation M^ν instead for the right twisted module structure without confusion.

Given two superalgebras A and B , the tensor product $A \otimes B$ has the superalgebra structure defined by the following twisted multiplication:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd, a, c \in A, b, d \in B$$

and $|a \otimes b| := |a| + |b|$ for any homogeneous $a \in A, b \in B$. Given an A -supermodule M and a B -supermodule N , the tensor product $M \otimes N$ has the $A \otimes B$ -supermodule structure defined by:

$$(a \otimes b)(m \otimes n) = (-1)^{|b||m|} am \otimes bn, a \in A, b \in B, m \in M, n \in N$$

and $|m \otimes n| := |m| + |n|$ for any homogeneous $m \in M, n \in N$.

The category $A\text{-mod}$ has the *underlying even subcategory* with the same objects but only *even* morphisms, which is an abelian category. Hence, we can define the corresponding *Grothendieck group* $K_0(A\text{-mod})$ to be the quotient of the free abelian group with all objects in $A\text{-mod}$ as a basis by the subgroup generated by

(1) $M_1 - M_2 + M_3$ for every short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ in the underlying even subcategory.

(2) $M - \Pi M$ for every $M \in A\text{-mod}$.

Similarly we define the Grothendieck group $K_0(A\text{-pmod})$. For any $M \in A\text{-mod}$ or $A\text{-pmod}$, its class in such Grothendieck group is denoted by $[M]$. Note that the natural embedding from $A\text{-pmod}$ to $A\text{-mod}$ induces the *Cartan map*

$$\chi : K_0(A\text{-pmod}) \rightarrow K_0(A\text{-mod}), \quad (2.12)$$

which describes the multiplicity of composition factors in projective modules. Also there is a canonical embedding

$$K_0(A\text{-mod}) \otimes_{\mathbb{Z}} K_0(B\text{-mod}) \rightarrow K_0((A \otimes B)\text{-mod}), [M] \otimes [N] \rightarrow [M \otimes N],$$

which is similarly defined on pmod 's and an isomorphism when changing the base ring to $\mathbb{Q}$.

There is a natural bilinear form

$$\langle \cdot, \cdot \rangle : K_0(A\text{-pmod}) \times K_0(A\text{-mod}) \rightarrow \mathbb{Z}, \langle [P], [M] \rangle = \dim_{\mathbb{K}} \text{Hom}_A(P, M). \quad (2.13)$$

For the pairing on $K_0((A \otimes B)\text{-pmod}) \times K_0((A \otimes B)\text{-mod}) \rightarrow \mathbb{Z}$, we have

$$\langle [P \otimes Q], [M \otimes N] \rangle = \langle [P], [M] \rangle \langle [Q], [N] \rangle. \quad (2.14)$$

Remark 2.3.1. The pairing defined in (2.13) only involves dimensions of homomorphism spaces but not their graded dimensions as in [25]. Such discrepancy is not weird, since the authors in [25] considered the structure of twisted dual Hopf algebras indeed.

If A is a finite dimensional superalgebra, let $V_1, \dots, V_r$ be a complete list of non-isomorphic simple A -supermodules. If P_i is the projective cover of V_i in $A\text{-mod}$ for $i = 1, \dots, r$, then $P_1, \dots, P_r$ is a complete list of non-isomorphic indecomposable projective A -supermodules and we have

$$K_0(A\text{-mod}) = \bigoplus_{i=1}^r \mathbb{Z}[V_i], \quad K_0(A\text{-pmod}) = \bigoplus_{i=1}^r \mathbb{Z}[P_i].$$

Note that

$$\langle [P_i], [V_j] \rangle = \begin{cases} 1, & \text{if } i = j \text{ and } V_i \text{ is of type M,} \\ 2, & \text{if } i = j \text{ and } V_i \text{ is of type Q,} \\ 0, & \text{otherwise.} \end{cases} \quad (2.15)$$

Lemma 2.3.2. (See [8, §2].) If M is a finite dimensional irreducible A -supermodule of type Q , then there exist bases $\{v_1, \dots, v_n\}$ for $M_{\bar{0}}$ and $\{\bar{v}_1, \dots, \bar{v}_n\}$ for $M_{\bar{1}}$ such that $\text{span}_{\mathbb{K}}\{v_1 + \bar{v}_1, \dots, v_n + \bar{v}_n\}$ and $\text{span}_{\mathbb{K}}\{v_1 - \bar{v}_1, \dots, v_n - \bar{v}_n\}$ form two non-isomorphic irreducible A -modules. Moreover, the linear map $J_M \in \text{End}_{\mathbb{K}}(M)$ defined by $v_i \mapsto \bar{v}_i$, $\bar{v}_i \mapsto -v_i$ is an odd A -supermodule automorphism of M .

Lemma 2.3.3. (See [8, §2] Schur's lemma.) If M is a finite dimensional irreducible A -supermodule, then

$$\text{End}_A(M) = \begin{cases} \text{span}_{\mathbb{K}}\{\text{id}_M\}, & \text{if } M \text{ is of type M,} \\ \text{span}_{\mathbb{K}}\{\text{id}_M, J_M\}, & \text{if } M \text{ is of type Q,} \end{cases}$$

where J_M is as in Lemma 2.3.2.

2.4. Towers of superalgebras

Now mainly for 0-Hecke–Clifford algebras, we introduce the following important definition (see [25, Def. 4.1]), which extends the original one in [5, §3]. Further discussion can be found in [4].

Definition 2.4.1. Let $A = \bigoplus_{n \geq 0} A_n$ be a graded superalgebra over $\mathbb{K}$ with multiplication $\mu : A \otimes A \rightarrow A$. Then A is called a *tower of superalgebras* if the following conditions hold:

- (1) Each graded component A_n is a finite dimensional superalgebra with unit 1_n , and $A_0 \cong \mathbb{K}$.
- (2) The restriction $\mu_{m,n} : A_m \otimes A_n \rightarrow A_{m+n}$ of multiplication μ is a homomorphism of superalgebras for all $m, n \geq 0$, sending $1_m \otimes 1_n$ to 1_{m+n} .
- (3) For all $m, n \geq 0$, $\mu_{m,n}$ induces a two-sided projective $A_m \otimes A_n$ -module structure on A_{m+n} defined by

$$(a \otimes b).c := \mu_{m,n}(a \otimes b)c, \quad c.(a \otimes b) := c\mu_{m,n}(a \otimes b)$$

for any $a \in A_m, b \in A_n, c \in A_{m+n}$.

For a tower $A = \bigoplus_{n \geq 0} A_n$ of superalgebras, we define the categories

$$A\text{-mod} := \bigoplus_{n \geq 0} A_n\text{-mod}, \quad A\text{-pmod} := \bigoplus_{n \geq 0} A_n\text{-pmod}$$

and the corresponding Grothendieck groups

$$\mathcal{G}(A) := \bigoplus_{n \geq 0} K_0(A_n\text{-mod}), \quad \mathcal{K}(A) := \bigoplus_{n \geq 0} K_0(A_n\text{-pmod}).$$

Both the Cartan map (2.12) and the pairing (2.13) can be linearly extended to one for towers of superalgebras. Define

$$\chi := \bigoplus_{n \geq 0} \chi_n : \mathcal{K}(A) \rightarrow \mathcal{G}(A), \quad (2.16)$$

with χ_n to be the Cartan map (2.12) of A_n ($n \in \mathbb{N}$), and

$$\langle \cdot, \cdot \rangle : \mathcal{K}(A) \times \mathcal{G}(A) \rightarrow \mathbb{Z}$$

by

$$\langle [P], [M] \rangle := \begin{cases} \dim_{\mathbb{K}} \operatorname{Hom}_{A_n}(P, M), & P \in A_n\text{-pmod}, M \in A_n\text{-mod for some } n, \\ 0, & \text{otherwise.} \end{cases} \quad (2.17)$$

For any $r \in \mathbb{N}$, write $A_{n_1, \dots, n_r} := A_{n_1} \otimes \cdots \otimes A_{n_r}$ and define

$$A\text{-mod}^{\otimes r} := \bigoplus_{n_1, \dots, n_r \geq 0} A_{n_1, \dots, n_r}\text{-mod}, \quad A\text{-pmod}^{\otimes r} := \bigoplus_{n_1, \dots, n_r \geq 0} A_{n_1, \dots, n_r}\text{-pmod}.$$

Note that the formula (2.14) can be further extended on $A\text{-pmod}^{\otimes r} \times A\text{-mod}^{\otimes r}$.

For $\sigma \in \mathfrak{S}_r$, define $\tau_\sigma : A\text{-mod}^{\otimes r} \rightarrow A\text{-mod}^{\otimes r}$ to be the functor twisting module structure by the following superalgebra isomorphism from $A_{n_1, \dots, n_r}$ to $A_{n_{\sigma^{-1}(1)}, \dots, n_{\sigma^{-1}(r)}}$:

$$a_1 \otimes \cdots \otimes a_r \mapsto (-1)^{d_\sigma} a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(r)}, \quad (2.18)$$

where $d_\sigma := \sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} |a_i| |a_j|$. Write $\tau_{ij} := \tau_{s_{ij}}$ for short. For any $M \in A_{n_{\sigma^{-1}(1)}, \dots, n_{\sigma^{-1}(r)}}\text{-mod}$, we denote its twisted $A_{n_1, \dots, n_r}$ -supermodule structure by ${}^\tau M$. Now we have the following result.

Lemma 2.4.2. *Given $M_i \in A_{n_i}\text{-mod}$ ($1 \leq i \leq r$) and $\sigma \in \mathfrak{S}_n$, there exists the following (even) $A_{n_1, \dots, n_r}$ -supermodule isomorphism*

$$\begin{aligned} \Psi : M_1 \otimes \cdots \otimes M_r &\rightarrow {}^\tau (M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(r)}), \\ m_1 \otimes \cdots \otimes m_r &\mapsto (-1)^{d'_\sigma} m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(r)}, \end{aligned}$$

where $d'_\sigma := \sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} |m_i| |m_j|$.

Proof. On one hand,

$$\begin{aligned} \Psi((a_1 \otimes \cdots \otimes a_r) \cdot (m_1 \otimes \cdots \otimes m_r)) &= (-1)^{\sum_{1 \leq i < j \leq r} |a_j| |m_i|} \Psi(a_1 m_1 \otimes \cdots \otimes a_r m_r) \\ &= (-1)^{\sum_{1 \leq i < j \leq r} |a_j| |m_i| + \sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} (|a_i| + |m_i|)(|a_j| + |m_j|)} \\ &\quad a_{\sigma^{-1}(1)} m_{\sigma^{-1}(1)} \otimes \cdots \\ &\quad \otimes a_{\sigma^{-1}(r)} m_{\sigma^{-1}(r)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (a_1 \otimes \cdots \otimes a_r) \cdot {}_\tau \Psi(m_1 \otimes \cdots \otimes m_r) &= (-1)^{\sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} (|a_i| |a_j| + |m_i| |m_j|)} \\ &\quad (a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(r)}) \cdot (m_{\sigma^{-1}(1)} \otimes \cdots \\ &\quad \otimes m_{\sigma^{-1}(r)}), \\ &= (-1)^{\sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} (|a_i| |a_j| + |m_i| |m_j|) + \sum_{1 \leq i < j \leq r} |a_{\sigma^{-1}(j)}| |m_{\sigma^{-1}(i)}|} \\ &\quad a_{\sigma^{-1}(1)} m_{\sigma^{-1}(1)} \otimes \cdots \\ &\quad \otimes a_{\sigma^{-1}(r)} m_{\sigma^{-1}(r)}. \end{aligned}$$

They are equal since

$$\begin{aligned} &\sum_{1 \leq i < j \leq r} |a_j| |m_i| + \sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} (|a_i| |m_j| + |a_j| |m_i|) \\ &= \sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) < \sigma(j)}} |a_j| |m_i| + \sum_{\substack{1 \leq i < j \leq r \\ \sigma(i) > \sigma(j)}} |a_i| |m_j| = \sum_{1 \leq i < j \leq r} |a_{\sigma^{-1}(j)}| |m_{\sigma^{-1}(i)}| \in \mathbb{Z}_2. \quad \square \end{aligned}$$

Given an even homomorphism $f : B \rightarrow A$ of superalgebras, the usual induction and restriction functors are defined by

$$\begin{aligned} \text{Ind}_B^A : B\text{-mod} &\rightarrow A\text{-mod}, \text{Ind}_B^A M := A^f \otimes_B N, N \in B\text{-mod}, \\ \text{Res}_B^A : A\text{-mod} &\rightarrow B\text{-mod}, \text{Res}_B^A M := \text{Hom}_A(A^f, M) \cong {}^f A \otimes_A M, M \in A\text{-mod}, \end{aligned}$$

where the left B -action on $\text{Hom}_A(A^f, M)$ is defined by $(bg)(a) = (-1)^{|b|(|a|+|g|)}g(af(b))$, $a \in A, b \in B, g \in \text{Hom}_A(A, M)$, and the above isomorphism is given by $g \mapsto 1_A \otimes g(1_A)$.

Now if a tower A of superalgebras is fixed, we abbreviate $\text{Ind}_{A_m \otimes A_n}^{A_{m+n}}$ as $\text{Ind}_{m,n}^{m+n}$ and $\text{Res}_{A_m \otimes A_n}^{A_{m+n}}$ as $\text{Res}_{m,n}^{m+n}$, which base on the multiplication $\mu_{m,n} : A_m \otimes A_n \rightarrow A_{m+n}$. Define

$$\text{Ind} := \bigoplus_{m,n \geq 0} \text{Ind}_{m,n}^{m+n}, \text{Res} := \bigoplus_{m,n \geq 0} \text{Res}_{m,n}^{m+n}.$$

We are interested in those pairs $(\mathcal{K}(A), \mathcal{G}(A))$, which form dual pairs of graded Hopf algebras via such induction and restriction. For this duality, we also need the notion of Frobenius superalgebras (see [25, §6]).

Definition 2.4.3. A finite dimensional superalgebra A is called a *Frobenius superalgebra* if one of the following three equivalent conditions holds:

- (a) There is an even left A -supermodule isomorphism $\rho : A \rightarrow A^*$.
- (b) There exists a nondegenerate invariant even $\mathbb{K}$ -bilinear form $(\cdot, \cdot) : A \times A \rightarrow \mathbb{K}$ such that $(a, b) = 0$ if $a \in A_{\bar{0}}, b \in A_{\bar{1}}$ or $a \in A_{\bar{1}}, b \in A_{\bar{0}}$, and also $(ab, c) = (a, bc)$, $a, b, c \in A$.
- (c) There exists an even $\mathbb{K}$ -linear map $\text{tr} : A \rightarrow \mathbb{K}$, called the *trace map*, such that Ker tr contains no non-zero left ideals of A .

The relationship between these three conditions is as follows:

$$\rho(b)(a) = (-1)^{|a||b|}(a, b), (a, b) = \text{tr}(ab), a, b \in A.$$

There exists an even automorphism φ of A satisfying $(a, b) = (-1)^{|a||b|}(\varphi(b), a)$, $a, b \in A$. This automorphism is called the *Nakayama automorphism* of A .

Lemma 2.4.4. (See [25, Lemma 6.4].) Let A_i ($i = 1, 2$) be two Frobenius superalgebras with trace maps tr_i ($i = 1, 2$) and Nakayama automorphisms φ_i ($i = 1, 2$) respectively. Then $A_1 \otimes A_2$ is also a Frobenius superalgebra with trace map $\text{tr}_1 \otimes \text{tr}_2$ and Nakayama automorphism $\varphi_1 \otimes \varphi_2$.

For a tower A of superalgebras such that each A_n is a Frobenius superalgebra with Nakayama automorphism φ_n . We abuse the notation φ_n to be the automorphism on $A_n\text{-mod}$ (resp. $A_n\text{-pmod}$) twisting modules by φ_n of A_n . Then $\varphi := \bigoplus_{n \geq 0} \varphi_n$ is an automorphism of $A\text{-mod}$ (resp. $A\text{-pmod}$), thus induces an automorphism on $\mathcal{K}(A)$ (resp. $\mathcal{G}(A)$).

Proposition 2.4.5. (See [25, Prop. 6.7].) Under the above assumption, the induction is conjugate right adjoint to restriction with conjugation φ , i.e. for any $m, n \geq 0$, $M \in A_m\text{-mod}$, $N \in A_n\text{-mod}$, $L \in A_{m+n}\text{-mod}$, the following functorial isomorphism holds:

$$\mathrm{Hom}_{A_{m+n}}(L, \mathrm{Ind}_{m,n}^{m+n}(M \otimes N)) \cong \mathrm{Hom}_{A_m \otimes A_n}(\varphi \mathrm{Res}_{m,n}^{m+n}(L), M \otimes N),$$

where $\varphi \mathrm{Res}_{m,n}^{m+n} := (\varphi_m \otimes \varphi_n) \circ \mathrm{Res}_{m,n}^{m+n} \circ \varphi_{m+n}^{-1}$.

3. Representation theory of 0-Hecke–Clifford algebras

In this section we further study the representation theory of 0-Hecke–Clifford algebras in detail, referring to the discussion in [3, §5]. The 0-Hecke–Clifford algebra $HCl_n(0)$ of type A is a superalgebra generated by T_i , $1 \leq i \leq n-1$; c_j , $1 \leq j \leq n$, where T_i 's generate the 0-Hecke algebra $H_n(0)$ with relations

$$\begin{aligned} T_i^2 &= -T_i, \quad 1 \leq i \leq n-1, \\ T_i T_j &= T_j T_i, \quad |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n-2, \end{aligned}$$

and c_j 's generate the Clifford algebra Cl_n with relations

$$c_i^2 = -1, \quad 1 \leq i \leq n; \quad c_i c_j = -c_j c_i, \quad i \neq j,$$

while the two parts satisfy the following cross-relations

$$\begin{aligned} T_i c_j &= c_j T_i, \quad j \neq i, i+1, \\ T_i c_i &= c_{i+1} T_i, \quad 1 \leq i \leq n-1, \\ (T_i + 1) c_{i+1} &= c_i (T_i + 1), \quad 1 \leq i \leq n-1. \end{aligned}$$

Let $\deg(T_i) = \bar{0}$, $\deg(c_j) = \bar{1}$, then $HCl_n(0)$ becomes a $\mathbb{Z}_2$ -graded superalgebra, with a linear basis $\{c_D T_w : D \subseteq [n], w \in \mathfrak{S}_n\}$, where $c_D := c_{i_1} \cdots c_{i_r}$ for $D = \{i_1 < \cdots < i_r\}$ and $T_w := T_{j_1} \cdots T_{j_s}$ for any reduced expression $w = s_{j_1} \cdots s_{j_s}$. Throughout this paper, we only consider 0-Hecke(–Clifford) algebras, thus write $\mathcal{H}_n := H_n(0)$, $\mathcal{H}\mathcal{C}_n := HCl_n(0)$ for short.

First recall the main result about representation theory of 0-Hecke–Clifford algebras discussed in [3, §5]. Note that the 0-Hecke algebra $\mathcal{H}_n$ of type A is a basic algebra whose simple modules are one-dimensional, and a complete set of non-isomorphic simple modules are indexed by the compositions of n . In fact, given $\alpha \models n$, the simple module $S_\alpha := \mathbb{K}\eta_\alpha$ is defined as follows.

$$T_i \cdot \eta_\alpha = \begin{cases} -\eta_\alpha, & i \in D(\alpha), \\ 0, & \text{otherwise.} \end{cases} \quad (3.1)$$

We denote by P_α the projective cover of S_α with a linear basis $\{u_w : w \in \mathfrak{D}_\alpha\}$, where $\mathfrak{D}_\alpha$ is the descent class in (2.1). P_α has the module structure given as follows [19, §5.3]:

$$T_i \cdot u_w = \begin{cases} -u_w, & i \in D(w^{-1}), \\ u_{s_i w}, & i \notin D(w^{-1}), s_i w \in \mathfrak{D}_\alpha, \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

Moreover, for the tower $\mathcal{H} := \bigoplus_{n \geq 0} \mathcal{H}_n$, Krob and Thibon have defined the following Frobenius isomorphism and its adjoint in [19]:

$$\text{Ch} : \mathcal{G}(\mathcal{H}) \rightarrow \text{QSym}, [S_\alpha] \mapsto F_\alpha, \text{Ch}^* : \text{NSym} \rightarrow \mathcal{K}(\mathcal{H}), R_\alpha \mapsto [P_\alpha]. \quad (3.3)$$

Both are graded Hopf algebra isomorphisms satisfying

$$\langle F, \text{Ch}([M]) \rangle = \langle \text{Ch}^*(F), [M] \rangle, F \in \text{NSym}, [M] \in \mathcal{G}(\mathcal{H}).$$

Due to Gessel, we know that the Cartan map χ of $\mathcal{H}$ satisfies the following commutative diagram [19, Prop. 5.9]:

$$\begin{array}{ccc} \mathcal{K}(\mathcal{H}) & \xrightarrow{\chi} & \mathcal{G}(\mathcal{H}) \\ \text{Ch}^* \uparrow & & \downarrow \text{Ch} \\ \text{NSym} & \xrightarrow{\pi} & \text{QSym} \end{array}$$

That is, $\chi([P_\alpha]) = \sum_\beta \langle R_\beta, r_\alpha \rangle [S_\beta]$, where $r_\alpha := \pi(R_\alpha) \in \Lambda$ is the *ribbon Schur function* of shape α .

3.1. Irreducible supermodules of 0-Hecke-Clifford algebras

A complete set of non-isomorphic simple supermodules of $\mathcal{HC}_n$ are parameterized by peak sets and thus denoted by $\{HClS_P\}$. They can be obtained from the induced modules

$$\tilde{S}_\alpha := \text{Ind}_{\mathcal{H}_n}^{\mathcal{HC}_n} S_\alpha, \alpha \models n.$$

Slightly modifying [3, Theorem 5.4], we have

Theorem 3.1.1. *Let $\alpha \models n$, $V := V(\alpha) \subseteq [n]$ and Cl_V be the subalgebra of Cl_n generated by $\{c_v\}_{v \in V}$. For any homogeneous $c \in Cl_V$, define $f_c \in \text{End}_{\mathbb{K}}(\tilde{S}_\alpha)$ by*

$$f_c(c_D \eta_\alpha) = (-1)^{|c| |D|} c_D c \eta_\alpha, D \subseteq [n].$$

If we linearly extend it to all $c \in Cl_V$, then the map $c \mapsto f_c$ is an even isomorphism from Cl_V to $\text{End}_{\mathcal{HC}_n}(\tilde{S}_\alpha)$.

By [3, Th. 5.5, Cor. 5.6] we know that $\tilde{S}_\alpha \cong \tilde{S}_\beta$ (even isomorphism) if and only if $P(\alpha) = P(\beta)$ and

$$\tilde{S}_\alpha \cong HClS_{P(\alpha)}^{\oplus 2^{l_\alpha}}, \quad l_\alpha := \left\lfloor \frac{|P(\alpha)|+1}{2} \right\rfloor. \quad (3.4)$$

In fact, we know that for $V(\alpha) = \{n_1, \dots, n_s\}$, if one defines the mutually orthogonal even idempotents

$$e_\alpha^\varepsilon := \frac{1}{2^{l_\alpha}} (1 + \varepsilon_1 \sqrt{-1} c_{n_1} c_{n_2}) (1 + \varepsilon_2 \sqrt{-1} c_{n_3} c_{n_4}) \cdots (1 + \varepsilon_{l_\alpha} \sqrt{-1} c_{n_{2l_\alpha-1}} c_{n_{2l_\alpha}}), \quad (3.5)$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{l_\alpha}) \in \{\pm 1\}^{\times l_\alpha}$, then $Cl_n e_\alpha^\varepsilon \eta_\alpha$ ($\varepsilon \in \{\pm 1\}^{\times l_\alpha}$) provide all the simple components of $\tilde{S}_\alpha$.

Proposition 3.1.2. *For any peak set P , the irreducible supermodule $HClS_P$ is of type M when $|P|$ is odd, and of type Q when $|P|$ is even.*

Proof. By Theorem 3.1.1, we know that $\text{End}_{\mathcal{H}e_n}(\tilde{S}_\alpha) \cong Cl_V$ for any $\alpha \models n$ and $V := V(\alpha) \subseteq [n]$. Since $|V(\alpha)| = |P(\alpha)| + 1$, by Schur's lemma we only need to prove that there exists an odd automorphism of $\tilde{S}_\alpha$ stabilizing any simple component of it if and only if $|V|$ is odd.

First for any even idempotent e_α^ε in (3.5), it is easy to check that

$$e_\alpha^\varepsilon c_i = \begin{cases} c_i e_\alpha^\varepsilon, & i \notin \{n_1, \dots, n_{2l_\alpha}\}, \\ c_i e_\alpha^{\varepsilon^{(i)}}, & i \in \{n_1, \dots, n_{2l_\alpha}\}, \end{cases}$$

where $V = \{n_1, \dots, n_s\}$ and $\varepsilon^{(i)}$ is obtained from ε by only changing the sign ε_j if $i \in \{n_{2j-1}, n_{2j}\} \subseteq V$. It means that the map f_{c_E} ($E \subseteq V$) defined in Theorem 3.1.1 stabilizes any simple component of $\tilde{S}_\alpha$ if and only if $|E \cap \{n_{2j-1}, n_{2j}\}|$ is even for all $j = 1, \dots, l_\alpha$. Hence, if we need f_{c_E} to be odd and satisfy the stability condition, then V must be odd too. $\square$

Denote

$$\mathcal{G} := \mathcal{G}(\mathcal{H}), \quad \mathcal{K} := \mathcal{K}(\mathcal{H}), \quad \tilde{\mathcal{G}} := \mathcal{G}(\mathcal{H}\mathbb{C}), \quad \tilde{\mathcal{K}} := \mathcal{K}(\mathcal{H}\mathbb{C}).$$

In [3] Bergeron, Hivert and Thibon nicely define the following Frobenius isomorphism for the tower of 0-Hecke–Clifford algebras:

$$\widetilde{\text{Ch}} : \tilde{\mathcal{G}} \rightarrow \text{Peak}^*, \quad [\tilde{S}_\alpha] \mapsto K_{P(\alpha)},$$

which satisfies the following commutative diagrams (as a categorification of the descent-to-peak map):

$$\begin{array}{ccc}
 \mathcal{G} & \xrightarrow{\text{Ind}_{\mathcal{H}}^{\mathcal{H}^c}} & \tilde{\mathcal{G}} \\
 \downarrow \text{Ch} & & \downarrow \widetilde{\text{Ch}} \\
 \text{QSym} & \xrightarrow{\vartheta} & \text{Peak}^*
 \end{array}
 , \quad
 \begin{array}{ccc}
 [S_{\alpha}] & \mapsto & [\tilde{S}_{\alpha}] \\
 \downarrow & & \downarrow \\
 F_{\alpha} & \mapsto & K_{P(\alpha)}
 \end{array}
 \quad (3.6)$$

$$\begin{array}{ccc}
 \tilde{\mathcal{G}} & \xrightarrow{\text{Res}_{\mathcal{H}}^{\mathcal{H}^c}} & \mathcal{G} \\
 \downarrow \widetilde{\text{Ch}} & & \downarrow \text{Ch} \\
 \text{Peak}^* & \longrightarrow & \text{QSym}
 \end{array}
 , \quad
 \begin{array}{ccc}
 [\tilde{S}_{\alpha}] & \xrightarrow{\quad} & [\text{Res}_{\mathcal{H}}^{\mathcal{H}^c} \tilde{S}_{\alpha}] \\
 \downarrow & & \downarrow \\
 K_{P(\alpha)} & \mapsto & 2^{|P(\alpha)|+1} \sum_{\substack{\beta \models n \\ P(\alpha) \subseteq D(\beta) \triangle (D(\beta)+1)}} F_{\beta}
 \end{array}
 \quad (3.7)$$

where $\text{Ind}_{\mathcal{H}}^{\mathcal{H}^c} := \bigoplus_{n \geq 0} \text{Ind}_{\mathcal{H}_n}^{\mathcal{H}_n^c}$ and $\text{Res}_{\mathcal{H}}^{\mathcal{H}^c} := \bigoplus_{n \geq 0} \text{Res}_{\mathcal{H}_n}^{\mathcal{H}_n^c}$.

3.2. Projective supermodules of 0-Hecke–Clifford algebras

In this subsection we study the category of finitely generated projective supermodules of 0-Hecke–Clifford algebras, which was not developed in [3]. So far it is not so easy to construct the indecomposable one directly, we similarly consider the induction from the projective modules of 0-Hecke algebras. Define

$$\tilde{P}_{\alpha} := \text{Ind}_{\mathcal{H}_n}^{\mathcal{H}_n^c} P_{\alpha}, \quad \alpha \models n,$$

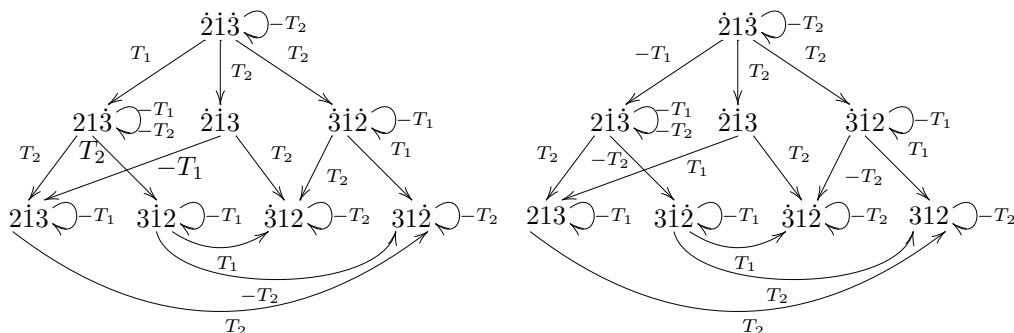
which becomes a projective $\mathcal{H}_n^c$ -supermodule.

Since $\mathcal{H}_n^c$ is a free right $\mathcal{H}_n$ -module of rank 2^n , we can write a basis for $\tilde{P}_{\alpha}$ as $\{c_D u_w : D \subseteq [n], w \in \mathfrak{D}_{\alpha}\}$ for short. Via the defining relation of $\mathcal{H}_n^c$, we get the following commutation relations:

$$T_i c_D = \begin{cases} c_D T_i, & i, i+1 \notin D, \\ c_{(D \setminus \{i\}) \cup \{i+1\}} T_i, & i \in D, i+1 \notin D, \\ c_{(D \setminus \{i+1\}) \cup \{i\}} (T_i + 1) - c_D, & i \notin D, i+1 \in D, \\ -c_D (T_i + 1) + c_{D \setminus \{i, i+1\}}, & i, i+1 \in D, \end{cases} \quad (3.8)$$

for any $D \subseteq [n]$, $i = 1, \dots, n-1$. Combining these relations and the module action (3.2) of P_{α} , one can describe explicitly the module structure of $\tilde{P}_{\alpha}$ for any $\alpha \models n$.

For example, the module structure of $\tilde{P}_{12}$ can be depicted explicitly by the following graphs (separated into two $\mathbb{Z}_2$ -graded components):



where we use $w \in \mathfrak{D}_\alpha$ to represent u_w and put dots on the heads of those numbers to represent basis elements $c_D u_w$, e.g. we abbreviate $c_{\{1,3\}} u_{213}$ as $\dot{2}\dot{1}\dot{3}$.

By relation (3.8), one easily gets the following result.

Lemma 3.2.1. *For any $w \in \mathfrak{S}_n$, $D \subseteq [n]$,*

$$T_w c_D = (-1)^{l_{w,D}} c_{w(D)} T_w + \sum_{\substack{E \subseteq [n] \\ v < w}} a_{E,v} c_E T_v$$

for some integers $a_{E,v}$, where $l_{w,D} := |\{i, j \in D : i < j, w(i) > w(j)\}|$, $w(D) := \{w(i) : i \in D\}$ and $<$ stands for the Bruhat order of $\mathfrak{S}_n$.

3.3. Dual Hopf algebras arising from 0-Hecke–Clifford algebras

In this subsection, we check a series of axioms for the tower of 0-Hecke–Clifford algebras in order to show that $(\tilde{\mathcal{K}}, \tilde{\mathcal{G}})$ forms a dual pair of graded Hopf algebras. One of the key steps is to define a proper Hopf pairing.

First the Mackey property of 0-Hecke–Clifford algebras is easy to proved by following the nice approach in [7, §2-h.] for affine Hecke–Clifford algebras. That guarantees both of the Grothendieck groups $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{G}}$ to be Hopf algebras via the induction and the restriction. One can also refer to [11, Theorem 2.7], [26, Prop. 4.3] for the case of Hecke algebras.

For the sake of completeness, we sketch the proof steps as follows. For any $\alpha = (\alpha_1, \dots, \alpha_r) \models n$, define $\mathcal{HC}_\alpha := \mathcal{HC}_{\alpha_1} \otimes \dots \otimes \mathcal{HC}_{\alpha_r}$, embedding as a parabolic subalgebra of $\mathcal{HC}_n$. Similarly define the subalgebra $\mathcal{H}_\alpha$ of $\mathcal{H}_n$.

Given $\alpha, \beta \models n$, let $\mathfrak{S}_\alpha := \langle s_i : i \notin D(\alpha) \rangle$ be the Young subgroup of $\mathfrak{S}_n$, $\mathfrak{R}_\alpha$ denote the set of minimal length left $\mathfrak{S}_\alpha$ -coset representatives in $\mathfrak{S}_n$, and $\mathfrak{R}_\alpha^{-1}$ for the one corresponding to right $\mathfrak{S}_\alpha$ -cosets. Then $\mathfrak{R}_{\alpha, \beta} := \mathfrak{R}_\alpha^{-1} \cap \mathfrak{R}_\beta$ is the set of minimal length $(\mathfrak{S}_\alpha, \mathfrak{S}_\beta)$ -double coset representatives in $\mathfrak{S}_n$. For any $x \in \mathfrak{R}_{\alpha, \beta}$, $\mathfrak{S}_\alpha \cap x\mathfrak{S}_\beta x^{-1}$ and $x^{-1}\mathfrak{S}_\alpha x \cap \mathfrak{S}_\beta$ are Young subgroups of $\mathfrak{S}_n$, thus we can let $\alpha \cap x\beta$ and $x^{-1}\alpha \cap \beta$ be the two compositions of n such that

$$\mathfrak{S}_\alpha \cap x\mathfrak{S}_\beta x^{-1} = \mathfrak{S}_{\alpha \cap x\beta}, \quad x^{-1}\mathfrak{S}_\alpha x \cap \mathfrak{S}_\beta = \mathfrak{S}_{x^{-1}\alpha \cap \beta}.$$

Now we need several technical lemmas as follows.

Lemma 3.3.1. For any $x \in \mathfrak{R}_{\alpha, \beta}$, the subspace $\mathcal{HC}_{\alpha} T_x \mathcal{H}_{\beta}$ of $\mathcal{HC}_n$ has a basis $\{C_D T_w : D \subseteq [n], w \in \mathfrak{S}_{\alpha} x \mathfrak{S}_{\beta}\}$. Moreover,

$$\mathcal{HC}_n = \bigoplus_{x \in \mathfrak{R}_{\alpha, \beta}} \mathcal{HC}_{\alpha} T_x \mathcal{H}_{\beta}.$$

Fix some total order $\prec$ refining the Bruhat order $<$ on $\mathfrak{R}_{\alpha, \beta}$. For $x \in \mathfrak{R}_{\alpha, \beta}$, let

$$\mathcal{B}_{\preceq x} := \bigoplus_{y \in \mathfrak{R}_{\alpha, \beta}, y \preceq x} \mathcal{HC}_{\alpha} T_y \mathcal{H}_{\beta}, \quad \mathcal{B}_{\prec x} := \bigoplus_{y \in \mathfrak{R}_{\alpha, \beta}, y \prec x} \mathcal{HC}_{\alpha} T_y \mathcal{H}_{\beta},$$

and $\mathcal{B}_x := \mathcal{B}_{\preceq x} / \mathcal{B}_{\prec x}$. By Lemma 3.2.1, we know that $\mathcal{B}_{\preceq x}$ (resp. $\mathcal{B}_{\prec x}$) is invariant under right multiplication by Cl_n . Hence, $\{\mathcal{B}_{\preceq x}\}_{x \in \mathfrak{R}_{\alpha, \beta}}$ is an $(\mathcal{HC}_{\alpha}, \mathcal{HC}_{\beta})$ -bimodule filtration of $\mathcal{HC}_n$.

Lemma 3.3.2. For any $x \in \mathfrak{R}_{\alpha, \beta}$, there exists an algebra isomorphism

$$\phi = \phi_x : \mathcal{HC}_{\alpha \cap x \beta} \rightarrow \mathcal{HC}_{x^{-1} \alpha \cap \beta}$$

with $\phi(T_w) = T_{x^{-1}wx}$, $\phi(c_i) = c_{x^{-1}(i)}$ for $w \in \mathfrak{S}_{\alpha \cap x \beta}$, $1 \leq i \leq n$.

Lemma 3.3.3. View $\mathcal{HC}_{\alpha}$ as an $(\mathcal{HC}_{\alpha}, \mathcal{HC}_{\alpha \cap x \beta})$ -bimodule and $\mathcal{HC}_{\beta}$ as an $(\mathcal{HC}_{x^{-1} \alpha \cap \beta}, \mathcal{HC}_{\beta})$ -bimodule. Then ${}^{\phi} \mathcal{HC}_{\beta}$ is an $(\mathcal{HC}_{\alpha \cap x \beta}, \mathcal{HC}_{\beta})$ -bimodule and

$$\Phi : \mathcal{B}_x \rightarrow \mathcal{HC}_{\alpha} \otimes_{\mathcal{HC}_{\alpha \cap x \beta}} {}^{\phi} \mathcal{HC}_{\beta}, \quad u T_x v + \mathcal{B}_{\prec x} \mapsto u \otimes v, \quad u \in \mathcal{HC}_{\alpha}, v \in \mathcal{HC}_{\beta}$$

is an isomorphism of $(\mathcal{HC}_{\alpha}, \mathcal{HC}_{\beta})$ -bimodules.

Theorem 3.3.4 (Mackey theorem). Let $\alpha, \beta \models n$ and M be an $\mathcal{HC}_{\beta}$ -module. Then the $\mathcal{HC}_{\alpha}$ -module $\text{Res}_{\mathcal{HC}_{\alpha}}^{\mathcal{HC}_n} \text{Ind}_{\mathcal{HC}_{\beta}}^{\mathcal{HC}_n} M$ admits an $\mathcal{HC}_{\alpha}$ -submodule filtration with subquotients isomorphic to $\text{Ind}_{\mathcal{HC}_{\alpha \cap x \beta}}^{\mathcal{HC}_{\alpha}} \left({}^{\phi} \text{Res}_{\mathcal{HC}_{x^{-1} \alpha \cap \beta}}^{\mathcal{HC}_{\beta}} M \right)$, one for each $x \in \mathfrak{R}_{\alpha, \beta}$.

Especially when α, β both have two parts, Theorem 3.3.4 implies that $\tilde{\mathcal{G}}$ has a graded Hopf algebra structure under induction and restriction. The case for $\tilde{\mathcal{K}}$ is similar, since $\mathcal{B}_x$ is projective as a left $\mathcal{HC}_{\alpha}$ -module.

In order to prove the Hopf duality between $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{G}}$ by Proposition 2.4.5, we continue to show that $\mathcal{HC}_n$ is a Frobenius superalgebra. It is straightforward to check that

Proposition 3.3.5. There exist two even algebra involutions φ, φ' for the 0-Hecke–Clifford algebra $\mathcal{HC}_n$, defined by

$$\begin{aligned} \varphi(T_i) &= T_{n-i} + c_{n-i} c_{n+1-i}, \quad i = 1, \dots, n-1, \\ \varphi(c_j) &= -c_{n+1-j}, \quad j = 1, \dots, n, \\ \varphi'(T_i) &= -(T_{n-i} + 1), \quad i = 1, \dots, n-1, \\ \varphi'(c_j) &= -c_{n+1-j}, \quad j = 1, \dots, n. \end{aligned} \tag{3.9}$$

There exist two unsigned even algebra anti-involutions ψ, ψ' of $\mathcal{HC}_n$ defined by

$$\begin{aligned}\psi(T_i) &= T_i + c_i c_{i+1}, \quad i = 1, \dots, n-1, \\ \psi(c_j) &= -c_j, \quad j = 1, \dots, n. \\ \psi'(T_i) &= -(T_i + 1), \quad i = 1, \dots, n-1, \\ \psi'(c_j) &= -c_j, \quad j = 1, \dots, n.\end{aligned}\tag{3.10}$$

Proposition 3.3.6. *The 0-Hecke-Clifford algebra $\mathcal{HC}_n$ is a Frobenius superalgebra with even trace map*

$$tr_n : \mathcal{HC}_n \rightarrow \mathbb{K}, \quad tr_n(c_D T_w) = \delta_{D, \emptyset} \delta_{w, w_0}, \quad D \subseteq [n], w \in \mathfrak{S}_n,$$

where w_0 is the longest element of $\mathfrak{S}_n$. Moreover, φ is the corresponding Nakayama automorphism.

Proof. We only need to prove that $\text{Ker } tr_n$ contains no non-zero left ideals. Suppose I is a non-zero left ideal of $\mathcal{HC}_n$. We choose an element $b = \sum_{\substack{D \subseteq [n] \\ w \in \mathfrak{S}_n}} b_{D, w} c_D T_w \in I \setminus \{0\}$ and let σ be a maximal length element in the set $\{w \in \mathfrak{S}_n : b_{D', w} \neq 0 \text{ for some } D' \subseteq [n]\}$. Then by Lemma 3.2.1 we have

$$\begin{aligned}tr_n(c_{w_0 \sigma^{-1}(D')} T_{w_0 \sigma^{-1}} b) &= b_{D', \sigma} tr_n(c_{w_0 \sigma^{-1}(D')} T_{w_0 \sigma^{-1}} c_{D'} T_\sigma) \\ &= (-1)^{l_{w_0 \sigma^{-1}, D'}} b_{D', \sigma} tr_n(c_{w_0 \sigma^{-1}(D')}^2 T_{w_0}) = (-1)^{l_{w_0 \sigma^{-1}, D'} + \binom{|D'|+1}{2}} b_{D', \sigma} \neq 0.\end{aligned}$$

Thus $I \not\subseteq \text{Ker } tr_n$. To show that φ is the corresponding Nakayama automorphism, it suffices to check that

- (1) $tr_n(c_D T_w T_i) = tr_n((T_{n-i} + c_{n-i} c_{n+1-i}) c_D T_w),$
- (2) $tr_n(c_D T_w c_j) = (-1)^{|D|} tr_n(-c_{n+1-j} c_D T_w)$

for all $i \in \{1, \dots, n-1\}, j \in \{1, \dots, n\}, D \subseteq [n]$ and $w \in \mathfrak{S}_n$.

For (1) we break the proof into four cases as in [26, Lemma 4.2]. When $w = w_0 s_i$, $tr_n(c_D T_w T_i) = tr_n(c_D T_{w_0}) = \delta_{D, \emptyset}$. On the other hand,

$$\begin{aligned}tr_n((T_{n-i} + c_{n-i} c_{n+1-i}) c_D T_w) &= tr_n(c_{s_{n-i}(D)} T_{n-i} T_w) + tr_n(c_{n-i} c_{n+1-i} c_D T_w) \\ &= tr_n(c_{s_{n-i}(D)} T_{w_0}) = \delta_{D, \emptyset},\end{aligned}$$

where we use relation (3.8) for the first equality and the identity $w_0 s_i = s_{n-i} w_0$ for the second last one.

When $w = w_0$, $tr_n(c_D T_{w_0} T_i) = -tr_n(c_D T_{w_0}) = -\delta_{D, \emptyset}$. On the other hand, if $D \neq \emptyset, \{n-i, n+1-i\}$, then $tr_n((T_{n-i} + c_{n-i} c_{n+1-i}) c_D T_{w_0}) = 0$ by relation (3.8). Otherwise,

for $D = \emptyset$,

$$\mathrm{tr}_n((T_{n-i} + c_{n-i}c_{n+1-i})T_{w_0}) = -\mathrm{tr}_n(T_{w_0}) = -1.$$

For $D = \{n-i, n+1-i\}$,

$$\begin{aligned} & \mathrm{tr}_n((T_{n-i} + c_{n-i}c_{n+1-i})c_{n-i}c_{n+1-i}T_{w_0}) \\ &= \mathrm{tr}_n((-c_{n-i}c_{n+1-i}(T_{n-i} + 1) + 1)T_{w_0}) + \mathrm{tr}_n(-T_{w_0}) \\ &= \mathrm{tr}_n(-c_{n-i}c_{n+1-i}(T_{n-i} + 1)T_{w_0}) = 0. \end{aligned}$$

The remaining two cases when $\ell(w) \leq \ell(w_0) - 2$ or $\ell(w) = \ell(w_0) - 1$ but $w \neq w_0 s_i$ are similar to check. For (2),

$$\begin{aligned} \mathrm{tr}_n(c_D T_w c_j) &= \mathrm{tr}_n(c_D c_{w(j)} T_w) = -\delta_{D, \{w(j)\}} \delta_{w, w_0} \\ &= -\delta_{D, \{n+1-j\}} \delta_{w, w_0} = (-1)^{|D|} \mathrm{tr}_n(-c_{n+1-j} c_D T_w). \quad \square \end{aligned}$$

Proposition 3.3.7. *For any $\alpha \models n$,*

(1) *we have a supermodule isomorphism between $\tilde{S}_{\alpha^*}$ and ${}^\varphi(\tilde{S}_{\alpha})$ of degree $\bar{n}$, sending $c_D \eta_{\alpha^*}$ to $(-1)^{n|D|} \varphi(c_D) c_{[n]} \eta_{\alpha}$. In particular, ${}^\varphi(HClS_{P(\alpha)}) \cong HClS_{P(\alpha^*)}$.*

(2) *we have an even supermodule isomorphism between $\tilde{S}_{\alpha^*}$ and ${}^{\varphi'}(\tilde{S}_{\alpha})$, sending $c_D \eta_{\alpha^*}$ to $\varphi'(c_D) \eta_{\alpha}$. In particular, ${}^{\varphi'}(HClS_{P(\alpha)}) \cong HClS_{P(\alpha^*)}$.*

(3) *we have an even supermodule isomorphism between $\tilde{S}_{\alpha}$ and $\psi(\tilde{S}_{\alpha}^*)$, sending $c_D \eta_{\alpha}$ to $c_{D \cdot \psi} \zeta_{\alpha}$, where ζ_{α} is the dual of η_{α} with respect to the standard basis $\{c_D \eta_{\alpha}\}$ of $\tilde{S}_{\alpha}$. In particular, $\psi(HClS_P^*) \cong HClS_P$ for any peak set P in $[n]$.*

(4) *we have a supermodule isomorphism between $\tilde{S}_{\alpha}$ and $\psi'(\tilde{S}_{\alpha}^*)$ of degree $\bar{n}$, sending $c_D \eta_{\alpha}$ to $(-1)^{n|D|} c_{D \cdot \psi'} \xi_{\alpha}$, where ξ_{α} is the dual of $c_{[n]} \eta_{\alpha}$ with respect to the standard basis $\{c_D \eta_{\alpha}\}$ of $\tilde{S}_{\alpha}$. In particular, $\psi'(HClS_P^*) \cong HClS_P$ for any peak set P in $[n]$.*

Proof. (1) From relations (3.1) and (3.8) we have

$$T_{i \cdot \varphi} c_{[n]} \eta_{\alpha} = (T_{n-i} + c_{n-i} c_{n+1-i}) \cdot c_{[n]} \eta_{\alpha} = -c_{[n]} (T_{n-i} + 1) \cdot \eta_{\alpha} = \begin{cases} -c_{[n]} \eta_{\alpha}, & \text{if } i \in D(\alpha^*), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, there exists an $\mathcal{H}_n$ -module homomorphism from S_{α^*} to $\mathrm{Res}_{\mathcal{H}_n}^{\mathcal{H}_n^{c_n}} {}^\varphi(\tilde{S}_{\alpha})$, sending η_{α^*} to $c_{[n]} \eta_{\alpha}$. By the adjointness between induction and restriction functors, we obtain an $\mathcal{H}_n^{c_n}$ -supermodule homomorphism from $\tilde{S}_{\alpha^*}$ to ${}^\varphi(\tilde{S}_{\alpha})$ sending $c_D \eta_{\alpha^*}$ to $(-1)^{n|D|} \varphi(c_D) c_{[n]} \eta_{\alpha}$. It is obviously surjective thus an isomorphism by dimension argument.

Meanwhile, for any $i \in [2, n-1]$,

$$i \in P(\alpha) \Leftrightarrow i \in D(\alpha), i-1 \notin D(\alpha) \Leftrightarrow n-i \notin D(\alpha^*), n+1-i \in D(\alpha^*) \Leftrightarrow n+1-i \in P(\alpha^*),$$

which means that $|P(\alpha)| = |P(\alpha^*)|$, thus $HClS_{P(\alpha^*)} \cong {}^\varphi(HClS_{P(\alpha)})$ by (3.4).

(2) From relation (3.1) we have

$$T_{i \cdot \varphi'} \eta_\alpha = -(T_{n-i} + 1) \cdot \eta_\alpha = \begin{cases} -\eta_\alpha, & \text{if } i \in D(\alpha^*), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, there exists an $\mathcal{H}_n$ -module homomorphism from S_{α^*} to $\text{Res}_{\mathcal{H}_n}^{\mathcal{HC}_n \varphi'}(\tilde{S}_\alpha)$, sending η_{α^*} to η_α . The rest of the proof is nearly the same as (1).

(3) By relation (3.8) we have

$$\begin{aligned} T_{i \cdot \psi} \zeta_\alpha (c_D \eta_\alpha) &= \zeta_\alpha ((T_i + c_i c_{i+1}) \cdot c_D \eta_\alpha) = \delta_{D, \emptyset} \zeta_\alpha (T_i \cdot \eta_\alpha) \\ &= \begin{cases} -1, & \text{if } D = \emptyset, i \in D(\alpha), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

That is, $T_{i \cdot \psi} \zeta_\alpha = -\zeta_\alpha$ if $i \in D(\alpha)$ and 0 otherwise. Hence, there exists an $\mathcal{H}_n$ -module homomorphism from S_α to $\text{Res}_{\mathcal{H}_n}^{\mathcal{HC}_n \psi}(\tilde{S}_\alpha^*)$, sending η_α to ζ_α . Again by the adjointness between induction and induction, we obtain an $\mathcal{HC}_n$ -supermodule homomorphism from $\tilde{S}_\alpha$ to $\psi(\tilde{S}_\alpha^*)$, sending $c_D \eta_\alpha$ to $c_{D \cdot \psi} \zeta_\alpha$. It is also obviously surjective thus an isomorphism by dimension argument.

(4) By relation (3.8) we have

$$\begin{aligned} T_{i \cdot \psi'} \xi_\alpha (c_D \eta_\alpha) &= -\xi_\alpha ((T_i + 1) \cdot c_D \eta_\alpha) = -\delta_{D, [n]} \xi_\alpha ((T_i + 1) \cdot c_{[n]} \eta_\alpha) \\ &= -\delta_{D, [n]} \xi_\alpha ((-c_{[n]} T_i + c_{[n] \setminus \{i, i+1\}}) \cdot \eta_\alpha) = \delta_{D, [n]} \xi_\alpha (c_{[n]} T_i \cdot \eta_\alpha) \\ &= \begin{cases} -1, & \text{if } D = [n], i \in D(\alpha), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

That is, $T_{i \cdot \psi'} \xi_\alpha = -\xi_\alpha$ if $i \in D(\alpha)$ and 0 otherwise. Hence, there exists an $\mathcal{H}_n$ -module homomorphism from S_α to $\text{Res}_{\mathcal{H}_n}^{\mathcal{HC}_n \psi'}(\tilde{S}_\alpha^*)$, sending η_α to ξ_α . The rest of the proof is nearly the same as (3). $\square$

For any $m, n \in \mathbb{N}$, denote $\mathcal{H}_{m, n} := \mathcal{H}_m \otimes \mathcal{H}_n$, $\mathcal{HC}_{m, n} := \mathcal{HC}_m \otimes \mathcal{HC}_n$. Let $\iota_n : \mathcal{H}_n \rightarrow \mathcal{HC}_n$, $\mu_{m, n} : \mathcal{H}_{m, n} \rightarrow \mathcal{H}_{m+n}$, $\tilde{\mu}_{m, n} : \mathcal{HC}_{m, n} \rightarrow \mathcal{HC}_{m+n}$ be the natural embeddings.

Proposition 3.3.8. *For the tower of 0-Hecke-Clifford superalgebras, we have an isomorphism of functors*

$$\varphi \text{Res} \cong \tau_{12} \circ \text{Res}$$

on $\mathcal{HC}$ -mod (hence also on $\mathcal{HC}$ -pmod).

Proof. For any $m, n \in \mathbb{N}$, let $\text{Res}_{m, n}^{m+n} := \text{Res}_{\mathcal{HC}_{m, n}}^{\mathcal{HC}_{m+n}}$ and $\varphi_{m, n} := \varphi_m \otimes \varphi_n$. It needs to prove that there exists an isomorphism of functors from $\mathcal{HC}_{m+n}$ -mod to $\mathcal{HC}_{m, n}$ -mod:

$$\varphi_{m,n} \circ \text{Res}_{m,n}^{m+n} \circ \varphi_{m+n}^{-1} \cong \tau_{12} \circ \text{Res}_{n,m}^{m+n}. \quad (3.11)$$

By the definition of the Nakayama automorphisms φ_n ($n \in \mathbb{N}$) in (3.9), we have

$$\varphi_{m+n} \circ \tilde{\mu}_{m,n} = \tilde{\mu}_{n,m} \circ \varphi_{n,m} \circ f_{12},$$

where $f_{12} : \mathcal{HC}_m \otimes \mathcal{HC}_n \rightarrow \mathcal{HC}_n \otimes \mathcal{HC}_m$ is the flip isomorphism (2.18). Hence, the LHS of (3.11) is

$$\begin{aligned} \tilde{\mu}_{m,n} \circ \varphi_{m,n} (\mathcal{HC}_{m+n})^{\varphi_{m+n}} \otimes_{\mathcal{HC}_{m+n}} - &\cong \varphi_{m+n} \circ \tilde{\mu}_{n,m} \circ f_{12} (\mathcal{HC}_{m+n})^{\varphi_{m+n}} \otimes_{\mathcal{HC}_{m+n}} - \\ &\cong \tilde{\mu}_{n,m} \circ f_{12} \mathcal{HC}_{m+n} \otimes_{\mathcal{HC}_{m+n}} -, \end{aligned}$$

which is exactly the RHS of (3.11). $\square$

Proposition 3.3.9. *For the tower of 0-Hecke-Clifford superalgebras, we have an isomorphism of functors $\text{Res} \cong \tau_{12} \circ \text{Res}$ on $\mathcal{HC}$ -pmod.*

Proof. For any $m, n \in \mathbb{N}$, let $\iota_{m,n} := \iota_m \otimes \iota_n$. First note that the following isomorphism of functors from $\mathcal{H}_{m+n}$ -mod to $\mathcal{HC}_{m,n}$ -mod holds:

$$\tilde{\mu}_{m,n} (\mathcal{HC}_{m+n})^{\iota_{m+n}} \otimes_{\mathcal{H}_{m+n}} - \cong (\mathcal{HC}_{m,n})^{\iota_{m,n}} \otimes_{\mathcal{H}_{m,n}} (\mu_{m,n} (\mathcal{H}_{m+n})^{\iota_{m+n}} \otimes_{\mathcal{H}_{m+n}} -).$$

In particular, for $\alpha \models m+n$, we choose $P_\alpha \in \mathcal{H}_{m+n}$ -pmod to get that

$$\begin{aligned} \text{Res}_{\mathcal{HC}_{m,n}}^{\mathcal{HC}_{m+n}} \tilde{P}_\alpha &\cong \tilde{\mu}_{m,n} (\mathcal{HC}_{m+n})^{\iota_{m+n}} \otimes_{\mathcal{H}_{m+n}} P_\alpha \\ &\cong (\mathcal{HC}_{m,n})^{\iota_{m,n}} \otimes_{\mathcal{H}_{m,n}} \text{Res}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} P_\alpha. \end{aligned} \quad (3.12)$$

Now by Lemma 2.4.2, we have

$$\begin{aligned} \tau_{12} \text{Res}_{\mathcal{HC}_{m,n}}^{\mathcal{HC}_{m+n}} \tilde{P}_\alpha &\cong \tau_{12} (\mathcal{HC}_{m,n})^{\iota_{m,n}} \otimes_{\mathcal{H}_{m,n}} \text{Res}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} P_\alpha \\ &\cong (\mathcal{HC}_{n,m})^{\tau_{12} \circ \iota_{m,n}} \otimes_{\mathcal{H}_{m,n}} \text{Res}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} P_\alpha \\ &\cong (\mathcal{HC}_{n,m})^{\iota_{n,m}} \otimes_{\mathcal{H}_{n,m}} \tau_{12} \text{Res}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} P_\alpha. \end{aligned}$$

As NSym is cocommutative, so is $\mathcal{K}(\mathcal{H})$ by the Frobenius map (3.3), i.e. $\text{Res} \cong \tau_{12} \circ \text{Res}$ on $\mathcal{H}$ -pmod, thus it also holds on $\mathcal{HC}$ -pmod by the above discussion. In fact, if we write $[\text{Res } P_\alpha] = \sum c_{\alpha_1, \alpha_2}^\alpha [P_{\alpha_1}] \otimes [P_{\alpha_2}]$ (The explicit formula is described by the shuffle product, see [21], [30, (16)]), then by (3.12) we simply have $[\text{Res } \tilde{P}_\alpha] = \sum c_{\alpha_1, \alpha_2}^\alpha [\tilde{P}_{\alpha_1}] \otimes [\tilde{P}_{\alpha_2}]$. $\square$

By Proposition 2.4.5, 3.3.6, 3.3.8, 3.3.9, we finally conclude that $(\tilde{\mathcal{K}}, \tilde{\mathcal{G}})$ forms a dual pair of graded Hopf algebras with respect to the pairing (2.17). That is, for $P, Q \in \mathcal{HC}$ -pmod, $M, N \in \mathcal{HC}$ -mod,

$$\begin{aligned}\langle \text{Ind}([P] \otimes [Q]), [M] \rangle &= \langle [P] \otimes [Q], \text{Res}[M] \rangle, \\ \langle [P], \text{Ind}([M] \otimes [N]) \rangle &= \langle \text{Res}[P], [M] \otimes [N] \rangle.\end{aligned}$$

It is easy to see that there exists an algebra involution $\bar{\varphi}$ of $\mathcal{H}_n$ defined by

$$\bar{\varphi}(T_i) = T_{n-i}, \quad i = 1, \dots, n-1.$$

$\bar{\varphi}$ is also the Nakayama automorphism of $\mathcal{H}_n$ [26, Lemma 4.2]. Now we have the following result

Proposition 3.3.10. *For any $\alpha \models n$,*

$$\bar{\varphi} S_\alpha \cong S_{\bar{\alpha}}, \quad \bar{\varphi} P_\alpha \cong P_{\bar{\alpha}}.$$

Proof. For the twisted module $\bar{\varphi} S_\alpha$, we have

$$T_{i \cdot \bar{\varphi}} \eta_\alpha = T_{n-i} \cdot \eta_\alpha = \begin{cases} -\eta_\alpha, & i \in D(\bar{\alpha}), \\ 0, & \text{otherwise,} \end{cases}$$

which implies the first isomorphism.

For the second one, we note that for any $w = w_1 \cdots w_n \in \mathfrak{S}_n$,

$$w \in \mathfrak{D}_\alpha \Leftrightarrow w_i > w_{i+1}, \quad i \in D(\alpha) \Leftrightarrow w_0 w w_0(i) > w_0 w w_0(i+1), \quad i \in D(\bar{\alpha}) \Leftrightarrow w_0 w w_0 \in \mathfrak{D}_{\bar{\alpha}}.$$

That is, $w_0 \mathfrak{D}_\alpha w_0 = \mathfrak{D}_{\bar{\alpha}}$ and also $i \in D(w) \Leftrightarrow n-i \in D(w_0 w w_0)$. Hence, by the module structure (3.2) of P_α , we know that for any $w \in \mathfrak{D}_\alpha$,

$$\begin{aligned}T_{i \cdot \bar{\varphi}} u_w = T_{n-i} \cdot u_w &= \begin{cases} -u_w, & n-i \in D(w^{-1}), \\ u_{s_{n-i}w}, & n-i \notin D(w^{-1}), s_{n-i}w \in \mathfrak{D}_\alpha, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} -u_w, & i \in D((w_0 w w_0)^{-1}), \\ u_{s_{n-i}w}, & i \notin D((w_0 w w_0)^{-1}), w_0 s_{n-i} w w_0 = s_i w_0 w w_0 \in \mathfrak{D}_{\bar{\alpha}}, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Hence, $u_w \mapsto u'_{w_0 w w_0}$, $w \in \mathfrak{D}_\alpha$ gives the second isomorphism, where $\{u'_w : w \in \mathfrak{D}_{\bar{\alpha}}\}$ is the standard basis of $P_{\bar{\alpha}}$. $\square$

4. From 0-Hecke–Clifford algebras to the Peak algebra of symmetric groups

In this section we base on the known result in [3] to clarify more explicitly the relation between (Peak, Peak^{*}) and the supermodule categories of 0-Hecke–Clifford algebras, especially the dual Hopf pair $(\tilde{\mathcal{K}}, \tilde{\mathcal{G}})$ discussed in the previous section. Then we consider the corresponding Heisenberg double in order to prove the freeness of Peak^{*} over Ω .

4.1. From $\widetilde{\mathcal{K}}$ to Peak

First of all, we define the adjoint map of the Frobenius isomorphism $\widetilde{\text{Ch}}$ relating two non-degenerate pairings $[\cdot, \cdot]$ in (2.10) and $\langle \cdot, \cdot \rangle$ in (2.17). That is a Hopf isomorphism $\widetilde{\text{Ch}}^* : \text{Peak} \rightarrow \widetilde{\mathcal{K}}$ satisfying

$$[F, \widetilde{\text{Ch}}([M])] = \langle \widetilde{\text{Ch}}^*(F), [M] \rangle, \quad F \in \text{Peak}, [M] \in \widetilde{\mathcal{G}}.$$

The following result provides the explicit form of $\widetilde{\text{Ch}}^*$:

Theorem 4.1.1. *For any composition α , we have*

$$\widetilde{\text{Ch}}^*(\Theta(R_\alpha)) = [\tilde{P}_\alpha]. \quad (4.1)$$

Moreover, the following commutative diagram of Hopf algebras holds:

$$\begin{array}{ccccc} \mathcal{K} & \xrightarrow{\text{Ind}_{\mathcal{H}}^{\mathcal{H}^e}} & \widetilde{\mathcal{K}} & \xrightarrow{\tilde{\chi}} & \widetilde{\mathcal{G}} \\ \uparrow \text{Ch}^* & & \uparrow \widetilde{\text{Ch}}^* & & \downarrow \widetilde{\text{Ch}} \\ \text{NSym} & \xrightarrow{\Theta} & \text{Peak} & \xrightarrow{\pi} & \text{Peak}^* \end{array}$$

where $\tilde{\chi} : \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{G}}$ is the Cartan map of 0-Hecke-Clifford algebras. In particular, $\text{Im } \tilde{\chi} = \Omega$.

Proof. Given compositions α, β , we have

$$\begin{aligned} \langle \widetilde{\text{Ch}}^*(\Theta(R_\alpha)), [\tilde{S}_\beta] \rangle &= [\Theta(R_\alpha), \widetilde{\text{Ch}}([\tilde{S}_\beta])] = \langle R_\alpha, \text{Ch}([\text{Res}_{\mathcal{H}}^{\mathcal{H}^e} \tilde{S}_\beta]) \rangle \\ &= \langle \text{Ch}^*(R_\alpha), [\text{Res}_{\mathcal{H}}^{\mathcal{H}^e} \tilde{S}_\beta] \rangle = \langle [P_\alpha], [\text{Res}_{\mathcal{H}}^{\mathcal{H}^e} \tilde{S}_\beta] \rangle \\ &= \langle [\text{Ind}_{\mathcal{H}}^{\mathcal{H}^e} P_\alpha], [\tilde{S}_\beta] \rangle = \langle [\tilde{P}_\alpha], [\tilde{S}_\beta] \rangle, \end{aligned}$$

where the second equality is due to (2.11) and (3.7), and the second last one bases on the fact that the induction functor is left adjoint to the restriction one. Since the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate, we get the desired formula, equivalent to the left commutative square.

For the right commutative square, we only need to prove that

$$\widetilde{\text{Ch}} \circ \tilde{\chi}([\tilde{P}_\alpha]) = \pi \circ \Theta(R_\alpha)$$

by formula (4.1). The module structure (3.2) of P_α ($\alpha \models n$) and Lemma 3.2.1 imply that if one fixes a total order $\prec$ refining the Bruhat order on $\mathfrak{D}_\alpha$, then $\tilde{P}_\alpha$ has a super submodule filtration $\{\tilde{P}_\alpha^w\}_{w \in \mathfrak{D}_\alpha}$, where $\tilde{P}_\alpha^w := \{c_D u_z : D \subseteq [n], z \in \mathfrak{D}_\alpha, w \preceq z\}$. Also, the subquotient $\tilde{P}_\alpha^w / \sum_{w \prec z} \tilde{P}_\alpha^z \cong \tilde{S}_{c(w^{-1})}$ for any $w \in \mathfrak{D}_\alpha$. Hence,

$$\begin{aligned}
\widetilde{\text{Ch}} \circ \widetilde{\chi}([\tilde{P}_\alpha]) &= \sum_{w \in \mathfrak{D}_\alpha} \widetilde{\text{Ch}}(\tilde{S}_{c(w^{-1})}) = \sum_{w \in \mathfrak{D}_\alpha} K_{P(w^{-1})} \\
&= \sum_{\beta} |\{w \in \mathfrak{S}_n : w \in \mathfrak{D}_\alpha, w^{-1} \in \mathfrak{D}_\beta\}| K_{P(\beta)} = \sum_{\beta} \langle R_\beta, r_\alpha \rangle K_{P(\beta)} \\
&= \sum_P \langle \sum_{P(\beta)=P} R_\beta, \pi(R_\alpha) \rangle K_P = \sum_P \langle \Xi_P, r_\alpha \rangle K_P \stackrel{(2.11)}{=} \sum_P [\Xi_P, \vartheta(r_\alpha)] K_P \\
&= \vartheta(r_\alpha) \stackrel{(2.9)}{=} \theta \circ \pi(R_\alpha) \stackrel{(2.9)}{=} \pi \circ \Theta(R_\alpha),
\end{aligned}$$

where the fourth equality is due to the following well-known formula of Gessel (see [19, Prop. 5.9]):

$$\langle R_\beta, r_\alpha \rangle = \langle R_\alpha, r_\beta \rangle = |\{w \in \mathfrak{S}_n : w \in \mathfrak{D}_\alpha, w^{-1} \in \mathfrak{D}_\beta\}|. \quad \square$$

Corollary 4.1.2. *For any composition α , we have the following decomposition formula:*

$$\tilde{P}_\alpha \cong \bigoplus_{P(\beta) \subseteq D(\alpha) \triangle (D(\alpha)+1)} HClP_{P(\beta)}^{\oplus 2^{l_\beta}}, \quad l_\beta := \left\lfloor \frac{|P(\beta)|+1}{2} \right\rfloor, \quad (4.2)$$

where we use $HClP_P$ to denote the projective cover of $HClS_P$ for any peak set P .

Proof. From [3, (6)] we know that

$$\Theta(R_\alpha) = \sum_{P \subseteq D(\alpha) \triangle (D(\alpha)+1)} 2^{|P|+1} \Xi_P. \quad (4.3)$$

Now by Theorem 4.1.1,

$$\begin{aligned}
\langle [\tilde{P}_\alpha], [\tilde{S}_\beta] \rangle &= [\Theta(R_\alpha), \widetilde{\text{Ch}}([\tilde{S}_\beta])] = [\Theta(R_\alpha), K_{P(\beta)}] \\
&= \begin{cases} 2^{|P(\beta)|+1}, & \text{if } P(\beta) \subseteq D(\alpha) \triangle (D(\alpha)+1), \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Combining it with (2.15), (3.4) and Proposition 3.1.2, we get the desired decomposition of $\tilde{P}_\alpha$. $\square$

In particular, the generator $Q_n = \Theta(R_{(n)})$ corresponds to the projective simple supermodule $[\tilde{P}_{(n)}] = [\tilde{S}_{(n)}] = [HClS_{\emptyset_n}] = [HClP_{\emptyset_n}]$ via $\widetilde{\text{Ch}}^*$ and

$$\langle [\tilde{P}_{(n)}], [\tilde{S}_\beta] \rangle = 2\delta_{P(\beta), \emptyset},$$

also due to Schur's lemma as the simple supermodule $HClS_{\emptyset_n}$ is of type Q. Note that in the classical case, $q_n \in \Omega$ corresponds to the *basic spin module* Cl_n of the Sergeev algebra $Cl_n \rtimes \mathbb{K}\mathfrak{S}_n$ under the Frobenius isomorphism [10, §3.3].

Now we abuse the notation to denote by $\bar{\varphi}$ the Hopf algebra anti-involution of NSym such that $\bar{\varphi}(H_n) = H_n$ ($n \in \mathbb{N}$). Note that $\bar{\varphi}(R_\alpha) = R_{\bar{\alpha}}$. Meanwhile, since $\bar{\varphi}(\text{Ker } \Theta) = \text{Ker } \Theta$, $\bar{\varphi}$ induces a Hopf algebra anti-involution of Peak , which we abuse to denote by φ , then $\varphi \circ \Theta = \Theta \circ \bar{\varphi}$.

We are in the position to prove the following restriction rule.

Theorem 4.1.3. *The following commutative diagram of Hopf algebras holds:*

$$\begin{array}{ccc} \tilde{\mathcal{K}} & \xrightarrow{\text{Res}_{\mathcal{H}}^{\mathcal{H}^c}} & \mathcal{K} \\ \uparrow \text{Ch}^* & & \uparrow \text{Ch}^* \\ \text{Peak} & \xrightarrow{\bar{\varphi} \circ i \circ \varphi} & \text{NSym} \end{array} \quad \text{or} \quad \begin{array}{ccc} \tilde{\mathcal{K}} & \xrightarrow{\bar{\varphi} \text{Res}_{\mathcal{H}}^{\mathcal{H}^c}} & \mathcal{K} \\ \uparrow \text{Ch}^* & & \uparrow \text{Ch}^* \\ \text{Peak} & \xrightarrow{i \circ \varphi} & \text{NSym} \end{array}$$

where $i : \text{Peak} \rightarrow \text{NSym}$ is the natural inclusion. Equivalently, for any composition α ,

$$[\text{Res}_{\mathcal{H}}^{\mathcal{H}^c} \tilde{P}_\alpha] = \sum_{P(\beta) \subseteq D(\bar{\alpha}) \Delta (D(\bar{\alpha})+1)} 2^{|P(\beta)|+1} [P_\beta]. \quad (4.4)$$

Proof. By Proposition 3.3.10, the above two diagrams are equivalent. Since $\text{Res}_{\mathcal{H}}^{\mathcal{H}^c} : \tilde{\mathcal{K}} \rightarrow \mathcal{K}$ is easily checked to be a Hopf algebra homomorphism, using formula (4.3) we only need to prove for the generators Q_n ($n \in \mathbb{N}$) that

$$[\text{Res}_{\mathcal{H}}^{\mathcal{H}^c} \tilde{P}_{(n)}] = \text{Ch}^*(\bar{\varphi} \circ i \circ \varphi(Q_n)) \stackrel{(2.6)}{=} 2 \sum_{P(\beta)=\emptyset_n} [P_\beta] = 2 \sum_{k=0}^{n-1} [P_{(n-k, 1^k)}], \quad n \in \mathbb{N}. \quad (4.5)$$

For $\tilde{P}_{(n)} = \tilde{S}_{(n)}$, $v_D := c_D \eta_{(n)}$ ($D \subseteq [n]$) form a basis. By (3.8)

$$T_i.v_D = \begin{cases} -v_D + v_{D \setminus \{i, i+1\}}, & \text{if } i, i+1 \in D, \\ -v_D + v_{(D \setminus \{i+1\}) \cup \{i\}}, & \text{if } i \notin D, i+1 \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Define a partial order $\triangleleft$ on $2^{[n]}$ such that D covers D' if $i, i+1 \notin D'$ and $D = D' \cup \{i, i+1\}$, or $i \in D', i+1 \notin D'$ and $D = (D' \setminus \{i\}) \cup \{i+1\}$ for some $i \in [n-1]$. We denote such covering relation by $D' \triangleleft_i D$. For any $0 \leq k \leq n-1$, take $D_{n,k} \subseteq [n]$ to be one of the two sets $\{n-k+1, \dots, n\} \setminus \{1\}$ and $\{1\} \cup \{n-k+1, \dots, n\}$ with odd cardinality. In particular, $D_{n,0} = \{1\}$. Define

$$v_{n,k} := v_{D_{n,k}} + \sum_D \epsilon_D v_D \in \tilde{P}_{(n)},$$

where the sum is over those $D \triangleleft D_{n,k}$ such that $D \triangleleft_{i_1} \dots \triangleleft_{i_r} D_{n,k}$ for some $i_1, \dots, i_r \in [n-k, n-1]$, and ϵ_D is the sign of length of any chain in $2^{[n]}$ from D to $D_{n,k}$. In

particular, $v_{n,0} = v_{\{1\}}$. Then

$$T_i.v_{n,k} = \begin{cases} 0, & \text{if } i \in \{1, \dots, n-k-2\}, \\ -v_{n,k}, & \text{if } i \in \{n-k, \dots, n-1\}. \end{cases}$$

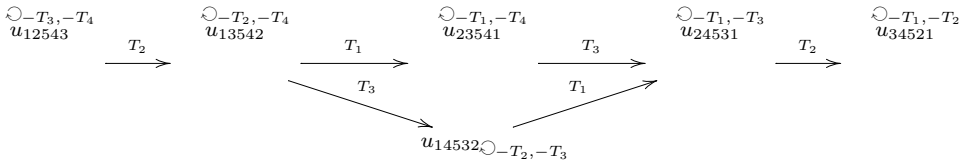
Meanwhile, $\mathfrak{D}_{(n-k,1^k)} = \{w_1 \cdots w_n \in \mathfrak{S}_n : w_1 < \cdots < w_{n-k} > w_{n-k+1} > \cdots > w_n\}$ for $0 \leq k \leq n-1$, and the projective $\mathcal{H}_n$ -module $P_{(n-k,1^k)}$ is generated by $u_{1 \cdots (n-k-1)n(n-1) \cdots (n-k)}$. Now one can check that the homogeneous component $(\tilde{P}_{(n)})_{\bar{1}} = \bigoplus_{k=0}^{n-1} \mathcal{H}_n.v_{n,k}$ and the isomorphism $\mathcal{H}_n.v_{n,k} \cong P_{(n-k,1^k)}$, $v_{n,k} \mapsto u_{1 \cdots (n-k-1)n(n-1) \cdots (n-k)}$ holds. Define $D'_{n,k}$ analogous to $D_{n,k}$ but with even cardinality and then $v'_{n,k}$ analogous to $v_{n,k}$. Then $(\tilde{P}_{(n)})_{\bar{0}} = \bigoplus_{k=0}^{n-1} \mathcal{H}_n.v'_{n,k}$ and $\mathcal{H}_n.v'_{n,k} \cong P_{(n-k,1^k)}$ as in the odd case. This proves formula (4.5) and thus the commutative diagrams.

In general, by formula (4.3)

$$\Theta(R_\alpha) = \sum_{P(\beta) \subseteq D(\alpha) \triangle (D(\alpha)+1)} 2^{|P(\beta)|+1} R_\beta.$$

Then by (4.1) and the commutative diagram, we get the desired restriction rule (4.4). $\square$

Example 4.1.4. For $n=5, k=2$, the diagram of $P_{(3,1^2)}$ is as follows.



Now $D_{5,2} = \{1, 4, 5\}$ and $v_{5,2} = v_{\{1,4,5\}} - v_{\{1,3,5\}} + v_{\{1,3,4\}} - v_{\{1\}}$. It is straightforward to check the isomorphism $\mathcal{H}_5.v_{5,2} \cong P_{(3,1^2)}$, $v_{5,2} \mapsto u_{12543}$.

Next we give another kind of restriction rule for the induced projective modules. Composing the natural inclusion from $\mathcal{H}_{n-1}$ (resp. $\mathcal{HC}_{n-1}$) to $\mathcal{H}_{n-1,1}$ (resp. $\mathcal{HC}_{n-1,1}$) with $\mu_{n-1,1}$ (resp. $\tilde{\mu}_{n-1,1}$), we have the embedding $\mu_n : \mathcal{H}_{n-1} \rightarrow \mathcal{H}_n$ (resp. $\tilde{\mu}_n : \mathcal{HC}_{n-1} \rightarrow \mathcal{HC}_n$).

Theorem 4.1.5. For any $\alpha = (\alpha_1, \dots, \alpha_r) \models n$, we have

$$\tilde{\mu}_n \tilde{P}_\alpha \cong \left(\bigoplus_{\substack{1 \leq i \leq r \\ \alpha_i > 1}} \tilde{P}_{\alpha_{(i)}} \right)^{\oplus 2} \oplus \left(\bigoplus_{\substack{1 \leq i \leq r-1 \\ \alpha_i > 1}} \tilde{P}_{\alpha'_{(i)}} \right)^{\oplus 2} \oplus \tilde{P}_{(\alpha_2, \dots, \alpha_r)}^{\oplus 2\delta_{\alpha_1,1}}, \quad (4.6)$$

where $\alpha_{(i)} := (\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_r)$, $\alpha'_{(i)} = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + \alpha_{i+1} - 1, \alpha_{i+2}, \dots, \alpha_r)$.

Proof. First note that we have the following isomorphism of functors on $\mathcal{H}_n\text{-mod}$.

$$\begin{aligned} \tilde{\mu}_n(\mathcal{HC}_n)^{\iota_n} \otimes_{\mathcal{H}_n} - &\cong ((\mathcal{HC}_{n-1})^{\iota_{n-1}} \otimes_{\mathcal{H}_{n-1}} \mu_n(\mathcal{H}_n) \otimes_{\mathcal{H}_n} -)^{\oplus 2} \\ c_D T_w \otimes - &\mapsto (c_D \otimes T_w \otimes -, 0), \quad c_D c_n T_w \otimes - \mapsto (0, c_D \otimes T_w \otimes -) \end{aligned}$$

for any $D \subseteq [n-1]$, $w \in \mathfrak{S}_n$. In particular,

$$\tilde{\mu}_n \tilde{P}_\alpha = \tilde{\mu}_n(\mathcal{HC}_n)^{\iota_n} \otimes_{\mathcal{H}_n} P_\alpha \cong ((\mathcal{HC}_{n-1})^{\iota_{n-1}} \otimes_{\mathcal{H}_{n-1}} \mu_n P_\alpha)^{\oplus 2},$$

which reduces formula (4.6) to

$$\mu_n P_\alpha \cong \left(\bigoplus_{\substack{1 \leq i \leq r \\ \alpha_i > 1}} P_{\alpha_{(i)}} \right) \oplus \left(\bigoplus_{\substack{1 \leq i \leq r-1 \\ \alpha_i > 1}} P_{\alpha'_{(i)}} \right) \oplus P_{(\alpha_2, \dots, \alpha_r)}^{\oplus \delta_{\alpha_1, 1}}. \quad (4.7)$$

For example, $\mu_5 P_{(1, 2, 2)} \cong P_{(2, 2)} \oplus P_{(1^2, 2)} \oplus P_{(1, 3)} \oplus P_{(1, 2, 1)}$.

Recall that P_α has basis $\{u_w : w \in \mathfrak{D}_\alpha\}$ with module action (3.2). Let $k_i := \alpha_1 + \dots + \alpha_i$, $i = 1, \dots, r$. It is easy to see that the j 's such that $w_j = n$ for some $w = w_1 \dots w_n \in \mathfrak{D}_\alpha$ are those satisfying $j = k_i$ with $i = 1$ or $1 < i \leq r, \alpha_i > 1$. Now for any such $j = k_i$, denote $\mathfrak{D}_\alpha^{(i)} := \{w \in \mathfrak{D}_\alpha : w_j = n\}$ and $P_\alpha^{(i)} := \text{span}_{\mathbb{K}}\{u_w : w \in \mathfrak{D}_\alpha^{(i)}\}$. We need to deal with three cases.

If $1 < j = k_i < n, \alpha_i > 1$, then $\{u_w : w \in \mathfrak{D}_\alpha^{(i)}, w_{j-1} > w_{j+1}\}$ spans the projective $\mathcal{H}_{n-1}$ -module $P_{\alpha_{(i)}}$. Modulo these vectors, the quotient space of $P_\alpha^{(i)}$ spanned by $\{u_w : w \in \mathfrak{D}_\alpha^{(i)}, w_{j-1} < w_{j+1}\}$ is isomorphic to another projective $\mathcal{H}_{n-1}$ -module $P_{\alpha'_{(i)}}$. If $j = k_1 = \alpha_1 = 1$, then $P_\alpha^{(1)}$ spans $P_{(\alpha_2, \dots, \alpha_r)}$. If $j = k_r = n, \alpha_r > 1$, then $P_\alpha^{(r)} \cong P_{\alpha_{(r)}} := P_{(\alpha_1, \dots, \alpha_{r-1}, \alpha_r - 1)}$. In summary, we get the desired formula. $\square$

Remark 4.1.6. By the adjointness between induction and restriction, we also have the following induction rule:

$$[\text{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} S_\alpha] = \sum_{\substack{\exists i \in [r] \\ \beta_{(i)} = \alpha}} [S_\beta] + \sum_{\substack{\exists i \in [r-1] \\ \beta'_{(i)} = \alpha}} [S_\beta] + [S_{(1, \alpha)}], \quad \alpha = (\alpha_1, \dots, \alpha_r) \models n-1,$$

from formula (4.7). It can be checked directly, though we do not find it in other references.

4.2. Application: Peak^* is free over Ω

Finally we consider the Heisenberg double arising from $(\tilde{\mathcal{K}}, \tilde{\mathcal{G}})$ in order to prove that Peak^* is a free Ω -module using the method of Savage and Yacobi in [26]. In general,

given a graded Hopf pair $(\mathcal{K}(A), \mathcal{G}(A))$, there exists a left action of $\mathcal{K}(A)$ on $\mathcal{G}(A)$ such that $\mathcal{G}(A)$ is a $\mathcal{K}(A)$ -module algebra. It is defined by

$$[P] \cdot [M] := \begin{cases} \left[\text{Hom}_{A_i} \left(P, \gamma_{n-i, i} \text{Res}_{A_{n-i} \otimes A_i}^{A_n} M \right) \right], & P \in A_i\text{-pmod}, M \in A_n\text{-mod}, i \leq n, \\ 0, & \text{otherwise,} \end{cases} \quad (4.8)$$

where $\gamma_{n-i, i} : A_i \rightarrow A_{n-i} \otimes A_i$, $a \mapsto 1_{n-i} \otimes a$ is the natural embedding. Note that for the Cartan map χ , $\mathcal{G}_{\text{proj}}(A) := \text{Im } \chi$ is stable under such action, thus a submodule of $\mathcal{G}(A)$. Now one can define the following two kinds of *Heisenberg doubles*:

$$\mathfrak{h}(A) := \mathcal{G}(A) \# \mathcal{K}(A), \quad \mathfrak{h}_{\text{proj}}(A) := \mathcal{G}_{\text{proj}}(A) \# \mathcal{K}(A). \quad (4.9)$$

The notation $\#$ means the smash product construction on $H \otimes B$ from a Hopf algebra H and an H -module algebra B .

Let

$$H^- := \mathcal{K}(A), \quad H^+ := \mathcal{G}(A), \quad H_{\text{proj}}^+ := \mathcal{G}_{\text{proj}}(A), \quad \mathfrak{h} := \mathfrak{h}(A), \quad \mathfrak{h}_{\text{proj}} := \mathfrak{h}_{\text{proj}}(A).$$

Then H^+ becomes a left $\mathfrak{h}$ -module, called the *lowest weight Fock representation*, where H^+ acts by left multiplication and H^- acts by formula (4.8). For any $\mathfrak{h}$ -module V , $v \in V$ is called a *lowest weight vacuum vector* if $H^-v = 0$.

From now on, we focus on the case when the tower $A = \mathcal{HC}$.

Lemma 4.2.1. *Suppose V is an $\mathfrak{h}_{\text{proj}}$ -module generated by a finite set of lowest weight vacuum vectors. Then V is a direct sum of lowest weight Fock representations.*

Proof. By the Stone–von Neumann Theorem for Heisenberg doubles [26, Theorem 2.11] and Theorem 4.1.1, we have $\mathfrak{h}_{\text{proj}} \cdot v \cong H_{\text{proj}}^+ \cong \Omega$ as an irreducible $\mathfrak{h}_{\text{proj}}$ -module over $\mathbb{K}$ for any lowest weight vacuum vector $v \in V$. Now the same argument in [26, Lemma 9.1] gives the complete reducibility of V . $\square$

For any $\beta = (\beta_1, \dots, \beta_r)$, $\gamma = (\gamma_1, \dots, \gamma_s)$, let $\beta \cdot \gamma := (\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s)$, then $\Delta(M_\alpha) = \sum_{\beta \cdot \gamma = \alpha} M_\beta \otimes M_\gamma$. Since ϑ is a Hopf algebra epimorphism, $N_\alpha := \vartheta(M_\alpha)(\alpha \in \mathbb{C})$ span the Stembridge algebra Peak^* and

$$\Delta(N_\alpha) = \sum_{\beta \cdot \gamma = \alpha} N_\beta \otimes N_\gamma.$$

Define an increasing filtration of $\mathfrak{h}_{\text{proj}}$ -submodules of Peak^* as follows. For $n \in \mathbb{N}_0$, let

$$(\text{Peak}^*)^{(n)} := \sum_{\ell(\alpha) \leq n} \mathfrak{h}_{\text{proj}} \cdot N_\alpha.$$

In particular, $(\text{Peak}^*)^{(0)} := \Omega$ and by convention we also let $(\text{Peak}^*)^{(-1)} := 0$.

Proposition 4.2.2. *The space Peak^* of peak quasisymmetric functions is free as an Ω -module.*

Proof. For any composition α such that $\ell(\alpha) = n$, we have

$$Q_m \cdot N_\alpha = \sum_{\beta \cdot \gamma = \alpha} [Q_m, N_\gamma] N_\beta \in (\text{Peak}^*)^{(n-1)}, \quad m \in \mathbb{N},$$

by the grading argument of $[\cdot, \cdot]$. Hence, in the quotient $V_n := (\text{Peak}^*)^{(n)} / (\text{Peak}^*)^{(n-1)}$, such N_α 's are lowest weight vacuum vectors. Clearly these vectors generate V_n and thus by Lemma 4.2.1,

$$V_n = \bigoplus_{v \in L_n} \Omega \cdot v,$$

where L_n is some collection of vacuum vectors in V_n .

Consider the short exact sequence of Ω -modules

$$0 \rightarrow (\text{Peak}^*)^{(n-1)} \rightarrow (\text{Peak}^*)^{(n)} \rightarrow V_n \rightarrow 0.$$

Since V_n is a free Ω -module, the above sequence splits. Now $(\text{Peak}^*)^{(0)} \cong \Omega$, so we know that all $(\text{Peak}^*)^{(n)}$ ($n \in \mathbb{N}_0$) are free over Ω by induction on n . It means that we can choose nested sets of vectors in Peak^*

$$\tilde{L}_0 \subseteq \tilde{L}_1 \subseteq \tilde{L}_2 \subseteq \cdots$$

such that, for any $n \in \mathbb{N}_0$, $(\text{Peak}^*)^{(n)} = \bigoplus_{\tilde{v} \in \tilde{L}_n} \Omega \cdot \tilde{v}$. Let $\tilde{L} = \bigcup_{n \in \mathbb{N}_0} \tilde{L}_n$. Then $\text{Peak}^* = \bigoplus_{\tilde{v} \in \tilde{L}} \Omega \cdot \tilde{v}$. $\square$

Remark 4.2.3. For the original proof of the freeness of QSym over the ring Λ of symmetric functions in [14], [15, §6.7], it even constructs a free commutative polynomial basis of QSym containing the elementary symmetric functions. However, we can not find a definite answer about the freeness of Peak^* over Ω in other references. And the existence problem for an analogous polynomial basis of Peak^* may be more interesting.

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