Nearly commuting matrices

Zhibek Kadyrsizova

Department of Mathematics, School of Science and Technology,
Nazarbayev University, 53 Kabanbay Batyr Ave, Astana, 010000, Kazakhstan

ARTICLE INFO

ABSTRACT

We prove that the algebraic set of pairs of matrices with a diagonal commutator over a field of positive prime characteristic, its irreducible components, and their intersection are F-pure when the size of matrices is equal to 3. Furthermore, we show that this algebraic set is reduced and the intersection of its irreducible components is irreducible in any characteristic for pairs of matrices of any size. In addition, we discuss various conjectures on the singularities of these algebraic sets and the system of parameters on the corresponding coordinate rings.

© 2017 Elsevier Inc. All rights reserved.

1. Introduction and preliminaries

In this paper we study algebraic sets of pairs of matrices such that their commutator is either nonzero diagonal or zero. We also consider some other related algebraic sets. First let us define relevant notions.

Let $X = (x_{ij})_{1 \leq i,j \leq n}$ and $Y = (y_{ij})_{1 \leq i,j \leq n}$ be $n \times n$ matrices of indeterminates over a field K. Let $R = K[X, Y]$ be the polynomial ring in $\{x_{ij}, y_{ij}\}_{1 \leq i,j \leq n}$ and let I denote the ideal generated by the off-diagonal entries of the commutator matrix $XY - YX$ and J denote the ideal generated by the entries of $XY - YX$. The ideal I defines the algebraic

E-mail address: zhibek.kadyrsizova@nu.edu.kz.

set of pairs of matrices with a diagonal commutator and is called *the algebraic set of nearly commuting matrices*. The ideal J defines the algebraic set of pairs of commuting matrices.

Let u_{ij} denote the (i,j)th entry of the matrix $XY - YX$. Then $I = (u_{ij} \mid 1 \leq i \neq j \leq n)$ and $J = (u_{ij} \mid 1 \leq i, j \leq n)$.

Theorem 1 *([3]). The algebraic set of commuting matrices is irreducible, i.e., it is a variety. Equivalently,* $\mathrm{Rad}(J)$ *is prime.*

The following results are due to A. Knutson [8], when the characteristic of the field is 0, and to H. Young [13] in all characteristics.

Theorem 2 *([8,13]). The algebraic set of nearly commuting matrices is a complete intersection, with the variety of commuting matrices as one of its irreducible components. In particular, the set* $\{u_{ij} \mid 1 \leq i \neq j \leq n\}$ *is a regular sequence and the dimension of R/I is $n^2 + n$.*

Theorem 3 *([8,13]). When K has characteristic zero, I is a radical ideal.*

A. Knutson in his paper [8] conjectured that $\mathbb{V}(I)$ has only two irreducible components and it was proved in all characteristics by H. Young in his thesis, [13].

Theorem 4 *([13]). If $n \geq 2$, the algebraic set of nearly commuting matrices has two irreducible components, one of which is the variety of commuting matrices and the other is the so-called skew component. That is, I has two minimal primes, one of which is* $\mathrm{Rad}(J)$.

Let $P = \mathrm{Rad}(J)$ and let Q denote the other minimal prime of I, i.e., $\mathrm{Rad}(I) = P \bigcap Q$. The following conjecture was made in 1982 by M. Artin and M. Hochster.

Conjecture 1. *J is reduced, i.e., $J = P$.*

It was answered positively by Mary Thompson in her thesis in the case of 3×3 matrices.

Theorem 5 *([12]). R/J is a Cohen–Macaulay domain when $n = 3$.*

Now let us go back to algebraic sets of nearly commuting matrices and their irreducible components. First, we take a look at what we have when $n = 1, 2$.

When $n = 1$, everything is trivial. More precisely, $I = P = Q = (0) \subset K[x_{11}, y_{11}]$.

When $n = 2$, without loss of generality we may replace X and Y by $X - x_{22}\mathrm{I}_n$ and $Y - y_{22}\mathrm{I}_n$ respectively. Here I_n is the identity matrix of size n. Denote $x'_{11} = x_{11} - x_{22}$, $y'_{11} = y_{11} - y_{22}$. Then the generators of I are 2 by 2 minors

$$u_{12} = \begin{vmatrix} x'_{11} & x_{12} \\ y'_{11} & y_{12} \end{vmatrix}, \quad u_{21} = -\begin{vmatrix} x'_{11} & x_{21} \\ y'_{11} & y_{21} \end{vmatrix}.$$

The diagonal entries of $XY - YX$ are

$$u_{11} = \begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix}, \quad u_{22} = -\begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix}.$$

Then J is the ideal generated by size 2 minors of $\begin{bmatrix} x'_{11} & x_{12} & x_{21} \\ y'_{11} & y_{12} & y_{21} \end{bmatrix}$ and therefore, $J = P$ is prime. We also have that $Q = (x'_{11}, y'_{11})$. Moreover, $I = P \bigcap Q$ is radical and $P + Q = (x_{12}y_{21} - x_{21}y_{12}, x'_{11}, y'_{11})$ is prime.

We have that

$$(u_{12}u_{21})^{p-1} = (x'_{11}y_{12} - x_{12}y'_{11})^{p-1}(x'_{11}y_{21} - x_{21}y'_{11})^{p-1} =$$

$$\sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{p-1}(-1)^{\alpha+\beta}\binom{p-1}{\alpha}\binom{p-1}{\beta}(x'_{11})^{\alpha+\beta}(y'_{11})^{2(p-1)-\alpha-\beta}x_{12}^{p-1-\alpha}y_{12}^{\alpha}x_{21}^{p-1-\beta}y_{21}^{\beta}.$$

Therefore, $(u_{12}u_{21})^{p-1}$ has a monomial term $(x'_{11}y'_{11}x_{12}y_{21})^{p-1}$ with coefficient $(-1)^{p-1}$. Since $I^{[p]} : I = (u_{12}u_{21})^{p-1} + I^{[p]}$, R/I is F-pure, see Fedder's criterion Lemma 2. Furthermore, determinantal rings R/P, R/Q, $R/(P+Q)$ are F-regular, see [6].

Therefore, for the rest of the paper we shall use the following notations.

Notation 1. Let $n \geq 3$ be an integer. Let $X = (x_{ij})_{1\leq i,j\leq n}$ and $Y = (y_{ij})_{1\leq i,j\leq n}$ be $n \times n$ matrices of indeterminates over a field K. Let $R = K[X, Y]$ be the polynomial ring in $\{x_{ij}, y_{ij}\}_{1\leq i,j\leq n}$ and let I denote the ideal generated by the off-diagonal entries of the commutator matrix $XY - YX$ and J denote the ideal generated by the entries of $XY - YX$. Let P denote the radical of J and Q be the other minimal prime of $\mathrm{Rad}(I)$.

We prove the following results in this paper.

Theorem 6. Let R be a ring as in Notation 1. Assume also that the field K has positive prime characteristic. Then R/I, R/P, R/Q and $R/(P+Q)$ are F-pure rings when $n = 3$. In other words, the algebraic set of nearly commuting matrices of size 3, its irreducible components and their intersection are F-pure. In particular, the skew component is reduced in this case.

Theorem 7. Let R be a ring as in Notation 1. Then R/I is reduced. In other words, the algebraic set of nearly commuting matrices is reduced for matrices of all sizes and in all characteristics.

Theorem 8. The intersection of the variety of commuting matrices and the skew component is irreducible, that is, $\mathrm{Rad}(P+Q)$ is prime.

2. *F*-purity

In this section we show that the coordinate ring of the algebraic set of pairs of matrices with a diagonal commutator is F-pure in the case of 3 by 3 matrices. Moreover, we also show that it implies the corresponding fact for its irreducible components, the variety of commuting matrices and the skew-component, and their intersection.

First we state two lemmas due to R. Fedder and they include a criterion for F-purity for finitely generated K-algebras and which has a particularly convenient form for complete intersections.

Lemma 1 *(Fedder [2]). Let S be a regular local ring or a polynomial ring over a field. If S has characteristic $p > 0$ and I is an unmixed proper ideal (homogeneous in the polynomial case) with the primary decomposition $I = \bigcap_{i=1}^{n} \mathfrak{A}_i$, then $I^{[p]} : I = \bigcap_{i=1}^{n} (\mathfrak{A}^{[p]} : \mathfrak{A})$.*

Lemma 2 *(Fedder's criterion [2]). Let $(S, \mathfrak{m})$ be a regular local ring or a polynomial ring over a field with its (homogeneous) maximal ideal. If S has characteristic $p > 0$ and I is a proper ideal (homogeneous in the polynomial case), then S/I is F-pure if and only if $I^{[p]} : I \not\subset \mathfrak{m}^{[p]}$.*

The next result is a straightforward consequence of the above two lemmas. It will prove to be quite useful for us.

Lemma 3. *Let S be a regular local ring or a polynomial ring over a field. Suppose that S has characteristic $p > 0$ and I is an ideal of S (homogeneous in the polynomial case). Suppose also that S/I is F-pure and $I = \bigcap_{i=1}^{n} \mathfrak{A}_i$ is the primary decomposition. Then $S/(\mathfrak{A}_{i_1} + \ldots + \mathfrak{A}_{i_m})$ is F-pure for all $1 \le i_1 < \ldots < i_m \le n$ and for all $1 \le m \le n$.*

Proof. Observe first that $(\mathfrak{A}_{i_1} + \ldots + \mathfrak{A}_{i_m})^{[p]} :_S (\mathfrak{A}_{i_1} + \ldots + \mathfrak{A}_{i_m}) \supseteq \bigcap_{j=1}^{m} (\mathfrak{A}_{i_j}^{[p]} : \mathfrak{A}_{i_j}) \supseteq \bigcap_{i=1}^{n} (\mathfrak{A}_i^{[p]} : \mathfrak{A}_i) = (I^{[p]} : I)$. The rest is immediate from Lemma 1 and Lemma 2.   □

The above lemma is closely related to results on compatibly split ideals, cf. [11]. Immediately we get the corresponding result for our algebraic set.

Corollary 1. *Suppose that the coordinate ring of the algebraic set of nearly commuting matrices R/I is F-pure. Then R/P, R/Q and $R/(P+Q)$ are F-pure.*

Next, we use Fedder's criterion to show F-purity of R/I in case when $n = 3$.

Theorem 9. *Let K be a field of characteristic $p > 0$ and let $n = 3$. Let R be a ring as in Notation 1. Then R/I is F-pure.*

Proof. Recall that I is generated by a regular sequence $\{u_{ij} | 1 \le i \ne j \le n\}$. Therefore, $I^{[p]} : I = \left(\prod_{1 \le i \ne j \le n} u_{ij}^{p-1} \right) R + I^{[p]}$. Thus by Fedder's criterion it is sufficient to prove

that $\prod_{1 \le i \ne j \le n} u_{ij}^{p-1} \notin \mathfrak{m}^{[p]}$, where $\mathfrak{m}$ is the homogeneous maximal ideal in R. We show this by proving the following claim.

Claim. *If* $\mu = x_{12}x_{13}x_{21}x_{23}x_{31}x_{33}y_{11}y_{12}y_{23}y_{31}y_{32}y_{33}$, *then* μ^{p-1} *is a monomial term of* $\prod_{1 \le i \ne j \le 3} u_{ij}^{p-1}$ *with a nonzero coefficient modulo p.*

Proof. We compute the coefficient of μ^{p-1}. It can be obtained by choosing a monomial from every u_{ij} in the following way:

$$
\begin{aligned}
u_{12}: \quad & (-x_{12}y_{11})^{\alpha_1}(x_{13}y_{32})^{\beta_1} \\
u_{13}: \quad & (-x_{23}y_{12})^{\alpha_2}(x_{12}y_{23})^{\beta_2}(-x_{13}y_{11})^{\gamma_2}(x_{13}y_{33})^{\delta_2} \\
u_{21}: \quad & (-x_{31}y_{23})^{\alpha_3}(x_{21}y_{11})^{\beta_3}(x_{23}y_{31})^{\gamma_3} \\
u_{23}: \quad & (x_{23}y_{33})^{\alpha_4}(-x_{33}y_{23})^{\beta_4} \\
u_{31}: \quad & (-x_{21}y_{32})^{\alpha_5}(x_{33}y_{31})^{\beta_5}(-x_{31}y_{33})^{\gamma_5}(x_{31}y_{11})^{\delta_5} \\
u_{32}: \quad & (x_{31}y_{12})^{\alpha_6}(-x_{12}y_{31})^{\beta_6}(x_{33}y_{32})^{\gamma_6}
\end{aligned}
$$

Then the exponents A_{st} and B_{st} of each x_{st} and y_{st} respectively are

$$
\begin{aligned}
A_{12} &= \alpha_1 + \beta_2 + \beta_6 & A_{23} &= \alpha_2 + \gamma_3 + \alpha_4 \\
A_{13} &= \beta_1 + \gamma_2 + \delta_2 & A_{31} &= \alpha_3 + \gamma_5 + \delta_5 + \alpha_6 \\
A_{21} &= \beta_3 + \alpha_5 & A_{33} &= \beta_4 + \beta_5 + \gamma_6 \\
B_{11} &= \alpha_1 + \gamma_2 + \beta_3 + \delta_5 & B_{31} &= \gamma_3 + \beta_5 + \beta_6 \\
B_{12} &= \alpha_2 + \alpha_6 & B_{32} &= \beta_1 + \alpha_5 + \gamma_6 \\
B_{23} &= \beta_2 + \alpha_3 + \beta_4 & B_{33} &= \delta_2 + \alpha_4 + \gamma_5
\end{aligned}
$$

In addition, denote

$$
\begin{aligned}
C_{12} &= \alpha_1 + \beta_1, & C_{23} &= \alpha_4 + \beta_4, \\
C_{13} &= \alpha_2 + \beta_2 + \gamma_2 + \delta_2, & C_{31} &= \alpha_5 + \beta_5 + \gamma_5 + \delta_5, \\
C_{21} &= \alpha_3 + \beta_3 + \gamma_3, & C_{32} &= \alpha_6 + \beta_6 + \gamma_6.
\end{aligned}
$$

Our goal is to find all nonnegative integer tuples $\alpha = (\alpha_1, \ldots, \alpha_6)$, $\beta = (\beta_1, \ldots, \beta_6)$, $\gamma = (\gamma_2, \gamma_3, \gamma_5, \gamma_6)$, $\delta = (\delta_5, \delta_6)$ such that $A_{st} = p - 1$, $B_{st} = p - 1$ for all $1 \le s, t \le 3$ and $C_{ij} = p - 1$ for all $1 \le i \ne j \le 3$.

Notice that the linear system does not have a nonzero determinant: the sum of the first 12 equations is twice the sum of the rest 6 equations. Therefore, there is not a unique solution.

The above linear system can be solved using standard methods from linear algebra and has the following solution:

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a & b & d & p-1-b-a+d & a & p-1-b \\ p-1-a & p-1-a-b & p-1-a & a+b-d & p-1-a-b+d & b \\ - & a-c & a-d & - & b+a-d-c & 0 \\ - & c & - & - & c-a & - \end{bmatrix}$$

where the column vector $[\alpha, \beta, \gamma, \delta]$ represents the matrix of solutions and a, b, c, d are arbitrary non-negative integers.

Since we look for non-negative integer solutions we must have that $a = c$ and $a, b \geq d$ and $a + b \leq p - 1$. Hence we have that

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a & b & d & p-1-b-a+d & a & p-1-b \\ p-1-a & p-1-a-b & p-1-a & a+b-d & p-1-a-b+d & b \\ - & 0 & a-d & - & b-d & 0 \\ - & a & - & - & 0 & - \end{bmatrix}$$

Therefore, the coefficient of μ^{p-1} is the sum of expressions of the form

$$(-1)^{\alpha_1+\alpha_2+\gamma_2+\alpha_3+\beta_4+\alpha_5+\gamma_5+\beta_6}((p-1)!)^6/(\alpha_1!\ldots\alpha_6!\beta_1!\ldots\beta_6!\gamma_2!\gamma_3!\gamma_5!\gamma_6!\delta_5!\delta_6!)$$

where $\alpha = (\alpha_1, \ldots, \alpha_6)$, $\beta = (\beta_1, \ldots, \beta_6)$, $\gamma = (\gamma_2, \gamma_3, \gamma_5, \gamma_6)$, $\delta = (\delta_5, \delta_6)$ run over all solutions of the linear system above. That is,

$$\sum_{d=0}^{(p-1)/2} \sum_{a,b\geq d,\, a+b\leq p-1} (-1)^{a-d} \binom{p-1}{a}^2 \binom{p-1}{b}^2 \binom{p-1}{a+b-d}^2 \binom{p-1-b}{a} \binom{a+b-d}{b} \binom{b}{d}$$

which modulo p is equivalent to

$$\sum_{d=0}^{(p-1)/2} \sum_{a,b\geq d,\, a+b\leq p-1} (-1)^{a-d} \binom{p-1-b}{a} \binom{a+b-d}{b} \binom{b}{d}$$

It also can be written as

$$\sum_{d=0}^{(p-1)/2} \sum_{a,b\geq d,\, a+b\leq p-1} (-1)^{a-d} \binom{p-1-b}{a} \binom{a+b-d}{a-d \quad b-d \quad d}$$

or

$$\sum_{b=0}^{p-1} \sum_{d=0}^{b} \sum_{a=d}^{p-1-b} (-1)^{a-d} \binom{p-1-b}{a} \binom{a+b-d}{b} \binom{b}{d}$$

The following lemma shows that the above expression is equal to 1 for all values of p. In fact, for this purpose p does not have to be prime. $\square$

Lemma 4. *Let $C_m = \sum_{b=0}^{m} \sum_{d=0}^{b} \sum_{a=d}^{m-b} (-1)^{a-d} \binom{m-b}{a} \binom{a+b-d}{b} \binom{b}{d}$. Then $C_m = 1$ for all $m \geq 1$.*

Proof. We prove a stronger statement.

Claim. *Let $B_{m,b} = \sum_{d=0}^{b} \sum_{a=d}^{m-b} (-1)^{a-d} \binom{m-b}{a} \binom{a+b-d}{b} \binom{b}{d}$. Then for all $m \geq 1$*

$$B_{m,b} = \begin{cases} 0 & \text{if } 0 \leq b \leq m-1; \\ 1 & \text{if } b = m. \end{cases}$$

Proof. First observe that $B_{m,m} = \sum_{d=0}^{m} \sum_{a=d}^{0} (-1)^{a-d} \binom{m-b}{a} \binom{a+b-d}{b} \binom{b}{d} = 1$ and $B_{m,0} = \sum_{a=0}^{m} (-1)^a \binom{m}{a} = 0$. Hence we may assume that $0 < b < m$.

Let $A_{m,b,d} = \sum_{a=d}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d}{b}$, then $B_{m,b} = \sum_{d=0}^{b} (-1)^d \binom{b}{d} A_{m,b,d}$. Consider the difference

$$A_{m,b,d} - A_{m,b,d+1} =$$

$$\sum_{a=d}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d}{b} - \sum_{a=d+1}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d-1}{b} =$$

$$(-1)^d \binom{m-b}{d} + \sum_{a=d+1}^{m-b} (-1)^a \binom{m-b}{a} \left(\binom{a+b-d}{b} - \binom{a+b-d-1}{b} \right) =$$

Using Pascal's identity, we get

$$(-1)^d \binom{m-b}{d} + \sum_{a=d+1}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d-1}{b-1} =$$

$$\sum_{a=d}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d-1}{b-1} =$$

$$\sum_{a=d}^{m-1-(b-1)} (-1)^a \binom{m-1-(b-1)}{a} \binom{a+(b-1)-d}{b-1}.$$

Thus we have that

$$A_{m,b,d} - A_{m,b,d+1} = A_{m-1,b-1,d} \text{ for all } m-1 \geq b \geq d+1 \text{ and } d \geq 0.$$

Therefore,

$$B_{m-1,b-1} = \sum_{d=0}^{b-1} (-1)^d \binom{b-1}{d} A_{m-1,b-1,d} = \sum_{d=0}^{b-1} (-1)^d \binom{b-1}{d} (A_{m,b,d} - A_{m,b,d+1}) =$$

$$\sum_{d=0}^{b-1}(-1)^d\binom{b-1}{d}A_{m,b,d} - \sum_{d=0}^{b-1}(-1)^d\binom{b-1}{d}A_{m,b,d+1} =$$

$$\sum_{d=0}^{b-1}(-1)^d\binom{b}{d}\frac{b-d}{b}A_{m,b,d} - \sum_{d=1}^{b}(-1)^{d-1}\binom{b-1}{d-1}A_{m,b,d} =$$

$$\sum_{d=0}^{b-1}(-1)^d\binom{b}{d}\frac{b-d}{b}A_{m,b,d} + \sum_{d=1}^{b}(-1)^d\binom{b}{d}\frac{d}{b}A_{m,b,d} =$$

$$\sum_{d=1}^{b-1}(-1)^d\binom{b}{d}A_{m,b,d} + A_{m,b,0} + (-1)^b A_{m,b,b} =$$

$$\sum_{d=0}^{b}(-1)^d\binom{b}{d}A_{m,b,d} = B_{m,b}.$$

Thus we have that $B_{m-1,b-1} = B_{m,b}$ for all $m \geq 1$ and $m - 1 \geq b \geq 1$.

In case $m = 1$, we only have $B_{1,0} = 0$. Finally, use induction on m to conclude that $B_{m,b} = 0$ for all $m \geq 1$ and $m - 1 \geq b$.

Thus, $C_m = \sum_{b=0}^{m} B_{m,b} = 1$. $\quad\square$

Finally, we complete the proof of Theorem 9. We have that $\prod_{1 \leq i \neq j \leq n} u_{ij}^{p-1} \notin \mathfrak{m}^{[p]}$ and R/I is F-pure when $n = 3$. $\quad\square$

Corollary 2. *Let R be a ring as in Notation 1. When $n = 3$, $R/P, R/Q$ and $R/(P + Q)$ are F-pure Cohen–Macaulay rings and $R/(P + Q)$ is Gorenstein.*

Proof. By [12], R/P is a Cohen–Macaulay ring when $n = 3$. Since the ideals P and Q are linked via I, that is $I : P = Q$ and $I : Q = P$, we have that R/Q is also Cohen–Macaulay, see [10]. Moreover, the theory of linkage also implies that $(P+Q)/P$ and $(P+Q)/Q$ are isomorphic to the canonical modules of R/P and R/Q, respectively. Hence $R/(P + Q)$ is Gorenstein of dimension $n^2 + n - 1$. $\quad\square$

Corollary 3. *Let R be a ring as in Notation 1. Then $P + Q$ is radical when $n = 3$.*

Remark. We prove in the next section that for all n, the radical of $P+Q$ is prime, which implies that $P + Q$ is prime when $n = 3$. In particular, we have that $R/(P + Q)$ is a domain when $n = 3$.

3. Irreducibility of $P + Q$

In this section we prove that the intersection of the variety of commuting matrices and the skew-component is irreducible. But first we define some notions.

Definition. Let X be an n by n matrix of indeterminates. Then $D(X)$ is an n by n matrix whose ith column is defined by the diagonal entries of X^{i-1} numbered from upper left corner to lower right corner. Let $\mathcal{P}(X)$ denote the determinant of $D(X)$.

Theorem 10 *([13]). $\mathcal{P}(X)$ is an irreducible polynomial.*

Remark. $\mathcal{P}(X) = \mathcal{P}(X - aI)$, where $a \in K$ and $I \in M_n(K)$ is the identity matrix.

The next two lemmas are due to H. Young. They give us the connection between the variety defined by $\mathcal{P}(X)$ and the algebraic set of nearly commuting matrices.

Lemma 5 *([13]). Given an $n \times n$ matrix A, if there exists a matrix B such that $[A, B]$ is a non-zero diagonal matrix, then $\mathcal{P}(A) = 0$.*

Lemma 6 *([13]). There is a dense open set $\mathcal{U}$ in the variety defined by $\mathcal{P}(X)$ where for every point A in $\mathcal{U}$, there exists a matrix B such that $[A, B]$ is a nonzero diagonal matrix.* $\square$

The following notion of a discriminant is of significant importance in matrix theory. We use it in this section in order to reduce our study to the case when commuting matrices have a particularly simple characterization.

Definition. Let $A \in M_n(K)$. Then the *discriminant* $\Delta(A)$ of A is the discriminant of its characteristic polynomial. That is, if K contains all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, then $\Delta(A) = \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2$.

Fact. Let $A \in M_n(K)$ be a matrix such that $\Delta(A) \ne 0$, or equivalently, A has distinct eigenvalues. Then a matrix B commutes with A if and only if B is a polynomial in A of degree at most $n - 1$, see Theorem 3.2.4.2 [7].

Remark. $\mathcal{P}(X)$ is an irreducible polynomial of degree $n(n-1)/2$ and $\Delta(X)$ is a polynomial of degree $n(n-1)$. Moreover, when $n \ge 3$, $\mathcal{P}(X)$ does not divide $\Delta(X)$. This can be proved by showing that there exists a matrix A with the property that $\mathcal{P}(A) = 0$ while $\Delta(A) \ne 0$. For example, for this purpose one can use the following matrices.

$$E_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \text{ if } p \nmid n, \text{ and } \widetilde{E}_n = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{n-1} \end{array} \right], \text{ otherwise.}$$

The characteristic polynomials are $x^n - 1$ for E_n and $x(x^{n-1} - 1)$ for $\widetilde{E}_n$.

Here is the outline for how we prove the main result of this section, that is, $\mathrm{Rad}(P+Q)$ is prime.

(1) $\dim \mathbb{V}(P+Q) = n^2 + n - 1$ and $\mathbb{V}(P+Q)$ is equidimensional.
(2) $\dim \mathbb{V}(P+Q, \Delta(X)) \leq n^2 + n - 2$, that is, $\Delta(X)$ is not in any of the minimal primes of $P+Q$.
(3) $\mathrm{Rad}(P+Q)R_{\Delta(X)}$ is a prime ideal.

We observe that $P+Q$ has no minimal primes of height larger than one over P and Q. First we need the following theorem due to R. Hartshorne.

Theorem 11 *([5, Proposition 2.1]). Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. If $\mathrm{Spec}(A) - \{\mathfrak{m}\}$ is disconnected, then the depth of A is at most 1.*

Lemma 7. *Let P and Q be ideals as in Notation 1. Then every minimal prime of $P+Q$ has height $n^2 - n + 1$.*

Proof. Suppose that there exists a minimal prime ideal T of $P+Q$ of height at least $\mathrm{ht}(I)+2$. Localize at T. Then $(P+Q)(R/I)_T$ is $T(R/I)_T$-primary. Moreover, $\mathbb{V}(P)$ and $\mathbb{V}(Q)$ are disjoint on the punctured spectrum $\mathrm{Spec}((R/I)_T) - \{T(R/I)_T\}$. However, the above theorem shows that this is not possible. □

Now let us define the set-up which we need to state and prove our next result.

Let m be a positive integer such that $m \leq n$. Fix a partition $(h_1, \ldots, h_m)$ of n, that is, choose positive integers $h_1, \ldots, h_m$ such that $h_1 + \ldots + h_m = n$. Let J_i be an upper triangular Jordan form of a nilpotent matrix of size h_i. For each h_i there are finitely many choices of J_i. Let $J = (J_1, \ldots, J_m)$ and let I_i denote the identity matrix of size h_i.

For any m-tuple $\underline{\lambda} = \lambda_1, \ldots, \lambda_m$ of distinct elements of K, let $J(\underline{\lambda}) = J(\lambda_1, \ldots, \lambda_m)$ be a matrix such that for all $1 \leq i \leq m$, the blocks $\lambda_i I_i + J_i$ are on the main diagonal. That is, $J(\underline{\lambda})$ is the direct sum of matrices $\lambda_i I_i + J_i$.

Let $\Lambda = \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{A}^m \,|\, \lambda_i \neq \lambda_j$ for all $1 \leq i \neq j \leq m\}$. It is an open subset of $\mathbb{A}^m$ and therefore is irreducible and has dimension m. Let

$$W_J = \{A \in M_n(K) \,|\ \text{A is similar to some } J(\lambda_1, \ldots, \lambda_m) \text{ with}$$

$$\lambda_1, \ldots, \lambda_m \in K \text{ distinct}\}.$$

Let c_J denote the dimension of the set of matrices that commute with $J(\underline{\lambda})$, for some $\underline{\lambda}$. This number is independent of the choice of $\underline{\lambda}$, since $J(\underline{\lambda})$ commutes with a matrix A if and only if A is a direct sum of matrices A_i such that each A_i has size h_i and A_i commutes with J_i. Moreover, c_J is the dimension of the set of invertible matrices that commute with $J(\underline{\lambda})$, for some $\underline{\lambda}$.

Lemma 8. *The dimension of W_J is $n^2 - c_J + m$.*

Proof. Define a surjective map of algebraic sets

$$\theta : GL_n(K) \times \Lambda \to W_J$$

such that

$$(U, \lambda_1, \ldots, \lambda_m) \to U^{-1} J(\lambda_1, \ldots, \lambda_m) U.$$

Fix $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$. Then

$$\theta^{-1}(J(\underline{\lambda})) = \{(U, \underline{\mu}) \in GL_n(K) \times \Lambda \,|\, U^{-1} J(\underline{\mu}) U = J(\underline{\lambda})\}$$

and it has the dimension of the set

$$\{U \in GL_n(K) \,|\, J(\underline{\lambda}) U = U J(\underline{\lambda})\},$$

that is, it is the set of all invertible matrices commuting with $J(\underline{\lambda})$.

Let $M = U^{-1} J(\underline{\lambda}) U \in W_J$ and let $(V, \underline{\mu}) \in \theta^{-1}(M)$. Then $U^{-1} J(\underline{\lambda}) U = V^{-1} J(\underline{\mu}) V$ for some $\underline{\mu}$ and $J(\underline{\mu}) = (UV^{-1})^{-1} J(\underline{\lambda})(UV^{-1})$. Hence, $(V, \underline{\mu}) \in \theta^{-1}(J(\underline{\lambda})) U$. Therefore, $\theta^{-1}(J(\underline{\lambda}))$ and $\theta^{-1}(M)$ have the same dimension. Since the dimension of W_J is the dimension of $GL_n(K) \times \Lambda$ minus the dimension of a generic fiber $\theta^{-1}(J(\underline{\lambda}))$, we have that the dimension of W_J is $n^2 - c_J + m$.

Moreover, the set of pairs of matrices $(A, B) \in M_n(K) \times M_n(K)$ such that A and B commute has dimension $(n^2 - c_J + m) + c_J = n^2 + m \leq n^2 + n$. $\quad\square$

Claim. *Let*

$$W = \{(A, B) \in M_n(K) \times M_n(K) |\, [A, B] = 0, \Delta(A) = 0, \mathcal{P}(A) = 0, \mathcal{P}(B) = 0\},$$

then there is an injective map

$$\Psi : \mathbb{V}(P + Q, \Delta(X)) \to W$$

so that

$$(A, B) \to (A, B).$$

Proof. Let $(A, B) \in \mathbb{V}(Q, \Delta(X)) - \mathbb{V}(P + Q, \Delta(X))$. Then by Lemma 5, $\mathcal{P}(A) = \mathcal{P}(B) = 0$. Therefore, $(A, B) \in W$. Since $\mathbb{V}(Q)$ is the closure of $\mathbb{V}(Q) - \mathbb{V}(P+Q)$ we have that $\mathcal{P}(A) = \mathcal{P}(B) = 0$ for all $(A, B) \in \mathbb{V}(Q, \Delta(X))$. Hence, $\mathbb{V}(P + Q, \Delta(X)) \subseteq W$. $\quad\square$

Claim. *The dimension of the set $W = \{(A, B) \in M_n(K) \times M_n(K) |\, [A, B] = 0, \ \Delta(A) = 0, \ \mathcal{P}(A) = 0, \ \mathcal{P}(B) = 0\}$ is at most $n^2 + n - 2$.*

Proof. Let $V = \{(A, B) \in M_n(K) \times M_n(K)|\, [A, B] = 0, \Delta(A) = 0\}$ and $V_m = \{(A, B) \in V|\, A$ has m distinct eigenvalues$\}$. Then we have that $\dim V_m = n^2 + m$ and $V = \bigcup_{m=1}^{n-1} V_m$. Therefore, $\dim V \leq n^2 + n - 1$. Notice that since $\Delta(A) = 0$, $m \leq n - 1$.

Similarly, let $W_m = \{(A, B) \in W|\, A$ has m distinct eigenvalues$\}$. Then $W = \bigcup_{m=1}^{n-1} W_m$. For each value of m, $W_m \subseteq V_m$. Therefore, the dimension of W is at most $n^2 + n - 1$. Moreover, W is a closed subset of V defined by the vanishing of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. To prove the claim we need to show that $\dim W$ cannot be $n^2 + n - 1$. We do this by showing that W does not contain any component of V of dimension $n^2 + n - 1$. In other words, we show that there are pairs of matrices $(A, B) \in V$ but not in W, i.e., either $\mathcal{P}(A) \neq 0$ or $\mathcal{P}(B) \neq 0$.

Let $A \in M_n(K)$ be a matrix with distinct eigenvalues $\lambda = \lambda_1 = \lambda_2, \lambda_3, \ldots, \lambda_n$. Then A is similar to a Jordan matrix in two possible forms.

Case 1. A is similar to $J = \begin{bmatrix} \lambda & 0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & 0 & \ldots & 0 \\ \ldots & & \ldots & & \ldots & \\ 0 & 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$

Take $B = \operatorname{diag}(a_1, \ldots, a_n)$ be a diagonal matrix with distinct entries on the diagonal. Then $[A, B] = 0$ and $\mathcal{P}(B) = \prod_{1 \leq i < j \leq n}(a_j - a_i) \neq 0$.

Case 2. A is similar to $J = \begin{bmatrix} \lambda & 1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & 0 & \ldots & 0 \\ \ldots & & \ldots & & \ldots & \\ 0 & 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$

Write $J = \begin{bmatrix} J_0 & 0 \\ \hline 0 & J_1 \end{bmatrix}$, where $J_0 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and $J_1 = \begin{bmatrix} \lambda_4 & 0 & \ldots & 0 \\ 0 & \lambda_5 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$.

Take an n by n block-diagonal matrix $U = \begin{bmatrix} U_0 & 0 \\ \hline 0 & U_1 \end{bmatrix}$ such that $U_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $U_1 \in M_{n-3}(K)$ is the identity matrix.

Then $U^{-1} = \begin{bmatrix} U_0^{-1} & 0 \\ \hline 0 & U_1^{-1} \end{bmatrix}$ and $U^{-1}JU = \begin{bmatrix} U_0^{-1}J_0U_0 & 0 \\ \hline 0 & J_1 \end{bmatrix}$.

Our goal is to show that $\mathcal{P}(U^{-1}JU) \neq 0$. First, we prove it for the case of 3 by 3 matrices, i.e., for $U_0^{-1}J_0U_0$.

Observe that $\mathcal{P}(U_0^{-1}J_0U_0) = \mathcal{P}(U_0^{-1}J_0U_0 - \lambda I) = \mathcal{P}(U_0^{-1}(J_0 - \lambda I)U_0)$.

Denote $M = U^{-1}(J - \lambda I)U$ and $M_0 = U_0^{-1}(J_0 - \lambda I)U_0$.

We have that

$$U_0^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$M_0 = U_0^{-1}(J_0 - \lambda I)U_0 = \begin{bmatrix} 1 & 1 - (\lambda_3 - \lambda) & -(\lambda_3 - \lambda) \\ -1 & -1 + (\lambda_3 - \lambda) & (\lambda_3 - \lambda) \\ 1 & 1 & 0 \end{bmatrix}.$$

Moreover,

$$M_0^2 = \begin{bmatrix} 0 & -(\lambda_3 - \lambda)^2 & -(\lambda_3 - \lambda)^2 \\ 0 & (\lambda_3 - \lambda)^2 & (\lambda_3 - \lambda)^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular, the diagonal $\mathrm{diag}(M_0^i) = (0, (\lambda_3 - \lambda)^i, 0)$ for all $i \geq 2$. Then

$$\mathcal{P}(M_0) = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 + (\lambda_3 - \lambda) & (\lambda_3 - \lambda)^2 \\ 1 & 0 & 0 \end{bmatrix} = (\lambda_3 - \lambda)^2 \neq 0.$$

Finally,

$$\mathcal{P}(M) = \det \begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -1 + \lambda_3 - \lambda & (\lambda_3 - \lambda)^2 & (\lambda_3 - \lambda)^3 & \ldots & (\lambda_3 - \lambda)^{n-1} \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & \lambda_4 - \lambda & (\lambda_4 - \lambda)^2 & (\lambda_4 - \lambda)^3 & \ldots & (\lambda_4 - \lambda)^{n-1} \\ \ldots & & \ldots & & & \\ 1 & \lambda_n - \lambda & (\lambda_n - \lambda)^2 & (\lambda_n - \lambda)^3 & \ldots & (\lambda_n - \lambda)^{n-1} \end{bmatrix} =$$

$$= \prod_{i=3}^{n} (\lambda_i - \lambda)^2 \prod_{3 \leq i < j \leq n} (\lambda_j - \lambda_i) \neq 0.$$

The final expression for the determinant is nonzero. Hence $\dim W \leq n^2 + n - 2$.

Thus we have that $\Delta(X)$ is not in any of the minimal primes of $P + Q$. $\quad \square$

Now we prove that $P + Q$ has only one minimal prime.

Theorem 12. *Let P and Q be as in [Notation 1](). Then $\mathbb{V}(P + Q)$ is irreducible, i.e., $\mathrm{Rad}(P + Q)$ is prime.*

Proof. Let $\mathcal{U}$ be a dense open subset in the algebraic set defined by $\mathcal{P}(X)$ as in Lemma 6. Let $A \in M_n(K)$ be such that $\mathcal{P}(A) = 0$. Suppose that $A \in \mathcal{U}$. Then by Lemma 6 there exists a matrix B such that (A, B) is in the skew-component of the algebraic set of nearly commuting matrices, that is $(A, B) \in \mathbb{V}(Q)$. Let $K[t]$ be a polynomial ring in one independent variable t. Fix any $f \in K[t]$. Then $(A, cB + f(A)) \in \mathbb{V}(Q)$ for all $c \in K - \{0\}$. Since Q defines a closed set, we must have that $(A, f(A)) \in \mathbb{V}(Q)$, i.e., when $c = 0$ as well. Since U is a dense subset in $\mathbb{V}(\mathcal{P}(X))$, $(A, f(A)) \in \mathbb{V}(Q)$ for all $A \in \mathbb{V}(\mathcal{P}(X))$. Recall that f was an arbitrary element of $K[t]$.

Now assume also that $\Delta(A) \neq 0$. Then every matrix B that commutes with A is a polynomial in A of degree at most $n - 1$. Thus

$$\mathbb{V}(P)_{\Delta(X)} = \{(A, f(A)) \mid \Delta(A) \neq 0 \text{ and } f \text{ is a polynomial of degree at most } n - 1\}.$$

Moreover, since $\mathbb{V}(P + Q) \subset \mathbb{V}(P)$, every element of $\mathbb{V}(P + Q)_{\Delta(X)}$ is of the form $(A, f(A))$, where $\mathcal{P}(A) = 0$ and f is a polynomial of degree at most $n - 1$.

Identify polynomials $f \in K[t]$ of degree at most $n - 1$ with $\mathbb{A}^n$. Then we can consider a map

$$\mathbb{V}(\mathcal{P}(X))_{\Delta(X)} \times \mathbb{A}^n \to \mathbb{V}(P + Q)_{\Delta(X)}$$

such that

$$(A, f) \to (A, f(A)).$$

Moreover, this map is a bijective morphism. Therefore, $\mathbb{V}(P + Q)_{\Delta(X)}$ is irreducible. If $\mathbb{V}(P + Q)$ is not irreducible, then its nontrivial irreducible decomposition will give us a nontrivial irreducible decomposition of $\mathbb{V}(P + Q)_{\Delta(X)}$, since $\Delta(X)$ is not in any of the minimal primes of $P + Q$. Thus the result. $\square$

Corollary 4. *Let P and Q be as in Notation 1. Then, when $n = 3$, $P + Q$ is prime.* $\square$

4. The ideal of nearly commuting matrices is a radical ideal

In this section we prove that I is a radical ideal in all characteristics. We know that $\mathrm{Rad}(I) = P \bigcap Q$ and I is unmixed as the heights of P and Q are equal to $n^2 - n$. To prove the result it is sufficient to show that I becomes prime or radical once we localize at P or Q.

Theorem 13. *The defining ideal of the algebraic set of nearly commuting matrices is radical.*

Proof. For simplicity of notation, let $\mathcal{P}$ denote $\mathcal{P}(X)$.

We have that $K[X] \bigcap P = (0)$, since every $f \in K[X] \bigcap P$ must vanish when we set $X = Y$. Therefore, $W = K[X] - \{0\}$ is disjoint from P and hence from I. Localize at P. Then we have an injective homomorphism of $K[X, Y]/I$-modules

$$(K[X, Y]/I)_P \hookrightarrow (K(X)[Y]/I)_P \cong (L[Y]/I)_P,$$

where $L = K(X)$ and now I is an ideal generated by $n^2 - n$ linear equations in the entries of Y with coefficients in L. We can always choose at least n variables y_{ij}, $(i, j) \in \Lambda$, and write the rest of them as L-linear combinations of the chosen ones. Thus $(K[X, Y]/I)_P \hookrightarrow L[y_{ij}]_{(i,j) \in \Lambda}$ and $IK[X, Y]_P$ is prime.

Next observe that $K[X] \bigcap Q = (\mathcal{P})$. Clearly, $(\mathcal{P}) \subseteq Q$. To prove the other direction, let $f \in K[X] \bigcap Q$ be nonzero. Then by Lemma 5, $f \in (\mathcal{P})$. In other words, for all $A \in M_n(K)$ such that $A \in \mathbb{V}(Q)$ and such that there exists a matrix B with the property that $[A, B]$ is nonzero diagonal, then $\mathcal{P}(A) = 0$.

Therefore, we have an injective homomorphism of $K[X, Y]/I$-modules

$$(K[X, Y]/I)_Q \hookrightarrow (V[Y]/I)_Q,$$

where $V = K[X]_{(\mathcal{P})}$ is a discrete valuation domain. Then generators of I become linear polynomials in the entries of Y with coefficients in V. Let $\mathcal{B}$ be the matrix of coefficients of this linear system such that its rows are indexed by (i, j) for $1 \le i \ne j \le n$ and columns are indexed by (h, k) for all $1 \le h, k \le n$. Then for $i \ne h$ and $k \ne j$ $\mathcal{B}$ has an entry x_{ih} in the $(i, k), (h, k)$ spot, has an entry $-x_{kj}$ in the $(i, j), (i, k)$ spot, and $x_{ii} - x_{jj}$ in the $(i, j)(i, j)$ spot and zero everywhere else. Let $y_1, \ldots, y_{n^2}$ denote the entries of Y such that $y_{(i-1)n+j} = y_{ij}$. In $V[Y]$, I is generated by the entries of the matrix

$$\mathcal{B} \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_{n^2} \end{bmatrix}.$$

By doing elementary row operations over V, we can transform $\mathcal{B}$ into a diagonal matrix $\mathcal{C}$. This gives new generators of I. To prove that $IV[Y]$ is radical, it is sufficient to show that the diagonal entries in $\mathcal{C}$ have order at most one in V. To this end it reduces to show that $\mathcal{C}$ has rank $n^2 - n$ and the ideal generated by the minors of $\mathcal{C}$ of size $n^2 - n$ cannot be contained in $\mathcal{P}^2 V$. But then it is sufficient to prove this for the original matrix $\mathcal{B}$. Hence it suffices to show:

Claim.

(1) *The submatrix $\mathcal{B}_0$ of $\mathcal{B}$ obtained from the first $n^2 - n$ columns has nonzero determinant in V.*

(2) *The determinant of $\mathcal{B}_0$ is in $(\mathcal{P}) - (\mathcal{P}^2)$.*

Proof. (1) It is sufficient to prove the first part of the claim over $K(X) = \text{frac}(V)$, i.e., after we invert $\mathcal{P}$. In this case, since X and Y nearly commute, they must commute, see Lemma 5. Moreover, X is a generic matrix, hence its discriminant is nonzero and is not divisible by $\mathcal{P}$. Thus X has distinct eigenvalues and Y is a polynomial in X of degree at most $n - 1$. Write $\mathcal{B} = [\mathcal{B}_0 | \mathcal{B}_1]$, then our equations become

$$
\mathcal{B}_0 \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{n^2-n} \end{bmatrix} + \mathcal{B}_1 \begin{bmatrix} y_{n^2-n+1} \\ y_{n^2-n+2} \\ \dots \\ y_{n^2} \end{bmatrix} = 0.
$$

Notice that $\mathcal{B}_0$ is invertible if and only if for every choice of the values for $[y_{n^2-n+1}, \dots, y_{n^2}]$ there is a unique solution for the above equation.

Furthermore, the bottom rows of $X^0, X, \dots, X^{n-1}$ are linearly independent for a generic matrix X. This is true because it even holds for the permutation matrix

$$
E = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & & & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}
$$

for which the bottom rows of $E^0, E, \dots, E^{n-1}$ are the standard basis vectors e_i for $1 \le i \le n$.

Thus, given any bottom row ρ of Y, there exist $\alpha_0, \dots, \alpha_{n-1} \in K(X)$ such that ρ equals the bottom row of $\alpha_0 + \alpha_1 X + \dots + \alpha_{n-1} X^{n-1}$. That is, such a Y is uniquely determined by the entries of its bottom row. Therefore, $\mathcal{B}_0$ is invertible in $K(X)$.

(2) First, let us show that $\det \mathcal{B}_0 \in (\mathcal{P})$. For any matrix A in an open dense subset of the closed set defined by $\mathcal{P}$, there exists a matrix A' such that the commutator $[A, A']$ is a nonzero diagonal matrix, see Lemma 6. Hence, for all $c \in K - \{0\}$ and for all $f \in K[X]$ polynomials of degree at most $n - 1$, $(A, cA' + f(A)) \in \mathbb{V}(I)$. Therefore, the space of solutions of

$$
\mathcal{B} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{n^2} \end{bmatrix} = 0
$$

has dimension $n + 1$, but we showed in (1) that it must be n. Therefore, the minors of $\mathcal{B}$ must vanish whenever $\mathcal{P}$ vanishes.

Notice that the degree of the polynomial $\mathcal{P}$ is $n(n-1)/2$, while the degree of $\det \mathcal{B}_0$ is $n(n-1)$. Therefore, to prove part (2) it is sufficient to show that $\det \mathcal{B}_0$ is not a K-scalar multiple of $\mathcal{P}^2$.

Now let us put grading on the entries of X and Y. Let $\deg x_{ij} = \deg y_{ij} = i - j$. Then their products XY and YX and sums have this property as well: $\deg (XY)_{ij} = i - j$ and $\deg (X + Y)_{ij} = i - j$. Therefore, so does the commutator matrix $XY - YX$. In fact, any polynomial in X and Y has this property. Notice that the diagonal entries have degree 0, thus $\mathcal{P}$ has degree 0. However, this is not the case for the determinant of the matrix $\mathcal{B}_0$. The nonzero entry corresponding to $(i, j), (k, h)$ has degree $i - j + h - k$. Therefore, if a product of the entries is a nonzero term of the determinant of $\mathcal{B}_0$, then its degree is $\sum_{1 \leq i \neq j \leq n} \sum_{1 \leq h \leq n} \sum_{1 \leq k < n} (i - j + h - k) = n^2(n-1)^2/2 \neq 0$ for all $n \geq 2$. Hence $\det \mathcal{B}_0$ cannot be a K-scalar multiple of $\mathcal{P}^2$. That is, when we factor out $\mathcal{P}$ from the minors of $\mathcal{B}$, the remaining expression is not divisible by $\mathcal{P}$. $\quad\square$

Now we are ready to finish our discussion. Let $\mathcal{C} = [\mathcal{C}_0 | \mathcal{C}_1]$ be a matrix that is obtained from B by elementary row transformations so that $\mathcal{C}_0 = \operatorname{diag}(c_{ii})_{1 \leq i \leq n^2 - n}$ is a diagonal matrix. We proved that there exists $1 \leq k \leq n^2 - n$ with the property that c_{ii} is a unit in V for all $i \neq k$ and $c_{kk} \in (\mathcal{P}) - (\mathcal{P}^2)$. Denote $c_{kk} = \alpha \mathcal{P}$, where α is a unit in V.

The ideal I is generated by the following equations

$$
\begin{bmatrix}
c_{11} & 0 & 0 & \ldots & & 0 \\
0 & c_{22} & 0 & \ldots & & 0 \\
& & \ldots & & \ldots & \\
0 & 0 & 0 & \ldots & c_{n^2-n-1,n^2-n-1} & 0 \\
0 & 0 & 0 & \ldots & 0 & c_{n^2-n,n^2-n}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\ldots \\
y_{n^2-n}
\end{bmatrix}
+ \mathcal{C}_1
\begin{bmatrix}
y_{n^2-n+1} \\
y_{n^2-n+2} \\
\ldots \\
y_{n^2}
\end{bmatrix}.
$$

Then $V[Y]/I \cong V[y_{n^2-n+1}, y_{n^2-n+2}, \ldots, y_{n^2}][y_{kk}]/(\alpha \mathcal{P} - \sum_{j=1}^{n} (\mathcal{C}_1)_{kj} y_{n^2-n+j})$ is reduced. To show this we consider two cases. If all $(\mathcal{C}_1)_{kj} \in \mathcal{P}$, then the last factor ring is isomorphic to $V[y_{n^2-n+1}, y_{n^2-n+2}, \ldots, y_{n^2}][z]/(z\mathcal{P})$. If there is j so that $(\mathcal{C}_1)_{kj}$ is a unit, then the factor ring is isomorphic to $V[y_i]_{n^2-n+1 \leq i \neq j \leq n^2}[y_{kk}]$. In either case, it is reduced. Therefore, since we have an injective map $(R/I)_Q \hookrightarrow (V[Y]/I)_Q$, IR_Q is radical. $\quad\square$

5. Conjectures

In this section we state conjectures that we have made while doing the research. Many of them appeared as a result of computations performed on a computer algebra program Macaulay2, [4].

Conjecture 2. *Let R be as in Notation 1. Then R/P, R/Q and $R/(P+Q)$ are F-regular.*

Remark. In the case when $n = 2$ the conjecture is true.

The following lemma allows us to reduce the above conjecture to the F-regularity of $R/(P + Q)$.

Lemma 9. *Let R be a Noetherian local or $\mathbb{N}$-graded ring of prime characteristic $p > 0$ and let I be an ideal (homogeneous in the graded case) generated by a regular sequence. Let P and Q be ideals of R of the same height such that P and Q are linked via $I = P \bigcap Q$. Let R/P be Cohen–Macaulay. Suppose that $R/(P + Q)$ is F-regular (or equivalently, F-rational). Then R/P and R/Q are F-regular.*

Proof. By [10], R/Q is Cohen–Macaulay and has the canonical module isomorphic to $(P + Q)/Q$. Similarly, the canonical module of R/P is $(P + Q)/P$. Then $R/(P + Q)$ is Gorenstein, hence it is F-rational if and only if it is F-regular.

Recall that a graded ring R is F-regular if and only if $R_{\mathfrak{m}}$ is F-regular, [9]. Then $R/(P + Q)$ is F-rational if and only if its localization at the homogeneous maximal ideal is F-rational. Then by applying Corollary 2.9 in [1] we have that F-rationality of $R/(P + Q)$ implies F-regularity of R/P and R/Q. □

Thus if we want to prove that the variety of commuting matrices and the skew component are F-regular, it is sufficient to prove the statement for their intersection. Of course we need to know whether R/P is Cohen–Macaulay.

Conjecture 3. *R/I is F-pure for all n.*

The above conjecture can be solved by proving the following one.

Conjecture 4. *Let $\mu = \dfrac{\prod_{i=1,j=1}^{n} x_{ij} y_{ij}}{\prod_{i=1}^{n-1} x_{ii} y_{i,n-i+1} \cdot x_{n,n-1} \cdot y_{n-1,1}}$.*

Then μ^{p-1} is a monomial term of $\prod_{1 \leq i \neq j \leq n} u_{ij}^{p}$ with coefficient equal to 1 modulo p.

Remark. The above monomial can be obtained taking the product of all the variables and dividing by the variables according to the following pattern: denote by $\star$ the variable to be divided out.

$$
X = \begin{vmatrix}
\star & x_{12} & \cdots & x_{1,n-2} & x_{1,n-1} & x_{1n} \\
x_{21} & \star & \cdots & x_{2,n-2} & x_{2,n-1} & x_{2n} \\
& \cdots & & & \cdots & \\
x_{n-2,1} & x_{n-2,2} & \cdots & \star & x_{n-2,n-1} & x_{n-1,n} \\
x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n-2} & \star & x_{n-1,n} \\
x_{n,1} & x_{n,2} & \cdots & x_{n,n-2} & \star & x_{n,n}
\end{vmatrix},
$$

$$
Y = \begin{vmatrix}
y_{11} & y_{12} & y_{1,3} & \cdots & y_{1,n-1} & \star \\
y_{21} & y_{22} & y_{2,3} & \cdots & \star & y_{2n} \\
& \cdots & & \cdots & & \\
y_{n-2,1} & y_{n-2,2} & \star & \cdots & y_{n-2,n-1} & y_{n-2,n} \\
\star & \star & y_{n-1,3} & \cdots & y_{n-1,n-1} & y_{n-1,n} \\
y_{n,1} & y_{n,2} & y_{n,3} & \cdots & y_{n,n-1} & y_{n,n}
\end{vmatrix}
$$

Conjecture 5. *Let X be a matrix of indeterminates of size n over a field K. Let $\mathcal{P}(X)$ be the irreducible polynomial as in Definition 3. Then $K[X]/\mathcal{P}(X)$ is F-regular.*

Conjecture 6. *The following is a regular sequence on R/I and hence a part of a system of parameters on R/J and R/Q.*

$$x_{st} - y_{t,\theta(s,t)}, \ x_{1n}, \ x_{nn}, \ x_{11} - y_{2n}$$

for all $1 \le s, t, \le n$ and where $\theta(s,t) = \begin{cases} (s+t) mod\ n, & \text{if } s+t \ne n; \\ n, & \text{if } s+t = n. \end{cases}$

Remark. The conjecture was verified by using Macaulay2 software when $n = 3, 4$ over $K = \mathbb{Q}$ and in some small prime characteristics.

In the case when $n = 3$, this is equivalent to the following identifications of variables in matrices X and Y

$$X = \begin{vmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & x_{22} \\ x_{31} & x_{32} & 0 \end{vmatrix}, \quad Y = \begin{vmatrix} x_{31} & x_{11} & x_{21} \\ x_{22} & x_{32} & x_{12} \\ 0 & x_{22} & 0 \end{vmatrix}.$$

Conjecture 7. *Let $Z \subseteq \{u_{ij} \,|\, 1 \le i \ne j \le n\}$ be any subset of cardinality at most $n^2 - n - 1$. Let $\mathcal{I}_Z$ be the ideal of R generated by the elements of Z. Then $R/\mathcal{I}_Z$ is F-regular. In particular, $\mathcal{I}_Z$ is a prime ideal.*

Acknowledgments

References

[1] F. Enescu, Applications of pseudocanonical covers to tight closure problems, J. Pure Appl. Algebra 178 (2) (2003) 159–167.
[2] R. Fedder, F-purity and rational singularity in graded complete intersection rings, Trans. Amer. Math. Soc. 301 (1) (1987) 47–62.
[3] M. Gerstenhaber, On dominance and varieties of commuting matrices, Ann. of Math. 73 (1961) 324–348.
[4] D. Grayson, M. Stillman, Macaulay 2: a computer algebra system for algebraic geometry and commutative algebra, available at http://www.math.uiuc.edu/Macaulay2.
[5] R. Hartshorne, Complete intersections and connectedness, Amer. J. Math. 84 (3) (1962) 497–508.

[6] M. Hochster, C. Huneke, Tight closure of parameter ideals and splitting in module-finite extensions, J. Algebraic Geom. 3 (4) (1994) 599–670.
[7] R. Horn, C. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, UK, 1985.
[8] A. Knutson, Some schemes related to the commuting variety, J. Algebraic Geom. 14 (2005) 283–294.
[9] G. Lyubeznik, K. Smith, Strong and weak F-regularity are equivalent for graded rings, Amer. J. Math. 121 (6) (1999) 1279–1290.
[10] C. Peskine, L. Szpiro, Liaison des variétés algébriques, I, Invent. Math. 26 (1974) 271–302.
[11] K. Schwede, K. Tucker, A survey of test ideals, in: Progress in Commutative Algebra 2, Closures, Finiteness and Factorization, Walter de Gruyter GmbH and Co. KG, Berlin, 2012, pp. 39–99.
[12] M. Thompson, Topics in the Ideal Theory of Commutative Noetherian Rings, Ph.D. thesis, University of Michigan, 1985.
[13] H. Young, Components of Algebraic Sets of Commuting and Nearly Commuting Matrices, Ph.D. thesis, University of Michigan, 2011.