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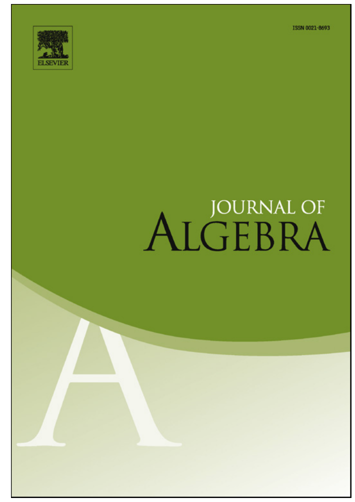

Ringel–Hall algebra construction of quantum Borcherds–Bozec algebras

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RINGEL-HALL ALGEBRA CONSTRUCTION OF QUANTUM BORCHERDS-BOZEC ALGEBRAS

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Abstract. We give the Ringel-Hall algebra construction of the positive half of quantum Borcherds-Bozec algebras as the generic composition algebras of quivers with loops.

Introduction

The *Hall algebra*, introduced by Steinitz [22] and rediscovered by Hall [8], is an associative algebra over $\mathbf{C}$ with a basis consisting of isomorphism classes of finite abelian p-groups. The finite abelian p-groups are parametrized by partitions and the structure coefficients of the Hall algebra are given by certain polynomials in p with integral coefficients, which are called the *Hall polynomials*. It turned out that there is a close connection between the Hall algebras and the theory of symmetric functions.

In [18], Ringel generalized the notion of Hall algebras to abelian categories with some finiteness conditions such as the category of representations of a quiver. The *Ringel-Hall algebra* is an associative algebra over $\mathbf{C}$ with a basis consisting of isomorphism classes of objects in a given abelian category, where the multiplication is defined in terms of the space of extensions. When we deal with the categories of representations of quivers without loops, the Ringel-Hall algebras provide a realization of the positive half of quantum groups associated with symmetric generalized Cartan matrices [18, 7]. The Ringel-Hall algebra construction of quantum groups is one of the main inspirations for the Kashiwara-Lusztig crystal/canonical basis theory [13, 14, 15].

Let us consider the quivers with loops. Then one can associate symmetric Borcherds-Cartan matrices, which yield *Borcherds algebras* or *generalized Kac-Moody algebras*. The Borcherds algebras were introduced by Borcherds in his study of the Monstrous Moonshine [1]. A special example of these algebras, the Monster Lie algebra, played an important role in the proof of the Moonshine Conjecture [2]. The quantum deformations of Borcherds algebras and their modules were constructed in [11]. In [12], the Ringel-Hall algebra

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construction and the Kashiwara-Lusztig crystal/canonical basis theory were generalized to the case of quantum Borcherds algebras (see also [10]).

The *Borcherds-Bozec algebras* are further generalizations of Borcherds algebras. They are also defined by the generators and relations coming from Borcherds-Cartan matrices, but they have far more generators than Borcherds algebras. That is, for each simple root, there are infinitely many generators whose degrees are positive integral multiples of the given simple root. Thus in addition to the Serre-type relations, we need to have the Drinfel'd-type relations. The quantum Boecherds-Bozec algebras arise as a natural algebraic structure behind the theory of perverse sheaves on the representation varieties of quivers with loops and a lot of interesting progresses are still under way ([3, 4, 5], etc.).

In this paper, we give the Ringel-Hall algebra construction of the positive half of quantum Borcherds-Bozec algebras as the generic composition algebras of quivers with loops. The main ingredients of our work are Green's Theorem on symmetric bilinear forms of Green-Lusztig algebras (Theorem 1.2) and the representations of quivers given in (3.11) that correspond to the higher degree generators of quantum Borcherds-Bozec algebras.

1. Green-Lusztig algebras

Let $\mathscr{A}$ be an integral domain containing $\mathbf{Z}$ and an invertible element v. Let X be a set of alphabets (possibly countably infinite) and let $\Lambda = \bigoplus_{x \in X} \mathbf{Z}\alpha_x$ be the free abelian group on X endowed with a symmetric bilinear form $(\ ,\ ) : \Lambda \times \Lambda \to \mathbf{Z}$. The quadruple $(X, (\ ,\ ), \mathscr{A}, v)$ is called a *Green-Lusztig datum.* We write $\Lambda^+ = \sum_{x \in X} \mathbf{Z}_{\geq 0}\alpha_x$.

Definition 1.1. Let $(X, (\ ,\ ), \mathscr{A}, v)$ be a Green-Lusztig datum. We say that an associative $\mathscr{A}$-algebra L is a *Green-Lusztig algebra belonging to the class* $\mathscr{L}(X, (\ ,\ ), \mathscr{A}, v)$ if the following conditions are satisfied.

(a) $L = \bigoplus_{\alpha \in \Lambda^+} L_\alpha$ is a Λ^+-graded algebra such that
  - (i) L is generated by the elements u_x $(x \in X)$,
  - (ii) $L_0 = \mathscr{A}\mathbf{1}$, where $\mathbf{1}$ is the identity element of L.

(b) There is an $\mathscr{A}$-bilinear map $\delta : L \to L \otimes_{\mathscr{A}} L$ such that
  - (i) $\delta(u_x) = u_x \otimes 1 + 1 \otimes u_x$ for all $x \in X$,
  - (ii) δ is an $\mathscr{A}$-algebra homomorphism, where the multiplication on $L \otimes_{\mathscr{A}} L$ is given by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) := v^{(\beta_2, \gamma_1)}(x_1 y_1 \otimes x_2 y_2) \quad \text{for } x_i \in L_{\beta_i},\ y_i \in L_{\gamma_i} \quad (i = 1, 2).$$

(c) There is a symmetric $\mathscr{A}$-bilinear form $(\ ,\ )_L : L \times L \to \mathscr{A}$ such that
  - (i) $(L_\alpha, L_\beta)_L = 0$ if $\alpha \neq \beta$,

(ii) $(\mathbf{1}, \mathbf{1})_L = 1$,
(iii) $(u_x, u_x)_L \neq 0$ for all $x \in X$,
(iv) $(a, bc)_L = (\delta(a), b \otimes c)_L$ for all $a, b, c \in L$, where

$$(x_1 \otimes x_2, y_1 \otimes y_2)_L := (x_1, y_1)_L \, (x_2, y_2)_L \quad \text{for } x_i, y_i \in L \ (i = 1, 2).$$

Let $\beta = \sum_{x \in X} d_x \alpha_x \in \Lambda^+$ with $ht(\beta) := \sum_{x \in X} d_x = r$. Set

$$X(\beta) := \{w = (x_1, \ldots, x_r) \mid \alpha_{x_1} + \cdots + \alpha_{x_r} = \beta\}.$$

If $L = \bigoplus_{\beta \in \Lambda^+} L_\beta$ is a Green-Lusztig algebra in $\mathscr{L}(X, (\ ,\ ), \mathscr{A}, v)$, then L_β is the $\mathscr{A}$-span of monomials of the form $u_w = u_{x_1} \cdots u_{x_r}$ such that $w = (x_1, \ldots, x_r) \in X(\beta)$. Note that if $w \in X(\beta)$, $w' \in X(\beta')$ with $\beta \neq \beta'$, by (c), we have $(u_w, u_{w'})_L = 0$.

Theorem 1.2. [7] Let $\beta = \sum_{x \in X} d_x \alpha_x \in \Lambda^+$ and $w, w' \in X(\beta)$. Then there exists a Laurent polynomial $P_{w,w'}(t) \in \mathbf{Z}[t, t^{-1}]$ such that for any Green-Lusztig datum $(X, (\ ,\ ), \mathscr{A}, v)$ and any Green-Lusztig algebra in $\mathscr{L}(X, (\ ,\ ), \mathscr{A}, v)$, we have

$$(u_w, u_{w'})_L = P_{w,w'}(v) \, B_\beta(L),$$

where $B_\beta(L) = \prod_{x \in X} (u_x, u_x)_L^{d_x}$.

Remark. The point is that $B_\beta(L)$ depends only on β and L.

Lemma 1.3. [7] Let L be a Green-Lusztig algebra in $\mathscr{L}(X, (\ ,\ ), \mathscr{A}, v)$ and let $u = \sum_{w \in X(\beta)} c_w u_w \in L$ ($c_w \in \mathscr{A}$). Then $u \in \mathrm{rad}(\ ,\ )_L$ if and only if

$$\sum_{w \in X(\beta)} c_w P_{w,w'}(v) = 0 \quad \text{for all } w' \in X(\beta), \ \beta \in \Lambda^+.$$

2. Quantum Borcherds-Bozec algebras

Let I be an index set (possibly countably infinite). A square matrix $A = (a_{ij})_{i,j \in I}$ is called an *even symmetrizable Borcherds-Cartan matrix* if

(i) $a_{ii} = 2, 0, -2, -4, \ldots$,
(ii) $a_{ij} \in \mathbf{Z}_{\leq 0}$ for $i \neq j$,
(iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$,
(iv) there is a diagonal matrix $D = \mathrm{diag}(s_i \in \mathbf{Z}_{>0} \mid i \in I)$ such that DA is symmetric.

Set $I^{\mathrm{re}} := \{i \in I \mid a_{ii} = 2\}$, the set of *real indices* and $I^{\mathrm{im}} := \{i \in I \mid a_{ii} \leq 0\}$, the set of *imaginary indices*. We denote by $I^{\mathrm{iso}} := \{i \in I \mid a_{ii} = 0\}$ the set of *isotropic indices*.

A *Borcherds-Cartan datum* consists of

(a) an even symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j\in I}$,
(b) a free abelian group P, the *weight lattice*,
(c) $P^\vee := Hom(P, \mathbf{Z})$, the *dual weight lattice*,
(d) $\Pi = \{\alpha_i \in P \mid i \in I\}$, the set of *simple roots*,
(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, the set of *simple coroots*

satisfying the following conditions

(i) $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$,
(ii) Π is linearly independent over $\mathbf{C}$,
(iii) for every $i \in I$, there is an element $\varpi_i \in P$ such that $\langle h_j, \varpi_i\rangle = \delta_{ij}$ for all $j \in I$.

We denote by $R := \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ the *root lattice* and set $R^+ := \sum_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i$.

Let $\mathfrak{h} := \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. Since A is symmetrizable and Π is linearly independent, there is a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ such that

$$(\alpha_i, \lambda) = s_i \langle h_i, \lambda\rangle \quad \text{for all } i \in I,\ \lambda \in \mathfrak{h}^*. \tag{2.1}$$

Let v be an indeterminate and set

$$v_i = v^{s_i}, \quad v_{(i)} = v^{(\alpha_i,\alpha_i)/2}, \quad [n]_i = \frac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}}.$$

Note that $v_i = v_{(i)}$ if $i \in I^{\mathrm{re}}$.

Let $I^\infty := (I^{\mathrm{re}} \times \{1\}) \cup (I^{\mathrm{im}} \times \mathbf{Z}_{>0})$. We will often identify $I^{\mathrm{re}} \times \{1\}$ with I^{re}. Let $\Lambda := \bigoplus_{(i,l)\in I^\infty} \mathbf{Z}\alpha_{il}$ be the free abelian group on I^∞. Then we have a symmetric bilinear form $(\ ,\ ) : \Lambda \times \Lambda \to \mathbf{Z}$ given by

$$(\alpha_{ik}, \alpha_{jl}) := kl(\alpha_i, \alpha_j) \quad \text{for all} \quad (i,k),(j,l) \in I^\infty. \tag{2.2}$$

Then $(I^\infty, (\ ,\ ), \mathbf{C}(v), v)$ is a Green-Lusztig datum.

Let $\mathscr{E}$ be the free associative algebra over $\mathbf{C}(v)$ generated by the symbols e_{il} for $(i,l) \in I^\infty$. Set $\deg e_{il} := l\alpha_i$ for $(i,l) \in I^\infty$. Then $\mathscr{E}$ becomes an R^+-graded algebra $\mathscr{E} = \bigoplus_{\beta\in R^+} \mathscr{E}_\beta$, where $\mathscr{E}_\beta$ is the $\mathbf{C}(v)$-span of monomials of the form $e_{i_1,l_1} \cdots e_{i_r,l_r}$ such that $l_1\alpha_{i_1} + \cdots + l_r\alpha_{i_r} = \beta$. We will denote by $|u|$ the degree of a homogeneous element u in $\mathscr{E}$.

Define a *twisted multiplication* on $\mathscr{E} \otimes \mathscr{E}$ by

$$(x_1 \otimes x_2)\,(y_1 \otimes y_2) = v^{(|x_2|,|y_1|)} x_1y_1 \otimes x_2y_2 \tag{2.3}$$

and a *co-multiplication* $\delta : \mathscr{E} \to \mathscr{E} \otimes \mathscr{E}$ by

$$\delta(e_{il}) = \sum_{m+n=l} v_{(i)}^{mn} e_{im} \otimes e_{in} \quad \text{for all} \quad (i,l) \in I^{\infty}. \tag{2.4}$$

Since $\mathscr{E}$ is the free associative algebra on $\{e_{il} \mid (i,l) \in I^{\infty}\}$, the map δ can be extended to a well-defined algebra homomorphism.

Proposition 2.1. [3, 4, 17, 19] For any family $\nu = (\nu_{il})_{(i,l)\in I^{\infty}}$ of non-zero elements in $\mathbf{C}(v)$, there exists a bilinear form $(\ ,\ )_L : \mathscr{E} \times \mathscr{E} \to \mathbf{C}(v)$ such that

(a) $(x, y)_L = 0$ if $|x| \neq |y|$,
(b) $(\mathbf{1}, \mathbf{1})_L = 1$,
(c) $(e_{il}, e_{il})_L = \nu_{il}$ for all $(i,l) \in I^{\infty}$,
(d) $(x, yz)_L = (\delta(x), y \otimes z)$ for all $x, y, z \in \mathscr{E}$.

We define $\widehat{U}$ to be the associative algebra over $\mathbf{C}(v)$ generated by the elements $K_i^{\pm 1}$ $(i \in I)$, e_{il}, f_{il} $((i,l) \in I^{\infty})$ with defining relations

$$\begin{aligned} & K_i K_i^{-1} = K_i^{-1} K_i = \mathbf{1}, \quad K_i K_j = K_j K_i \qquad (i,j \in I), \\ & K_i e_{jl} K_i^{-1} = v_i^{l a_{ij}} e_{jl}, \quad K_i f_{jl} K_i^{-1} = v_i^{-l a_{ij}} f_{jl} \qquad (i \in I, (j,l) \in I^{\infty}), \\ & \sum_{k=0}^{1-l a_{ij}} (-1)^k e_i^{(k)} e_{jl}\, e_i^{(1-l a_{ij}-k)} = 0 \quad \text{for} \quad i \in I^{\mathrm{re}},\ i \neq (j,l), \\ & \sum_{k=0}^{1-l a_{ij}} (-1)^k f_i^{(k)} f_{jl}\, f_i^{(1-l a_{ij}-k)} = 0 \quad \text{for} \quad i \in I^{\mathrm{re}},\ i \neq (j,l), \\ & [e_{ik}, e_{jl}] = 0 \ \text{ if } a_{ij} = 0. \end{aligned} \tag{2.5}$$

Here, we use the notation $e_i^{(k)} = e_i^k/[k]_i!$, $f_i^{(k)} = f_i^k/[k]_i!$ for $i \in I^{\mathrm{re}}$.

The algebra $\widehat{U}$ is endowed with the co-multiplication $\Delta : \widehat{U} \to \widehat{U} \otimes \widehat{U}$ given by

$$\begin{aligned} & \Delta(K_i) = K_i \otimes K_i, \\ & \Delta(e_{il}) = \sum_{m+n=l} v_{(i)}^{mn}\, e_{im} K_i^n \otimes e_{in}, \\ & \Delta(f_{il}) = \sum_{m+n=l} v_{(i)}^{-mn} f_{im} \otimes K_i^{-m} f_{in}. \end{aligned} \tag{2.6}$$

We will use Sweedler's notation to write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \quad \text{for} \quad x \in \widehat{U}.$$

Let $\widehat{U}^+$ be the subalgebra of $\widehat{U}$ generated by e_{il}'s ($(i,l) \in I^\infty$).

Proposition 2.2. [3, 17, 20]
(a) If $i \in I^{\mathrm{re}}$, $i \neq (j,l)$, then the elements

$$\sum_{k=0}^{1-la_{ij}} (-1)^k e_i^{(k)} e_{jl} e_i^{(1-la_{ij}-l)}$$

lie in the radical of $(\ ,\ )_L$.

(b) If $a_{ij} = 0$, then the elements $[e_{ik}, e_{jl}]$ $(k, l \geq 1)$ lie in the radical of $(\ ,\ )_L$.

Hence the bilinear form $(\ ,\ )_L$ is well-defined on $\widehat{U}^+$.

Let $\widehat{U}^{\geq 0}$ be the subalgebra of $\widehat{U}$ generated by $\widehat{U}^+$ and $K_i^{\pm 1}$ $(i \in I)$. We extend the bilinear form $(\ ,\ )_L$ to $\widehat{U}^{\geq 0}$ via

$$(xK_i, yK_j)_L = v_i^{a_{ij}} (x,y)_L = v_j^{a_{ji}} (x,y)_L \ \text{ for all } x, y \in \widehat{U}^+,\ i, j \in I. \tag{2.7}$$

Let $\omega : \widehat{U} \to \widehat{U}$ be the involution defined by

$$e_{il} \mapsto f_{il}, \quad f_{il} \mapsto e_{il}, \quad K_i \mapsto K_i^{-1}. \tag{2.8}$$

Then the subalgebra $\widehat{U}^-$ generated by f_{il}'s ($(i,l) \in I^\infty$) is endowed with a symmetric bilinear form $(\ ,\ )_L$ by setting

$$(x,y)_L = (\omega(x), \omega(y))_L \ \text{ for all } x, y \in \widehat{U}^-. \tag{2.9}$$

Following the Drinfel'd double process, we take the algebra $\widetilde{U}$ to be the quotient of $\widehat{U}$ by the relations

$$\sum (a_{(1)}, b_{(2)})_L\, \omega(b_{(1)})\, a_{(2)} = \sum (a_{(2)}, b_{(1)})_L\, a_{(1)}\, \omega(b_{(2)}) \ \text{ for all } \ a, b \in \widehat{U}^{\geq 0}. \tag{2.10}$$

Definition 2.3. The *quantum Borcherds-Bozec algebra* $U_v(\mathfrak{g})$ associated with the Borcherds-Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ is the quotient algebra of $\widetilde{U}$ by the radical of $(\ ,\ )_L$ restricted to $\widetilde{U}^- \times \widetilde{U}^+$.

Thus we have $U_v^{\pm}(\mathfrak{g}) = \widetilde{U}^{\pm}/\mathrm{rad}(\ ,\ )_L$, where $U_v^+(\mathfrak{g})$ (resp. $U_v^-(\mathfrak{g})$) is the subalgebra of $U_v(\mathfrak{g})$ generated by e_{il}'s (resp. f_{il}'s) for $(i,l) \in I^\infty$.

From now on, we assume that

$$(e_{il}, e_{il})_L \in 1 + v^{-1}\mathbf{Z}_{\geq 0}[[v^{-1}]] \ \text{ for all } i \in I^{\mathrm{im}} \setminus I^{\mathrm{iso}},\ l \geq 1. \tag{2.11}$$

Then $(\ ,\ )_L$ is non-degenerate on $\mathscr{E}(i) := \bigoplus_{l\geq 1} \mathscr{E}_{l\alpha_i}$.

Proposition 2.4. [3, 4] For each $i \in I^{\mathrm{im}}$ and $l \geq 1$, there exists a unique element $s_{il} \in \mathscr{E}_{l\alpha_i}$ such that

(i) $\langle s_{i,1}, \ldots, s_{i,l}\rangle = \langle e_{i,1}, \ldots, e_{i,l}\rangle$ as algebras,

(ii) $(s_{il}, z)_L = 0$ for all $z \in \langle e_{i,1}, \ldots, e_{i,l-1}\rangle$,

(iii) $s_{il} - e_{il} \in \langle e_{i,1}, \ldots, e_{i,l-1}\rangle$,

(iv) $\delta(s_{il}) = s_{il} \otimes 1 + 1 \otimes s_{il}$,

(v) $\Delta(s_{il}) = s_{il} \otimes 1 + K_i^l \otimes s_{il}$.

Proposition 2.5. [3, 4] $U_v^{\pm}(\mathfrak{g}) = \widetilde{U}^{\pm}$. In particular, $(\ ,\ )_L$ is non-degenerate on $U_v^{\pm}(\mathfrak{g})$.

Combining Proposition 2.4 and Proposition 2.5, we obtain

Corollary 2.6. The algebra $U_v^+(\mathfrak{g})$ is a non-degenerate Green-Lusztig algebra belonging to the class $\mathscr{L}(I^\infty, (\ ,\ )_L, \mathbf{C}(v), v)$.

Proof. Note that $U_v^+(\mathfrak{g})$ is generated by s_{il} and that $\delta(s_{il}) = s_{il}\otimes 1 + 1 \otimes s_{il}$ for $(i,l) \in I^\infty$. Since $(\ ,\ )_L$ is non-degenerate on $\mathscr{E}(i)$ for each $i \in I^{\mathrm{im}}$, we have

$$(s_{il}, s_{il})_L = (s_{il}, e_{il})_L \neq 0,$$

which proves our claim. □

Remark. The algebra $\mathscr{E}$ is also a (degenerate) Green-Lusztig algebra belonging to the class $\mathscr{L}(I^\infty, (\ ,\ )_L, \mathbf{C}(v), v)$.

3. Ringel-Hall algebras

Let I be an index set (possibly countably infinite) and let $R = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ be the free abelian group on I. Let $Q = (I, \Omega)$ be a quiver, where I is the set of vertices and Ω is the set of arrows. We have the functions $\mathrm{out}, \mathrm{in} : \Omega \to I$ defined by

$$\mathrm{out}(h) \xrightarrow{h} \mathrm{in}(h) \quad \text{for} \quad h \in \Omega.$$

Definition 3.1. Let $\mathbf{k}$ be a field and let $Q = (I, \Omega)$ be a quiver. A *representation of Q over* $\mathbf{k}$ consists of

(i) a family of finite dimensional $\mathbf{k}$-vector spaces $M = (M_i)_{i\in I}$ such that $M_i = 0$ for all but finitely many i,

(ii) a family of $\mathbf{k}$-linear maps $x = (x_h : M_{\mathrm{out}(h)} \to M_{\mathrm{in}(h)})_{h\in\Omega}$.

For simplicity, we often write (M, x) for a representation of Q.

Definition 3.2. Let (M, x) and (N, y) be representations of a quiver $Q = (I, \Omega)$. A *morphism* $\phi : (M, x) \to (N, y)$ is a family of $\mathbf{k}$-linear maps $\phi = (\phi_i : M_i \to N_i)_{i\in I}$ such that for all $h \in \Omega$, the following diagram is commutative.

$$\begin{array}{ccc} M_{\mathrm{out}(h)} & \xrightarrow{\phi_{\mathrm{out(h)}}} & N_{\mathrm{out}(h)} \\ \downarrow{\scriptstyle x_h} & & \downarrow{\scriptstyle y_h} \\ M_{\mathrm{in}(h)} & \xrightarrow{\phi_{\mathrm{in}(h)}} & N_{\mathrm{in}(h)} \end{array} \tag{3.1}$$

Let $M = (M_i)_{i\in I}$ be a representation of Q. We define the *dimension vector* of M by

$$\underline{\dim}\, M = \sum_{i\in I} (\dim_{\mathbf{k}} M_i)\, \alpha_i \in R^+. \tag{3.2}$$

Let M and N be representations of Q. The (non-symmetric) *Euler form* of M and N is defined by

$$\langle M, N\rangle = \dim_{\mathbf{k}} Hom_{\mathbf{k}Q}(M, N) - \dim_{\mathbf{k}} Ext^1_{\mathbf{k}Q}(M, N). \tag{3.3}$$

On the other hand, for $\alpha = \sum_i d_i\alpha_i$, $\beta = \sum_i d'_i\alpha_i \in R^+$, we define

$$\langle \alpha, \beta\rangle = \sum_i (1 - g_i)d_i d'_i - \sum_{\substack{i\neq j\\ i\to j}} c_{ij} d_i d'_j, \tag{3.4}$$

where g_i is the number of loops at i and c_{ij} denotes the number of arrows from i to j in Ω.

The following lemma is well-known (see, for example, [6, 9]).

Lemma 3.3. Let M and N be representations of Q. Then we have

$$\langle M, N\rangle = \langle \underline{\dim}\, M, \underline{\dim}\, N\rangle.$$

For $\alpha, \beta \in R^+$, we define

$$(\alpha, \beta) := \langle \alpha, \beta\rangle + \langle \beta, \alpha\rangle. \tag{3.5}$$

In particular, we have

$$(\alpha_i, \alpha_j) = \begin{cases} 2(1-g_i) & \text{if } i = j, \\ -c_{ij} - c_{ji} & \text{if } i \neq j. \end{cases} \tag{3.6}$$

Hence we obtain a symmetric Borcherds-Cartan matrix $A_Q = (a_{ij})_{i,j\in I} = ((\alpha_i, \alpha_j))_{i,j\in I}$ with R as the root lattice. We will denote by $U_v(\mathfrak{g}_Q)$ the quantum Borcherds-Bozec algebra associated with A_Q.

Let $\mathbf{k}$ be a finite field with q elements and choose a complex number $v = v_{\mathbf{k}} \in \mathbf{C}$ such that $v^2 = q$. Then $(I, (\ ,\ ), \mathbf{C}, v)$ is a Green-Lusztig datum.

Definition 3.4. The *Ringel-Hall algebra* $H_{\mathbf{k}}(Q)$ is the associative algebra over $\mathbf{C}$ with a basis consisting of isomorphism classes of representations of Q endowed with the multiplication defined by

$$[M]\,[N] := \sum_L v^{\langle \dim M, \dim N\rangle} \alpha^L_{M,N}\,[L], \tag{3.7}$$

where $[M]$ denotes the isomorphism class of M and

$$\alpha^L_{M,N} = \#\{X \subset L \mid X \cong N,\ L/X \cong M\}. \tag{3.8}$$

Let $\alpha \in R^+$ and let $H_{\mathbf{k}}(Q)_\alpha$ be the $\mathbf{C}$-span of the isomorphism classes with $\underline{\dim}\, M = \alpha$. Then $H_{\mathbf{k}}(Q) = \bigoplus_{\alpha\in R^+} H_{\mathbf{k}}(Q)_\alpha$ becomes an R^+-graded algebra ([7, 19], etc).

We define a *twisted* algebra structure on $H_{\mathbf{k}}(Q) \otimes H_{\mathbf{k}}(Q)$ by

$$([M_1] \otimes [M_2])\,([N_1] \otimes [N_2]) = v^{(\underline{\dim}\, M_2,\, \underline{\dim}\, N_1)}([M_1]\,[N_1] \otimes [M_2]\,[N_2]) \tag{3.9}$$

and a $\mathbf{C}$-linear map $\delta : H_{\mathbf{k}}(Q) \to H_{\mathbf{k}}(Q) \otimes H_{\mathbf{k}}(Q)$ by

$$\delta([L]) = \sum_{M,N} v^{\langle \underline{\dim}\, M,\, \underline{\dim}\, N\rangle} \alpha^L_{M,N} \frac{a_M a_N}{a_L}([M] \otimes [N]), \tag{3.10}$$

where $a_M = \#(Aut_{\mathbf{k}Q}(M))$.

Proposition 3.5. [7]

(a) $\delta : H_{\mathbf{k}}(Q) \to H_{\mathbf{k}}(Q) \otimes H_{\mathbf{k}}(Q)$ is a $\mathbf{C}$-algebra homomorphism.

(b) There exists a non-degenerate symmetric bilinear form $(\ ,\ )_G : H_{\mathbf{k}}(Q) \times H_{\mathbf{k}}(Q) \to \mathbf{C}$ defined by

$$([M], [N])_G = \delta_{[M],[N]} \frac{1}{a_M}.$$

(c) We have

$$(x, yz)_G = (\delta(x), y \otimes z)_G \quad \text{for all} \quad x, y, z \in H_{\mathbf{k}}(Q).$$

Let $(i,l) \in I^\infty$. If $i \in I^{\mathrm{re}}$, we define E_i to be the unique simple representation of Q with dimension vector α_i. By (3.10), we see that

$$\delta([E_i]) = [E_i] \otimes 1 + 1 \otimes [E_i].$$

Assume that $i \in I^{\mathrm{im}}$. For each $l \geq 1$, we define a representation $(E_{i,l}, x)$ of Q by setting

$$\begin{gathered} (E_{i,l})_j = \begin{cases} \mathbf{k}^l & \text{if } \ j = i, \\ 0 & \text{if } \ j \neq i, \end{cases} \\ x_h = 0 \ \text{ for all } h \in \Omega. \end{gathered} \tag{3.11}$$

Note that

$$\begin{aligned} & a_{E_{i,l}} = \#(GL_l(\mathbf{k})) = (q^l - 1)(q^l - q)\cdots(q^l - q^{l-1}) = v^{\frac{3}{2}l(l-1)}(v^2-1)^l\,[l]!, \\ & \alpha_{E_{i,m},\,E_{i,n}}^{E_{i,m+n}} = \#Gr_{\mathbf{k}}\binom{m+n}{m} = v^{mn}\begin{bmatrix} m+n \\ m \end{bmatrix}. \end{aligned} \tag{3.12}$$

Therefore we obtain

$$\begin{aligned} \delta([E_{i,l}]) &= \sum_{m+n=l} v^{\langle \underline{\dim}\, E_{i,m},\, \underline{\dim}\, E_{i,n}\rangle} \alpha_{E_{i,m},\, E_{i,n}}^{E_{i,m+n}} \frac{a_{E_{i,m}} a_{E_{i,n}}}{a_{E_{i,m+n}}}\, [E_{i,m}] \otimes [E_{i,n}] \\ &= \sum_{m+n=l} v^{mn\langle \alpha_i, \alpha_i\rangle} v^{mn} \begin{bmatrix} m+n \\ m \end{bmatrix} \frac{v^{\frac{3}{2}(m(m-1)+n(n-1))}(v^2-1)^{m+n}\,[m]!\,[n]!}{v^{\frac{3}{2}((m+n)(m+n-1))}(v^2-1)^{m+n}\,[m+n]!} \\ &= \sum_{m+n=l} v^{mn(1-g_i-2)}\, [E_{i,m}] \otimes [E_{i,n}] \\ &= \sum_{m+n=l} v^{mn(-1-g_i)}\, [E_{i,m}] \otimes [E_{i,n}]. \end{aligned}$$

Hence for all $(i,l) \in I^\infty$, we have

$$\delta([E_{i,l}]) = \sum_{m+n=l} v^{mn(-1-g_i)} [E_{i,m}] \otimes [E_{i,n}]. \tag{3.13}$$

Moreover, by (3.12), we see that

$$([E_{i,l}], [E_{i,l}])_G = \frac{1}{a_{E_{i,l}}} \in v^{-2l^2}(1 + v^{-1}\mathbf{Z}[[v^{-1}]]) \quad \text{for all} \quad (i,l) \in I^\infty. \tag{3.14}$$

Set $\mathbf{e}_{il} := v^{l^2}[E_{i,l}] \in H_{\mathbf{k}}(Q)$. Then by (3.13), we obtain

$$\delta(\mathbf{e}_{il}) = \sum_{m+n=l} v_{(i)}^{mn}\, \mathbf{e}_{im} \otimes \mathbf{e}_{in}, \tag{3.15}$$

where $v_{(i)} = v^{\langle \alpha_i, \alpha_i \rangle} = v^{1-g_i}$. Moreover, it is easy to see that

$$\mathbf{e}_{il}, \mathbf{e}_{il})_G \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]] \quad \text{for all} \quad (i,l) \in I^{\infty}. \tag{3.16}$$

Definition 3.6. The subalgebra $C_{\mathbf{k}}(Q)$ of $H_{\mathbf{k}}(Q)$ generated by $\mathbf{e}_{i,l}$ ($(i,l) \in I^{\infty}$) is called the *composition algebra of Q over* $\mathbf{k}$.

By (3.15), we see that $\delta(C_{\mathbf{k}}(Q)) \subset C_{\mathbf{k}}(Q) \otimes_{\mathbf{k}} C_{\mathbf{k}}(Q)$ and hence $C_{\mathbf{k}}(Q)$ is a bi-algebra.

For each $i \in I$, set $H_{\mathbf{k}}(i) := \bigoplus_{l \geq 1} H_{\mathbf{k}}(Q)_{l\alpha_i}$. Then the restriction of $(\ ,\ )_G$ to $H_{\mathbf{k}}(i)$ is non-degenerate. Hence, as in [3, Proposition 2.16], we have:

Proposition 3.7. For each $(i,l) \in I^{\infty}$, there exists a unique element $\mathbf{s}_{il} \in H_{\mathbf{k}}(Q)$ such that

(a) $\langle \mathbf{s}_{i,1}, \ldots, \mathbf{s}_{i,l} \rangle = \langle \mathbf{e}_{i,1}, \ldots, \mathbf{e}_{i,l} \rangle$ as algebras,
(b) $(\mathbf{s}_{il},\, x)_G = 0$ for all $x \in \langle \mathbf{e}_{i,1}, \ldots, \mathbf{e}_{i,l-1} \rangle$,
(c) $\mathbf{s}_{il} - \mathbf{e}_{il} \in \langle \mathbf{e}_{i,1}, \ldots, \mathbf{e}_{i,l-1} \rangle$,
(d) $\delta(\mathbf{s}_{il}) = \mathbf{s}_{il} \otimes 1 + 1 \otimes \mathbf{s}_{il}$.

As in the proof of Corollary 2.6, by (b) and (c), we see that

$$(\mathbf{s}_{il}, \mathbf{s}_{il})_G \neq 0 \quad \text{for all } (i,l) \in I^{\infty}.$$

Therefore we obtain:

Proposition 3.8. The composition algebra $C_{\mathbf{k}}(Q)$ is a Green-Lusztig algebra belonging to the class $\mathscr{L}(I^{\infty}, (\ ,\ )_G, \mathbf{C}, v_{\mathbf{k}})$.

The following proposition and its corollary show that the quantum Serre relations hold in the composition algebra $C_{\mathbf{k}}(Q)$.

Proposition 3.9. For every finite field $\mathbf{k}$, the following relations hold.

(a) If $a_{ij} = 0$, then

$$[E_{i,k}][E_{j,l}] = [E_{j,l}][E_{i,k}].$$

(b) If $i \in I^{\mathrm{re}}$ and $i \neq (j,l)$, then we have

$$\sum_{k=0}^{1-la_{ij}} (-1)^k [E_i]^{(k)} [E_{j,l}] [E_i]^{(1-la_{ij}-k)} = 0,$$

where $[E_i]^{(k)} := [E_i]^k / [k]!$.

Proof. Set $v = v_{\mathbf{k}}$. If $a_{ii} = 0$, by the duality, we have

$$[E_{i,k}][E_{i,l}] = \sum_L \alpha^L_{E_{i,k},\, E_{i,l}} [L] = \sum_L \alpha^{L*}_{E^*_{i,l},\, E^*_{i,k}} [L^*] = \sum_L \alpha^L_{E_{i,l},\, E_{ik}} [L] = [E_{i,l}][E_{i,k}].$$

If $i \neq j$, $a_{ij} = 0$ implies $c_{ij} = c_{ji} = 0$. Thus $Hom_{\mathbf{k}Q}(E_{i,k}, E_{j,l}) = 0$ and

$$[E_{i,k}][E_{j,l}] = [E_{i,k} \oplus E_{j,l}] = [E_{j,l}][E_{i,k}],$$

which proves (a).

To prove (b), by induction, we first verify

$$[E_i]^{(k)} = \frac{1}{[k]!}[E_i]^k = v^{k(k-1)}[E_i^{\oplus k}].$$

Now we have

$$\begin{aligned}
[E_i]^{(k)}[E_{j,l}] &= v^{k(k-1)} v^{\langle k\alpha_i, l\alpha_j \rangle} \sum_L \alpha^L_{E_i^{\oplus k},\, E_{j,l}} [L] \\
&= v^{k(k-1)-klc_{ij}} \sum_L \alpha^L_{E_i^{\oplus k},\, E_{j,l}} [L],
\end{aligned}$$

where L runs over $\mathbf{k}Q$-modules containing a submodule X such that

$$X \cong E_{j,l}, \quad L/X \cong E_i^{\oplus k}.$$

Since $Hom_{\mathbf{k}Q}(E_i, E_{j,l}) = 0$, such a submodule X is unique and hence

$$\alpha^L_{E_i^{\oplus k},\, E_{j,l}} = 1 \ \text{ for all } L.$$

It follows that

$$[E_i]^{(k)}[E_{j,l}] = v^{k(k-1)-klc_{ij}} \sum_L [L],$$

where L contains a (unique) submodule X such that $X \cong E_{j,l}$, $L/X \cong E_i^{\oplus k}$.

Hence for any $n \geq 0$, we have

$$\begin{aligned}
&[E_i]^{(k)}[E_{j,l}][E_i]^{(n)} = (v^{k(k-1)-klc_{ij}} \sum_L [L])\, E_i^{(n)} \\
&= v^{k(k-1)-klc_{ij}}\, v^{\langle k\alpha_i + l\alpha_j, n\alpha_i\rangle}\, v^{n(n-1)} \sum_L \sum_P \alpha^P_{L,E_i^{\oplus n}}\, [P] \\
&= v^{k(k-1)+n(n+1)+kn-klc_{ij}-lnc_{ji}} \sum_P (\sum_L \alpha^P_{L,E_i^{\oplus n}})\, [P],
\end{aligned}$$

where

$$\alpha^P_{L,E_i^{\oplus n}} = \#\{Y \subset P \mid Y \cong E_i^{\oplus n},\, P\big/Y \cong L\}.$$

Set

$$\begin{aligned}
&K_P := \bigcap_{h:i\to j} \mathrm{Ker}(x_h : \mathbf{k}^{\oplus(k+n)} \to \mathbf{k}^l) \subset P_i, \\
&J_P := \sum_{h':j\to i} \mathrm{Im}(x_{h'} : \mathbf{k}^l \to \mathbf{k}^{\oplus(k+n)}) \subset P_i, \\
&m_P := \dim K_P, \quad n_P := \dim J_P.
\end{aligned}$$

Note that $P\big/Y \cong L$ if and only if

(i) $\underline{\dim}\, Y = n\alpha_i$,

(ii) $x_h = 0$ for all $h : i \to j$,

(iii) $\mathrm{Im} x_{h'} \subset Y$ for all $h' : j \to i$.

Hence we have

$$\begin{aligned}
\beta_{P,n} &:= \sum_L \alpha^P_{L,E_i^{\oplus n}} = \sum_L \#\{Y \subset P \mid Y \cong E_i^{\oplus n},\ P/Y \cong L\} \\
&= \#\{n\text{-dimensional subspaces } Y \text{ of } K_P \text{ containing } J_P\} \\
&= \#\{(n-n_P)\text{-dimensional subspaces of } K_P/J_P\} \\
&= \#Gr_{\mathbf{k}}\binom{m_P - n_P}{n - n_P} = v^{(m_P-n)(n-n_P)} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix},
\end{aligned}$$

which implies

$$[E_i]^{(k)}[E_{j,l}][E_i]^{(n)} = v^{k(k-1)+n(n-1)+kn-klc_{ij}-lnc_{ji}} \sum_P v^{(m_P-n)(n-n_P)} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix} [P].$$

By setting $n = 1 - la_{ij} - k$ and summing up, we obtain

$$\sum_{k=0}^{1-la_{ij}} (-1)^k [E_i]^{(k)} [E_{j,l}] [E_i]^{(1-la_{ij}-k)} = \sum_{P\,:\,J_P \subset K_P} \gamma_P [P],$$

where

$$\begin{aligned}
\gamma_P &= \sum_{k=0}^{1-la_{ij}} (-1)^k v^{k(k-1)+n(n-1)+kn-klc_{ij}-lnc_{ji}+(m_P-n)(n-n_P)} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix} \\
&= \sum_{n=0}^{1-la_{ij}} (-1)^{1-laij-n} v^{lc_{ji}(1-la_{ij})+n(-2lc_{ji}+m_P+n_P-1)-m_P n_P} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix} \\
&= (-1)^{1-la_{ij}} v^{lc_{ji}(1-la_{ij})-m_P n_P} \sum_{n=n_P}^{m_P} (-1)^n v^{n(-2lc_{ji}+m_P+n_P-1)} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix}.
\end{aligned}$$

Let

$$\gamma_P^0 := \sum_{n=n_P}^{m_P} (-1)^n v^{n(-2lc_{ji}+m_P+n_P-1)} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix}.$$

Note that $\dim \operatorname{Im} x_h \le lc_{ij}$ and $n_P = \dim J_P \le lc_{ji}$. Hence we have

$$m_P = \dim K_P \ge 1 - la_{ij} - lc_{ij} = 1 + lc_{ji} > n_P$$

and obtain

$$\begin{aligned}
&(m_P - n_P - 1) - (-2lc_{ji} + m_P + n_P - 1) = 2(lc_{ji} - n_P) \ge 0, \\
&(-2lc_{ji} + m_P + n_P - 1) - (-m_P + n_P + 1) = 2(m_P - lc_{ji} - 1) \ge 0,
\end{aligned}$$

which yield

$$-m_P + n_P + 1 \le -2lc_{ji} + m_P + n_P - 1 \le m_P - n_P - 1.$$

It is well-known that

$$\sum_{k=0}^{m} (-1)^k v^{dk} \begin{bmatrix} m \\ k \end{bmatrix} = 0$$

for all $m \ge 1$, $-m+1 \le d \le m-1$, $d \equiv m-1 \pmod 2$ (see, for example, [13]).

Therefore, since $-2lc_{ji} + m_P + n_P - 1 \equiv m_P - n_P - 1 \pmod 2$, we have

$$\begin{aligned}
\gamma_P^0 &= \sum_{n=n_P}^{m_P} (-1)^n v^{n(-2lc_{ji}+m_P+n_P-1)} \begin{bmatrix} m_P - n_P \\ n - n_P \end{bmatrix} \\
&= \sum_{r=0}^{m_P-n_P} (-1)^{r+n_P} v^{r+n_P(-2lc_{ji}+m_P+n_P-1)} \begin{bmatrix} m_P - n_P \\ r \end{bmatrix} \\
&= (-1)^{n_P} v^{n_P(-2lc_{ji}+m_P+n_P-1)} \sum_{r=0}^{m_P-n_P} (-1)^r v^{r(-2lc_{ji}+m_P+n_P-1)} \begin{bmatrix} m_P - n_P \\ r \end{bmatrix} \\
&= 0.
\end{aligned}$$

Hence we conclude $\gamma_P = 0$ for all P, which proves our assertion. □

Corollary 3.10. For every finite field $\mathbf{k}$, the following relations hold.

(a) If $a_{ij} = 0$, then

$$\mathbf{e}_{ik}\, \mathbf{e}_{jl} = \mathbf{e}_{jl}\, \mathbf{e}_{ik}.$$

(b) If $i \in I^{\mathrm{re}}$ and $i \neq (j, l)$, then we have

$$\sum_{k=0}^{1-la_{ij}} (-1)^k \mathbf{e}_i^{(k)} \mathbf{e}_{jl}\, \mathbf{e}i^{(1-la_{ij}-k)} = 0,$$

where $\mathbf{e}_i^{(k)} := \mathbf{e}_i^k / [k]!$.

4. Ringel-Hall algebra construction of $U_v^+(\mathfrak{g}_Q)$

Let K be an infinite set of mutually non-isomorphic finite fields. For each $\mathbf{k} \in K$, choose $v_{\mathbf{k}} \in \mathbf{C}$ such that $v_{\mathbf{k}}^2 = \#(\mathbf{k})$ and set

$$H(Q) := \prod_{\mathbf{k} \in K} H_{\mathbf{k}}(Q), \tag{4.1}$$

the *generic Ringel-Hall algebra*.

Let v be an indeterminate. Then $H(Q)$ can be regarded as a $\mathbf{C}[v, v^{-1}]$-module via

$$v^{\pm 1} \longmapsto (v_{\mathbf{k}}^{\pm 1})_{\mathbf{k} \in K}.$$

For each $(i,l) \in I^\infty$, let $E_{i,l;\mathbf{k}}$ be the representation of Q over $\mathbf{k}$ defined in (3.11) and let $\mathbf{s}_{i,l;\mathbf{k}}$ be the element in $H_{\mathbf{k}}(Q)$ given in Proposition 3.7. Set

$$\mathbf{E}_{i,l} := (\mathbf{e}_{i,l;\mathbf{k}})_{\mathbf{k}\in K} = (v_{\mathbf{k}}^{l^2}[E_{i,l;\mathbf{k}}])_{\mathbf{k}\in K}, \qquad \mathbf{S}_{i,l} := (\mathbf{s}_{i,l;\mathbf{k}})_{\mathbf{k}\in K}. \tag{4.2}$$

Definition 4.1. The *generic composition algebra of* Q is the $\mathbf{C}[v,v^{-1}]$-subalgebra $C(Q)$ of $H(Q)$ generated by $\mathbf{E}_{i,l}$ for all $(i,l) \in I^\infty$.

By Proposition 3.8, the generic composition algebra $C(Q)$ is a Green-Lusztig algebra belonging to the class $\mathscr{L}(I^\infty, (\ ,\ )_G, \mathbf{C}[v,v^{-1}], v)$. We now state and prove the main theorem of this paper.

Theorem 4.2. There exists a natural isomorphism of $\mathbf{C}(v)$-bialgebras

$$\Phi : U_v^+(\mathfrak{g}_Q) \to \mathbf{C}(v) \otimes_{\mathbf{C}[v,v^{-1}]} C(Q)$$

given by

$$e_{il} \longmapsto \mathbf{E}_{i,l} \ \text{ for all } (i,l) \in I^\infty.$$

Proof. By Corollary 3.10, Φ defines a surjective $\mathbf{C}(v)$-bialgebra homomorphism.

To prove the injectivity of Φ, we will use Theorem 1.2. Let $\Lambda := \bigoplus_{(i,l)\in I^\infty} \mathbf{Z}\alpha_{i,l}$ and let $\Lambda^+ := \sum_{(i,l)\in I^\infty} \mathbf{Z}_{\geq 0}\alpha_{i,l}$. For $\beta = \sum d_{i,l}\,\alpha_{i,l} \in \Lambda^+$, set

$$I^\infty(\beta) := \{w = ((i_1,l_1),\ldots,(i_r,l_r)) \mid \alpha_{i_1,l_1} + \cdots + \alpha_{i_r,l_r} = \beta\}.$$

For each $w = ((i_1,l_1),\ldots,(i_r,l_r)) \in I^\infty(\beta)$, we denote the generating monomials by

$$\begin{aligned} s_w &:= s_{i_1,l_1}\cdots s_{i_r,l_r} \in U_v^+(\mathfrak{g}_Q)_\beta, \\ \mathbf{s}_{w;\mathbf{k}} &:= \mathbf{s}_{i_1,l_1;\mathbf{k}}\cdots \mathbf{s}_{i_r,l_r;\mathbf{k}} \in C_{\mathbf{k}}(Q)_\beta, \\ \mathbf{S}_w &:= (\mathbf{s}_{w;\mathbf{k}})_{\mathbf{k}\in K} = \mathbf{S}_{i_1,l_1}\cdots\mathbf{S}_{i_r,l_r} \in C(Q)_\beta. \end{aligned}$$

Then one can see that s_w is mapped onto $\mathbf{S}_w$ under the homomorphism Φ.

By Theorem 1.2, for all $\beta = \sum d_{i,l}\,\alpha_{i,l} \in \Lambda^+$, $w, w' \in I^\infty(\beta)$, there exists a polynomial $P_{w,w'}(t) \in \mathbf{Z}[t,t^{-1}]$ such that for all $\mathbf{k} \in K$, we have

(i) $(s_w, s_{w'})_L = P_{w,w'}(v) \prod_{(i,l)\in I^\infty} (s_{i,l}, s_{i,l})_L^{d_{i,l}}$,

(ii) $(\mathbf{s}_{w;\mathbf{k}}, \mathbf{s}_{w';\mathbf{k}})_G = P_{w,w'}(v_{\mathbf{k}}) \prod_{(i,l)\in I^\infty} (\mathbf{s}_{i,l;\mathbf{k}}, \mathbf{s}_{i,l;\mathbf{k}})_G^{d_{i,l}}$.

Let $u = \sum_w c_w(v) s_w \in \operatorname{Ker}\Phi \subset U_v^+(\mathfrak{g}_Q)$. Thus $\sum_w c_w(v)\,\mathbf{S}_w = 0$, which implies

$$\sum_w c_w(v_{\mathbf{k}})\,\mathbf{s}_{w;\mathbf{k}} = 0 \ \text{ for all } \mathbf{k} \in K.$$

Then, for any $w' \in I^{\infty}(\beta)$, we have

$$0 = \sum_{w} c_w(v_{\mathbf{k}})(\mathbf{s}_{w;\mathbf{k}},\, \mathbf{s}_{w';\mathbf{k}})_G = \sum_{w} c_w(v_{\mathbf{k}}) P_{w,w'}(v_{\mathbf{k}}) \prod_{(i,l)\in I^{\infty}} (\mathbf{s}_{i,l;\mathbf{k}}, \mathbf{s}_{i,l;\mathbf{k}})_G^{d_{i,l}}.$$

It follows that

$$\sum_{w} c_w(v_{\mathbf{k}}) P_{w,w'}(v_{\mathbf{k}}) = 0 \quad \text{for all } w' \in I^{\infty}(\beta),\ \ \mathbf{k} \in K.$$

Therefore $\sum_w c_w(v) P_{w,w'}(v) = 0$ for all $w' \in I^{\infty}(\beta)$.

By Lemma 1.3, we have

$$u = \sum_{w} c_w(v) s_w \in \mathrm{rad}(\ ,\ )_L.$$

Since $(\ ,\ )_L$ is non-degenerate on $U_v^+(\mathfrak{g}_Q)$, we conclude $u = 0$ and hence Φ is injective. □

Remark. Using the theory of perverse sheaves on the representation varieties of quivers with loops [3], it is expected that one can construct the isomorphism Φ as the Frobenius trace map given in [16]. (See [21] for the details.)

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