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On certain tilting modules for SL_2

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ABSTRACT

We give a complete picture of when the tensor product of an induced module and a Weyl module is a tilting module for the algebraic group SL_2 over an algebraically closed field of characteristic p. Whilst the result is recursive by nature, we give an explicit statement in terms of the p-adic expansions of the highest weight of each module.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and let G be the group $SL_2(k)$. In this article we investigate the tensor product $\nabla(r) \otimes \Delta(s)$ of the induced module of highest weight r and the Weyl module of highest weight s. Similar tensor products for $SL_2(k)$ have been studied before, in particular the product $L(r) \otimes L(s)$ of corresponding simple modules, by Doty and Henke in 2005 [3]. Motivated by their results utilising tilting modules, we describe exactly when the product $\nabla(r) \otimes \Delta(s)$ is a tilting module.

By an argument of Donkin given in [6, Lemma 3.3], it's known already that when $|r - s| \leq 1$ the module $\nabla(r) \otimes \Delta(s)$ is tilting. Some other special cases are also known,

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for example the tensor product of Steinberg modules $\nabla(p^n - 1) \otimes \Delta(p^m - 1)$ is tilting, since $\nabla(p^k - 1) = \Delta(p^k - 1)$ for all $k \in \mathbb{N}$, as is the tensor product $\nabla(a) \otimes \Delta(b)$ for $a, b \in \{0, \ldots, p-1\}$.

Before stating the main theorem of this paper, we introduce some notation. Let $r \in \mathbb{N}$ and p a prime. We write the base p expansion

$$r = \sum_{i=0}^{n} r_i p^i,$$

where each $r_i \in \{0, \ldots, p-1\}$, $r_n \neq 0$ and for all $j > n$ we set $r_j = 0$. We say that r has p-length n (or just length n if the prime is clear), and write $\mathrm{len}_p(r) = n$. We define $\mathrm{len}_p(0) = -1$. Now given any pair $(r, s) \in \mathbb{N}^2$ we can write

$$r = \sum_{i=0}^{n} r_i p^i, \quad s = \sum_{i=0}^{n} s_i p^i$$

where $n = \max(\mathrm{len}_p(r), \mathrm{len}_p(s))$ so that at least one of r_n and s_n is non zero. If $r \neq s$, let m be the largest integer such that $r_m \neq s_m$ and let

$$\hat{r} = \sum_{i=0}^{m} r_i p^i, \quad \hat{s} = \sum_{i=0}^{m} s_i p^i$$

so that if $r > s$ we have $r_m > s_m$ and $\hat{r} > \hat{s}$. In the case $r = s$, we define $\hat{r} = \hat{s} = 0$. We call the pair $(\hat{r}, \hat{s})$ the primitive of (r, s), and say that (r, s) is a primitive pair if $(r, s) = (\hat{r}, \hat{s})$.

Theorem 1.1. *Let the pair $(\hat{r}, \hat{s})$ be the primitive of (r, s). The module $\nabla(r) \otimes \Delta(s)$ is a tilting module if and only if one of the following*

1. $\hat{r} = p^n - 1 + ap^n$ *for some* $a \in \{0, \ldots, p-2\}$, $n \in \mathbb{N}$, *and* $\hat{s} < p^{n+1}$,
2. $\hat{s} = p^n - 1 + bp^n$ *for some* $b \in \{0, \ldots, p-2\}$, $n \in \mathbb{N}$, *and* $\hat{r} < p^{n+1}$.

Fig. 1 illustrates which of the modules $\nabla(r) \otimes \Delta(s)$ are tilting for $r, s \leq 31$ and $p = 2$.

1.1. Terminology

In this section, we fix some terminology. Throughout, k will be an algebraically closed field of characteristic $p > 0$, and G will be the affine algebraic group $SL_2(k)$. Let B be the Borel subgroup of G consisting of lower triangular matrices and containing the maximal torus T of diagonal matrices. Let $X(T)$ be the weight lattice, which we associate with $\mathbb{Z}$ in the usual manner. Under this association the set of dominant weights X^+ corresponds to the set $\mathbb{N} \cup \{0\}$.

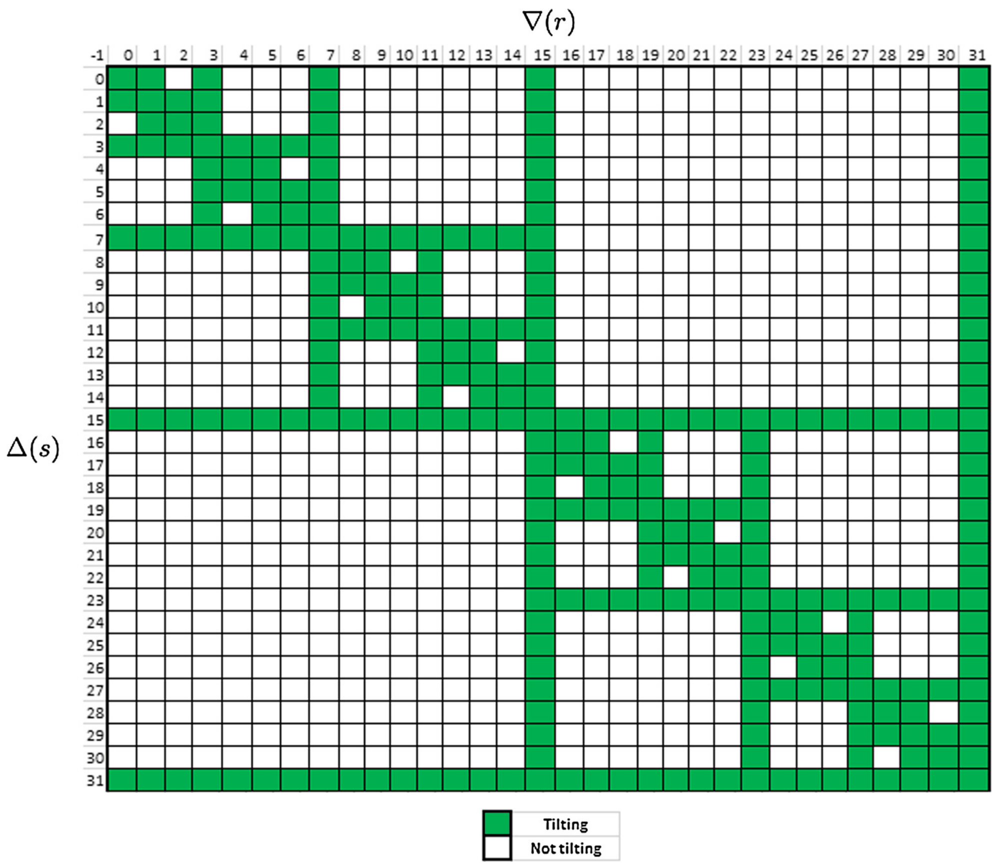

Fig. 1. The modules $\nabla(r) \otimes \Delta(s)$ when $\mathrm{char}(k) = 2$.

Whenever we refer to a module, we will always mean a finite dimensional, rational G-module. Let $F : G \longrightarrow G$ denote the usual Frobenius morphism, and denote by G_1 its kernel. For any module V, we denote by V^F the Frobenius twist of V.

Let k_r be the one dimensional B module on which T acts via $r \in \mathbb{Z}$, and let $\nabla(r)$ be the induced module $\mathrm{Ind}_B^G(k_r)$. Then $\nabla(r)$ is finite dimensional and is zero when r is not dominant. It is well known that when r is dominant we have $\nabla(r) = S^r E$, the rth symmetric power of the natural module E, although we will not need to use this. Let $\Delta(s)$ be the Weyl module of highest weight s, for which we have $\Delta(s) = \nabla(s)^*$. By a tilting module we mean a module which has both a ∇-filtration (or good filtration) and a Δ-filtration (or Weyl filtration) as defined in [1]. We denote by $T(r)$ the unique indecomposable tilting module of highest weight $r \in X^+$, and remark that the tensor product of two tilting modules is also a tilting module [1, Proposition 1.2(i)].

We will make use of the character $\mathrm{Ch}(V)$ of a module V. This is given by

$$\mathrm{Ch}(V) = \sum_{r \in X(T)} (\dim V^r) x^r$$

inside the ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials, where V^r is the r weight space of V. For $r \geq 0$ we write $\chi(r)$ for $\mathrm{Ch}(\nabla(r)) = \mathrm{Ch}(\Delta(r))$, and note that $\chi(1) = x + x^{-1}$. From the action of the Weyl group on each weight space, we have in fact that $\mathrm{Ch}(V) \in \mathbb{Z}[\chi(1)]$, which is a unique factorization domain.

The objects of interest in this article are the modules $\nabla(r)\otimes\Delta(s)$, for dominant weights r and s. The character of these modules is given by the well known Clebsch–Gordan formula (assuming $r \geq s$)

$$\mathrm{Ch}(\nabla(r) \otimes \Delta(s)) = \chi(r)\chi(s) = \sum_{i=0}^{s} \chi(r + s - 2i).$$

Furthermore, we have the following result from [6].

Lemma 1.2 *([6, Lemma 3.3]). The module $\nabla(r) \otimes \Delta(s)$ has a good filtration if $r \geq s - 1$, and a Weyl filtration if $r \leq s + 1$.*

The sections of each filtration are given by the character, so if $r \geq s - 1$ then the module $\nabla(r) \otimes \Delta(s)$ has a good filtration with sections

$$\nabla(r + s), \nabla(r + s - 2), \ldots, \nabla(r - s). \tag{1}$$

It follows immediately from Lemma 1.2 that if $|r-s| \leq 1$, then the module $\nabla(r)\otimes\Delta(s)$ is a tilting module.

2. Tilting modules

We now give several useful results which will be used in the proceeding sections. In particular we will make extensive use of the following well known result, for which we have outlined a proof for the reader's convenience.

Proposition 2.1. *There exists a short exact sequence given by*

$$0 \longrightarrow \nabla(r - 1) \longrightarrow \nabla(r) \otimes E \longrightarrow \nabla(r + 1) \longrightarrow 0,$$

and this is split if and only if p does not divide $r + 1$.

Proof. That the sequence exists is clear by considering the ∇-filtration of $\nabla(r) \otimes E = \nabla(r)\otimes\Delta(1)$. If p does not divide $r+1$, the result follows by considering the blocks (see [5, II.7.1]) for $SL_2(k)$. On the other hand, if p does divide $r + 1$, then the module $E \otimes \nabla(r)$ is projective as a G_1-module, while neither $\nabla(r - 1)$ nor $\nabla(r + 1)$ are, so the sequence cannot be split. $\square$

As mentioned above, if $|r - s| \leq 1$ then the module $\nabla(r) \otimes \Delta(s)$ is a tilting module. The next result extends this.

Lemma 2.2. *If $r, s \in \{np-1, np, np+1, \dots, np+p-1\}$ for some fixed $n \in \mathbb{N}$, then $\nabla(r) \otimes \Delta(s)$ is tilting.*

Proof. Suppose, for a contradiction, that we have $\nabla(r) \otimes \Delta(s)$ is not tilting for some $r, s \in \{np-1, np, np+1, \dots, np+p-1\}$, choosing r and s so that $r+s$ is minimal. If $r \notin \{np-1, np\}$ then by Proposition 2.1 we have

$$\big(\nabla(r-1) \otimes E\big) \otimes \Delta(s) = \nabla(r) \otimes \Delta(s) \oplus \nabla(r-2) \otimes \Delta(s). \tag{2}$$

Since r and s were chosen so that $r+s$ was minimal, we have that $\nabla(r-1) \otimes \Delta(s)$ is tilting, so the tensor product on the left hand side of eq. (2) is tilting. It follows that each summand on the right hand side of eq. (2) is tilting, giving a contradiction. We must have then, that $r \in \{np-1, np\}$, and similarly, that $s \in \{np-1, np\}$. But then we have that $|r-s| \leq 1$, so by Lemma 1.2 we have $\nabla(r) \otimes \Delta(s)$ is tilting, contradicting our initial assumption. □

For the following lemma, G may be an arbitrary semisimple, simply connected algebraic group, over an algebraically closed field of prime characteristic. We denote by $(\ ,\ )$ the usual positive definite, symmetric, bilinear form on the Euclidean space in which the root system of G lies.

Lemma 2.3. *Let T_1 and T_2 be tilting modules where T_1 is projective as a G_1-module, then the tensor product $T_1 \otimes T_2^F$ is also a tilting module.*

Proof. First, since each tilting module has a unique decomposition (up to isomorphism) into indecomposable tilting modules, it's sufficient to prove the lemma in the case that T_1 is indecomposable. Now, let ρ be the half sum of all positive roots, and take T_1 to be the Steinberg module $\nabla((p-1)\rho) = \mathrm{St}$. In this case the result holds by [1, Proposition 2.1].

Next let $\lambda \in X^+$ be such that $T(\lambda)$ is projective as a G_1-module, so that $(\lambda, \check{\alpha}) \geq p-1$ for all simple roots α (where, as usual $\check{\alpha} = 2\alpha/(\alpha, \alpha)$) [1, Proposition 2.4]. Then, since $(\lambda - (p-1)\rho, \check{\alpha}) \geq 0$ for all simple roots α, we may write $\lambda = (p-1)\rho + \mu$ for some $\mu \in X^+$. It follows that the tilting module $\mathrm{St} \otimes T(\mu)$ has highest weight λ, and so $T(\lambda)$ is a summand of this module. Then $T(\lambda) \otimes T_2^F$ is a summand of the tilting module $(\mathrm{St} \otimes T_2^F) \otimes T(\mu)$, and is thus tilting itself. □

We will use this lemma throughout the article, in conjunction with the facts that, for $G = SL_2(k)$, we have that $\nabla(p-1) = \Delta(p-1)$ is a projective G_1-module [5, Proposition II.10.1], and that the tensor product of a projective G_1-module with another G_1-module is again projective. Next we return to the case $G = SL_2(k)$.

Lemma 2.4. *Let V be a tilting module, and define the module W by $H^0(G_1, V) = W^F$. Then W is a tilting module.*

Proof. As in the previous lemma, it suffices to prove this result for the case $V = T(m)$, for some $m \in \mathbb{N}$. We can split this into three separate cases, the first of which deals with $0 \leq m \leq p-1$. For such m we have $T(m) = L(m)$ and so

$$H^0(G_1, T(m)) = \begin{cases} L(0), & m = 0 \\ \\ 0, & 1 \leq m \leq p-1. \end{cases}$$

Next we consider the case $m = p-1+t$ for $1 \leq t \leq p-1$. Here $T(m)$, considered as a G_1-module, is the injective envelope of $L(p-1-t)$ [1, Example 2.2.1]. In particular $L(p-1-t)$ is the socle of $T(p-1+t)$ so if $H^0(G_1, T(p-1+t)) \neq 0$ then $H^0(G_1, L(p-1-t)) \neq 0$. Considering the case $t = p-1$ separately we get

$$H^0(G_1, T(m)) = \begin{cases} L(0), & t = p-1 \\ \\ 0, & 1 \leq t \leq p-2. \end{cases}$$

For the remaining cases we use induction by writing $m = p-1+t+pn$ for some $n \in \mathbb{N}$ and $0 \leq t \leq p-1$ so that we can write $T(m) = T(p-1+t) \otimes T(n)^F$. Taking the G_1 fixed points we get $H^0(G_1, T(m)) = H^0(G_1, T(p-1+t)) \otimes T(n)^F$ which by the previous case gives us

$$H^0(G_1, T(m)) = \begin{cases} T(n)^F, & t = p-1 \\ \\ 0, & 0 \leq t \leq p-2, \end{cases}$$

so that

$$W = \begin{cases} T(n), & t = p-1 \\ \\ 0, & 0 \leq t \leq p-2, \end{cases}$$

and is thus tilting. □

3. Lemmas

In order to prove Theorem 1.1, we gather some elementary results on the modules $\nabla(r) \otimes \Delta(s)$. First we make an important observation.

Remark 3.1. Since the dual of a tilting module is also a tilting module, and we have the relation $(\nabla(r) \otimes \Delta(s))^* = \nabla(s) \otimes \Delta(r)$, it's clear that $\nabla(s) \otimes \Delta(r)$ is tilting if and only if $\nabla(r) \otimes \Delta(s)$ is tilting. Hence, for many of the results in this section, it's sufficient to only prove the result for $r \geq s$.

Lemma 3.2. *Let $t, u \in \mathbb{N}$. The module $\nabla(p-1+pt) \otimes \Delta(p-1+pu)$ is tilting if and only if the module $\nabla(t) \otimes \Delta(u)$ is tilting.*

Proof. First recall the identities $\nabla(p-1+pt)) = \nabla(p-1) \otimes \nabla(t)^F$ and $\Delta(p-1+pu) = \Delta(p-1) \otimes \Delta(u)^F$, found in [5, Proposition II.3.19]. Using these we may rewrite $\nabla(p-1+pt) \otimes \Delta(p-1+pu)$ as $\nabla(p-1) \otimes \Delta(p-1) \otimes (\nabla(t) \otimes \Delta(u))^F$.

Using Lemma 2.4 we easily obtain the forward implication. The reverse implication is also clear since $\nabla(p-1) \otimes \Delta(p-1)$ is tilting and projective as a G_1-module, so we can apply Lemma 2.3. □

Lemma 3.3. *Let $r = r_0 + pt$, $s = p-1+pu$ for some $0 \leq r_0 \leq p-2$ and $t, u \in \mathbb{N}$. Then $\nabla(r) \otimes \Delta(s)$ is tilting if and only if both $\nabla(t) \otimes \Delta(u)$ and $\nabla(t-1) \otimes \Delta(u)$ are tilting.*

Proof. First we assume that both $\nabla(t) \otimes \Delta(u)$ and $\nabla(t-1) \otimes \Delta(u)$ are tilting, and show that $\nabla(r) \otimes \Delta(s)$ is tilting. We will use the identity $\Delta(s) = \Delta(p-1) \otimes \Delta(u)^F$ as above, and the short exact sequence

$$0 \longrightarrow \nabla(r_0) \otimes \nabla(t)^F \longrightarrow \nabla(r) \longrightarrow \nabla(p-2-r_0) \otimes \nabla(t-1)^F \longrightarrow 0, \tag{3}$$

which can be found in [4, Satz 3.8, Bemerkung 2], in its dual form for Weyl modules. Tensoring sequence (3) with $\Delta(s)$ gives the following short exact sequence

$$\begin{aligned} 0 \longrightarrow \nabla(r_0) \otimes \Delta(p-1) \otimes (\nabla(t) \otimes \Delta(u))^F \longrightarrow \nabla(r) \otimes \Delta(s) \\ \longrightarrow \nabla(p-2-r_0) \otimes \Delta(p-1) \otimes (\nabla(t-1) \otimes \Delta(u))^F \longrightarrow 0. \end{aligned} \tag{4}$$

Now, for $0 \leq r_0 \leq p-2$, both $\nabla(r_0) \otimes \Delta(p-1)$ and $\nabla(p-2-r_0) \otimes \Delta(p-1)$ are tilting and projective as G_1-modules, so if both $\nabla(t) \otimes \Delta(u)$ and $\nabla(t-1) \otimes \Delta(u)$ are also tilting then by Lemma 2.3 both the second and fourth terms in sequence (4) are tilting. Hence we have that $\nabla(r) \otimes \Delta(s)$ is an extension of tilting modules. The only such extensions are split (e.g. by [5, Proposition II.4.16]), so we obtain $\nabla(r) \otimes \Delta(s)$ as a direct sum of two tilting modules, and hence is tilting itself.

For the converse statement, we claim that if $\nabla(r) \otimes \Delta(s) = \nabla(r_0 + pt) \otimes \Delta(s)$ is tilting for some $r_0 \in \{0, 1, \ldots, p-2\}$, then each module $\nabla(v + pt) \otimes \Delta(s)$ for $v \in \{-1, 0, 1, \ldots, p-1\}$ is tilting, so in particular the modules $\nabla(p-1+p(t-1)) \otimes \Delta(s)$ and $\nabla(p-1+pt) \otimes \Delta(s)$ are tilting. We prove the claim by induction on v, taking $v = r_0$ for the base case and using Proposition 2.1 for the induction step we obtain

$$\big(\nabla(v+pt) \otimes E\big) \otimes \Delta(s) = \big(\nabla(v+1+pt) \otimes \Delta(s)\big) \oplus \big(\nabla(v-1+pt) \otimes \Delta(s)\big),$$

so that both $\nabla(v+1+pt) \otimes \Delta(s)$ and $\nabla(v-1+pt) \otimes \Delta(s)$ are tilting. The result now follows from Lemma 3.2 applied to $\nabla(p-1+p(t-1)) \otimes \Delta(s)$ and $\nabla(p-1+pt) \otimes \Delta(s)$. □

Remark 3.4. Note that by duality we obtain the corresponding result for when $r = p-1+pt$ and $s = s_0+pu$ for some $0 \leq s_0 \leq p-2$, and $t, u \in \mathbb{N}$. In this case we have that $\nabla(r) \otimes \Delta(s)$ is tilting if and only if both $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$ are tilting.

It remains to determine which of the modules $\nabla(r) \otimes \Delta(s)$ are tilting when neither r nor s is congruent to $p-1$ modulo p. It turns out that this only occurs in the cases given in Lemma 2.2.

Lemma 3.5. *Let G be a semisimple, simply connected algebraic group over k, and let T be a G-module that is projective as a G_1-module. Then $\chi((p-1)\rho)$ divides $Ch(T)$.*

Proof. This follows immediately from [2, 1.2(2)], since T must also be a projective B_1 module. □

We now revert to the case $G = SL_2(k)$ and obtain the following corollary.

Corollary 3.6. *For all $r \geq p-1$, the character of the Steinberg module $\nabla(p-1)$ divides that of the indecomposable tilting module $T(r)$ of highest weight r.*

Proof. By [1, Proposition 2.4] we have that for all $r \geq p-1$, the module $T(r)$ is a projective G_1-module. □

Now let's consider the character $\chi(r) \in \mathbb{Z}[x, x^{-1}]$. We have that

$$\begin{aligned}\chi(r) &= x^r + x^{r-2} + \ldots + x^0 + \ldots + x^{-r} \\ &= \frac{1}{x^r}(x^{2r} + x^{2r-2} + \ldots + 1) \\ &= \frac{1}{x^r}\left(\frac{x^{2r+2}-1}{x^2-1}\right),\end{aligned}$$

so the roots of this equation are the $(2r+2)$th roots of unity, except ± 1. If $\chi(p-1)$ divides $\chi(r)$ then, we must have that the $2p$th roots of unity are also $(2r+2)$th roots of unity, which would imply that p divides $r+1$, i.e. that r is congruent to $p-1$ modulo p.

Hence we have shown that if both r and s are not congruent to $p-1$ modulo p, the character $\chi(p-1)$ does not divide $\mathrm{Ch}(\nabla(r) \otimes \Delta(s)) = \chi(r)\chi(s)$. Now suppose that $\nabla(r) \otimes \Delta(s)$ is tilting, and that $|r-s| > p-1$. By considering its good filtration (given in (1)), we see that the decomposition of $\nabla(r) \otimes \Delta(s)$ into indecomposable tilting modules cannot contain any $T(j)$ for $j = 0, \ldots, p-1$. By Corollary 3.6 its character is divisible by $\chi(p-1)$ but the above calculation contradicts this. In summary:

Lemma 3.7. *For r and s both not congruent to $p-1$ modulo p, and $|r-s| > p-1$, the module $\nabla(r) \otimes \Delta(s)$ is not tilting.* □

There are now only a few more cases which we have not considered, which we deal with in the following lemma.

Lemma 3.8. *Let $r = r_0 + pt$ and $s = s_0 + pu$ with $r_0, s_0 \in \{0, 1, \ldots, p-2\}$. Then $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $t = u$.*

Proof. Assume that $\nabla(r) \otimes \Delta(s)$ is tilting, and suppose for a contradiction that $t \neq u$ with r and s chosen so that $r + s$ is minimal. Since $s \not\equiv p - 1 (\operatorname{mod} p)$ we have, by Proposition 2.1

$$\nabla(r) \otimes \big(E \otimes \Delta(s)\big) = \nabla(r) \otimes \Delta(s-1) \oplus \nabla(r) \otimes \Delta(s+1)$$

is tilting, and so the module $\nabla(r) \otimes \Delta(s-1)$ is tilting. Now, if $s_0 \neq 0$ then $s-1 = s_0 - 1 + pu$ with $s_0 - 1 \geq 0$. Since r and s were chosen so that $r + s$ was minimal we must have that $t = u$, contradicting our initial assumption. We must have then, that $s_0 = 0$.

Similarly, if $r_0 \neq 0$, we obtain a contradiction. But if $s_0 = r_0 = 0$, then since $t \neq u$ we must have $|r - s| \geq p$, so by Lemma 3.7 we obtain a contradiction.

For the converse, we assume $t = u$, so that we have $r, s \in \{np, np+1, \ldots, np+p-2\}$ for some $n \in \mathbb{N}$. Then by Lemma 2.2 the module $\nabla(r) \otimes \Delta(s)$ is tilting. □

Remark 3.9. Note that Lemma 3.8 shows us that if $\nabla(r) \otimes \Delta(s)$ is a tilting module, then we must have either at least one of r and s congruent to $p-1$ modulo p, or both r and s lie in the set $\{np, np+1, \ldots, np+p-2\}$ for some $n \in \mathbb{N}$.

We are now in a position where, given any r and s we could determine whether the module $\nabla(r) \otimes \Delta(s)$ is tilting by repeated application of the lemmas from this section. What remains is to use these results to prove Theorem 1.1, the statement of which is just a closed form of this procedure, based on the p-adic expansions of r and s. (See Fig. 2.)

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 in two steps. The first is to show that for a primitive pair $(\hat{r}, \hat{s})$, we have that $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is a tilting module if and only if $\hat{r}$ and $\hat{s}$ are as described in the statement of Theorem 1.1. The second step is to show that for any pair (r, s) with primitive pair $(\hat{r}, \hat{s})$, we have that $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. By the duality argument in Remark 3.1, we may assume that $r \geq s$ throughout.

Proposition 4.1. *Let (r, s) be a primitive pair. Then the module $\nabla(r) \otimes \Delta(s)$ is tilting if and only if*

$$r = p^n - 1 + ap^n, \quad s < p^{n+1},$$

or

$$s = p^n - 1 + bp^n, \quad r < p^{n+1},$$

for some $n \in \mathbb{N}$ and $a, b \in \{0, \ldots, p-2\}$.

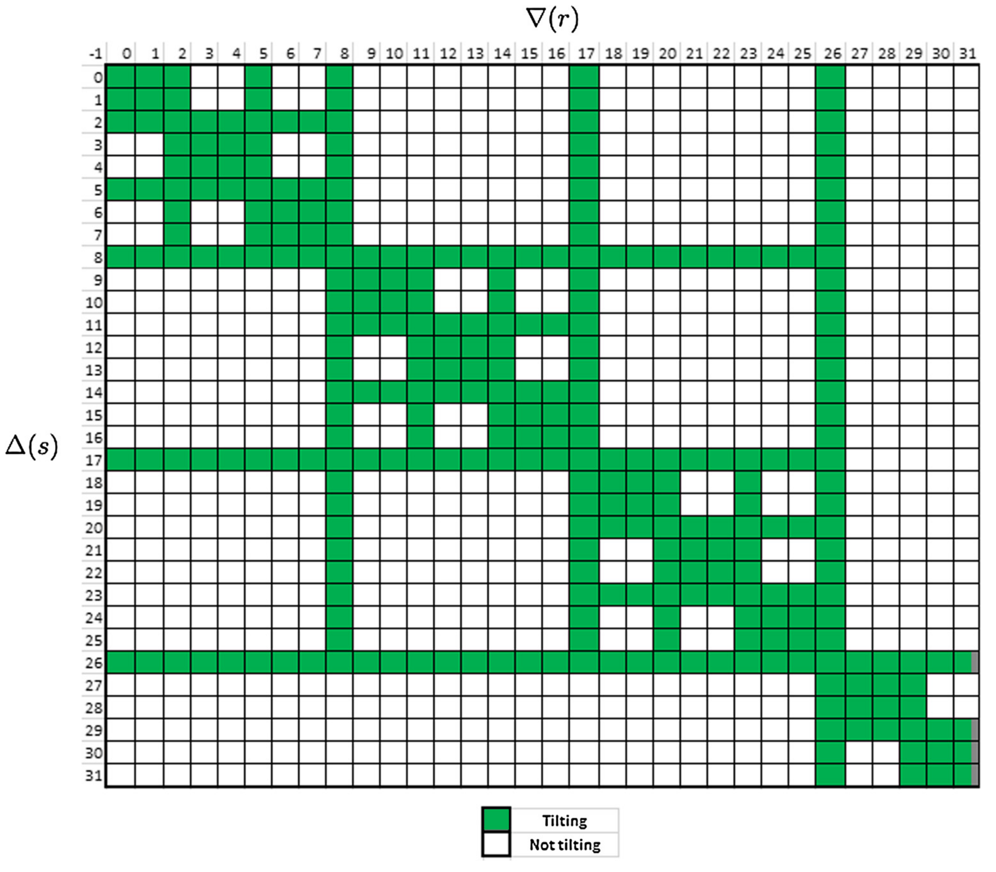

Fig. 2. The modules $\nabla(r) \otimes \Delta(s)$ when $\mathrm{char}(k) = 3$.

Proof. ($\Rightarrow$) We assume that for a primitive pair (r, s), we have that $\nabla(r) \otimes \Delta(s)$ is tilting, and proceed by induction on $\mathrm{len}_p(r) = N$. For $N = 0$ we have that $r \leq p - 1$ and so we may write r in the form $r = ap^N + p^N - 1$ for $a = 0, \ldots, p-2$, or in the case $r = p-1$ we have $r = p^{N+1} - 1$. In each case we have that r is of the desired form, and $s < r < p^{N+1}$.

Next let's write $r = r_0 + pt$ and $s = s_0 + pu$ for some $u, t \in \mathbb{N}$ and $0 \leq r_0, s_0 < p$, so that $\mathrm{len}_p(t) = \mathrm{len}_p(r) - 1$, and $\mathrm{len}_p(u) = \mathrm{len}_p(s) - 1$. Since $\nabla(r) \otimes \Delta(s)$ is tilting, by Remark 3.9 we must have that either r_0 or s_0 is equal to $p - 1$, or r and s both lie in the set $\{np, np+1, \ldots, np+p-2\}$ for some $n \in \mathbb{N}$. However, since we are assuming that the pair (r, s) is primitive, we cannot have the second case. Hence either $r_0 = p - 1$ or $s_0 = p - 1$. Let's prove the statement for $r_0 = p - 1$, and note that the case $s_0 = p - 1$ is proved similarly.

Now we have two further cases to consider, the first is that $s_0 = p - 1$, and the second that $s_0 \neq p - 1$.

i.) Suppose that $s_0 = p-1$, then by Lemma 3.2 we have that $\nabla(t) \otimes \Delta(u)$ is tilting. By induction we must have that t and u are of the form given in the statement of the theorem. If $t = p^{N-1} - 1 + ap^{N-1}$ for some $a \in \{0, \dots, p-2\}$, and $u \leq p^N - 1$, then

$$r = p(p^{N-1} - 1 + ap^{N-1}) + p - 1 = p^N - 1 + ap^N,$$

and $s \leq p^N - p + s_0$, which is strictly less than p^{N+1} since $s_0 < p$. On the other hand, if we have that $u = p^{N-1} - 1 + bp^{N-1}$ and $t < p^N - 1$, then we obtain, in a similar manner, $s = p^N - 1 + bp^N$ and $r < p^{N+1}$.

ii.) For the second case, we suppose that $s_0 \neq p-1$, so that by Lemma 3.3 we have that $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$ are tilting. By induction we have that the pairs (t, u) and $(t, u-1)$ are both of the form in the theorem. Since we cannot have that both u and $u-1$ are of the form $p^{N-1} - 1 + bp^{N-1}$, we must have that $t = p^{N-1} - 1 + ap^{N-1}$ and $u \leq p^N - 1$, so we obtain $r = p^N - 1 + ap^N$ and $s \leq p^{N+1}$ as above.

($\Leftarrow$) We prove the converse statement for the case that $r = p^n - 1 + ap^n$ for some $a \in \{0, \dots, p-2\}$, $n \in \mathbb{N}$, and $s < p^{n+1}$, and note that other case is proved similarly. Once again, we use induction on n, with the case $n = 0$ being clear. For the inductive step, we have that if $r = p^n - 1 + ap^n = p - 1 + pt$ and $s = s_0 + pu < p^{n+1}$ for some t and u, then $t = p^{n-1} - 1 + ap^{n-1}$ and $u < p^n$. By induction the modules $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$ are tilting, so by Lemma 3.3 (or Lemma 3.2 if $s_0 = p-1$) we have that $\nabla(r) \otimes \Delta(s)$ is tilting too. □

Proposition 4.2. *Let $(\hat{r}, \hat{s})$ be the primitive of (r, s). Then $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting.*

Proof. Following Remark 3.9, we will first look at the case where at least one of r and s (and hence $\hat{r}$ and $\hat{s}$) is congruent to $p-1$. Note that if $r = s$, then we have $(\hat{r}, \hat{s}) = (0, 0)$, so in this case the result holds. Let's suppose then, that $r = p - 1 + pt$ and $s = s_0 + pu$, so that we have $\hat{r} = p - 1 + p\hat{t}$ and $\hat{s} = s_0 + p\hat{u}$. We remark that the other case, when $s = p - 1 + pu$ and $r = r_0 + pt$, is obtained in an identical manner.

As before, there are two cases to consider: $s_0 = p-1$ and $s_0 \neq p-1$. In both cases we will proceed by induction on $\operatorname{len}_p(r)$. Let's first consider the case $s_0 = p-1$, where, when $\operatorname{len}_p(r) = 0$, we have that $r = s = p-1$ which we have already covered. Now by Lemma 3.2 we have $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(t) \otimes \Delta(u)$ is tilting. By induction then we have that this is tilting if and only if $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting. Applying Lemma 3.2 again we find that $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting.

Next, we consider the case $s_0 \neq p-1$, where we may assume $r > s$. Again, the base case is easily obtained since this time the pair $(p-1, s_0)$ is primitive. For the inductive step, we will consider separately the cases $u \not\equiv 0 \mod p$ and $u \equiv 0 \mod p$. If $u \not\equiv 0 \mod p$ then, since $t > u$ it's clear that the pair $(\hat{t}, \widehat{u-1})$ is equal to the pair $(\hat{t}, \hat{u} - 1)$. We then have that $\nabla(r) \otimes \Delta(s)$ is tilting if and only if $\nabla(t) \otimes \Delta(u)$ and $\nabla(t) \otimes \Delta(u-1)$

are tilting by Lemma 3.3. By induction, these are tilting if and only if both $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ and $\nabla(\hat{t}) \otimes \Delta(\widehat{u-1})$ are tilting. Now $\nabla(\hat{t}) \otimes \Delta(\widehat{u-1}) = \nabla(\hat{t}) \otimes \Delta(\hat{u}-1)$, so we apply Lemma 3.3 again to obtain that these are tilting if and only if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting.

For the case $u \equiv 0 \mod p$, we treat each direction separately. If $\nabla(r) \otimes \Delta(s)$ is tilting, then by Lemma 3.3 we have that $\nabla(t) \otimes \Delta(u)$ is tilting, and by induction we have that $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting. Now $\hat{u} \equiv 0 \mod p$, so by Proposition 2.1 we obtain

$$\nabla(\hat{t}) \otimes E \otimes \Delta(\hat{u}) = \nabla(\hat{t}) \otimes \Delta(\hat{u}+1) \oplus \nabla(\hat{t}) \otimes \Delta(\hat{u}-1).$$

The module on the left hand side is a tilting module, so $\nabla(\hat{t}) \otimes \Delta(\hat{u}-1)$ is also a tilting module. We apply Lemma 3.3 again to obtain that $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. For the reverse direction we have that if $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting, then $\nabla(\hat{t}) \otimes \Delta(\hat{u})$ is tilting, so by induction $\nabla(t) \otimes \Delta(u)$ is also tilting. Now, as above, since $u \equiv 0 \mod p$ we have that $\nabla(t) \otimes \Delta(u-1)$ is also tilting, so we apply Lemma 3.3 to obtain that $\nabla(r) \otimes \Delta(s)$ is tilting.

What remains is to prove the result when both r and s lie in the set $\{np, np+1, \dots, (n+1)p-2\}$ for some $n \in \mathbb{N}$. From Lemma 2.2 we know already that for such r and s the module $\nabla(r) \otimes \Delta(s)$ is tilting, so it's sufficient to show that $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. However, it's clear that in this case $\hat{r}$ and $\hat{s}$ lie in the set $\{0, \dots, p-2\}$, and so $\nabla(\hat{r}) \otimes \Delta(\hat{s})$ is tilting. $\square$

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