Irreducible modules for equivariant map superalgebras and their extensions

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A R T I C L E   I N F O

A B S T R A C T

Let Γ be a group acting on a scheme X and on a Lie superalgebra $\mathfrak{g}$, both defined over an algebraically closed field of characteristic zero $\Bbbk$. The corresponding equivariant map superalgebra $M(\mathfrak{g}, X)^\Gamma$ is the Lie superalgebra of Γ-equivariant regular maps from X to $\mathfrak{g}$. In this paper we complete the classification of finite-dimensional irreducible $M(\mathfrak{g}, X)^\Gamma$-modules when $\mathfrak{g}$ is a finite-dimensional simple Lie superalgebra, X is of finite type and Γ is a finite abelian group acting freely on the rational points of X, by classifying these $M(\mathfrak{g}, X)^\Gamma$-modules in the case where $\mathfrak{g}$ is a periplectic Lie superalgebra. We also describe extensions between irreducible modules in terms of homomorphisms and extensions between modules for certain finite-dimensional Lie superalgebras. As an application, one obtains the block decomposition of the category of finite-dimensional $M(\mathfrak{g}, X)^\Gamma$-modules in terms of blocks and spectral characters of finite-dimensional Lie superalgebras.

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Contents

1. Introduction

Lie superalgebras have a wide range of applications in many areas of physics and mathematics, such as supersymmetry, string theory, conformal field theory and number theory. In the study of symmetries, for instance, while Lie algebras describe bosonic degrees of freedom, Lie superalgebras also allow fermonic degrees of freedom [32]. In number theory, affine Kac–Moody superalgebras and their representations can be used to study problems related to sums of squares and sums of triangular numbers [19]. For more examples, see for instance [9,10,14,30].

It is usually the case that the representation theory of Lie superalgebras is more complicated than that of their Lie algebra counterparts. For instance, the category of finite-dimensional modules for a finite-dimensional simple Lie algebra is always semisimple, while the category of finite-dimensional modules for a finite-dimensional simple Lie superalgebra is not necessarily so. It is thus important to describe extensions between their irreducible modules. Despite being a subject of intense study since the birth of supersymmetry, these extensions are not known in general. And in contrast with extensions for Lie algebras, their study is often done case by case. See, for instance, [1,5,11,12,15,16,26,28,29] for some of these results.

The main goal of the current paper is to develop the representation theory of certain Lie superalgebras known as equivariant map superalgebras. Equivariant map superalgebras generalize, on the one hand, simple Lie superalgebras, and on the other hand, current and loop Lie algebras. They are constructed in the following way. Consider a scheme X, a Lie superalgebra $\mathfrak{g}$, both defined over an algebraically closed field of characteristic zero $\Bbbk$, and a group Γ that acts on X and $\mathfrak{g}$ by automorphisms. The corresponding

equivariant map superalgebra $M(X, \mathfrak{g})^\Gamma$ is the Lie superalgebra of Γ-equivariant regular maps from X to $\mathfrak{g}$. If one denotes by A the coordinate ring of X, then $M(X, \mathfrak{g})^\Gamma$ can be identified with the Lie subsuperalgebra $(\mathfrak{g} \otimes_\Bbbk A)^\Gamma$ of $\mathfrak{g} \otimes_\Bbbk A$ that consists of its Γ-fixed points under the diagonal action.

In the particular case where $\mathfrak{g}$ is a Lie algebra, $M(X, \mathfrak{g})^\Gamma$ is called equivariant map algebra. The representation theory of these Lie algebras has been a subject of intense research for the last thirty years (see, for instance, the survey [23]). One reason is that representations of $M(\Bbbk^\times, \mathfrak{g})^\Gamma$, known as twisted loop algebras, and $M(\Bbbk, \mathfrak{g})^\Gamma$, known as twisted current algebras, are closely related to those of affine Kac–Moody Lie algebras. In fact, when $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and Γ is the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$, the twisted current algebra is a parabolic subalgebra of the affine Kac–Moody Lie algebra associated to $\mathfrak{g}$ and Γ, and the twisted loop algebra is its centerless derived subalgebra (see [21, Section 13.1]).

Finite-dimensional irreducible representations of equivariant map algebras were classified by Neher, Savage and Senesi [25] in the case where $\mathfrak{g}$ is a finite-dimensional Lie algebra and Γ is a finite group. Finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero and finite-dimensional irreducible representations of the so-called basic classical Lie superalgebras were classified by Kac in [17] and [18]. In [27], Savage classified irreducible finite-dimensional representations of equivariant map superalgebras in the case where $\mathfrak{g}$ is a basic classical Lie superalgebra (or $\mathfrak{sl}(n, n)$, $n \geq 1$, if Γ is trivial), X has a finitely-generated coordinate ring and Γ is a finite abelian group acting freely on the rational points of X. Moving beyond basic classical Lie superalgebras, the first author, Moura and Savage classified finite-dimensional irreducible representations of equivariant map queer Lie superalgebras in [7]. While in the basic classical setting those irreducible representations were isomorphic to tensor products of generalized evaluation representations, in the queer case they are irreducible products of evaluation representations (see Section 2.3 for details). In [2], Bagci extended this classification to equivariant map superalgebras where $\mathfrak{g}$ is of Cartan type.

In the current paper, we complete the classification of finite-dimensional irreducible representations of equivariant map superalgebras by describing these modules for $M(\mathfrak{g}, X)^\Gamma$ when $\mathfrak{g} = \mathfrak{p}(n)$, a periplectic Lie superalgebra, X has a finitely-generated coordinate ring, Γ is a finite abelian group acting on $\mathfrak{g}$ and X, and such that the induced action of Γ on the rational points of X is free. In Theorem 3.9 we prove that, similarly to the case where $\mathfrak{g}$ is basic classical of type II, all irreducible finite-dimensional $M(\mathfrak{g}, X)^\Gamma$-modules are evaluation modules. The technique used to prove this result is similar to the one used in [27]. Behind this technique are the facts that one can choose a Cartan subalgebra of $\mathfrak{p}(n)$ that is purely even and abelian, that the root space decomposition of $\mathfrak{p}(n)$ with respect to such a subalgebra is relatively similar to that of the basic case, and that $\mathfrak{p}(n)_{\bar{0}}$ is a semisimple Lie algebra.

Equipped with a complete classification of irreducible finite-dimensional $M(\mathfrak{g}, X)^\Gamma$-modules, one can inquire about their extensions. This is the second problem that we address in the current paper. In Theorem 5.7 we describe extensions between finite-

dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules in terms of homomorphisms and extensions between finite-dimensional indecomposable (irreducible, in some cases) modules for a finite-dimensional Lie superalgebra of the form $\mathfrak{g} \otimes A/\mathfrak{m}^n$, where $\mathfrak{m}$ is a maximal ideal of A and n is a positive integer. The technique used here is similar to that used in [23] and [4] for the nonsuper case. The main difference in this super setting is that one needs to describe the kernel of certain transgression maps, which we do in Section 4.

Using these formulas for extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules, we are able to describe the block decomposition of the category of finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-modules in terms of blocks and spectral characters of certain finite-dimensional Lie superalgebras. This description extends results of Kodera [20], Neher and Savage [24].

Organization of the paper

In Section 2 we fix the notation and state some results that will be used throughout the paper. Most of the results in this section are known and we provide a source for their proofs. However, as far as we know, Proposition 2.5, Lemmas 2.10, 2.11, and Corollary 2.12 have not appeared in the literature before, thus we provide complete proofs for these results. In Section 3 we describe the structure of periplectic Lie superalgebras, construct and classify the finite-dimensional irreducible $M(\mathfrak{g}, X)^\Gamma$-modules when $\mathfrak{g}$ is of periplectic type, X has a finitely-generated coordinate ring, Γ is a finite abelian group acting on $\mathfrak{g}$ and X, and such that the action of Γ on the rational points of X is free (see Section 3.3). In Section 4 we detail the construction of Lyndon–Hochschild–Serre spectral sequences and describe the kernel of certain transgression maps that are relevant to the computation of extensions between finite-dimensional irreducible $M(\mathfrak{g}, X)^\Gamma$-modules (see Proposition 4.2). Using the results of Sections 3 and 4, in Section 5 we develop a general technique to describe p-extensions between finite-dimensional irreducible $M(\mathfrak{g}, X)^\Gamma$-modules, and describe 1-extensions between these modules in terms of homomorphisms and extensions between finite-dimensional indecomposable (irreducible, in some cases) modules for a finite-dimensional Lie superalgebra of the form $\mathfrak{g} \otimes A/\mathfrak{m}^n$, where $\mathfrak{m}$ is a maximal ideal of A and n is a positive integer (see Theorem 5.7 for the main result of this section). We finish the paper with some examples and applications.

Acknowledgments

2. Notation and preliminaries

2.1. Notation

Let $\Bbbk$ denote an algebraically closed field of characteristic zero, $\mathbb{Z}_2$ denote the finite field with two elements $\{\bar{0}, \bar{1}\}$, and $\mathbb{Z}_{>0}$ denote the set of positive integers. All vector spaces, algebras, and tensor products will be considered over the field $\Bbbk$ (unless otherwise specified). A vector space V is said to be a super space if it is $\mathbb{Z}_2$-graded; that is, there exist subspaces $V_{\bar{0}}, V_{\bar{1}} \subseteq V$ such that $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We denote by $|\cdot|$ the degree of a homogeneous element in a super space V; that is, $|v| = z$ for all $v \in V_z$ and $z \in \mathbb{Z}_2$.

2.2. Finite-dimensional Lie superalgebras

In this subsection we follow [13,22].

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a $\mathbb{Z}_2$-graded linear transformation $[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that satisfies $\mathbb{Z}_2$-graded versions of anticommutativity and Jacobi identity. Given two Lie superalgebras $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ and $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$, every homomorphism of Lie superalgebras $\phi\colon \mathfrak{a} \to \mathfrak{b}$ is assumed to be even; that is, $\phi(\mathfrak{a}_{\bar{0}}) \subseteq \mathfrak{b}_{\bar{0}}$ and $\phi(\mathfrak{a}_{\bar{1}}) \subseteq \mathfrak{b}_{\bar{1}}$. Similarly, given $\mathfrak{a}$-modules $M = M_{\bar{0}} \oplus M_{\bar{1}}$ and $N = N_{\bar{0}} \oplus N_{\bar{1}}$, every homomorphism of $\mathfrak{a}$-modules $\psi\colon M \to N$ is assumed to be even; that is, $\psi(M_{\bar{0}}) \subseteq N_{\bar{0}}$ and $\psi(M_{\bar{1}}) \subseteq N_{\bar{1}}$.

The following examples will be used throughout the paper.

Example 2.1. Consider a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The associative algebra $\mathrm{End}(V)$, consisting of linear endomorphisms of V admits a $\mathbb{Z}_2$-grading defined by

$$\mathrm{End}(V)_z = \{\phi \in \mathrm{End}(V) \mid \phi(v) \in V_{z+z'} \text{ for all } v \in V_{z'}, z' \in \mathbb{Z}_2\},$$

and a Lie superbracket defined by $[\phi, \psi] = \phi \circ \psi - (-1)^{|\phi||\psi|}\psi \circ \phi$. This Lie superalgebra will be denoted by $\mathfrak{gl}(V)$.

Example 2.2. Consider a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Its exterior algebra $\Lambda(V)$ admits a $\mathbb{Z}_2$-grading given by $|v_1 \wedge \cdots \wedge v_n| = |v_1| + \cdots + |v_n|$ for all homogeneous elements $v_1, \ldots, v_n \in V$, and a unique Lie superbracket satisfying $[v, w] = v \wedge w - (-1)^{|v||w|}w \wedge v$ for all homogeneous $v, w \in \Lambda(V)$. It follows that

$$\Lambda^n(V) = \bigoplus_{i+j=n} \Lambda^i(V_{\bar{0}}) \otimes S^j(V_{\bar{1}}) \qquad \text{for all } n \geq 0.$$

For any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, the subspace $\mathfrak{g}_{\bar{0}}$ inherits the structure of a Lie algebra and the subspace $\mathfrak{g}_{\bar{1}}$ inherits the structure of a $\mathfrak{g}_{\bar{0}}$-module. A finite-dimensional simple Lie superalgebra $\mathfrak{g}$ is said to be classical if the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is completely reducible. Otherwise it is said to be of Cartan type. When $\mathfrak{g}$ is a classical Lie superalgebra,

Table 1
Classification of finite-dimensional simple Lie superalgebras with nonzero odd parts.

Lie superalgebra	Classification
$A(m,n)$, $m > n \geq 0$	Basic, type I
$A(n,n)$, $n \geq 1$	Basic, type I
$B(m,n)$, $m \geq 0$, $n \geq 1$	Basic, type II
$C(n+1)$, $n \geq 1$	Basic, type I
$D(m,n)$, $m \geq 2$, $n \geq 1$	Basic, type II
$D(2,1;\alpha)$, $\alpha \neq 0, -1$	Basic, type II
$F(4)$	Basic, type II
$G(3)$	Basic, type II
$\mathfrak{p}(n)$, $n \geq 2$	Strange, type I
$\mathfrak{q}(n)$, $n \geq 2$	Strange, type II
$H(n)$, $n \geq 4$	Cartan type
$S(n)$, $n \geq 3$	Cartan type
$\tilde{S}(n)$, $n = 2m, m \geq 2$	Cartan type
$W(n)$, $n \geq 2$	Cartan type

the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is either irreducible or a direct sum of two irreducible modules. In the first case, $\mathfrak{g}$ is said to be of type II, and in the second case, $\mathfrak{g}$ is said to be of type I. A classical Lie superalgebra is said to be basic if it admits an even nondegenerate supersymmetric invariant bilinear form. Otherwise, it is said to be strange. Table 1 summarizes the classification of finite-dimensional simple Lie superalgebras.

When $\mathfrak{g}$ is a classical Lie superalgebra, a Cartan subalgebra of $\mathfrak{g}$ is defined to be a Cartan subalgebra of the Lie algebra $\mathfrak{g}_{\bar{0}}$. When $\mathfrak{g}$ is a Lie superalgebra of Cartan type, it admits a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{-1 \leq i} \mathfrak{g}_i$ that is compatible with the $\mathbb{Z}_2$-grading; that is, $\mathfrak{g}_{\bar{0}} = \bigoplus_{i \in 2\mathbb{Z}} \mathfrak{g}_i$ and $\mathfrak{g}_{\bar{1}} = \bigoplus_{i \in 2\mathbb{Z}} \mathfrak{g}_{i+1}$, and such that $\mathfrak{g}_0$ is a reductive Lie algebra. In this case, a Cartan subalgebra of $\mathfrak{g}$ is defined to be a Cartan subalgebra of the Lie algebra $\mathfrak{g}_0$. (It is worth noting that when $\mathfrak{g}$ is of type $\tilde{S}$, this is only a $\mathbb{Z}$-grading as a vector space, not as Lie superalgebras.)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The action of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalizable and we have a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha, \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Let $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ denote the set of roots of $\mathfrak{g}$ and Q denote the subgroup of $\mathfrak{h}^*$ generated by Δ.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra. Then the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{g}$-modules is in bijection with a subset Λ^+ of $\mathfrak{h}^*$ (see [17, Theorem 8]). We denote by Λ the subgroup of $\mathfrak{h}^*$ generated by Λ^+, and by $V(\lambda)$ the irreducible $\mathfrak{g}$-module corresponding to $\lambda \in \Lambda^+$, namely the unique finite-dimensional irreducible $\mathfrak{g}$-module of highest weight λ. Let $\lambda^* \in \Lambda^+$ denote the highest weight of the dual $\mathfrak{g}$-module $V(\lambda)^*$.

For a Lie superalgebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, we define

$$\text{End}_{\mathfrak{g}}(V) = \{\varphi \in \text{End}(V) \mid \varphi(xv) = x\varphi(v), \text{ for all } x \in \mathfrak{g}, \ v \in V\},$$

$$\operatorname{End}_{\mathfrak{g}}(V)_{\bar{0}} = \operatorname{End}_{\mathfrak{g}}(V) \cap \operatorname{End}(V)_{\bar{0}} \text{ and } \operatorname{End}_{\mathfrak{g}}(V)_{\bar{1}} = \operatorname{End}_{\mathfrak{g}}(V) \cap \operatorname{End}(V)_{\bar{1}}.$$

Let $\mathfrak{g}^1$ and $\mathfrak{g}^2$ be two finite-dimensional Lie superalgebras, and V_1 and V_2 be irreducible finite-dimensional modules for $\mathfrak{g}^1$ and $\mathfrak{g}^2$ respectively. The $(\mathfrak{g}^1 \oplus \mathfrak{g}^2)$-module $V_1 \otimes V_2$ is reducible only if $\operatorname{End}_{\mathfrak{g}^1}(V_1)_{\bar{1}} \neq 0$ and $\operatorname{End}_{\mathfrak{g}^2}(V_2)_{\bar{1}} \neq 0$ (see [6, Proposition 8.4]). In this case, by Schur's Lemma for Lie superalgebras, $\operatorname{End}_{\mathfrak{g}^i}(V_i)_{\bar{1}} = \Bbbk\varphi_i$ for some $\varphi_i^2 = 1$, $i \in \{1, 2\}$, and

$$\widehat{V} = \left\{ v \in V_1 \otimes V_2 \mid \left(\sqrt{-1}\varphi_1 \otimes \varphi_2\right) v = v \right\}$$

is an irreducible $(\mathfrak{g}^1 \oplus \mathfrak{g}^2)$-submodule satisfying $V_1 \otimes V_2 \cong \widehat{V} \oplus \widehat{V}$ (see [6, p. 27]). Define the irreducible product $V_1 \widehat{\otimes} V_2$ to be

$$V_1 \widehat{\otimes} V_2 = \begin{cases} V_1 \otimes V_2, & \text{if } V_1 \otimes V_2 \text{ is irreducible,} \\ \widehat{V}, & \text{otherwise.} \end{cases}$$

If $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are finite-dimensional simple Lie superalgebras not of type $\mathfrak{q}$, then the irreducible product is always equal to the tensor product.

Given $\ell > 1$, finite-dimensional Lie superalgebras $\mathfrak{g}^1, \ldots, \mathfrak{g}^\ell$, and for each $i \in \{1, \ldots, \ell\}$, an irreducible finite-dimensional $\mathfrak{g}^i$-module V_i, define the $(\mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^\ell)$-module $V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_\ell$ to be

$$V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_\ell = (V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_{\ell-1}) \widehat{\otimes} V_\ell.$$

Up to isomorphism, $\widehat{\otimes}$ is associative and, when $\mathfrak{g}^1 = \cdots = \mathfrak{g}^\ell$, $\widehat{\otimes}$ is also commutative (see [7, Lemma 6.2]). Also, define

$$\kappa(V_1, \ldots, V_\ell) = \sum_{i=2}^{\ell} \dim \operatorname{End}_{(\mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^{i-1})} \left(V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_{i-1}\right)_{\bar{1}} \dim \operatorname{End}_{\mathfrak{g}^i}(V_i)_{\bar{1}}. \qquad (2.1)$$

One can prove by induction that $\kappa(V_1, \ldots, V_\ell) \leq \ell - 1$ and that

$$V_1 \otimes \cdots \otimes V_\ell \cong (V_1 \widehat{\otimes} \cdots \widehat{\otimes} V_\ell)^{\oplus 2^k} \qquad \text{for} \quad k = \kappa(V_1, \ldots, V_\ell).$$

2.3. Equivariant map superalgebras

In this subsection, we review results proved in [2,7,27].

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra and A be an associative, commutative, finitely-generated $\Bbbk$-algebra with unit. The map superalgebra $\mathfrak{g} \otimes A$ is the Lie superalgebra with underlying vector space $\mathfrak{g} \otimes_\Bbbk A$, with $\mathbb{Z}_2$-grading given by $(\mathfrak{g} \otimes A)_z = \mathfrak{g}_z \otimes A$, $z \in \mathbb{Z}_2$, and with Lie superbracket extending bilinearly

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

Let Γ be a group acting on $\mathfrak{g}$ and A by automorphisms. One can induce an action of Γ on $\mathfrak{g} \otimes A$ by extending linearly

$$\gamma(x \otimes a) = \gamma(x) \otimes \gamma(a), \quad \text{for all } \gamma \in \Gamma, x \in \mathfrak{g} \text{ and } a \in A.$$

The equivariant map superalgebra $(\mathfrak{g} \otimes A)^\Gamma$ is the Lie subsuperalgebra of $\mathfrak{g} \otimes A$ consisting of its Γ-fixed points:

$$(\mathfrak{g} \otimes A)^\Gamma = \{x \in \mathfrak{g} \otimes A \mid \gamma x = x \text{ for all } \gamma \in \Gamma\}.$$

Let $\mathrm{MaxSpec}(A)$ denote the set of maximal ideals of A. Notice that the action of Γ on A induces an action of Γ on $\mathrm{MaxSpec}(A)$ that is explicitly given by $\gamma\mathsf{m} = \{\gamma a \mid a \in \mathsf{m}\} \in \mathrm{MaxSpec}(A)$ for all $\mathsf{m} \in \mathrm{MaxSpec}(A)$ and $\gamma \in \Gamma$. For most finite-dimensional simple Lie superalgebras $\mathfrak{g}$, if Γ is a finite abelian group acting freely on $\mathrm{MaxSpec}(A)$, then every finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-module can be described in terms of generalized evaluation modules. These generalized evaluation modules are defined in the following way. Given $\mathsf{m} \in \mathrm{MaxSpec}(A)$ and $n \in \mathbb{Z}_{>0}$, define $\mathrm{ev}_{\mathsf{m}^n}$ to be the homomorphism of Lie superalgebras given by the composition

$$\mathrm{ev}_{\mathsf{m}^n} : \mathfrak{g} \otimes A \to \mathfrak{g} \otimes A/\mathfrak{g} \otimes \mathsf{m}^n \xrightarrow{\cong} \mathfrak{g} \otimes A/\mathsf{m}^n.$$

For the rest of this subsection, assume that Γ is a finite group acting freely on $\mathrm{MaxSpec}(A)$.

In this case, the restriction of $\mathrm{ev}_{\mathsf{m}^n}$ to $(\mathfrak{g} \otimes A)^\Gamma$ is surjective (see [27, Lemma 5.6]), and induces a surjective homomorphism of Lie superalgebras

$$\mathrm{ev}_{\mathsf{m}^n}^\Gamma : (\mathfrak{g} \otimes A)^\Gamma \to \mathfrak{g} \otimes A/\mathsf{m}^n.$$

Given a $\mathfrak{g} \otimes A/\mathsf{m}^n$-module V with associated representation $\rho \colon \mathfrak{g} \otimes A/\mathsf{m}^n \to \mathfrak{gl}(V)$, define $\mathrm{ev}_{\mathsf{m}^n}^{\Gamma*}(V)$ to be the $(\mathfrak{g} \otimes A)^\Gamma$-module with associated representation given by the pull-back of ρ along $\mathrm{ev}_{\mathsf{m}^n}^\Gamma$,

$$\mathrm{ev}_{\mathsf{m}^n}^{\Gamma*}(\rho) \colon (\mathfrak{g} \otimes A)^\Gamma \xrightarrow{\mathrm{ev}_{\mathsf{m}^n}^\Gamma} \mathfrak{g} \otimes A/\mathsf{m}^n \xrightarrow{\rho} \mathfrak{gl}(V).$$

Given $\mathsf{m} \in \mathrm{MaxSpec}(A)$ and $n \in \mathbb{Z}_{>0}$, denote by $\mathrm{Irred}(\mathfrak{g} \otimes A/\mathsf{m}^n)$ the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathsf{m}^n$-modules, and denote by $\mathcal{R}$ the disjoint union

$$\mathcal{R} = \bigsqcup_{\substack{n \in \mathbb{Z}_{>0} \\ \mathsf{m} \in \mathrm{MaxSpec}(A)}} \mathrm{Irred}(\mathfrak{g} \otimes A/\mathsf{m}^n).$$

Notice that the action of Γ on $\mathfrak{g} \otimes A$ induces an action of Γ on $\mathcal{R}$. Namely, let $[V]$ be an element in $\mathcal{R}$, and let V be a representative of the class $[V]$ with associated representation

$\rho\colon \mathfrak{g}\otimes A/\mathfrak{m}^n \to \mathfrak{gl}(V)$. For each $\gamma \in \Gamma$, define $\gamma \cdot [V]$ in $\mathcal{R}$ to be the isomorphism class of the $\mathfrak{g}\otimes A/(\gamma\mathfrak{m})^n$-module V^γ, whose associated representation $\rho^\gamma\colon \mathfrak{g}\otimes A/(\gamma\mathfrak{m})^n \to \mathfrak{gl}(V)$ is given by $\rho^\gamma(x) = \rho(\gamma^{-1}x)$, for all $x \in \mathfrak{g}\otimes A/(\gamma\mathfrak{m})^n$.

Denote by $\mathcal{P}$ the set of Γ-equivariant functions $\pi\colon \operatorname{MaxSpec}(A) \to \mathcal{R}$ satisfying the following conditions:

- For each $\mathfrak{m} \in \operatorname{MaxSpec}(A)$, $\pi(\mathfrak{m}) \in \operatorname{Irred}(\mathfrak{g}\otimes A/\mathfrak{m}^n)$ for some $n > 0$;
- $\pi(\mathfrak{m})$ is the isomorphism class of the trivial (one-dimensional irreducible) module for all but finitely many $\mathfrak{m} \in \operatorname{MaxSpec}(A)$.

Notice that for any two representatives V and V' of $\pi(\mathfrak{m}) \in \operatorname{Irred}(\mathfrak{g}\otimes A/\mathfrak{m}^n)$, there is an isomorphism of $(\mathfrak{g}\otimes A)^\Gamma$-modules $\operatorname{ev}_{\mathfrak{m}^n}^{\Gamma}{}^{*}V \cong \operatorname{ev}_{\mathfrak{m}^n}^{\Gamma}{}^{*}V'$. Thus we will abuse notation and for each maximal ideal $\mathfrak{m} \subseteq A$, we will denote by $\pi(\mathfrak{m})$ an arbitrary but fixed choice of $(\mathfrak{g}\otimes A/\mathfrak{m}^n)$-module representative of $\pi(\mathfrak{m})$.

Given $\pi \in \mathcal{P}$, define its support to be $\operatorname{Supp}(\pi) = \{\mathfrak{m} \in \operatorname{MaxSpec}(A) \mid \pi(\mathfrak{m})$ is nontrivial$\}$, and let $\operatorname{Supp}_*(\pi)$ be a subset of $\operatorname{MaxSpec}(A)$ which contains exactly one element of each Γ-orbit in $\operatorname{Supp}(\pi)$. Since every $\pi \in \mathcal{P}$ is Γ-equivariant, up to isomorphism, the $(\mathfrak{g}\otimes A)^\Gamma$-module $\widehat{\bigotimes}_{\mathfrak{m}\in\operatorname{Supp}_*(\pi)}\operatorname{ev}_{\mathfrak{m}^n}^{\Gamma}{}^{*}\pi(\mathfrak{m})$ is independent of the choice of $\operatorname{Supp}_*(\pi)$ (see [27, Lemma 5.9]). Thus for every $\pi \in \mathcal{P}$, we fix an arbitrary subset $\operatorname{Supp}_*(\pi)$ as above and define $\mathcal{V}(\pi)$ to be the $(\mathfrak{g}\otimes A)^\Gamma$-module

$$\mathcal{V}(\pi) = \widehat{\bigotimes_{\mathfrak{m}\in\operatorname{Supp}_*(\pi)}}\operatorname{ev}_{\mathfrak{m}^n}^{\Gamma}{}^{*}\pi(\mathfrak{m}).$$

For most finite-dimensional simple Lie superalgebras, it is known that every finite-dimensional irreducible $(\mathfrak{g}\otimes A)^\Gamma$-module is isomorphic to $\mathcal{V}(\pi)$ for a unique $\pi \in \mathcal{P}$. This was proved by Savage when $\mathfrak{g}$ is a basic Lie superalgebra (see [27, §7]), by Bagci when $\mathfrak{g}$ is a Lie superalgebra of Cartan type (see [2, Theorem 4.3]), and by the first author, Moura and Savage when $\mathfrak{g}$ is a queer Lie superalgebra (see [7, Theorem 7.1]). In Section 3, we will show that, if $\mathfrak{g}$ is of type $\mathfrak{p}$, then every finite-dimensional irreducible $(\mathfrak{g}\otimes A)^\Gamma$-module is also isomorphic to $\mathcal{V}(\pi)$ for a unique $\pi \in \mathcal{P}$, and that $n = 1$ for all $\mathfrak{m} \in \operatorname{Supp}_*(\pi)$. This completes the classification of finite-dimensional irreducible modules for equivariant map superalgebras associated to finite-dimensional simple Lie algebras.

Moreover, if $\mathfrak{g}$ is of type II, H, S or $\tilde{S}$, then every finite-dimensional irreducible $(\mathfrak{g}\otimes A)^\Gamma$-module is isomorphic to $\mathcal{V}(\pi)$ for a unique $\pi \in \mathcal{P}$ with $n = 1$ for all $\mathfrak{m} \in \operatorname{Supp}_*(\pi)$ (these modules are called evaluation modules by Savage, see [27, Definition 5.2]). It is important to remark that if $\mathfrak{g}$ is of type I, then there exist finite-dimensional irreducible $(\mathfrak{g}\otimes A)^\Gamma$-modules which are isomorphic to generalized evaluation modules but not to evaluation modules [8, §4].

Remark 2.3. Let $n, n' \in \mathbb{Z}_{>0}$ with $n' \leq n$, let $\mathfrak{m}$ be a maximal ideal in A, and let V be a $\mathfrak{g}\otimes A/\mathfrak{m}^{n'}$-module with corresponding representation $\rho\colon \mathfrak{g}\otimes A/\mathfrak{m}^{n'} \to \mathfrak{gl}(V)$. Since $n' \leq n$,

then $\mathsf{m}^n \subseteq \mathsf{m}^{n'}$, and therefore we have a canonical projection $\pi \colon \mathfrak{g} \otimes A/\mathsf{m}^n \twoheadrightarrow \mathfrak{g} \otimes A/\mathsf{m}^{n'}$. We can thus regard V as a $\mathfrak{g} \otimes A/\mathsf{m}^n$-module with representation given by $\rho \circ \pi$.

Notice that as representations of $(\mathfrak{g} \otimes A)^\Gamma$, $\mathrm{ev}_{\mathsf{m}^{n'}}^{\Gamma *}(\rho \circ \pi)$ and $\mathrm{ev}_{\mathsf{m}^n}^{\Gamma *}(\rho)$ are the same. Hence, the $(\mathfrak{g} \otimes A)^\Gamma$-modules $\mathrm{ev}_{\mathsf{m}^{n'}}^{\Gamma *}V$ and $\mathrm{ev}_{\mathsf{m}^n}^{\Gamma *}V$ are isomorphic. Moreover, if ρ is an irreducible representation of $\mathfrak{g} \otimes A/\mathsf{m}^{n'}$, then $\rho \circ \pi$ is an irreducible representation of $\mathfrak{g} \otimes A/\mathsf{m}^n$. As a consequence, given any two finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules V and V', one loses no generality in assuming that there exist integers $\ell, n \in \mathbb{Z}_{>0}$, maximal ideals $\mathsf{m}_1, \ldots, \mathsf{m}_\ell \in \mathrm{MaxSpec}(A)$ in distinct Γ-orbits, and for each $i \in \{1, \ldots, \ell\}$, irreducible $\mathfrak{g} \otimes A/\mathsf{m}_i^n$-modules V_i, V_i', such that $V \cong \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma *} V_i$ and $V' \cong \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma *} V_i'$.

Given $\pi \in \mathcal{P}$, let $\ell, n_1, \ldots, n_\ell \in \mathbb{Z}_{>0}$, $\mathsf{m}_1, \ldots, \mathsf{m}_\ell$ be maximal ideals of A in distinct Γ-orbits, and for each $i \in \{1, \ldots, \ell\}$, let V_i be an irreducible $\mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}$-module such that $\mathcal{V}(\pi) \cong \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^{n_i}}^{\Gamma *} V_i$. Recall from (2.1) the definition of $\kappa(V_1, \ldots, V_\ell)$, and define $\kappa(\pi)$ to be

$$\kappa(\pi) = \kappa(V_1, \ldots, V_\ell), \tag{2.2}$$

and notice that $\mathcal{V}(\pi)^{\oplus 2^{\kappa(\pi)}} \cong \bigotimes_{\mathsf{m} \in \mathrm{Supp}\,(\pi)} \mathrm{ev}_{\mathsf{m}^n}^{\Gamma *} \pi(\mathsf{m})$ for all $\pi \in \mathcal{P}$.

2.4. Ideals

Throughout this subsection, let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra and A be an associative, commutative, finitely-generated $\Bbbk$-algebra with unit. Define the support of an ideal $I \subseteq A$ to be

$$\mathrm{Supp}\,(I) = \{\mathsf{m} \in \mathrm{MaxSpec}\, A \mid I \subseteq \mathsf{m}\}.$$

Lemma 2.4. *Let I and J be ideals of A.*

(a) *For any $n > 0$, we have $\mathrm{Supp}\,(I) = \mathrm{Supp}\,(I^n)$.*
(b) *If A is finitely generated, then $\mathrm{Supp}\,(I)$ is finite if and only if I has finite codimension in A.*
(c) *If $\mathrm{Supp}\,(I) \cap \mathrm{Supp}\,(J) = \varnothing$, then $I + J = A$ and $IJ = I \cap J$. Moreover, $I^m + J^n = A$ and $I^m J^n = I^m \cap J^n$ for any $m, n > 0$.*
(d) *If A is Noetherian, then every ideal $I \subseteq A$ contains a power of its radical. In particular, $\mathrm{rad}\, I = \prod_{\mathsf{m} \in \mathrm{Supp}\,(I)} \mathsf{m}$.*

Proof. To prove part (a), fix $n > 0$. It is clear that $\mathrm{Supp}\,(I) \subseteq \mathrm{Supp}\,(I^n)$. The reverse inclusion follows from the fact that maximal ideals are also prime. Indeed, suppose m is a maximal ideal containing I^n. Then for all $a \in I$, we have $a^n \in \mathsf{m}$. Since maximal ideals are prime ideals, this implies that $a \in \mathsf{m}$, showing that $I \subseteq \mathsf{m}$. The proofs of parts (b), (c) and (d) can be found in [27, §2.1]. $\quad\square$

The next result is a generalization of [27, Corollary 4.17] and will be used in the proof of Lemma 5.3.

Proposition 2.5. *Let* Γ *be a finite group acting freely on* $\mathrm{MaxSpec}(A)$ *and* M *be a finite-dimensional* $(\mathfrak{g} \otimes A)^{\Gamma}$*-module. Then there exist* $\ell, n \in \mathbb{Z}_{>0}$ *and* $\mathsf{m}_1, \ldots, \mathsf{m}_\ell \in \mathrm{MaxSpec}(A)$, *such that* $\mathsf{m}_1^n \cdots \mathsf{m}_\ell^n$ *is a* Γ*-invariant finite-codimensional ideal of* A *and* $(\mathfrak{g} \otimes \mathsf{m}_1^n \cdots \mathsf{m}_\ell^n)^{\Gamma} M = 0$.

Proof. We begin by proving the case where Γ is trivial. Let $\rho\colon \mathfrak{g} \otimes A \to \mathfrak{gl}(M)$ be the representation of $\mathfrak{g} \otimes A$ corresponding to M. Notice that $\ker \rho$ is a finite-codimensional ideal of $\mathfrak{g} \otimes A$, since M is finite dimensional. Thus, by [27, Proposition 8.1], $\ker \rho$ must be of the form $\mathfrak{g} \otimes I$, for some finite-codimensional ideal $I \subseteq A$. Now, by Lemma 2.4(b), the fact that I is finite-codimensional implies that I has finite support; that is, $\mathrm{Supp}(I) = \{\mathsf{m}_1, \ldots, \mathsf{m}_\ell\}$ for some $\ell > 0$. Since these maximal ideals are pairwise distinct, the radical of I is given by $\mathrm{rad}\, I = \mathsf{m}_1 \cdots \mathsf{m}_\ell$. Moreover, since A is assumed to be finitely generated, by Lemma 2.4(d), there exists $n > 0$, such that $(\mathrm{rad}\, I)^n \subseteq I$; that is, such that $\mathsf{m}_1^n \cdots \mathsf{m}_\ell^n \subseteq I$. We thus conclude that there exist $n, \ell \in \mathbb{Z}_{>0}$ and $\mathsf{m}_1, \ldots, \mathsf{m}_\ell \in \mathrm{MaxSpec}(A)$ such that $(\mathfrak{g} \otimes \mathsf{m}_1^n \cdots \mathsf{m}_\ell^n) M = 0$.

For the case where Γ acts non trivially on $\mathfrak{g} \otimes A$, we recall that, by [27, Proposition 8.5], M is the restriction of a finite-dimensional $\mathfrak{g} \otimes A$-module M'. From the case where Γ is trivial, since M' is finite dimensional, there exist $n, k \in \mathbb{Z}_{>0}$ and maximal ideals $\mathsf{m}_1, \ldots, \mathsf{m}_k \subseteq A$, such that $(\mathfrak{g} \otimes \mathsf{m}_1^n \cdots \mathsf{m}_k^n) M = 0$. Consider all ideals of the form $\gamma \mathsf{m}_i$, with $\gamma \in \Gamma$ and $i \in \{1, \ldots, k\}$. Since Γ is a finite group, we can enumerate the elements of $\{\gamma \mathsf{m}_i \mid \gamma \in \Gamma, i \in \{1, \ldots, k\}\}$ as $\mathsf{m}_1, \ldots, \mathsf{m}_\ell$, with $\ell \leq |\Gamma| k$. (Notice that $\ell \leq |\Gamma| k$ only if $\mathsf{m}_i \notin \Gamma \mathsf{m}_j$, for $1 \leq i \neq j \leq k$.) Moreover, notice that $\mathsf{m}_1^n \cdots \mathsf{m}_\ell^n$ is a Γ-invariant finite-codimensional ideal and

$$(\mathfrak{g} \otimes \mathsf{m}_1^n \cdots \mathsf{m}_\ell^n)^{\Gamma} M = (\mathfrak{g} \otimes \mathsf{m}_1^n \cdots \mathsf{m}_k^n)^{\Gamma} M' \subseteq (\mathfrak{g} \otimes \mathsf{m}_1^n \cdots \mathsf{m}_k^n) M' = 0. \quad \square$$

2.5. Cohomology of Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra and V be a $\mathfrak{g}$-module. Since $\mathfrak{g}$ and V are $\mathbb{Z}_2$-graded, $\Lambda^n \mathfrak{g} \otimes V$ and $\mathrm{Hom}_{\Bbbk}(\Lambda^n \mathfrak{g}, V)$ are naturally $\mathbb{Z}_2$-graded. Namely

$$(\Lambda^n \mathfrak{g} \otimes V)_z = \{x_1 \wedge \cdots \wedge x_n \otimes v \mid |x_1| + \cdots + |x_n| + |v| = z\} \quad \text{and}$$

$$\mathrm{Hom}_{\Bbbk}(\Lambda^n \mathfrak{g}, V)_z = \{f\colon \Lambda^n \mathfrak{g} \to V \mid f(\lambda) \in V_{w+z}, \ \lambda \in (\Lambda^n \mathfrak{g})_w, w \in \mathbb{Z}_2\},$$

for all $n \geq 0$ and $z \in \mathbb{Z}_2$.

Given two $\mathfrak{g}$-modules V and U, consider $\mathrm{Hom}_{\Bbbk}(V, U)$ as a $\mathfrak{g}$-module, where, for homogeneous elements $x \in \mathfrak{g}$ and $f \in \mathrm{Hom}_{\Bbbk}(V, U)$, we have

$$(xf)(v) = x(f(v)) - (-1)^{|x||f|} f(xv), \text{ for all } v \in V.$$

Define $\mathrm{Ext}_{\mathfrak{g}}^{\bullet}(V, U)$ to be the cohomology of the cocomplex

$$0 \to \mathrm{Hom}_{\Bbbk}(V, U) \xrightarrow{\partial^0} \mathrm{Hom}_{\Bbbk}(\mathfrak{g}, \mathrm{Hom}_{\Bbbk}(V, U)) \xrightarrow{\partial^1} \mathrm{Hom}_{\Bbbk}(\Lambda^2 \mathfrak{g}, \mathrm{Hom}_{\Bbbk}(V, U)) \xrightarrow{\partial^2} \cdots ,$$
(2.3)

where, for all $f \in \mathrm{Hom}_{\Bbbk}(\Lambda^n \mathfrak{g}, \mathrm{Hom}_{\Bbbk}(V, U))$, its image under the differential ∂^n extends linearly

$$\begin{aligned}
\partial^n &f(x_0 \wedge \cdots \wedge x_n) \\
&= \sum_{0 \leq i \leq n} (-1)^{i + |x_i|(|f| + |x_0| + \cdots + |x_{i-1}|)} x_i f(x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n) \\
&+ \sum_{0 \leq i < j \leq n} (-1)^{j + |x_j|(|x_{i+1}| + \cdots + |x_{j-1}|)} f(x_0 \wedge \ldots x_{i-1} \wedge [x_i, x_j] \wedge x_{i+1} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n),
\end{aligned}$$
(2.4)

for all homogeneous elements $x_0, \ldots, x_n \in \mathfrak{g}$, $n \geq 0$. In particular, define $\mathrm{H}^{\bullet}(\mathfrak{g}, U)$, the cohomology of $\mathfrak{g}$ with coefficients in U, to be $\mathrm{Ext}_{\mathfrak{g}}^{\bullet}(\Bbbk, U)$. Notice that, the differentials $\partial^{\bullet}$ respect the $\mathbb{Z}_2$-grading on $\mathrm{Hom}_{\Bbbk}(\Lambda^{\bullet} \mathfrak{g}, \mathrm{Hom}_{\Bbbk}(V, U))$, thus it induces a $\mathbb{Z}_2$-grading on $\mathrm{Ext}_{\mathfrak{g}}^{\bullet}(V, U)$. We denote $\mathrm{H}^{\bullet}(\mathfrak{g}, U)_z$ by $\mathrm{H}_z^{\bullet}(\mathfrak{g}, U)$ for all $z \in \mathbb{Z}_2$.

We now state some homological techniques that will be used in this paper. This first lemma reduces the computation of extensions between finite-dimensional modules to the computation of cohomologies.

Lemma 2.6. *If $\mathfrak{s}$ is a Lie superalgebra and U, V and W are finite-dimensional $\mathfrak{s}$-modules, then we have the following isomorphisms of $\mathfrak{s}$-modules:*

$$U \otimes V \cong V \otimes U, \quad (U \otimes V)^* \cong U^* \otimes V^*,$$

$$\mathrm{Hom}_{\Bbbk}(U \otimes V, W) \cong \mathrm{Hom}_{\Bbbk}(U, V^* \otimes W), \text{ and}$$

$$\mathrm{Ext}_{\mathfrak{s}}^n(V, U) \cong \mathrm{H}^n(\mathfrak{s}, V^* \otimes U), \text{ for all } n > 0.$$

Proof. The proof is similar to that of [21, Lemma 3.1.13]. $\square$

The following lemma reduces the computation of the cohomology of any trivial $\mathfrak{s}$-module; that is, any $\mathfrak{s}$-module V such that $\mathfrak{s} \cdot V = 0$, to that of the trivial $\mathfrak{s}$-module $\Bbbk$. Its proof follows directly from the definition of the differentials (2.4).

Lemma 2.7. *If $\mathfrak{s}$ is a Lie superalgebra and V is a finite-dimensional trivial $\mathfrak{s}$-module, then*

$$\mathrm{H}^{\bullet}(\mathfrak{s}, V) \cong \mathrm{H}^{\bullet}(\mathfrak{s}, \Bbbk) \otimes V.$$

The following proposition is a special case of the well known Künneth formula.

Proposition 2.8. *If $\mathfrak{s}^1, \mathfrak{s}^2$ are Lie superalgebras, U_1, V_1 are $\mathfrak{s}^1$-modules and U_2, V_2 are $\mathfrak{s}^2$-modules, then*

$$\mathrm{Ext}^n_{\mathfrak{s}^1 \oplus \mathfrak{s}^2}(U_1 \otimes U_2, V_1 \otimes V_2) \cong \bigoplus_{p+q=n} \mathrm{Ext}^p_{\mathfrak{s}^1}(U_1, V_1) \otimes \mathrm{Ext}^q_{\mathfrak{s}^2}(U_2, V_2), \quad n \geq 0.$$

Proof. The proof follows from [33, Theorem 3.6.3] using standard arguments. □

The following result is a graded version of Lyndon–Hochschild–Serre spectral sequence. In the superalgebra setting, a filtration by powers of an ideal turns out to be $\mathbb{Z}_2$-graded, thus yielding two spectral sequences that converge respectively to even and odd cohomologies (see Section 4 for further details).

Proposition 2.9 *([12, Chapter 1, §6.5]).* *If $\mathfrak{s}$ is a Lie superalgebra, V is a $\mathfrak{s}$-module and $\mathfrak{i} \subseteq \mathfrak{s}$ is an ideal, then there exist first-quadrant cohomology convergent spectral sequences*

$$E_2^{p,q} \cong \mathrm{H}^p_z(\mathfrak{s}/\mathfrak{i}, \mathrm{H}^q_z(\mathfrak{i}, V)) \Rightarrow \mathrm{H}^{p+q}_z(\mathfrak{s}, V), \quad z \in \mathbb{Z}_2.$$

The following result is a superalgebra generalization of a well-known result for Lie algebra cohomology. We record it as it will be used to prove Lemma 2.11, and we have not yet seen in the literature.

Lemma 2.10. *For any Lie superalgebra $\mathfrak{s}$, we have*

$$\mathrm{H}^1_{\bar{0}}(\mathfrak{s}, \Bbbk) \cong (\mathfrak{s}_{\bar{0}}/([\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}] + [\mathfrak{s}_{\bar{1}}, \mathfrak{s}_{\bar{1}}]))^* \quad \text{and} \quad \mathrm{H}^1_{\bar{1}}(\mathfrak{s}, \Bbbk) \cong \mathrm{H}^0(\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{1}}^*) \cong (\mathfrak{s}_{\bar{1}}/[\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{1}}])^*.$$

Proof. Recall that the cocomplex $\Lambda^\bullet \mathfrak{s}^*$ is $\mathbb{Z}_2$-graded, that the differential $\partial^\bullet$ preserves this grading, and that it induces a $\mathbb{Z}_2$-grading on $\mathrm{H}^\bullet(\mathfrak{s}, \Bbbk)$. We will compute each graded component of $\mathrm{H}^1(\mathfrak{s}, \Bbbk)$. First notice that the restriction of ∂^1 to the even part is $\partial^1_{\bar{0}} \colon \mathfrak{s}^*_{\bar{0}} \to \Lambda^2 \mathfrak{s}^*_{\bar{0}} \oplus S^2 \mathfrak{s}^*_{\bar{1}}$, and that the restriction of ∂^1 to the odd part is $\partial^1_{\bar{1}} \colon \mathfrak{s}^*_{\bar{1}} \to \mathfrak{s}^*_{\bar{0}} \otimes \mathfrak{s}^*_{\bar{1}}$.

By definition, $\mathrm{H}^1(\mathfrak{s}, \Bbbk) = \ker(\partial^1)/\mathrm{im}(\partial^0)$, where $\partial^0 \colon \Bbbk \to \mathfrak{s}^*$ is given by $\partial^0(\lambda)(x) = x \cdot \lambda = 0$ for all $\lambda \in \Bbbk, x \in \mathfrak{s}$, and $\partial^1 \colon \mathfrak{s}^* \to \Lambda^2 \mathfrak{s}^*$ is given by $\partial^1(\varphi)(x \wedge y) = -\varphi([x,y])$ for all $\varphi \in \mathfrak{s}^*, x, y \in \mathfrak{s}$. So, in order to compute $\mathrm{H}^1(\mathfrak{s}, \Bbbk)$ it is enough to determine $\ker(\partial^1)$.

From the formula of ∂^1, it follows that $\partial^1_{\bar{0}}(\varphi) = 0$ if and only if $\varphi([\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}] + [\mathfrak{s}_{\bar{1}}, \mathfrak{s}_{\bar{1}}]) = 0$. Thus $\mathrm{H}^1_{\bar{0}}(\mathfrak{s}, \Bbbk) \cong (\mathfrak{s}_{\bar{0}}/[\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}] + [\mathfrak{s}_{\bar{1}}, \mathfrak{s}_{\bar{1}}])^*$. Also from the formula of ∂^1, one can see that $\ker(\partial^1_{\bar{1}})$ is the kernel of ∂^0 in the cocomplex used for computing the cohomology of the Lie algebra $\mathfrak{s}_{\bar{0}}$ with coefficients in $\mathfrak{s}^*_{\bar{1}}$ (see Section 2.5). Thus $\mathrm{H}^1_{\bar{1}}(\mathfrak{s}, \Bbbk) \cong \mathrm{H}^0(\mathfrak{s}_{\bar{0}}, \mathfrak{s}^*_{\bar{1}})$. □

The next result follows from the previous one. It will be used in the proof of Lemma 5.3.

Lemma 2.11. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra, A be an associative, commutative algebra with unit, Γ be an abelian group acting on $\mathfrak{g}$ and A by automorphisms, and I be a Γ-invariant ideal of A. If $\mathfrak{g}_{\bar{1}}$ is nonzero, then*

$$\mathrm{H}^1_{\bar{0}}((\mathfrak{g} \otimes I)^{\Gamma}, \Bbbk) \cong \left(\left(\mathfrak{g}_{\bar{0}} \otimes I/I^2 \right)^{\Gamma} \right)^* \quad \text{and} \quad \mathrm{H}^1_{\bar{1}}((\mathfrak{g} \otimes I)^{\Gamma}, \Bbbk) \cong \left(\left(\mathfrak{g}_{\bar{1}} \otimes I/I^2 \right)^{\Gamma} \right)^*.$$

Proof. First notice that $\mathfrak{g}_{\bar{0}} \oplus [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \oplus \mathfrak{g}_{\bar{1}}$ are ideals of $\mathfrak{g}$. When $\mathfrak{g}$ is simple and $\mathfrak{g}_{\bar{1}}$ is nonzero, this implies that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$. Hence:

$$\left[(\mathfrak{g}_z \otimes I)^{\Gamma}, (\mathfrak{g}_{\bar{1}} \otimes I)^{\Gamma} \right] = [\mathfrak{g}_z \otimes I, \mathfrak{g}_{\bar{1}} \otimes I]^{\Gamma} = \left([\mathfrak{g}_z, \mathfrak{g}_{\bar{1}}] \otimes I^2 \right)^{\Gamma} = (\mathfrak{g}_{z+\bar{1}} \otimes I^2)^{\Gamma}, \quad z \in \mathbb{Z}_2.$$

The result follows from Lemma 2.10. □

The next corollary follows from Lemma 2.11 and, as far as we know, it is a new result.

Corollary 2.12. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra with $\mathfrak{g}_{\bar{1}} \neq 0$, A be an associative, commutative algebra with unit, and Γ be an abelian group acting on $\mathfrak{g}$ and A by automorphisms. If M and N are finite-dimensional trivial $(\mathfrak{g} \otimes A)^{\Gamma}$-modules, then $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(M, N) = 0$. In particular, $\mathrm{H}^1((\mathfrak{g} \otimes A)^{\Gamma}, \Bbbk) = 0$.*

Proof. By Lemma 2.6, $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(M, N) \cong \mathrm{H}^1((\mathfrak{g} \otimes A)^{\Gamma}, M^* \otimes N)$. Since M and N are finite-dimensional trivial $(\mathfrak{g} \otimes A)^{\Gamma}$-modules, by Lemma 2.7,

$$\mathrm{H}^1((\mathfrak{g} \otimes A)^{\Gamma}, M^* \otimes N) \cong \mathrm{H}^1((\mathfrak{g} \otimes A)^{\Gamma}, \Bbbk) \otimes (M^* \otimes N).$$

Since A is an associative, commutative algebra with unit, then $A^2 = A$. Thus, by Lemma 2.11, $\mathrm{H}^1((\mathfrak{g} \otimes A)^{\Gamma}, \Bbbk) = 0$. The result follows. □

3. Equivariant periplectic map Lie superalgebras

3.1. Structure of periplectic Lie superalgebras

Given $n \geq 2$, the periplectic Lie superalgebra $\mathfrak{p}(n)$ is the Lie subalgebra of $\mathfrak{gl}(n + 1, n + 1)$ whose elements are matrices of the form

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right), \tag{3.1}$$

where $A \in \mathfrak{sl}_{n+1}$, $B = B^t$ and $C^t = -C$.

Throughout this section, we will denote $\mathfrak{p}(n)$ by $\mathfrak{g}$.

The even part, $\mathfrak{g}_{\bar{0}}$, is isomorphic to the Lie algebra $\mathfrak{sl}_{n+1}$. The structure of $\mathfrak{g}_{\bar{1}}$ is the following. Let $S^2(\Bbbk^{n+1})$ (resp. $\Lambda^2(\Bbbk^{n+1})^*$) denote the second symmetric (resp. exterior) power of $\Bbbk^{n+1}$ (resp. $(\Bbbk^{n+1})^*$), with the action of $\mathfrak{sl}_{n+1}$ induced by matrix multiplication, and let $\mathfrak{g}_{\bar{1}}^+$ (resp. $\mathfrak{g}_{\bar{1}}^-$) be the set of all matrices of the form (3.1) such that $A = C = 0$

(resp. $A = B = 0$). As a $\mathfrak{g}_{\bar{0}}$-module, we have: $\mathfrak{g}_{\bar{1}} \cong \mathfrak{g}_{\bar{1}}^+ \oplus \mathfrak{g}_{\bar{1}}^-, \mathfrak{g}_{\bar{1}}^+ \cong S^2(\Bbbk^{n+1})$ and $\mathfrak{g}_{\bar{1}}^- \cong \Lambda^2(\Bbbk^{n+1})^*$.

If we set $\mathfrak{g}_{-1} = \mathfrak{g}_{\bar{1}}^-$, $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_1 = \mathfrak{g}_{\bar{1}}^+$, then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-grading of $\mathfrak{g}$ which is compatible with the $\mathbb{Z}_2$-grading; that is, $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Let $\mathfrak{h} \subseteq \mathfrak{g}_0$ be the Cartan subalgebra of $\mathfrak{g}_0$ formed by diagonal matrices. Since $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}_{n+1}$, we can identify $\mathfrak{h}$ with its dual via the bilinear, nondegenerate, $\mathfrak{g}_0$-invariant form $(A_1, A_2) = \mathrm{tr}(A_1 A_2)$. For $i \in \{1, \ldots, n+1\}$, let ε_i be the unique element in $\mathfrak{h}^*$ satisfying

$$\varepsilon_i(E_{j,j} + E_{k,k} - E_{n+j+1,n+j+1} - E_{n+k+1,n+k+1}) = \delta_{i,j} + \delta_{i,k} \qquad \text{for all } 1 \leq j \neq k \leq n+1.$$

The roots of $\mathfrak{g}$ are described as follows:

- Roots of $\mathfrak{g}_{-1}$: $-\varepsilon_i - \varepsilon_j$, where $1 \leq i < j \leq n+1$.
- Roots of $\mathfrak{g}_0$: $\varepsilon_i - \varepsilon_j$, where $1 \leq i \neq j \leq n+1$.
- Roots of $\mathfrak{g}_1$: $\varepsilon_i + \varepsilon_j$, where $1 \leq i \leq j \leq n+1$.

Choose a triangular decomposition $\mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ of the Lie algebra $\mathfrak{g}_0$ such that the positive roots are: $\varepsilon_i - \varepsilon_j$, with $i < j$. We fix a triangular decomposition $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{p}(n)$, where $\mathfrak{n}^\pm = \mathfrak{g}_{\pm 1} \oplus \mathfrak{n}_0^\pm$, and denote by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ a Borel subalgebra of $\mathfrak{p}(n)$. Notice that all the roots of $\mathfrak{g}_1$ are positive.

3.2. Construction of finite-dimensional irreducible modules

Recall that throughout this section we are denoting $\mathfrak{g} = \mathfrak{p}(n)$, $n \geq 2$. Let A be an associative commutative algebra with unit. In this subsection we will construct some finite-dimensional irreducible $\mathfrak{p}(n) \otimes A$-modules which will be used in the next subsection.

For each $\psi \in (\mathfrak{h} \otimes A)^*$, denote by $\Bbbk_\psi$ the one-dimensional $\mathfrak{b} \otimes A$-module where the action of $\mathfrak{n}^+ \otimes A$ is trivial and the action of $\mathfrak{h} \otimes A$ is given by ψ. Define a Verma type module $M(\psi)$ to be

$$M(\psi) = U(\mathfrak{g} \otimes A) \otimes_{U(\mathfrak{b} \otimes A)} \Bbbk_\psi.$$

Since a submodule of $M(\psi)$ is proper if and only if its intersection with $\Bbbk_\psi$ is trivial, $M(\psi)$ admits a unique maximal non-proper submodule. Let $V(\psi)$ denote the unique irreducible quotient of $M(\psi)$ by such a submodule.

Proposition 3.1. *Every finite-dimensional irreducible $\mathfrak{g} \otimes A$-module is isomorphic to $V(\psi)$, for some $\psi \in (\mathfrak{h} \otimes A)^*$.*

Proof. The proof of [27, Proposition 4.5] only requires the existence of a nonzero weight vector $v \in V$. Since $\mathfrak{h} \otimes A$ is abelian, such a vector always exists. $\square$

The annihilator $\mathrm{Ann}_A(V)$ of a $\mathfrak{g} \otimes A$-module V is, by definition, the sum of all ideals I of A such that $(\mathfrak{g} \otimes I)V = 0$. The support of V is defined to be

$$\mathrm{Supp}\,(V) = \mathrm{Supp}\,(\mathrm{Ann}_A(V)).$$

Proposition 3.2. *The tensor product of two irreducible finite-dimensional $\mathfrak{g} \otimes A$-modules with disjoint supports is irreducible as well.*

Proof. This proof is similar to that of [27, Proposition 4.12]. □

The proof of the next result follows ideas contained in [27, Theorem 4.16] (see also [2, Lemma 2.3] and [3, Proposition 4.3]).

Proposition 3.3. *Let $\psi \in (\mathfrak{h} \otimes A)^*$. The weight spaces of $V(\psi)$ are finite dimensional if and only if there exists an ideal I of A of finite codimension such that $(\mathfrak{g} \otimes I)V(\psi) = 0$.*

Proof. Suppose all the weight spaces of $V(\psi)$ are finite dimensional and let v be a highest weight vector of $V(\psi)$. Let Δ^- denote the set $\{-\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$. For each $\alpha \in \Delta^-$, define I_α to be $\{a \in A \mid (\mathfrak{g}_\alpha \otimes a)v = 0\}$. Since each weight space of $V(\psi)$ is finite dimensional, I_α is an ideal of A of finite codimension. Let I be $\bigcap_{\alpha \in \Delta^-} I_\alpha$. Since $\mathfrak{g}$ is finite dimensional, I is an intersection of finitely many finite-codimensional ideals. In particular, I is also a finite-codimensional ideal of A. We claim that $(\mathfrak{g} \otimes I)V(\psi) = 0$. Indeed, the fact that $(\mathfrak{n}^+ \otimes A)v = 0$ follows from the fact that v is a highest weight vector. The fact that $(\mathfrak{n}^- \otimes I)v = 0$ follows from the construction of I. Finally, notice that $\mathfrak{h} \subseteq [\mathfrak{n}^-, \mathfrak{n}^+]$, and so $(\mathfrak{h} \otimes I)v \subseteq [\mathfrak{n}^- \otimes I, \mathfrak{n}^+ \otimes A]v = 0$. Thus $(\mathfrak{g} \otimes I)v = 0$ and hence $W = \{w \in V(\psi) \mid (\mathfrak{g} \otimes I)v = 0\}$ is a non-trivial submodule of $V(\psi)$. Since $V(\psi)$ is irreducible, we conclude that $W = V(\psi)$.

Suppose now there exists a finite-codimensional ideal I of A such that $(\mathfrak{g} \otimes I)V(\psi)$. Then the action of $\mathfrak{g} \otimes A$ on $V(\psi)$ factors through an action of the finite-dimensional Lie superalgebra $\mathfrak{g} \otimes A/I$. Thus, by standards arguments using the PBW Theorem, all the weight spaces of $V(\psi)$ are finite dimensional. □

For the remainder of this section, we will assume that A is finitely generated.

Proposition 3.4. *Let $\psi \in (\mathfrak{h} \otimes A)^*$. The support of $V(\psi)$ is finite if and only if there exists a finite-codimensional ideal I of A such that $(\mathfrak{g} \otimes I)V(\psi) = 0$.*

Proof. Recall that $\mathrm{Supp}\, V(\psi) = \mathrm{Supp}\, \mathrm{Ann}_A(V(\psi))$. So $\mathrm{Ann}_A(V(\psi))$ has finite codimension if and only if there exists a finite-codimensional ideal I of A such that $(\mathfrak{g} \otimes I)V(\psi) = 0$. Since A is finitely generated, $\mathrm{Ann}_A(V(\psi))$ has finite codimension if and only if its support is finite. □

Before stating the next result, recall that $\mathfrak{g}_{\bar{0}}$ is a finite-dimensional simple Lie algebra.

Proposition 3.5. *If V is an irreducible finite-dimensional $\mathfrak{g} \otimes A$-module, then $(\mathfrak{g} \otimes J)V = 0$ for some radical ideal J of A of finite codimension.*

Proof. Since V is irreducible and has finite dimension, Proposition 3.3 implies that $(\mathfrak{g} \otimes I)V = 0$ for some finite-codimensional ideal I of A. Let $J = \sqrt{I}$ be the radical of I. We will show that $(\mathfrak{g} \otimes J)V = 0$. Since we are assuming that A is finitely generated (and in particular, Noetherian), there exists some power of J that is contained in I. Then $\mathfrak{g} \otimes (I/J)$ is a solvable Lie superalgebra satisfying the following property:

$$[(\mathfrak{g} \otimes (J/I))_{\bar{1}}, (\mathfrak{g} \otimes (J/I))_{\bar{1}}] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \otimes (J^2/I) \subseteq \mathfrak{g}_{\bar{0}} \otimes (J^2/I)$$
$$= [(\mathfrak{g} \otimes (J/I))_{\bar{0}}, (\mathfrak{g} \otimes (J/I))_{\bar{0}}],$$

where the last equality follows from the fact that $\mathfrak{g}_{\bar{0}}$ is a simple Lie algebra. Therefore, it follows from [17, Proposition 5.2.4], that any irreducible finite-dimensional representation of $\mathfrak{g} \otimes J$ is one-dimensional. Then there exists a nonzero vector $w \in V$, and an element $\mu \in (\mathfrak{g} \otimes J)^*$, such that $xw = \mu(x)w$ for all $x \in \mathfrak{g} \otimes J$. We claim that $\mu = 0$. Since V is finite dimensional, for any $z \in \mathfrak{n}^{\pm} \otimes J$, there exists $m \geq 0$ such that $z^m w = \mu(z)^m w = 0$. In other words, $\mu(\mathfrak{n}^{\pm} \otimes J) = 0$, and hence $(\mathfrak{n}^{\pm} \otimes J)w = 0$. Let μ' denote the restriction of μ to $\mathfrak{g}_{\bar{0}} \otimes J$. Since $\mathfrak{g}_{\bar{0}}$ is a simple Lie algebra, it follows that the kernel of μ' must be $\mathfrak{g}_{\bar{0}} \otimes J$. In particular, $\mu'(\mathfrak{h} \otimes J) = 0$, since $\mathfrak{h} \subseteq \mathfrak{g}_{\bar{0}}$. We have thus proved that $(\mathfrak{g} \otimes J)w = 0$. Now the result follows from the irreducibility of V along with the fact that $W = \{v \in V \mid (\mathfrak{g} \otimes J)v = 0\}$ is a nonzero submodule of V. $\quad\square$

3.3. Classification of finite-dimensional irreducible modules

Recall that we are assuming that $\mathfrak{g} = \mathfrak{p}(n)$, $n \geq 2$, and that A is an associative, commutative, finitely-generated algebra with unit. In this subsection we will classify all the finite-dimensional irreducible $(\mathfrak{g} \otimes A)^{\Gamma}$-modules in the case where Γ is a finite abelian group acting on $\mathfrak{g}$ and A by automorphisms and such that the induced action of Γ on $\mathrm{MaxSpec}(A)$ is free.

Recall from Section 2.3 that for every $\mathsf{m} \in \mathrm{MaxSpec}(A)$ one may consider the composition

$$\mathrm{ev}_{\mathsf{m}} \colon \mathfrak{g} \otimes A \to \mathfrak{g} \otimes A / \mathfrak{g} \otimes \mathsf{m} \xrightarrow{\cong} \mathfrak{g}.$$

Furthermore for a $\mathfrak{g}$-module V with associated representation $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$, the module $\mathrm{ev}_{\mathsf{m}}^*(V)$ is defined to be the $\mathfrak{g} \otimes A$-module with associated representation given by the pull-back of ρ along ev_{m}:

$$\mathrm{ev}_{\mathsf{m}}^*(\rho) \colon \mathfrak{g} \otimes A \xrightarrow{\mathrm{ev}_{\mathsf{m}}} \mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(V).$$

The $\mathfrak{g} \otimes A$-module $\mathrm{ev}_{\mathsf{m}}^*(V)$ is called an evaluation module and its associated representation $\mathrm{ev}_{\mathsf{m}}^*(\rho)$ is called an evaluation representation. Define $\mathrm{ev}_{\mathsf{m}}^{\Gamma *}(\rho)$ to be the restriction of $\mathrm{ev}_{\mathsf{m}}^*(\rho)$ to $(\mathfrak{g} \otimes A)^\Gamma$.

Theorem 3.6. *Every irreducible finite-dimensional $\mathfrak{g} \otimes A$-module V is isomorphic to a tensor product of evaluation modules.*

Proof. Since V is finite dimensional, by Proposition 3.5, there exists a radical ideal I of A of finite codimension such that $(\mathfrak{g} \otimes I)V = 0$. Since A is finitely generated and I has finite codimension, Lemma 2.4 (b) implies that the support of I is finite. If $\mathrm{Supp}\,(I) = \{\mathsf{m}_1, \dots, \mathsf{m}_n\} \subseteq \mathrm{MaxSpec}(A)$, then $I = \sqrt{I} = \mathsf{m}_1 \cdots \mathsf{m}_n$. Therefore the action of $\mathfrak{g} \otimes A$ on V factors through the map

$$\mathfrak{g} \otimes A \twoheadrightarrow \mathfrak{g} \otimes A/I \xrightarrow{\cong} \bigoplus_{i=1}^n \mathfrak{g} \otimes A/\mathsf{m}_i \xrightarrow{\cong} \mathfrak{g}^{\oplus n}. \tag{3.2}$$

In other words, V is isomorphic to the pull-back of an irreducible finite-dimensional $\mathfrak{g}^{\oplus n}$-module W along (3.2). Notice that W is also an irreducible finite-dimensional module for $U(\mathfrak{g}^{\oplus n}) \cong U(\mathfrak{g})^{\otimes n}$. By [6, Proposition 8.4], there exist irreducible finite-dimensional modules $V_1, \dots, V_n$ for $U(\mathfrak{g})$ such that W is either isomorphic to $V_1 \otimes \cdots \otimes V_n$ or to a proper submodule of $V_1 \otimes \cdots \otimes V_n$. Since $V_1 \otimes \cdots \otimes V_n$ is irreducible by Proposition 3.2, $W \cong V_1 \otimes \cdots \otimes V_n$, and V is isomorphic to a tensor product of evaluation modules. $\square$

> From now on we will assume that Γ is a finite abelian group acting on $\mathfrak{g}$ and A by automorphisms and such that the induced action of Γ on $\mathrm{MaxSpec}(A)$ is free.

Let $\mathrm{Irred}(\mathfrak{g})$ (resp. $\mathrm{Irred}(\mathfrak{g} \otimes A)^\Gamma$) be the set of isomorphism classes of irreducible finite-dimensional modules for $\mathfrak{g}$ (resp. $(\mathfrak{g} \otimes A)^\Gamma$). Let $[V] \in \mathrm{Irred}(\mathfrak{g})$ denote the isomorphism class of a $\mathfrak{g}$-module V. Notice that, if V and V' are isomorphic $\mathfrak{g}$-modules, then $\mathrm{ev}_{\mathsf{m}}^{\Gamma *}(V)$ and $\mathrm{ev}_{\mathsf{m}}^{\Gamma *}(V')$ are isomorphic $(\mathfrak{g} \otimes A)^\Gamma$-modules. Therefore, for each $[V] \in \mathrm{Irred}(\mathfrak{g})$, we define $\mathrm{ev}_{\mathsf{m}}^{\Gamma *}[V]$ to be the isomorphism class of $\mathrm{ev}_{\mathsf{m}}^{\Gamma *}(V)$ in $\mathrm{Irred}(\mathfrak{g} \otimes A)^\Gamma$.

Also recall that the action of Γ on $\mathfrak{g}$ induces an action of Γ on $\mathrm{Irred}(\mathfrak{g})$. Namely, if V is a $\mathfrak{g}$-module representative of $[V] \in \mathrm{Irred}(\mathfrak{g})$ with associated representation $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$, then $\gamma[V] = [V^\gamma]$, where V^γ is a $\mathfrak{g}$-module with underlying vector space V and associated representation $\rho' \colon \mathfrak{g} \to \mathfrak{gl}(V)$ given by $\rho'(x) = \rho(\gamma^{-1}x)$ for all $x \in \mathfrak{g}$.

Let $\mathcal{P}$ be the set of Γ-equivariant functions $\pi \colon \mathrm{MaxSpec}(A) \to \mathrm{Irred}(\mathfrak{g})$ such that $\pi(\mathsf{m}) = [\Bbbk]$ for all but finitely many distinct $\mathsf{m} \in \mathrm{MaxSpec}(A)$. Given $\pi \in \mathcal{P}$, recall that its support is defined to be $\mathrm{Supp}\,(\pi) = \{\mathsf{m} \in \mathrm{MaxSpec}(A) \mid \pi(\mathsf{m}) \text{ is nontrivial}\}$. Let X_* denote the set of all finite subsets $\mathsf{M} \subseteq \mathrm{MaxSpec}(A)$ satisfying the following property: if m and m' are distinct elements in M, then $\mathsf{m} \notin \Gamma\mathsf{m}'$. As in Section 2.3, for each $\pi \in \mathcal{P}$, fix an element $\mathrm{Supp}_*(\pi)$ in X_* containing one element of each Γ-orbit in $\mathrm{Supp}\,(\pi)$, and define $\mathcal{V}(\pi)$ to be the $(\mathfrak{g} \otimes A)^\Gamma$-module

$$\mathcal{V}(\pi) = \bigotimes_{\mathsf{m} \in \mathrm{Supp}_*(\pi)} \mathrm{ev}_{\mathsf{m}}^{\Gamma}{}^*\pi(\mathsf{m}).$$

Lemma 3.7. *With the above notation, the following hold:*

(a) *For every $\pi \in \mathcal{P}$, the isomorphism class of $\mathcal{V}(\pi)$ does not depend on the choice of $\mathrm{Supp}_*(\pi)$.*

(b) *For every $\pi \in \mathcal{P}$, the $(\mathfrak{g} \otimes A)^{\Gamma}$-module $\mathcal{V}(\pi)$ is irreducible.*

(c) *The map $\mathcal{P} \to \mathrm{Irred}(\mathfrak{g} \otimes A)^{\Gamma}$ given by $\pi \mapsto \mathcal{V}(\pi)$ is injective.*

Proof. Part (a) follows from [27, Lemma 5.9]. Part (b) follows from Proposition 3.2 along with the fact that the map $\mathrm{ev}_{\mathsf{m}}^{\Gamma}{}^*$ is surjective for all $\mathsf{m} \in \mathrm{MaxSpec}(A)$. Part (c) follows from [27, Proposition 5.11]. Notice that the condition of $\mathfrak{g}$ being basic is not used in the proofs of the results cited from [27]. □

Proposition 3.8. *Every finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$-module V is isomorphic to the restriction of a finite-dimensional $\mathfrak{g} \otimes A$-module V' whose support is in X_*. Moreover, V is irreducible if and only if V' is.*

Proof. The proof of this fact for any finite-dimensional simple Lie superalgebra is the same as the proof of [27, Proposition 8.5]. Notice that, although the fact that $\mathrm{Supp}(V')$ is an element of X_* is not stated, it is also proved there. □

Theorem 3.9. *Let A be an associative, commutative, finitely-generated algebra with unit, Γ be a finite abelian group acting on A and $\mathfrak{g}$ by automorphisms, and such that the induced action of Γ on $\mathrm{MaxSpec}(A)$ is free. The map $\mathcal{P} \to \mathrm{Irred}(\mathfrak{g} \otimes A)^{\Gamma}$ given by $\pi \mapsto \mathcal{V}(\pi)$ is a bijection.*

Proof. Recall from Lemma 3.7(c) that the map $\mathcal{P} \to \mathrm{Irred}(\mathfrak{g} \otimes A)^{\Gamma}$ given by $\pi \mapsto \mathcal{V}(\pi)$ is injective. Let V be an irreducible finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$-module. By Proposition 3.8, V is isomorphic to the restriction of an irreducible finite-dimensional $\mathfrak{g} \otimes A$-module V', whose support is in X_*. Hence, by Theorem 3.6, $V' \cong \bigotimes_{i=1}^{n} \mathrm{ev}_{\mathsf{m}_i}^*(V_i)$ for some $n \geq 0$, $\{\mathsf{m}_1, \ldots, \mathsf{m}_n\} \in X_*$ and irreducible finite-dimensional $\mathfrak{g}$-modules $V_1, \ldots, V_n$. Thus, V is isomorphic to $\mathcal{V}(\pi)$, where $\pi(\mathsf{m}_i) = [V_i]$ for all $i \in \{1, \ldots, n\}$, and $\pi(\mathsf{m}) = [\mathbb{k}]$ for all $\mathsf{m} \notin \mathrm{Supp}(V')$. □

4. Transgression maps

Let $\mathfrak{s}$ be a Lie superalgebra and consider the exterior algebra $\Lambda^{\bullet}\mathfrak{s}$ with $\mathfrak{s}$-module structure induced by the adjoint representation. Given an ideal $\mathfrak{i} \subsetneq \mathfrak{s}$, define an increasing filtration $0 = \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda = \Lambda^{\bullet}\mathfrak{s}$ by:

$$\Lambda_p^n = \mathrm{span}_{\mathbb{k}}\{x_1 \wedge \cdots \wedge x_n \mid x_{i_1}, \ldots, x_{i_k} \in \mathfrak{i} \text{ for some } k > n - p, \ 1 \leq i_1 < \ldots < i_k \leq n\},$$

$$\Lambda_p^n = \Lambda^n \mathfrak{s} \quad \text{if } p > n \text{ and}$$

$$\text{for all } 0 < p \le n.$$

Now, given a $\mathfrak{s}$-module M, define a decreasing filtration $0 \subsetneq \cdots \subsetneq C_1 \subsetneq C_0 = \mathrm{Hom}_{\Bbbk}(\Lambda^{\bullet}\mathfrak{s}, M)$ by

$$C_p = \{f \colon \Lambda \to M \mid f(\Lambda_p) = 0\} \cong \mathrm{Hom}_{\Bbbk}(\Lambda/\Lambda_p, M).$$

Recall that by definition, $\mathrm{H}^{\bullet}(\mathfrak{s}, M)$, the cohomology of $\mathfrak{s}$ with coefficients in M, is the cohomology of the cocomplex $(\mathrm{Hom}_{\Bbbk}(\Lambda^{\bullet}\mathfrak{s}, M), \partial^{\bullet})$ where

$$
\begin{aligned}
\partial^n &f(x_0 \wedge \cdots \wedge x_n) \\
&= \sum_{0 \le i \le n} (-1)^{i + |x_i|(|f| + |x_0| + \cdots + |x_{i-1}|)} x_i f(x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n) \\
&+ \sum_{0 \le i < j \le n} (-1)^{j + |x_j|(|x_{i+1}| + \cdots + |x_{j-1}|)} f(x_0 \wedge \ldots x_{i-1} \wedge [x_i, x_j] \wedge x_{i+1} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n),
\end{aligned}
\tag{4.1}
$$

for all $n \ge 0$, $f \colon \Lambda^n \mathfrak{s} \to M$ and $x_0, \ldots, x_n \in \mathfrak{s}$. Moreover, notice that $\partial(C_p) \subseteq C_p$ for all $p \ge 0$. We will use this filtration to construct a spectral sequence converging to $\mathrm{H}^{\bullet}(\mathfrak{s}, M)$.

4.1. E_0-page

Let $E_0^{p,q} = C_p^{p+q}/C_{p+1}^{p+q}$ $(p, q \ge 0)$. Since C_p^n consists of maps in $\mathrm{Hom}_{\Bbbk}(\Lambda^n \mathfrak{s}, M)$ that vanish on Λ_p^n (and similarly for C_{p+1}^n), we have

$$E_0^{p,q} \cong \mathrm{Hom}_{\Bbbk}\left(\Lambda_{p+1}^{p+q}/\Lambda_p^{p+q}, M\right) \quad \text{for all } p, q \ge 0,$$

where $f \in \mathrm{Hom}_{\Bbbk}(\Lambda_{p+1}^{p+q}/\Lambda_p^{p+q}, M)$ corresponds to an equivalence class of maps in $\mathrm{Hom}_{\Bbbk}(\Lambda^{p+q}\mathfrak{s}, M)$ that vanish on Λ_p^{p+q} modulo those maps that vanish on Λ_{p+1}^{p+q}. Now recall that, as a vector space, $\mathfrak{s} \cong \mathfrak{i} \oplus (\mathfrak{s}/\mathfrak{i})$ and thus $\Lambda^n \mathfrak{s} \cong \bigoplus_{j=0}^n \Lambda^{n-j}\mathfrak{i} \otimes \Lambda^j(\mathfrak{s}/\mathfrak{i})$. Using this isomorphim and the definition of Λ_p^n, we see that

$$\Lambda_p^n \cong \bigoplus_{j=0}^{p-1} \Lambda^{n-j}\mathfrak{i} \otimes \Lambda^j(\mathfrak{s}/\mathfrak{i}) \quad \text{and}$$

$$\Lambda_{p+1}^{p+q}/\Lambda_p^{p+q} \cong \Lambda^q\mathfrak{i} \otimes \Lambda^p(\mathfrak{s}/\mathfrak{i}) \quad \text{for all } p, q, n \ge 0.
\tag{4.2}$$

From now on, we will fix a (vector space) splitting of the map $\mathfrak{s} \to \mathfrak{s}/\mathfrak{i}$. We will thus identify $\mathfrak{s}/\mathfrak{i}$ with a subspace of $\mathfrak{s}$ (namely, its image under the splitting) and, given an element $x \in \mathfrak{s}/\mathfrak{i}$, we will denote by $\widetilde{x}$ its corresponding representative in $\mathfrak{s}$.

Consider the map $d_0^{p,q} \colon E_0^{p,q} \to E_0^{p,q+1}$ explicitly given by $d_0^{p,q}(f + C_{p+1}^{p+q}) = \partial^{p+q}f + C_{p+1}^{p+q+1}$. Using the explicit formula for $\partial^{p+q}f$ (4.1) and the isomorphisms (4.2), we see that

$$\partial^{p+q} f(i_0 \wedge \cdots \wedge i_q \otimes \widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_p) \tag{4.3}$$

$$= \sum_{0 \leq \ell \leq q} (-1)^{\ell + |x_\ell|(|f| + |x_0| + \cdots + |x_{\ell-1}|)} i_\ell f(i_0 \wedge \cdots \wedge \widehat{i_\ell} \wedge \cdots \wedge i_q \otimes \widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_p)$$

$$+ \sum_{0 \leq j < k \leq q} (-1)^{\sigma(j,k)} f(i_0 \wedge \ldots i_{j-1} \wedge [i_j, i_k] \wedge i_{j+1} \wedge \cdots \wedge \widehat{i_k} \wedge \cdots \wedge i_q \otimes \widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_p),$$

where $\sigma(j,k) = k + |x_k|(|x_{j+1}| + \cdots + |x_{k-1}|)$, for all $i_0, \ldots, i_q \in \mathfrak{i}$ and representatives $\widetilde{x}_1, \ldots, \widetilde{x}_p \in \mathfrak{s}$ of corresponding elements in $\mathfrak{s}/\mathfrak{i}$. (Notice that (4.3) does not depend on the choice of these representatives, as any other choice would differ from this one by elements in $\mathfrak{i}$, and f vanishes when more than q of its arguments are in $\mathfrak{i}$.)

Thus, for each $p, q \geq 0$, we have an isomorphism $E_0^{p,q} \cong \mathrm{Hom}_{\Bbbk} \left(\Lambda^q \mathfrak{i}, \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M) \right)$ and we can identify $d_0^{p,q}$ with the q-th differential in the cocomplex $\left(\mathrm{Hom}_{\Bbbk} \left(\Lambda^\bullet \mathfrak{i}, \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M) \right), \partial_{\mathfrak{i}}^\bullet \right)$, where $\mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M)$ is viewed as an $\mathfrak{i}$-module via $(if)(x) = i(f(x))$ for all $i \in \mathfrak{i}$, $x \in \Lambda^p(\mathfrak{s}/\mathfrak{i})$ and $f \in \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M)$. (The $\mathfrak{s}$-module structure on $\mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M)$ is given by $(yf)(x) = y(f(x)) - (-1)^{|y||f|} f(y \cdot x)$ for all $y \in \mathfrak{s}$, $f \in \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M)$ and $x \in \Lambda^p(\mathfrak{s}/\mathfrak{i})$. Since $[\mathfrak{i}, \mathfrak{s}] \subseteq \mathfrak{i}$, we have that the action of $\mathfrak{i}$ on $\Lambda^p(\mathfrak{s}/\mathfrak{i})$ is trivial, and thus $i \cdot x = 0$ for all $i \in \mathfrak{i}$ and $x \in \Lambda^p(\mathfrak{s}/\mathfrak{i})$.)

4.2. E_1-page

Let $E_1^{p,q} = \mathrm{H}^q \left(E_0^{p,\bullet}, d_0^{p,\bullet} \right)$ $(p, q \geq 0)$. As a direct consequence of the conclusions of Section 4.1, we see that

$$E_1^{p,q} \cong \mathrm{H}^q(\mathfrak{i}, \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M)) \quad \text{for all } p, q \geq 0.$$

Moreover, notice that every element in $\mathrm{H}^q(\mathfrak{i}, \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M))$ is an equivalence class (modulo $\mathrm{im}\, \partial^{p+q-1}$) of linear maps $f \colon \Lambda^q \mathfrak{i} \otimes \Lambda^p(\mathfrak{s}/\mathfrak{i}) \to M$ such that (4.3) $= 0$. Via the standard linear isomorphism $\mathrm{Hom}_{\Bbbk}(\Lambda^q \mathfrak{i} \otimes \Lambda^p(\mathfrak{s}/\mathfrak{i}), M) \cong \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), \mathrm{Hom}_{\Bbbk}(\Lambda^q \mathfrak{i}, M))$, f corresponds to a linear map $F \colon \Lambda^p(\mathfrak{s}/\mathfrak{i}) \to \mathrm{Hom}_{\Bbbk}(\Lambda^q \mathfrak{i}, M)$. Since (4.3) $= 0$, then $\partial_{\mathfrak{i}}^q \left(F(\widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_p) \right) = 0$; and since $\mathrm{im}\, \partial^{p+q-1}$ corresponds to $\mathrm{im}\, \partial_{\mathfrak{i}}^{q-1}$ via this standard isomorphism, we see that $F(\widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_p)$ is an element in $\mathrm{H}^q(\mathfrak{i}, M)$, for all $\widetilde{x}_1, \ldots, \widetilde{x}_p \in \mathfrak{s}/\mathfrak{i}$. This shows that $\mathrm{H}^q(\mathfrak{i}, \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), M)) \cong \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), \mathrm{H}^q(\mathfrak{i}, M))$, and thus

$$E_1^{p,q} \cong \mathrm{Hom}_{\Bbbk}(\Lambda^p(\mathfrak{s}/\mathfrak{i}), \mathrm{H}^q(\mathfrak{i}, M)) \quad \text{for all } p, q \geq 0.$$

Now, consider $\mathrm{H}^q(\mathfrak{i}, M)$ as a $\mathfrak{s}/\mathfrak{i}$-module (that is, a $\mathfrak{s}$-module with trivial $\mathfrak{i}$-action) with the structure induced from the $\mathfrak{s}$-module structure on $\mathrm{Hom}_{\Bbbk}(\Lambda^q \mathfrak{i}, M)$ explicitly given by $(xf)(\lambda) = x(f(y)) - (-1)^{|x||f|} f(x \cdot y)$ for all $x \in \mathfrak{s}$, $y \in \Lambda^q \mathfrak{i}$ and $f \in \mathrm{Hom}_{\Bbbk}(\Lambda^q \mathfrak{i}, M)$. Then, denote by $\partial_{\mathfrak{s}/\mathfrak{i}}^\bullet$ the differential of the cocomplex $\mathrm{Hom}_{\Bbbk}(\Lambda^\bullet(\mathfrak{s}/\mathfrak{i}), \mathrm{H}^q(\mathfrak{i}, M))$. Now we consider the map $d_1^{p,q} \colon E_1^{p,q} \to E_1^{p+1,q}$ defined by $d_1^{p,q}(f + \mathrm{im}\, d_0^{p,q-1}) = \partial^{p+q} f +$

im $d_0^{p+1,q-1}$. Using the explicit formula for $\partial^{p+q} f$ (4.1), the isomorphisms above and (4.3), we see that

$$\partial^{p+q} f(\widetilde{x}_0 \wedge \cdots \wedge \widetilde{x}_p \otimes i_1 \wedge \cdots \wedge i_q) \tag{4.4}$$
$$= \left(\partial_{\mathfrak{s}/\mathfrak{i}}^p f(\widetilde{x}_0 \wedge \cdots \wedge \widetilde{x}_p) \right) (i_1 \wedge \cdots \wedge i_q) + d_0^{p+1,q-1} f(\widetilde{x}_0 \wedge \cdots \wedge \widetilde{x}_p \otimes i_1 \wedge \cdots \wedge i_q),$$

for all $i_1, \ldots, i_{q-1} \in \mathfrak{i}$ and representatives $\widetilde{x}_0, \ldots, \widetilde{x}_{p+1} \in \mathfrak{s}$ of corresponding elements in $\mathfrak{s}/\mathfrak{i}$. Thus, we can identify $d_1^{p,q}$ with the p-th differential of the cocomplex $\left(\mathrm{Hom}_{\Bbbk} \left(\Lambda^\bullet(\mathfrak{s}/\mathfrak{i}), \mathrm{H}^q(\mathfrak{i}, M) \right), \partial_{\mathfrak{s}/\mathfrak{i}}^\bullet \right)$.

4.3. E_2-page and the transgression map

Let $E_2^{p,q} = \mathrm{H}^p \left(E_1^{\bullet,q}, d_1^{\bullet,q} \right)$ for $p, q \geq 0$. As a direct consequence of the conclusions from Section 4.2, we see that

$$E_2^{p,q} \cong \mathrm{H}^p \left(\mathfrak{s}/\mathfrak{i}, \mathrm{H}^q(\mathfrak{i}, M) \right) \qquad \text{for all } p, q \geq 0.$$

It is known (see, for instance, [12, Chapter 1, §6.5]) that the spectral sequence thus obtained converges to the cohomology of $\mathfrak{s}$ with coefficients in M; that is,

$$\mathrm{H}^p \left(\mathfrak{s}/\mathfrak{i}, \mathrm{H}^q(\mathfrak{i}, M) \right) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{s}, M). \tag{4.5}$$

Now, consider the map $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ that is explicitly given by $d_2^{p,q}(f + \mathrm{im}\, d_1^{p-1,q}) = \partial^{p+q} f + \mathrm{im}\, d_1^{p+1,q-1}$. Using the explicit formula (4.1) for $\partial^{p+q} f$ and (4.2), we can choose a representative of f such that

$$\partial^{p+q} f(\widetilde{x}_0 \wedge \cdots \wedge \widetilde{x}_{p+1} \otimes i_1 \wedge \cdots \wedge i_{q-1}) \tag{4.6}$$
$$= \sum_{0 \leq j < k \leq p+1} (-1)^{\sigma(j,k)} f(\widetilde{x}_0 \wedge \cdots \wedge \widetilde{x}_{j-1} \wedge [\widetilde{x}_j, \widetilde{x}_k] \wedge \widetilde{x}_{j+1} \wedge \cdots \wedge \widehat{\widetilde{x}_k} \wedge \cdots \wedge \widetilde{x}_{p+1} \otimes i_1 \wedge \cdots \wedge i_{q-1}),$$

where $\sigma(j,k) = k + |x_k|(|x_{j+1}| + \cdots + |x_{k-1}|)$, for all representatives $\widetilde{x}_0, \ldots, \widetilde{x}_{p+1}$ in $\mathfrak{s}$ of corresponding elements in $\mathfrak{s}/\mathfrak{i}$ and $i_1, \ldots, i_{q-1} \in \mathfrak{i}$. (Notice that every other term in (4.1) will vanish because we can choose a representative f that vanishes if there is anything other than p entries in $\mathfrak{s}/\mathfrak{i}$ and q entries in $\mathfrak{i}$.) Moreover, we see that $d_2^{p,q} f \neq 0$ only if there exist representatives $\widetilde{x}_j, \widetilde{x}_k \in \mathfrak{s}$ of elements $x_j, x_k \in \mathfrak{s}/\mathfrak{i}$ such that $[\widetilde{x}_j, \widetilde{x}_k] \in \mathfrak{i}$.

Definition 4.1. The transgression map t_M of the spectral sequence (4.5) is defined to be

$$d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}.$$

In Proposition 4.2 we describe the kernel of this transgression map in two particular cases that are important to the proof of Theorem 5.7.

Proposition 4.2. *Let $\mathfrak{s}$ be a Lie superalgebra and $\mathfrak{i} \subsetneq \mathfrak{s}$ be an ideal.*

(a) *If $\mathfrak{s}/\mathfrak{i}$ is a Lie subalgebra of $\mathfrak{s}$, that is, $[\widetilde{x}_j, \widetilde{x}_k] \in \mathfrak{s} \setminus \mathfrak{i}$ for all $\widetilde{x}_j, \widetilde{x}_k \in \mathfrak{s} \setminus \mathfrak{i}$, then $d_2^{0,1} = 0$. In particular, if M is a finite-dimensional $\mathfrak{g}$-module, $\mathfrak{s} = (\mathfrak{g} \otimes A)^\Gamma$ and $\mathfrak{i} = \mathfrak{g} \otimes I$, $I = \prod_{\gamma \in \Gamma} \gamma\mathfrak{m}$, $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, then the kernel of the transgression map*

$$t_M \colon \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma} \left((\mathfrak{g} \otimes I/I^2)^\Gamma, \mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*} M \right) \to \mathrm{H}^2 \left((\mathfrak{g} \otimes A/I)^\Gamma, \mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*} M \right)$$

is isomorphic to $\mathrm{Hom}_{\mathfrak{g}} \left(\mathfrak{g}, M \right)^{\oplus \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2}$.

(b) *If $[\mathfrak{s}/\mathfrak{i}, \mathfrak{s}/\mathfrak{i}] = \mathfrak{i}$, then $d_2^{0,1} f(\widetilde{x}_0 \wedge \widetilde{x}_1) = 0$ for all $\widetilde{x}_0, \widetilde{x}_1$, if and only if $f = 0$. In particular, if $\mathfrak{s} = (\mathfrak{g} \otimes A)^\Gamma$ and $\mathfrak{i} = \mathfrak{g} \otimes I$, $I = \prod_{\gamma \in \Gamma} (\gamma\mathfrak{m})^n$, $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, $n > 1$, then the kernel of the transgression map $t_M \colon \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma} \left((\mathfrak{g} \otimes I/I^2)^\Gamma, \mathrm{ev}_{\mathfrak{m}^n}^{\Gamma\,*} M \right) \to \mathrm{H}^2 \left((\mathfrak{g} \otimes A/I)^\Gamma, \mathrm{ev}_{\mathfrak{m}^n}^{\Gamma\,*} M \right)$ is 0.*

Proof. Part (a) follows from (4.6), the facts that $(\mathfrak{g} \otimes A/I)^\Gamma \cong \mathfrak{g}$ and $(\mathfrak{g} \otimes I/I^2)^\Gamma \cong \mathfrak{g} \otimes \mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{g}^{\oplus \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2}$. To prove part (b), notice that, by (4.6), we have

$$\partial^1 f(\widetilde{x}_0 \wedge \widetilde{x}_1) = (-1)^{1 + |x_1||x_0|} f([\widetilde{x}_0, \widetilde{x}_1])$$

for all representatives $\widetilde{x}_0, \widetilde{x}_1$ in $\mathfrak{s}$ of corresponding elements in $\mathfrak{s}/\mathfrak{i}$. Since $[\mathfrak{s}/\mathfrak{i}, \mathfrak{s}/\mathfrak{i}] = \mathfrak{i}$, it follows that $\partial^1 f(\widetilde{x}_0 \wedge \widetilde{x}_1) = 0$ for all $\widetilde{x}_0, \widetilde{x}_1$ if and only if $f = 0$. $\quad\square$

5. Extensions

Throughout this section, we will assume that $\mathfrak{g}$ is a finite-dimensional simple Lie superalgebra, that A is an associative, commutative, finitely-generated algebra with unit, that Γ is a finite abelian group acting on $\mathfrak{g}$ and A by automorphisms, and that the action of Γ on $\mathrm{MaxSpec}(A)$ is free.

We begin by using Lemma 2.6 to reduce the problem of describing p-extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules to that of describing the p-th cohomology of $(\mathfrak{g} \otimes A)^\Gamma$ with coefficients in indecomposable modules.

Proposition 5.1. *Let $\pi, \pi' \in \mathcal{P}$. There exist $n, \ell, q_1, \ldots, q_\ell \in \mathbb{Z}_{>0}$, maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell \subseteq A$ in distinct Γ-orbits, and, for each $i \in \{1, \ldots, \ell\}$, $j \in \{1, \ldots, q_i\}$, a finite-dimensional indecomposable $\mathfrak{g} \otimes A/\mathfrak{m}_i^n$-module $M_{i,j}$, such that*

$$\mathrm{Hom}_{\Bbbk}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong \bigoplus_{\substack{1 \le j_i \le q_i \\ 1 \le i \le \ell}} \mathrm{ev}_{\mathfrak{m}_1^n}^{\Gamma\,*} M_{1,j_1} \otimes \cdots \otimes \mathrm{ev}_{\mathfrak{m}_\ell^n}^{\Gamma\,*} M_{\ell, j_\ell} \quad \text{and}$$

$$\mathrm{Ext}^p_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong \bigoplus_{\substack{1 \le j_i \le q_i \\ 1 \le i \le \ell}} \mathrm{H}^p \left((\mathfrak{g} \otimes A)^\Gamma, \mathrm{ev}_{\mathfrak{m}_1^n}^{\Gamma\,*} M_{1,j_1} \otimes \cdots \otimes \mathrm{ev}_{\mathfrak{m}_\ell^n}^{\Gamma\,*} M_{\ell, j_\ell} \right), \quad p > 0.$$

Proof. From Remark 2.3, there exist $n, \ell \in \mathbb{Z}_{>0}$, maximal ideals $\mathsf{m}_1, \ldots, \mathsf{m}_\ell \subseteq A$ in distinct Γ-orbits, and finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathsf{m}_i^n$-modules V_i and V_i' such that

$$\mathcal{V}(\pi) \cong \widetilde{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} V_i \quad \text{and} \quad \mathcal{V}(\pi') \cong \widetilde{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} V_i'.$$

Thus, there are isomorphisms of $(\mathfrak{g} \otimes A)^\Gamma$-modules

$$\mathrm{Hom}_{\Bbbk} \left(\mathcal{V}(\pi), \mathcal{V}(\pi') \right)^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong \mathrm{Hom}_{\Bbbk} \left(\mathcal{V}(\pi)^{\oplus 2^{\kappa(\pi)}}, \mathcal{V}(\pi')^{\oplus 2^{\kappa(\pi')}} \right)$$

$$\cong \left(\bigotimes_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} V_i^* \right) \otimes \left(\bigotimes_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} V_i' \right)$$

$$\cong \bigotimes_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} \left(V_i^* \otimes V_i' \right).$$

For each $i \in \{1, \ldots, \ell\}$, since V_i^* and V_i' are finite dimensional, there exist $q_i > 0$ and indecomposable $\mathfrak{g} \otimes A/\mathsf{m}_i^n$-modules, $M_{i,1}, \ldots, M_{i,q_i}$, such that $V_i^* \otimes V_i' \cong \bigoplus_{j=1}^{q_i} M_{i,j}$. Thus there exist isomorphisms of $(\mathfrak{g} \otimes A)^\Gamma$-modules

$$\bigotimes_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} \left(V_i^* \otimes V_i' \right) \cong \bigotimes_{i=1}^{\ell} \left(\bigoplus_{j_i=1}^{q_i} \mathrm{ev}_{\mathsf{m}_i^n}^{\Gamma \, *} \left(M_{i,j_i} \right) \right) \cong \bigoplus_{\substack{1 \le j_i \le q_i \\ 1 \le i \le \ell}} \mathrm{ev}_{\mathsf{m}_1^n}^{\Gamma \, *} M_{1,j_1} \otimes \cdots \otimes \mathrm{ev}_{\mathsf{m}_\ell^n}^{\Gamma \, *} M_{\ell, j_\ell}.$$

This proves the first statement. The second statement follows from the first one and Lemma 2.6. $\quad\square$

This next result is a particular case of Proposition 5.1 and will be used to prove Corollary 5.6.

Corollary 5.2. *Let V and V' be finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules. If the supports of V and V' are disjoint, then $V^* \widehat{\otimes} V'$ is irreducible and, for all $p > 0$,*

$$\mathrm{Ext}^p_{(\mathfrak{g} \otimes A)^\Gamma}(V, V') \cong \begin{cases} \mathrm{H}^p((\mathfrak{g} \otimes A)^\Gamma, V^* \widehat{\otimes} V'), & \text{if } V^* \otimes V' \text{ is irreducible,} \\ \mathrm{H}^p((\mathfrak{g} \otimes A)^\Gamma, V^* \widehat{\otimes} V')^{\oplus 2}, & \text{otherwise.} \end{cases}$$

Our next goal is to reduce the problem of determining 1-extensions between finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-modules to that of determining homomorphisms and extensions between finite-dimensional $\mathfrak{g} \otimes A/\mathsf{m}^n$-modules, where m is a maximal ideal of A and n is a positive integer. We start with a general result regarding first cohomology. Recall the definition of transgression map (Definition 4.1).

Lemma 5.3. *If M is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module, then there exists a finite-codimensional Γ-invariant ideal $I \subseteq A$, such that*

$$\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, M\right) \cong \mathrm{H}^1\left((\mathfrak{g} \otimes A/I)^\Gamma, M\right) \oplus K,$$

where K is the kernel of the transgression map

$$t_M \colon \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}\left((\mathfrak{g} \otimes I/I^2)^\Gamma, M\right) \to \mathrm{H}^2\left((\mathfrak{g} \otimes A/I)^\Gamma, M\right).$$

Proof. Since M is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module, by Proposition 2.5, there exists a Γ-invariant finite-codimensional ideal $I \subseteq A$ such that $(\mathfrak{g} \otimes I)^\Gamma M = 0$. By Proposition 2.9, there exists a first-quadrant cohomology spectral sequence associated to $(\mathfrak{g} \otimes A)^\Gamma$ and $(\mathfrak{g} \otimes I)^\Gamma$, namely

$$E_2^{p,q} \cong \mathrm{H}^p\left((\mathfrak{g} \otimes A/I)^\Gamma, \mathrm{H}^q\left((\mathfrak{g} \otimes I)^\Gamma, M\right)\right) \Rightarrow \mathrm{H}^{p+q}\left((\mathfrak{g} \otimes A)^\Gamma, M\right). \qquad (5.1)$$

Since (5.1) is a first-quadrant cohomology spectral sequence, we have an isomorphism of vector spaces $\mathrm{H}^1((\mathfrak{g} \otimes A)^\Gamma, M) \cong E_\infty^{1,0} \oplus E_\infty^{0,1}$. Moreover,

$$E_\infty^{1,0} = E_2^{1,0} \cong \mathrm{H}^1\left((\mathfrak{g} \otimes A/I)^\Gamma, M\right) \quad \text{and} \quad E_\infty^{0,1} = E_3^{0,1} \cong \ker(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0}),$$

where $d_2^{0,1}$ is the transgression map t_M.

In order to finish the proof, we only need to describe $E_2^{0,1}$ and $E_2^{2,0}$. By (5.1),

$$E_2^{0,1} \cong \mathrm{H}^0\left((\mathfrak{g} \otimes A/I)^\Gamma, \mathrm{H}^1\left((\mathfrak{g} \otimes I)^\Gamma, M\right)\right) \quad \text{and} \quad E_2^{2,0} \cong \mathrm{H}^2((\mathfrak{g} \otimes A/I)^\Gamma, M).$$

Since $(\mathfrak{g} \otimes I)^\Gamma$ acts trivially on M, there is an isomorphism of $(\mathfrak{g} \otimes A/I)^\Gamma$-modules $\mathrm{H}^\bullet\left((\mathfrak{g} \otimes I)^\Gamma, M\right) \cong \mathrm{H}^\bullet\left((\mathfrak{g} \otimes I)^\Gamma, \Bbbk\right) \otimes M$ by Lemma 2.7. Moreover, by Lemma 2.11, $\mathrm{H}^1\left((\mathfrak{g} \otimes I)^\Gamma, \Bbbk\right)$ is isomorphic to $((\mathfrak{g} \otimes I/I^2)^\Gamma)^*$ as a $(\mathfrak{g} \otimes A/I)^\Gamma$-module. Thus $E_2^{0,1} \cong \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}\left((\mathfrak{g} \otimes I/I^2)^\Gamma, M\right)$. $\quad\square$

As a consequence of Lemmas 2.6 and 5.3, we obtain the following result.

Corollary 5.4. *If V, V' are finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules, then $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(V, V')$ is finite dimensional.*

Proof. Since V and V' are finite-dimensional modules, by Lemma 2.6, $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(V, V')$ is isomorphic to $\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V'\right)$. By Lemma 5.3, $\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V'\right)$ is isomorphic to a subspace of

$$\mathrm{H}^1\left((\mathfrak{g} \otimes A/I)^\Gamma, V^* \otimes V'\right) \oplus \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}\left((\mathfrak{g} \otimes I/I^2)^\Gamma, V^* \otimes V'\right), \qquad (5.2)$$

where I is a finite-codimensional Γ-invariant ideal of A. Since $\mathfrak{g}$, A/I, I/I^2, $V^* \otimes V'$ are finite dimensional, both terms in (5.2) are finite dimensional. This proves that $\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V'\right)$ is finite dimensional, and finishes the proof. $\quad\square$

We emphasize the relevance of Corollary 5.4 by contrasting it with a case where Ext^1 is not finite dimensional. Namely, let Γ be a group acting by automorphisms on an abelian Lie superalgebra $\mathfrak{a}$ and on an associative commutative algebra with unit B. For any finite-dimensional trivial $(\mathfrak{a} \otimes B)^\Gamma$-modules M and M',

$$\mathrm{Ext}^1_{(\mathfrak{a} \otimes B)^\Gamma}(M, M') \cong \mathrm{Hom}_\Bbbk\left((\mathfrak{a} \otimes B)^\Gamma, M^* \otimes M'\right)$$

is finite dimensional if and only if $(\mathfrak{a} \otimes B)^\Gamma$ is finite dimensional. In particular, when Γ is trivial and B is infinite dimensional, $\mathrm{Ext}^1_{(\mathfrak{a} \otimes B)^\Gamma}(M, M')$ is infinite dimensional.

Recall from Corollary 2.12 that $\mathrm{H}^1((\mathfrak{g} \otimes A)^\Gamma, \Bbbk) = 0$. The next result gives a vanishing condition for $\mathrm{H}^1((\mathfrak{g} \otimes A)^\Gamma, M)$ when M is a finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-module of the form $\bigotimes_{i=1}^\ell \mathrm{ev}_{\mathsf{m}_i}^{\Gamma\ *}{}^{n_i} M_i$. It also generalizes [20, Theorem 3.6] and [24, Theorem 3.7].

Proposition 5.5. *Let* $\ell, n_1, \ldots, n_\ell \in \mathbb{Z}_{>0}$, $\mathsf{m}_1, \ldots, \mathsf{m}_\ell \subseteq A$ *be maximal ideals in distinct* Γ-*orbits, for each* $i \in \{1, \ldots, \ell\}$, *let* M_i *be a finite-dimensional* $\mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}$-*module, and* $M = \bigotimes_{i=1}^\ell \mathrm{ev}_{\mathsf{m}_i}^{\Gamma\ *}{}^{n_i} M_i$.

(a) *If* $\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}}(\Bbbk, M_i) = 0$ *for more than one index* i, *then* $\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, M\right) = 0$.

(b) *If* $\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}}(\Bbbk, M_i) = 0$ *for exactly one index* i, *then*

$$\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, M\right) \cong \mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, \mathrm{ev}_{\mathsf{m}_i}^{\Gamma\ *}{}^{n_i} M_i\right) \otimes \bigotimes_{j \neq i} \mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_j^{n_j}}(\Bbbk, M_j).$$

(c) *If* $\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}}(\Bbbk, M_i) \neq 0$ *for all indices* i, *then*

$$\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, M\right) \cong \bigoplus_{i=1}^\ell \left(\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, \mathrm{ev}_{\mathsf{m}_i}^{\Gamma\ *}{}^{n_i} M_i\right) \otimes \bigotimes_{j \neq i} \mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_j^{n_j}}(\Bbbk, M_j)\right).$$

Proof. By Proposition 2.8, we have

$$\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, M\right) \cong \bigoplus_{i=1}^\ell \left(\mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, \mathrm{ev}_{\mathsf{m}_i}^{\Gamma\ *}{}^{n_i} M_i\right) \otimes \bigotimes_{j \neq i} \mathrm{Hom}_{(\mathfrak{g} \otimes A)^\Gamma}\left(\Bbbk, \mathrm{ev}_{\mathsf{m}_j}^{\Gamma\ *}{}^{n_j} M_j\right)\right).$$

Now, notice that $\mathrm{Hom}_{(\mathfrak{g} \otimes A)^\Gamma}\left(\Bbbk, \mathrm{ev}_{\mathsf{m}_k}^{\Gamma\ *}{}^{n_k} M_k\right) \cong \mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_k^{n_k}}(\Bbbk, M_k)$ for all $k \in \{1, \ldots, \ell\}$. This proves part (c). If $\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_k^{n_k}}(\Bbbk, M_k) = 0$ for more than one index k, then for each i, there exists $j \neq i$ such that $\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_j^{n_j}}(\Bbbk, M_j) = 0$. This proves part (a). If $\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_i^{n_i}}(\Bbbk, M_i) = 0$ for exactly one index i, then $\bigotimes_{j \neq k} \mathrm{Hom}_{\mathfrak{g} \otimes A/\mathsf{m}_j^{n_j}}(\Bbbk, M_j) = 0$ for all $k \neq i$. This proves part (b). $\square$

The next result generalizes [20, Lemma 3.3] and [24, Proposition 3.6]. It follows directly from Corollary 5.2 (for $p = 1$), the classification of finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules (see Section 3) and Proposition 5.5(a), thus we omit its proof.

Corollary 5.6. *Let V and V' be nontrivial, finite-dimensional, irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules. If the supports of V and V' are disjoint, then $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(V, V') = 0$.*

Recall that we are assuming that $\mathfrak{g}$ is a finite-dimensional simple Lie superalgebra, A is an associative, commutative, finitely-generated algebra with unit, and Γ is a finite abelian group acting on $\mathfrak{g}$ and A by automorphisms, such that the induced action of Γ on $\mathrm{MaxSpec}(A)$ is free. Now we state, and prove, the main result of this section. It describes 1-extensions between finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules in terms of homomorphisms and extensions between finite-dimensional $\mathfrak{g} \otimes A/\mathfrak{m}^n$-modules.

Theorem 5.7. *Let $\pi, \pi' \in \mathcal{P}$, $\ell, n_1, \ldots, n_\ell \in \mathbb{Z}_{>0}$, $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell \subseteq A$ be maximal ideals in distinct Γ-orbits, and V_i, V_i' be finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}$-modules such that $\mathcal{V}(\pi) = \widehat{\bigotimes}_{i=1}^\ell \mathrm{ev}^{\Gamma}_{\mathfrak{m}_i^{n_i}} {}^* V_i$ and $\mathcal{V}(\pi') = \widehat{\bigotimes}_{i=1}^\ell \mathrm{ev}^{\Gamma}_{\mathfrak{m}_i^{n_i}} {}^* V_i'$. For each $i \in \{1, \ldots, \ell\}$, let d_i denote $\delta_{1,n_i} \dim_{A/\mathfrak{m}_i} \mathfrak{m}_i/\mathfrak{m}_i^2$.*

(a) *If V_i is not isomorphic to V_i' for two or more indices i, then $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi')) = 0$.*

(b) *If V_i is isomorphic to V_i' for all but one index i, then*

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong \mathrm{Ext}^1_{\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}}(V_i, V_i') \oplus \mathrm{Hom}_\mathfrak{g}\left(\mathfrak{g} \otimes V_i, V_i'\right)^{\oplus d_i}.$$

(c) *If V_i is isomorphic to V_i' for all $i \in \{1, \ldots, \ell\}$, then*

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong \bigoplus_{i=1}^\ell \left(\mathrm{Ext}^1_{\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}}(V_i, V_i') \oplus \mathrm{Hom}_\mathfrak{g}\left(\mathfrak{g} \otimes V_i, V_i'\right)^{\oplus d_i} \right).$$

Proof. Denote $\left(\bigotimes_{i=1}^\ell \mathrm{ev}^{\Gamma}_{\mathfrak{m}_i^{n_i}} {}^* V_i\right)$ by V, $\left(\bigotimes_{i=1}^\ell \mathrm{ev}^{\Gamma}_{\mathfrak{m}_i^{n_i}} {}^* V_i'\right)$ by V', and recall from Section 2.3 that $V \cong \mathcal{V}(\pi)^{\oplus 2^{\kappa(\pi)}}$ and $V' \cong \mathcal{V}(\pi')^{\oplus 2^{\kappa(\pi')}}$. Thus, by Lemma 2.6, we have

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi) + \kappa(\pi')}} \cong \mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, V^* \otimes V'\right).$$

Now, notice that $V^* \otimes V' \cong \bigotimes_{i=1}^\ell \mathrm{ev}^{\Gamma}_{\mathfrak{m}_i^{n_i}} {}^* (V_i^* \otimes V_i')$, that $V_i^* \otimes V_i'$ are finite dimensional, and that

$$\mathrm{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}}(\Bbbk, V_i^* \otimes V_i') \cong \mathrm{Hom}_{\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}}(V_i, V_i') \cong \begin{cases} 0, & \text{if } V_i \not\cong V_i', \\ \Bbbk, & \text{if } V_i \cong V_i'. \end{cases}$$

(a) To prove part (a), notice that, if $V_i \not\cong V_i'$ for two or more indices i, then by Proposition 5.5(a),

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi')) = 0.$$

(b) If $V_i \not\cong V_i'$ for exactly one index i, then by Proposition 5.5(b) we have

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi)+\kappa(\pi')}} \cong \mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, \mathrm{ev}_{\mathfrak{m}_i^{n_i}}^{\Gamma\,*}(V_i^* \otimes V_i')\right).$$

Now, let $I = \prod_{\gamma \in \Gamma}(\gamma \mathfrak{m}_i)^{n_i}$. By Lemma 5.3, we have

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi)+\kappa(\pi')}} \cong \mathrm{H}^1\left(\mathfrak{g} \otimes A/\mathfrak{m}_i^{n_i}, (V_i^* \otimes V_i')\right) \oplus K_i,$$

where K_i is the kernel of the transgression map

$$t_{V_i^* \otimes V_i} \colon \mathrm{Hom}_{(\mathfrak{g} \otimes A/I)^\Gamma}\left((\mathfrak{g} \otimes I/I^2)^\Gamma, \mathrm{ev}_{\mathfrak{m}_i^{n_i}}^{\Gamma\,*}(V_i^* \otimes V_i')\right) \to \mathrm{H}^2\left((\mathfrak{g} \otimes A/I)^\Gamma, \mathrm{ev}_{\mathfrak{m}_i^{n_i}}^{\Gamma\,*}(V_i^* \otimes V_i')\right).$$

To finish the proof of part (b), notice that, $K_i \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V_i, V_i')^{\oplus d_i}$ by Proposition 4.2.

(c) If $V_i \cong V_i'$ for all $i \in \{1, \dots, \ell\}$, then by Proposition 5.5(c), we have

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi)+\kappa(\pi')}} \cong \bigoplus_{i=1}^{\ell} \mathrm{H}^1\left((\mathfrak{g} \otimes A)^\Gamma, \mathrm{ev}_{\mathfrak{m}_i^{n_i}}^{\Gamma\,*}(V_i^* \otimes V_i')\right).$$

The rest of the proof of part (c) follows from Lemma 5.3 and Proposition 4.2 using, for each $i \in \{1, \dots, \ell\}$, the same arguments that we used to prove part (b). $\quad\square$

Theorem 5.7 generalizes [23, Theorem 3.7] and [4, Theorem 4.2.1] to the super setting.

Example 5.8 (*Evaluation modules*). Let $\mathcal{V}(\pi) = \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathfrak{m}_i}^{\Gamma\,*} V(\lambda_i)$ and $\mathcal{V}(\pi') = \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}_{\mathfrak{m}_i}^{\Gamma\,*} V(\mu_i)$ be irreducible finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-modules. In this case, $d_i = \dim_{A/\mathfrak{m}_i} \mathfrak{m}_i/\mathfrak{m}_i^2$ and Theorem 5.7 yields:

(i) $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi')) = 0$, when $\lambda_i \neq \mu_i$ for two or more indices i.

(ii) $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi)+\kappa(\pi')}} \cong \mathrm{Ext}^1_{\mathfrak{g}}(V(\lambda_i), V(\mu_i)) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_i), V(\mu_i))^{\oplus d_i}$, when $\lambda_i = \mu_i$ for all but one index i.

(iii) $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi)+\kappa(\pi')}} \cong \bigoplus_{i=1}^{\ell} \mathrm{Ext}^1_{\mathfrak{g}}(V(\lambda_i), V(\mu_i)) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_i), V(\mu_i))^{\oplus d_i}$, when $\lambda_i = \mu_i$ for all $i \in \{1, \dots, \ell\}$.

Thus, $\mathcal{V}(\pi)$ and $\mathcal{V}(\pi')$ are in the same block if and only if, for each $i \in \{1, \dots, \ell\}$, either: $\lambda_i - \mu_i \in Q$, or $V(\lambda_i)$ and $V(\mu_i)$ are in the same block in the category of finite-dimensional

$\mathfrak{g}$-modules. In particular, if $\mathfrak{g}$ is of type II, $\mathfrak{p}(n)$, $\mathfrak{q}(n)$, $S(n)$, $\tilde{S}(n)$ or $H(n)$, since all finite-dimensional irreducible $(\mathfrak{g} \otimes A)^{\Gamma}$-modules are evaluation modules, this gives a description of the block decomposition of the category of finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$-modules.

Moreover, if either λ or μ are *typical*, then $\mathrm{Ext}^1_{\mathfrak{g}}(V(\lambda), V(\mu)) = 0$ (see [18, Theorem 1]). Thus, if either λ_i or μ_i is *typical* for each $i \in \{1, \ldots, \ell\}$, then $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(\mathcal{V}(\pi), \mathcal{V}(\pi'))$ depends only on $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_i), V(\mu_i))$. In particular, if $\mathfrak{g}$ is of type $B(0, n)$, since all the weights are typical, the blocks of the category of finite-dimensional $(\mathfrak{g} \otimes A)^{\Gamma}$-modules are parametrized by the so-called spectral character (compare with [20, Proposition 4.5] and [24, Theorem 5.19]).

Now, let $\mathfrak{g}$ be a Lie superalgebra of type A or C (type I), $\Bbbk$ be the trivial $\mathfrak{g}$-module and $V(\lambda)$ be an irreducible finite-dimensional $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}^*$. Let ρ_1 be the half sum of the odd positive roots of $\mathfrak{g}$, $\alpha_{\max}$ (resp. $\alpha_{\min}$) be the maximal (resp. minimal) root of $\mathfrak{g}$, and $\mu^{(j)}$ be as in [31, Theorem 1.1]. For any maximal ideal $\mathfrak{m} \in \mathrm{MaxSpec}\, A$, we have

$$\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(\mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}\Bbbk, \mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}V(\lambda)) \cong \mathrm{Ext}^1_{\mathfrak{g}}(\Bbbk, V(\lambda)) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, V(\lambda))^{\oplus d},$$

where $d = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. Thus $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(\mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}\Bbbk, \mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}V(\lambda)) \neq 0$ if and only if $\mathrm{Ext}^1_{\mathfrak{g}}(\Bbbk, V(\lambda)) \neq 0$ or $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, V(\lambda)) \neq 0$. From [31, Theorem 1.1], we obtain that:

- in type A, $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(\mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}\Bbbk, \mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}V(\lambda)) \neq 0$ if and only if $\lambda \in \{-\alpha_{\min}, \alpha_{\max}, \mu^{(0)}, \ldots, \mu^{(n-1)}\}$,
- in type C, $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(\mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}\Bbbk, \mathrm{ev}_{\mathfrak{m}}^{\Gamma\,*}V(\lambda)) \neq 0$ if and only if $\lambda \in \{-\alpha_{\min}, 2\rho_1, \alpha_{\max}\}$.

Example 5.9 *(Generalized evaluation modules).* Suppose $\mathfrak{g}$ is either of type I or $W(n)$. Let $n > 1$, $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, V, V' be finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathfrak{m}^n$-modules, $\mathcal{V} = \mathrm{ev}_{\mathfrak{m}^n}^{\Gamma\,*}V$ and $\mathcal{V}' = \mathrm{ev}_{\mathfrak{m}^n}^{\Gamma\,*}V'$. By Theorem 5.7, $\mathrm{Ext}^1_{(\mathfrak{g} \otimes A)^{\Gamma}}(\mathcal{V}, \mathcal{V}') \cong \mathrm{Ext}^1_{\mathfrak{g} \otimes A/\mathfrak{m}^n}(V, V')$, as the kernel of the transgression map is zero (or $d_1 = 0$; cf. Proposition 4.2). Therefore, $\mathcal{V}$ and $\mathcal{V}'$ are in the same block if and only if $\mathrm{Ext}^1_{\mathfrak{g} \otimes A/\mathfrak{m}^n}(V, V') \neq 0$.

For instance, let $\mathfrak{g} = \mathfrak{sl}(1, 2)$ (type I), $A = \Bbbk[t]$ and $\mathfrak{m} = \langle t \rangle \in \mathrm{MaxSpec}\,\Bbbk[t]$. Recall that $\mathfrak{g}$ admits a $\mathbb{Z}$-grading, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \cong \mathfrak{gl}_2$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Write $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}$, where $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ and $\mathfrak{z}$ is the 1-dimensional center of $\mathfrak{g}_0$. Let $z \in \mathfrak{z} \setminus \{0\}$ and $\theta \in (\mathfrak{z} \otimes A/\mathfrak{m}^2)^*$ be such that $\theta(z) = \theta(z \otimes t) \neq 0$. Consider the 1-dimensional $(\mathfrak{g}_0 \otimes A/\mathfrak{m}^2)$-module $\Bbbk \otimes \theta$, where $\Bbbk$ is the trivial 1-dimensional $(\mathfrak{g}_0^{ss} \otimes A/\mathfrak{m}^2)$-module and θ is the 1-dimensional $(\mathfrak{z} \otimes A/\mathfrak{m}^2)$-module whose action is given by θ. Inflate $\Bbbk \otimes \theta$ to a $(\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes A/\mathfrak{m}^2$-module by setting $(\mathfrak{g}_1 \otimes A/\mathfrak{m}^2) \cdot (\Bbbk \otimes \theta) = 0$, and consider the Kac module:

$$K(\Bbbk \otimes \theta) := U(\mathfrak{g} \otimes A/\mathfrak{m}^2) \otimes_{U((\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes A/\mathfrak{m}^2)} (\Bbbk \otimes \theta).$$

One can check that $K(\Bbbk \otimes \theta)$ is an irreducible finite-dimensional $(\mathfrak{g} \otimes A/\mathfrak{m}^2)$-module on which $\mathfrak{g} \otimes \mathfrak{m}$ does not act trivially (see [27] for details on Kac modules). Thus $\mathrm{ev}_{\mathfrak{m}^2}^{\Gamma\,*}K(\Bbbk \otimes \theta)$

is a generalized evaluation $(\mathfrak{g} \otimes A)^\Gamma$-module which is not isomorphic to an evaluation module. By Theorem 5.7,

$$\operatorname{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma} (\operatorname{ev}^{\Gamma}_{\mathsf{m}^2}{}^* \Bbbk, \operatorname{ev}^{\Gamma}_{\mathsf{m}^2}{}^* K(\Bbbk \otimes \theta)) \cong \operatorname{H}^1 \left(\mathfrak{g} \otimes A/\mathsf{m}^2, K(\Bbbk \otimes \theta) \right).$$

Using the LHS spectral sequence

$$E_2^{p,q} \cong \operatorname{H}^p \left(\mathfrak{g}, \operatorname{H}^q \left(\mathfrak{g} \otimes \mathsf{m}/\mathsf{m}^2, K(\Bbbk \otimes \theta) \right) \right) \Rightarrow \operatorname{H}^{p+q} \left(\mathfrak{g} \otimes A/\mathsf{m}^2, K(\Bbbk \otimes \theta) \right),$$

along with the fact that $\operatorname{H}^0(\mathfrak{g} \otimes \mathsf{m}, K(\Bbbk \otimes \theta)) = 0$ (as $K(\Bbbk \otimes \theta)$ is irreducible and is not an evaluation module), one can check that $\operatorname{Ext}^1_{(\mathfrak{g} \otimes A)^\Gamma} (\operatorname{ev}^{\Gamma}_{\mathsf{m}^2}{}^* \Bbbk, \operatorname{ev}^{\Gamma}_{\mathsf{m}^2}{}^* K(\Bbbk \otimes \theta)) \neq 0$.

It is worth noting however, that irreducible finite-dimensional $\mathfrak{g} \otimes A/\mathsf{m}^n$-modules are not well understood in general, as their only known realization is as quotients of Kac and Verma modules.

References

[1] I. Bagci, On cohomology and support varieties for Lie superalgebras, J. Algebra 351 (2012) 204–219.

[2] I. Bagci, On representations of Cartan map Lie superalgebras, ArXiv e-prints, arXiv:1508.00877, 2015.

[3] D. Britten, M. Lau, F. Lemire, Weight modules for current algebras, J. Algebra 440 (2015) 245–263.

[4] B. Boe, C. Drupieski, T. Macedo, D. Nakano, Extensions for generalized current algebras, ArXiv e-prints, arXiv:1511.00024, 2015.

[5] B.D. Boe, J.R. Kujawa, D.K. Nakano, Cohomology and support varieties for Lie superalgebras, Trans. Amer. Math. Soc. 362 (12) (2010) 6551–6590.

[6] S.-J. Cheng, Differentiably simple Lie superalgebras and representations of semisimple Lie superalgebras, J. Algebra 173 (1) (1995) 1–43.

[7] L. Calixto, A. Moura, A. Savage, Equivariant map queer Lie superalgebras, Canad. J. Math. 68 (2016) 258–279.

[8] S. Eswara Rao, Finite dimensional modules for multiloop superalgebras of types $A(m,n)$ and $C(m)$, Proc. Amer. Math. Soc. 141 (10) (2013) 3411–3419.

[9] D. Fattori, V.G. Kac, Classification of finite simple Lie conformal superalgebras, J. Algebra 258 (1) (2002) 23–59, Special issue in celebration of Claudio Procesi's 60th birthday.

[10] B.L. Feĭgin, D.A. Leĭtes, New Lie superalgebras of string theories, in: Group Theoretical Methods in Physics, vols. 1–3, Zvenigorod, 1982, Harwood Academic Publ., Chur, 1985, pp. 623–629.

[11] D.B. Fuch, D.A. Leĭtes, Cohomology of Lie superalgebras, C. R. Acad. Bulgare Sci. 37 (12) (1984) 1595–1596.

[12] D.B. Fuks, Cohomology of infinite-dimensional Lie algebras, in: Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986, Translated from the Russian by A.B. Sosinskiĭ.

[13] F. Gavarini, Algebraic supergroups of Cartan type, Forum Math. 26 (5) (2014) 1473–1564.

[14] P. Grozman, D. Leites, I. Shchepochkina, Lie superalgebras of string theories, Acta Math. Vietnam. 26 (1) (2001) 27–63.

[15] C. Gruson, Sur la cohomologie des super algèbres de Lie étranges, Transform. Groups 5 (1) (2000) 73–84.

[16] C. Gruson, Cohomologie des modules de dimension finie sur la super algèbre de Lie $\mathfrak{osp}(3,2)$, J. Algebra 259 (2) (2003) 581–598.

[17] V. Kac, Lie superalgebras, Adv. Math. 26 (1) (1977) 8–96.

[18] V. Kac, Representations of classical Lie superalgebras, in: Differential Geometrical Methods in Mathematical Physics, II, Proc. Conf., Univ. Bonn, 1977, in: Lecture Notes in Math., vol. 676, Springer, Berlin, 1978, pp. 597–626.

[19] V.G. Kac, M. Wakimoto, Integrable Highest Weight Modules over Affine Superalgebras and Number Theory, Lie Theory and Geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 415–456.

[20] R. Kodera, Extensions between finite-dimensional simple modules over a generalized current Lie algebra, Transform. Groups 15 (2) (2010) 371–388.

[21] S. Kumar, Kac–Moody Groups, Their Flag Varieties and Representation Theory, Progr. Math., vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002.

[22] I.M. Musson, Lie Superalgebras and Enveloping Algebras, Graduate Studies in Mathematics, vol. 131, American Mathematical Society, Providence, RI, 2012.

[23] E. Neher, A. Savage, A Survey of Equivariant Map Algebras with Open Problems, Recent Developments in Algebraic and Combinatorial Aspects of Representation Theory, Contemp. Math., vol. 602, Amer. Math. Soc., Providence, RI, 2013, pp. 165–182.

[24] E. Neher, A. Savage, Extensions and block decompositions for finite-dimensional representations of equivariant map algebras, Transform. Groups 20 (1) (2015) 183–228.

[25] E. Neher, A. Savage, P. Senesi, Irreducible finite-dimensional representations of equivariant map algebras, Trans. Amer. Math. Soc. 364 (5) (2012) 2619–2646.

[26] E.I. Poletaeva, Spenser cohomology connected with some Lie superalgebras, in: Problems in Group Theory and Homological Algebra, Matematika, Yaroslav. Gos. Univ., Yaroslavl', 1988, pp. 162–167 (Russian).

[27] A. Savage, Equivariant map superalgebras, Math. Z. 277 (1–2) (2014) 373–399.

[28] M. Scheunert, R.B. Zhang, Cohomology of Lie superalgebras and their generalizations, J. Math. Phys. 39 (9) (1998) 5024–5061.

[29] M. Scheunert, R.B. Zhang, The second cohomology of sl$(m|1)$ with coefficients in its enveloping algebra is trivial, Lett. Math. Phys. 47 (1) (1999) 33–48.

[30] V.V. Serganova, Automorphisms of Lie superalgebras of string theories, Funktsional. Anal. i Prilozhen. 19 (3) (1985) 75–76.

[31] Y. Su, R.B. Zhang, Cohomology of Lie superalgebras $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$, Proc. Lond. Math. 94 (1) (2007) 91–136.

[32] V.S. Varadarajan, Supersymmetry for mathematicians: an introduction, in: Courant Lecture Notes in Mathematics, vol. 11, New York University, Courant Institute of Mathematical/Sciences American Mathematical Society, New York/Providence, RI, 2004.

[33] C.A. Weibel, An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.