Journal of Algebra 537 (2019) 26–38

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A categorification of Hecke algebras with parameters 1 and v

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ARTICLE INFO

Article history:
Received 13 June 2018
Available online 31 July 2019
Communicated by Volodymyr Mazorchuk

MSC:
primary 17B10

Keywords:
Soergel bimodules
Hecke algebras
Unequal parameters

ABSTRACT

We categorify the Hecke algebra with parameters 1 and v using a variation of the category of Soergel bimodules.

Introduction

0.1. Let (W, S) be a Coxeter system, where S denotes the set of simple reflections. Let $m_{s_1,s_2} \in \mathbb{N} \cup \{\infty\}$, such that $(s_1 s_2)^{m_{s_1,s_2}} = 1$ for $s_1, s_2 \in S$. We denote by $\ell(\cdot)$ the length function on W. Let $T = \bigcup_{w \in W} wSw^{-1}$ be the set of reflections. We denote by $\leq$ the Bruhat order on W. We denote by V (over $\mathbb{R}$) the geometric representation of W and denote by $\Phi = \Phi^+ \cup \Phi^-$ the root system of W in the sense of [8, Section 5.4]. Let

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https://doi.org/10.1016/j.jalgebra.2019.06.031

$\{\beta_s | s \in S\}$ be the collection of simple roots. Let $n(w) = \mathrm{Card}(\Phi^+ \cap w(\Phi^-))$. We know that $n(w) = \ell(w)$ for any $w \in W$.

A weight function $L : W \to \mathbb{Z}$ is a function on W such that $L(w_1w_2) = L(w_1)+L(w_2)$ whenever $\ell(w_1w_2) = \ell(w_1)+\ell(w_2)$ for $w_1, w_2 \in W$. It follows that $L(s_1) = L(s_2)$ for any $s_1, s_2 \in S$ such that m_{s_1,s_2} is odd.

In this note, we assume that a weight function L is fixed such that

$$L(s) = 0 \text{ or } 1, \qquad \text{for } s \in S.$$

We define $S_e = S \cap L^{-1}(e)$ for $e \in \{0, 1\}$.

0.2. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with a generic parameter v. For $s \in S$, we set $v_s = v^{L(s)} \in \mathcal{A}$. Let $\mathcal{H} = \mathcal{H}_{(W,S,L)}$ be the $\mathcal{A}$-algebra generated by $T_s (s \in S)$ subject to the relations:

$$(T_s - v_s^{-1})(T_s + v_s) = 0, \qquad \text{for } s \in S$$
$$T_sT_{s'}T_s \cdots = T_{s'}T_sT_{s'} \cdots ,$$

where both products in the second relation have $m_{s,s'}$ factors for any $s \neq s' \in S$ such that $m_{s,s'} \neq \infty$. This algebra is called the Hecke algebra with unequal parameters associated to a weight function L. It was introduced and studied in [9].

We write

$$T_w = T_{s_1} \cdots T_{s_n}, \qquad \text{for any reduced expression } w = s_1 \cdots s_n \text{ with } s_i \in S. \tag{0.1}$$

The set $\{T_w | w \in W\}$ forms an $\mathcal{A}$-basis of $\mathcal{H}$. Let $^- : \mathcal{H} \to \mathcal{H}$ be the $\mathcal{A}$-semilinear bar involution such that $\overline{T_s} = T_s^{-1}$ and $\overline{v} = v^{-1}$. Note that for any $s \in S_0$, we have $T_s^2 = 1$ and $\overline{T_s} = T_s$.

It is shown in [9, Chap 5] that, for any $w \in W$, there is a unique element c_w such that

(1) $\overline{c_w} = c_w$;
(2) $c_w = \sum_{y \in W} p_{y,w} T_y$ where
- $p_{y,w} = 0$ unless $y \leq w$;
- $p_{w,w} = 1$;
- $p_{y,w} \in v\mathbb{Z}[v]$ if $y < w$.

The set $\{c_w | w \in W\}$ forms an $\mathcal{A}$-basis of $\mathcal{H}$, called the canonical (or Kazhdan-Lusztig) basis.

Let $s, r \in S$ with $L(s) = 0$ and $L(r) = 1$. Then we have $c_s = T_s$ and $c_r = T_r + v$. In particular, we have $c_s^2 = 1$ and $c_r^2 = (v + v^{-1})c_r$.

0.3. Soergel ([10]) categorified the Hecke algebras with equal parameters (that is, $L = \ell$) in terms of the category of Soergel bimodules. We briefly recall the construction here.

Following [5, §3.1], we fix a Soergel realization $(\mathfrak{h}, \{\alpha_s\}, \{\alpha_s^\vee\})$ of (W, S) over a field k (char $k \neq 2$). This realization is faithful and Soergel's techniques can be applied. Let $R = \bigoplus_{m \geq 0} S^m(\mathfrak{h}^*)$, which we view as a graded k-algebra with $\deg(\mathfrak{h}^*) = 2$. For any $s \in S$, let $R^s \subset R$ be the subring of s-invariants.

We work in the abelian category of finitely generated graded R-bimodules, where morphisms preserve gradings. For any $s \in S$, we define the graded R-bimodule $B_s = R \otimes_{R^s} R(1)$, where (1) denotes the grading shift. For any $w \in W$, we denote the standard bimodule associated with w by R_w. Recall that R_w is isomorphic to R as k-modules and the R-bimodule structure is defined as: $f \cdot a = a \cdot w^{-1}(f)$ for $f \in R$ and $a \in R_w$.

For any (not necessarily reduced) expression $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_n} \in W$, we define the Bott-Samelson bimodule as the tensor product

$$B_{\underline{w}} = B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \cdots \otimes_R B_{s_{i_n}},$$

viewed as a graded R-bimodule. Let $\mathbb{BS}\mathrm{Bim}$ denote the full monoidal subcategory of graded R-bimodules whose objects are Bott-Samelson bimodules. Let $\mathbb{S}\mathrm{Bim}$ denote the Karoubi envelope of $\mathbb{BS}\mathrm{Bim}$, which is nowadays called the category of Soergel bimodules. Following [6,10], we know that $\mathbb{S}\mathrm{Bim}$ categorifies the Hecke algebra $\mathcal{H}_{(W,S,\ell)}$ with equal parameters. We have an algebra isomomorphism from the split Grothendieck group $[\mathbb{S}\mathrm{Bim}]$ to the Hecke algebra $\mathcal{H}_{(W,S,\ell)}$, where the images of the indecomposable objects up to degree shift are the canonical basis elements.

0.4. Now let $\mathbb{BS}\mathrm{Bim}^L$ be the full monoidal subcategory of the category of graded R-bimodules generated by R_s $(s \in S_0)$ and $B_{s'}$ $(s' \in S_1)$. For any expression $\underline{w} = s_{i_1} \cdots s_{i_n} \in W$, we define the graded R-bimodule $B^L_{\underline{w}}$ as the tensor product

$$B^L_{\underline{w}} = B^L_{s_{i_1}} \otimes_R B^L_{s_{i_2}} \otimes_R \cdots \otimes_R B^L_{s_{i_n}}, \text{ where } B^L_{s_{i_j}} = \begin{cases} R_{s_{i_j}}, & \text{if } s_{i_j} \in S_0; \\ B_{s_{i_j}}, & \text{if } s_{i_j} \in S_1. \end{cases}$$

We denote by $\mathbb{S}\mathrm{Bim}^L$ the Karoubi envelope of $\mathbb{BS}\mathrm{Bim}^L$. We prove the following theorem in this note (which follows from Proposition 1.14 and Proposition 1.17).

Theorem 1. *For any $w \in W$, there exists a unique indecomposable bimodule (up to isomorphism) B^L_w which occurs as a summand of $B^L_{\underline{w}}$ for any reduced expression $\underline{w}$ of w such that $R_w(-L(w))$ occurs in its Δ-filtration. The set $\{B^L_w(\nu) | w \in W, \nu \in \mathbb{Z}\}$ gives a complete list of indecomposable bimodules in $\mathbb{S}Bim^L$.*

There is a unique isomorphism of $\mathcal{A}$-algebras

$$\varepsilon : \mathcal{H} \longrightarrow [\mathbb{S}Bim^L],$$
$$c_w \mapsto B^L_w.$$

The inverse of ε is given by the character map $ch_\Delta : [\mathbb{S}Bim^L] \to \mathcal{H}$ defined in (1.3).

0.5. The only interesting case in this paper is when the weight function L is not constant. When L is constant, we either obtain the Hecke algebra with equal parameters ($L = \ell$), or simply the group algebra $\mathcal{A}[W]$ ($L = 0$). Note that L is necessarily constant for simply-laced Coxeter groups.

In the paper [7], Gobet and Thiel studied the generalized category of Soergel bimodules, with focus on type A_2. The category $\mathbb{S}\mathrm{Bim}^L$ we constructed here is a subcategory of their category $\mathcal{C}$.

Elias [4] studied the categorifications of Hecke algebras with unequal parameters via folding. The Hecke algebras categorified by Elias are different from this paper.

Acknowledgments: The author would like to thank Xuhua He for helpful comments. The author would like to thank the referee to careful reading and very helpful comments.

1. Proof of the theorem

1.1. Coxeter groups

In this section we review basics of Coxeter groups and their reflection subgroups. We refer to [8] for more details. Let $e \in \{0, 1\}$ and $w \in W$. We define

$$S_e = S \cap L^{-1}(e), \quad T_e = \bigcup_{w \in W} w S_e w^{-1}, \quad \Phi_e = \{w(\beta_s) \in V | w \in W, s \in S_e\},$$

$$\Phi_e^{\pm} = \Phi_e \cap \Phi^{\pm}, \quad n_e(w) = \mathrm{Card}\{\beta \in \Phi_e^+ \cap w(\Phi_e^-)\}.$$

Let W_{S_0} be the parabolic subgroup of W generated by $s \in S_0$. We have $T_0 \cap T_1 = S_0 \cap S_1 = \emptyset$. We are interested in the reflection subgroup $W' = \langle T_1 \rangle$ of W. For $\beta \in \Phi^+$, we denote by $s_\beta \in T$ the reflection of V sending the root β to $-\beta$.

Proposition 1.1. *[2,3] The subgroup W' of W is itself a Coxeter group with simple reflections $S' = \{w \in T_1 | n_1(w) = 1\}$. Moreover the restriction of n_1 on W' coincides with the (new) length function of (W', S').*

It is clear that the subspace spanned by Φ_1 equipped with the natural W'-action coincides with the geometric representation of W'. We shall generally use n_1 to denote the (new) length function on W' and reserve $\ell(\cdot)$ (or $n(\cdot)$) for the length function on W. Let $m_{r_1,r_2} \in \mathbb{N} \cup \{\infty\}$ such that $(r_1 r_2)^{m_{r_1,r_2}} = 1$ for $r_1, r_2 \in S'$.

We first prove that the Bruhat order on W' (as a Coxeter group itself) is compatible with the Bruhat order on W. It follows from [3, Theorem 3.3] that the set of reflections (with respect to the Coxeter system (W', S')) in W' is exactly T_1. One can also see this fact from Corollary 1.4.

Lemma 1.2. *Let $w' \in W'$ such that $n_1(w' s_\beta) > n_1(w')$ for some $s_\beta \in T_1$ ($\beta \in \Phi^+$). Then*

(1) *we have* $\ell(w's_\beta) > \ell(w')$;
(2) *for any* $g_1, g_2 \in W_{S_0}$, *we have* $\ell(g_1 w' s_\beta) > \ell(g_1 w')$ *and* $\ell(g_1 w' s_\beta g_2) > \ell(g_1 w' g_2)$.

Proof. Thanks to Proposition 1.1, we know that $n_1(\cdot)$ coincides with the length function on W'. Then thanks to [8, Proposition 5.7], we see that $w'(\beta) \in \Phi_1^+ \subset \Phi^+$. Therefore we also have $\ell(w's_\beta) > \ell(w')$ by [8, Proposition 5.7]. The first claim follows.

Now since $w'(\beta) \in \Phi_1^+$ and $g_1 \in W_{S_0}$, we must also have $g_1 w'(\beta) \in \Phi^+$, hence $\ell(g_1 w' s_\beta) > \ell(g_1 w')$. On the other hand, we equivalently have $g_1 w' g_2 \Big(g_2^{-1}(\beta) \Big) \in \Phi^+$, which means $\ell(g_1 w' g_2 g_2^{-1} s_\beta g_2) = \ell(g_1 w' s_\beta g_2) > \ell(g_1 w' g_2)$. The second claim follows. □

We then give a description of the set S'.

Proposition 1.3. *Let* $r \in T_1$. *Then* $r \in S'$ *if and only if* $r = gsg^{-1}$ *for some* $s \in S_1$ *and* $g \in W_{S_0}$.

Moreover, the generator $r \in S'$ *has a reduced expression (as an element in* W*) of the form* $s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1$, *with* $s_k \in S_1$ *and* $s_1, s_2, \dots, s_{k-1} \in S_0$.

Proof. The necessary condition follows from the second statement, which we shall prove now. Let $s_\beta = r \in S'$ be a reflection of V sending the root β ($\in \Phi^+$) to $-\beta$ with reduced expression $s_\beta = s_1 \cdots s_n$ with $s_i \in S$. We know that

$$\Phi^+ \cap s_\beta(\Phi^-) = \{\beta_{s_n}, s_n(\beta_{s_{n-1}}), \dots, s_n s_{n-1} \cdots s_2(\beta_{s_1})\}.$$

By the definition of S', we have $\Phi_1^+ \cap s_\beta(\Phi^-) = \{\beta\}$. Assume $s_n s_{n-1} \cdots s_{k+1}(\beta_{s_k}) = \beta$ with $s_k \in S_1$. It also follows that $s_n, s_{n-1}, \dots, s_{k+1}, s_{k-1}, \dots, s_1 \in S_0$.

Now we can write $s_\beta = s_n s_{n-1} \cdots s_{k+1} s_k s_{k+1} \cdots s_{n-1} s_n = s_1 \cdots s_n$. We obtain that $s_n s_{n-1} \cdots s_{k+1} = s_1 \cdots s_{k-1}$. Since we have reduced expressions on both sides, we must have $n = 2k - 1$ and $r = s_\beta = s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1$ being a reduced expression.

Then we prove the sufficient condition. By the deletion condition of Coxeter groups (cf. [8, Corollary 5.8]), $r = gsg^{-1}$ must have the reduced expression of the form $r = s_1 s_2 \cdots s_l s s_{l+1} \cdots s_m$ with $s_1, \dots s_m \in S_0$. Hence we have

$$\Phi_1^+ \cap s_\beta(\Phi^-) = \{s_m s_{m-1} \cdots s_{l+1}(\beta_s)\},$$

that is $n_1(r) = 1$. Then the statement follows from Proposition 1.1. □

Corollary 1.4. *Let* $w \in W_{S_0}$. *The conjugation action by* w *preserves the set* S'.

1.2. Hecke algebras

We denote by $\mathcal{H}' = \mathcal{H}_{(W', S', L)}$ the Hecke algebra associated with the Coxeter subgroup W' of W with generators T'_r ($r \in S'$) subject to the relations (we write $v_r = v^{L(r)}$ for $r \in S'$):

$$(T'_r - v_r^{-1})(T'_r + v_r) = 0, \qquad \text{for } r \in S'$$
$$T'_r T'_t T'_r \cdots = T'_t T'_r T'_t \cdots,$$

where both products in the second relation have $m_{r,t}$ factors for any $r \neq t \in S'$ such that $m_{r,t} \neq \infty$. Note that we have $v_r = v_t = v$ for any $r, t \in S'$, thanks to the definition of the function L and Proposition 1.3. So this is a Hecke algebra with the weight function $L(r) = n_1(r) = 1$ for all $r \in S'$.

We write the canonical basis element in $\mathcal{H}'$ as c'_w for any $w \in W'$ to distinguish it from the canonical basis element c_w in $\mathcal{H}$ (since $W' \subset W$). But we shall see very soon that they actually coincide.

Lemma 1.5. *For any $w \in W'$ and $s \in S_0$, we have $T_w T_s = T_{ws}$ and $T_s T_w = T_{sw}$.*

Proof. Let us prove the first identity. The second one is entirely similar.

If $\ell(w) + \ell(s) = \ell(ws)$, then the statement is well-known ([9, §3.2]). If $\ell(w) + \ell(s) > \ell(ws)$, then w admits a reduced expression ending with s, that is $w = s_1 s_2 \cdots s_n s$. Hence we have

$$T_w T_s = T_{s_1} T_{s_2} \cdots T_{s_n} T_s T_s = T_{s_1} T_{s_2} \cdots T_{s_n} = T_{ws},$$

since $T_s^2 = 1$ for $s \in S_0$. □

Lemma 1.6. *For any $w \in W'$ and $r \in S'$, we have $T_w T_r = T_{wr}$ if $n_1(w) + n_1(r) = n_1(wr)$.*

Proof. Thanks to Proposition 1.3, we have a reduced expression

$$r = s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1 \in W,$$

with $s_k \in S_1$ and $s_1, s_2, \ldots, s_{k-1} \in S_0$. Therefore we have $T_r = T_{s_1} \cdots T_{s_k} \cdots T_{s_1}$. Thanks to Lemma 1.5, we have

$$T_w T_{s_1} \cdots T_{s_{k-1}} = T_{w s_1 s_2 \cdots s_{k-1}}.$$

Note that since

$$\begin{aligned} n_1(w s_1 s_2 \cdots s_{k-1}) + n_1(s_k) &= n_1(w) + n_1(r) \\ &= n_1(w s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1) \\ &= n_1(w s_1 s_2 \cdots s_{k-1} s_k), \end{aligned}$$

we have $\ell(w s_1 s_2 \cdots s_{k-1}) + \ell(s_k) = \ell(w s_1 s_2 \cdots s_{k-1} s_k)$ thanks to Lemma 1.2. Therefore we have

$$T_{w s_1 s_2 \cdots s_{k-1}} T_{s_k} = T_{w s_1 s_2 \cdots s_{k-1} s_k}.$$

Then thanks to Lemma 1.5 again, we have

$$T_{ws_1 s_2 \cdots s_{k-1} s_k} T_{s_{k-1}} \cdots T_{s_1} = T_{wr}.$$

The lemma follows. □

Theorem 1.7. *We have the $\mathcal{A}$-algebra embedding $\rho : \mathcal{H}' \to \mathcal{H}$ such that*

$$\rho(T'_r) = T_r, \quad \textit{for } r \in S',$$

where T_r is the element in $\mathcal{H}$ defined in (0.1). Moreover, we have $\rho(T'_w) = T_w$ and $\rho(c'_w) = c_w$ for any $w \in W' \subset W$.

Proof. Let $r \in S'$. Thanks to Proposition 1.3, we have a reduced expression

$$r = s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1 \in W, \tag{1.1}$$

with $s_k \in S_1$ and $s_1, s_2, \ldots, s_{k-1} \in S_0$. Therefore we have $L(r) = L(s_k)$, hence $v_r = v_{s_k}$.

We write $g = s_1 s_2 \cdots s_{k-1}$. Note that $T_{g^{-1}} = T_g^{-1}$. We first check the quadratic relations. We have

$$\rho\Big((T'_r - v_r^{-1})(T'_r + v_r)\Big) = T_g (T_{s_k} - v_{s_k}^{-1})(T_{s_k} + v_{s_k}) T_g^{-1} = 0.$$

It follows from Lemma 1.6 that $\rho(T'_w) = T_w$ hence the braid relations hold.

Following the reduced expression of r in (1.1), it is easy to see that we have $\rho(\overline{T'_r}) = \overline{T_r}$, that is ρ commutes with the bar involutions. Now in order to prove that $\rho(c'_w) = c_w$, it suffices to prove that the Bruhat order on W' is compatible with the Bruhat order on W, which follows from Lemma 1.2. □

So from now on we shall omit the superscripts $'$ for the generators $T'_r \in \mathcal{H}'$ and for the elements $c'_w \in \mathcal{H}'$.

The following corollary is also proved in [9, §6.1] and [1, §5.4.C].

Corollary 1.8. *For any $w \in W$ and $g \in W_{S_0}$, we have $T_g c_w = c_{gw}$ and $c_w T_g = c_{wg}$.*

1.3. Soergel bimodules

For any reflection $r = wsw^{-1}$ in W with $s \in S$, we define $\alpha_r = w(\alpha_s) \in \mathfrak{h}$ and $\alpha_r^\vee = w(\alpha_s^\vee) \in \mathfrak{h}^*$. We require that $(\mathfrak{h}, \{\alpha_r\}, \{\alpha_r^\vee\})$ for $r \in S'$ is also a Soergel realization of the Coxeter system (W', S'). This is certainly true if we take $(\mathfrak{h}, \{\alpha_s\}, \{\alpha_s^\vee\})$ with $s \in S$ to be Soergel's realization [10, Proposition 2.1]. Note that if $\mathfrak{h}$ is a reflection faithful representation of W, then $\mathfrak{h}$ is also a reflection faithful representation of W', since the set of reflections in W' is exactly T_1 by [3, Theorem 3.3].

We refer to [5, §3.1] for more discussion on realizations.

1.3.1. We summarize here Soergel's construction of the Δ-filtration and the ∇-filtration of Soergel bimodules. We follow the short summary in [6, §3.5], while details can be found in [10]. We make minor modification on the grading shifts to accommodate the weight function L.

We identify $R \otimes_k R$ with the regular functions on $\mathfrak{h} \times \mathfrak{h}$ and any R-bimodule with a quasi-coherent sheaf on $\mathfrak{h} \times \mathfrak{h}$. For any $x \in W$ and $A \subset W$, we define:

$$\mathrm{Gr}(x) = \{(xv, v)|v \in \mathfrak{h}\} \subset \mathfrak{h} \times \mathfrak{h} \text{ and } \mathrm{Gr}(A) = \cup_{x\in A}\mathrm{Gr}(x) \subset \mathfrak{h} \times \mathfrak{h}.$$

For any R-bimodule M and $A \subset W$, we define the subbimodule

$$\Gamma_A(M) = \{m \in M| \text{ supp } m \subset \mathrm{Gr}(A)\} \subset M.$$

For any $x \in W$, we shall abuse the notation and write $\leq x = \{y \in W|y \leq x, x \in W\}$, and we shall similarly define the sets $< x$, $> x$, and $\geq x$ for $x \in W$. For any $x \in W$, we define the graded R-bimodules $\Delta_x = R_x(-L(x))$ and $\nabla_x = R_x(L(x))$.

In the case of equal parameters ($L = \ell$), each $M \in \mathbb{S}\mathrm{Bim}^L(= \mathbb{S}\mathrm{Bim})$ admits a Δ-filtration (resp. ∇-filtration), that is, we have

$$\Gamma_{\geq x}(M)/\Gamma_{>x}(M) \cong \Delta_x^{\oplus h_x^\Delta(M)} \quad (\text{resp. } \Gamma_{\leq x}(M)/\Gamma_{<x}(M) \cong \nabla_x^{\oplus h_x^\nabla(M)}), \tag{1.2}$$

with $h_x^\Delta(M) \in \mathbb{N}[v, v^{-1}]$ (resp. $h_x^\nabla(M) \in \mathbb{N}[v, v^{-1}]$) describing the graded multiplicity. This is proved in [10, Proposition 5.7 & 5.9].

For any $M \in \mathbb{S}\mathrm{Bim}^L$ with a Δ-filtration (resp. ∇-filtration) as in (1.2), we define the Δ-character (resp. ∇-character) of M in $\mathcal{H}_{(W,S,L)}$ as follows:

$$ch_\Delta M = \sum_{x\in W} h_x^\Delta(M)T_x \qquad (\text{resp. } ch_\nabla M = \sum_{x\in W} \overline{h_x^\nabla}(M)T_x). \tag{1.3}$$

We shall see in Corollary 1.13 that any $M \in \mathbb{S}\mathrm{Bim}^L$ admits both a Δ-filtration and a ∇-filtration.

1.3.2. We shall establish here the categorification of the Hecke algebra $\mathcal{H}_{(W',S',L)}$ with equal parameters.

Lemma 1.9. *Let $s \in S$ and $w \in W$. We have $R_w \otimes_R B_s \otimes_R R_{w^{-1}} \cong R \otimes_{R^{wsw^{-1}}} R(1)$ as R-bimodules.*

Proof. Let $r = wsw^{-1}$ be a reflection in W. We define $\partial_r = w\partial_s w^{-1} : R \to R^r(-2)$ such that

$$\partial_r(f) = \frac{f - r(f)}{\alpha_r}.$$

We obtain that $R \otimes_{R^r} R(1)$ is a free left (or right) R-module with basis

$$d_1^r = 1 \otimes 1 \in R \otimes_{R^r} R(1), \qquad d_r^r = \frac{1}{2}(\alpha_r \otimes 1 + 1 \otimes \alpha_r) \in R \otimes_{R^r} R(1),$$

such that

$$x d_r^r = d_r^r x, \qquad x d_1^r = d_1^r r(x) + \partial_r(x) d_r^r, \quad x \in R.$$

On the other hand, we know that B_s is a free left (or right) R-module with basis

$$d_1^s = 1 \otimes 1 \in R \otimes_{R^s} R(1), \qquad d_s^s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in R \otimes_{R^s} R(1),$$

such that

$$x d_s^s = d_s^s x, \qquad x d_1^s = d_1^s s(x) + \partial_s(x) d_s^s, \quad x \in R.$$

Hence by direct computation we see $R_w \otimes_R B_s \otimes_R R_{w^{-1}} \cong R \otimes_{R^r} R(1)$ as R-bimodules, where the isomorphism simply sends d_1^r to $1 \otimes d_1^s \otimes 1$ and d_r^r to $1 \otimes d_s^s \otimes 1$. $\square$

Now for any $r \in S'$ such that $r = wsw^{-1}$ with $w \in W_{S_0}$ and $s \in S_1$, we define the R-bimodule (independent of the reduced expression of r)

$$B'_r = R \otimes_{R^r} R(1) \cong R_w \otimes_R B_s \otimes_R R_{w^{-1}}.$$

Recall the definitions of $\mathbb{BS}\mathrm{Bim}^L$ and $\mathbb{S}\mathrm{Bim}^L$ in §0.4. Let $\mathbb{S}\mathrm{Bim}_{W'}$ be the Karoubi envelope of the monoidal subcategory of $\mathbb{S}\mathrm{Bim}^L$ generated by B'_r for $r \in S'$.

For any (not necessarily reduced) expression $\underline{w} = r_{i_1} r_{i_2} \cdots r_{i_n} \in W'$ with $r_{i_j} \in S$, we define the graded R-bimodule:

$$B'_{\underline{w}} = B'_{r_{i_1}} \otimes_R B'_{r_{i_2}} \otimes_R \cdots \otimes_R B'_{r_{i_n}}.$$

Proposition 1.10. *There is a unique isomorphism of $\mathcal{A}$-algebras*

$$\varepsilon : \mathcal{H}' \longrightarrow [\mathbb{S}Bim_{W'}],$$
$$c_r \longmapsto B'_r, \quad \text{for } r \in S'.$$

The inverse of ε is given by the character map $ch_\Delta : [\mathbb{S}Bim_{W'}] \to \mathcal{H}'$.

For any $w \in W'$, there exists a unique indecomposable bimodule (up to isomorphism) B'_w which occurs as a summand of $B'_{\underline{w}}$ for any reduced expression $\underline{w}$ of w (in W') such that $R_w(-L(w))$ occurs in its Δ-filtration. The set $\{B'_w(\nu) | w \in W', \nu \in \mathbb{Z}\}$ gives a complete list of indecomposable bimodules in $\mathbb{S}Bim_{W'}$.

Proof. This is exactly Soergel's categorification theorem of the Hecke algebra $\mathcal{H}' = \mathcal{H}_{(W',S',L)}$ with the Soergel realization $(\mathfrak{h}, \{\alpha_r\}, \{\alpha_r^\vee\})$ of the Coxeter system (W', S'), since $L(\cdot)$ is the length function of W'. □

1.3.3. The following lemma is the categorical analog of Corollary 1.4.

Lemma 1.11. *Let $w \in W_{S_0}$. The functor $R_w \otimes - \otimes R_{w^{-1}} : \mathbb{S}Bim^L \to \mathbb{S}Bim^L$ is an equivalence of monoidal category. In particular, the functor restricts to an equivalence $R_w \otimes - \otimes R_{w^{-1}} : \mathbb{S}Bim_{W'} \to \mathbb{S}Bim_{W'}$.*

Proof. The first statement is obvious. The second statement follows from the fact that the conjugation action by w preserves the set S'. □

Lemma 1.12. *We have the following equivalence of additive categories*

$$\mathbb{S}Bim^L \cong \bigoplus_{w\in W_{S_0}} R_w \otimes \mathbb{S}Bim_{W'} \cong \bigoplus_{w\in W_{S_0}} \mathbb{S}Bim_{W'} \otimes R_w.$$

Proof. Thanks to Lemma 1.11, any tensor product of $R_s (s \in S_0)$ and $B_{s'} (s' \in S_1)$ lies in $R_w \otimes \mathbb{S}\mathrm{Bim}_{W'}$ (or $\mathbb{S}\mathrm{Bim}_{W'} \otimes R_w$) for some uniquely determined $w \in W_{S_0}$. On the other hand, since $R_w \otimes -$ (and $- \otimes R_w$) preserves indecomposibility, the category $R_w \otimes \mathbb{S}\mathrm{Bim}_{W'}$ (and $\mathbb{S}\mathrm{Bim}_{W'} \otimes R_w$) equals to its own Karoubi envelop. The lemma follows. □

Corollary 1.13. *Any $M \in \mathbb{S}Bim^L$ admits both a Δ-filtration and a ∇-filtration.*

Proof. We prove the existence of the Δ-filtration. Thanks to Lemma 1.12, we write $M \cong R_w \otimes_R M'$ with $M' \in \mathbb{S}\mathrm{Bim}_{W'}$ and $w \in W_{S_0}$. Thanks to Proposition 1.10, M' admits a Δ-filtration. Hence we have

$$\Gamma_{\geq x\cap \mathrm{Gr}(W')}(M')/\Gamma_{>x\cap \mathrm{Gr}(W')}(M') \cong \Delta_x^{\oplus h_x^\Delta(M)}, \quad x \in W'.$$

Note that since $\mathrm{supp}\ M' \subset \mathrm{Gr}(W')$, we have $\Gamma_{\geq x}(M') = \Gamma_{\geq x\cap \mathrm{Gr}(W')}(M')$ and $\Gamma_{>x}(M') = \Gamma_{>x\cap \mathrm{Gr}(W')}(M')$. Now thanks to Lemma 1.2, we have

$$R_w \otimes_R \Gamma_{\geq x}(M') \cong \Gamma_{\geq wx}(R_w \otimes_R M'), \quad R_w \otimes_R \Gamma_{>x}(M') \cong \Gamma_{>wx}(R_w \otimes_R M').$$

Now the corollary follows from the fact that $R_w \otimes_R -$ is exact. □

Proposition 1.14. *There is a unique isomorphism of $\mathcal{A}$-algebras*

$$\varepsilon : \mathcal{H} \longrightarrow [\mathbb{S}Bim^L],$$

$$c_s \longmapsto \begin{cases} B_s, & \text{if } s \in S_1; \\ R_s, & \text{if } s \in S_0. \end{cases}$$

The inverse of ε is given by the character map $ch_\Delta : [\mathbb{S}Bim^L] \to \mathcal{H}$.

For any $w \in W$, there exists a unique indecomposable bimodule (up to isomorphism) B^L_w which occurs as a summand of $B^L_{\underline{w}}$ for any reduced expression $\underline{w}$ of w such that $R_w(-L(w))$ occurs in its Δ-filtration. The set $\{B^L_w(\nu)|w \in W, \nu \in \mathbb{Z}\}$ gives a complete list of indecomposable bimodules in $\mathbb{S}Bim^L$.

Proof. Note that we have $B_s \cong B^L_s$ for $s \in S_1$. The statement about indecomposable bimodules follows from Lemma 1.12 and Proposition 1.10. One can also establish those statements following [10] thanks to Corollary 1.13. The statement about the character map ch_Δ follows from Corollary 1.13 and a similar argument to [10, Proposition 5.7]. Let us prove that ε is an algebra homomorphism.

It suffices to consider the case when W is a dihedral group with $S = \{s, r\}$. We can assume $L(s) = 0$ and $L(r) = 1$. The other two cases are either trivial (when $L(s) = L(r) = 0$) or a consequence of [10] (when $L(s) = L(r) = 1$).

We first prove the quadratic relation. Algebraically we have

$$c_s^2 = 1, \text{ and } c_r^2 = (v + v^{-1})c_r.$$

Categorically, we first have $B^L_s \otimes_R B^L_s = R_s \otimes_R R_s \cong R$. Secondly, since $R \cong R^r \oplus R^r(-2)$ as R^r-modules, we have

$$\begin{aligned} &B^L_r \otimes_R B^L_r = R \otimes_{R^r} R \otimes_R R \otimes_{R^r} R(2) \cong R \otimes_{R^r} R \otimes_{R^r} R(2) \\ \cong &\Big(R \otimes_{R^r} (R^r \oplus R^r(-2)) \otimes_{R^r} \otimes_{R^r} R\Big)(2) = R \otimes_{R^r} R(2) \oplus R \otimes_{R^r} R = B^L_r(1) \oplus B^L_r(-1). \end{aligned}$$

This finishes the proof of the quadratic relations.

We then prove the braid relation

$$T_r T_s T_r T_s \cdots = T_s T_r T_s T_r \cdots ,$$

where both products have $m_{r,s}$ factors. Note that $m_{r,s}$ is necessarily even, since $L(s) \neq L(r)$.

Let us first consider the subalgebra $\mathcal{H}'$ generated by $T_{srs} = T_s T_r T_s$ and T_r. Note that $L(r) = L(srs) = 1$. Using the relation $T_s^2 = 1$, we rewrite the braid relation as

$$T_{srs} T_r T_{srs} T_r \cdots = T_r T_{srs} T_r T_{srs} \cdots ,$$

where both products have $m_{srs,r} = m_{r,s}/2$ factors. Thanks to Proposition 1.10, we have the algebra homomorphism

$$\begin{aligned} \varepsilon : \mathcal{H}' &\longrightarrow [\mathbb{S}\mathrm{Bim}_{W'}] \hookrightarrow [\mathbb{S}\mathrm{Bim}^L], \\ T_r + v &\longmapsto [B'_r] \cong [B_r]; \\ T_{srs} + v &\longmapsto [B'_{srs}]. \end{aligned}$$

Thanks to Lemma 1.9, we have $R_s \otimes_R B_r \otimes_R R_s \cong R \otimes_{R^{srs}} R(1)$. This finishes the proof that ε is an algebra homomorphism. $\square$

The following corollary can be regarded as the categorical analog of Corollary 1.8.

Corollary 1.15. *For $g \in W_{S_0}$, we have $R_g \otimes B^L_w \cong B^L_{gw}$ and $B^L_w \otimes R_g \cong B^L_{wg}$ in $\mathbb{S}Bim^L$.*

Proof. We prove the first isomorphism. Since $R_g \otimes - : \mathbb{S}\mathrm{Bim}^L \to \mathbb{S}\mathrm{Bim}^L$ is an equivalence of additive categories, $R_g \otimes B^L_w$ is indecomposable. On the other hand, $R_g \otimes B^L_w$ is a direct summand of $R_g \otimes B^L_{\underline{w}} \cong B^L_{\underline{gw}}$ for any reduced expression $\underline{gw}$ of gw where $R_{gw}(-n_1(gw))$ (recall $n_1(gw) = n_1(w)$) occurs in the standard filtration. Then by Proposition 1.14, we have $R_g \otimes B^L_w \cong B^L_{gw}$. $\square$

1.3.4. Recall that we have identified $\mathbb{S}\mathrm{Bim}_{W'}$ with a full subcategory of $\mathbb{S}\mathrm{Bim}^L$, thanks to the isomorphism of bimodules in Lemma 1.9. It is straightforward that we have $B'_w \cong B^L_w \in \mathbb{S}\mathrm{Bim}^L$ for $w \in W'$.

Corollary 1.16. *We have the following commutative diagram of $\mathcal{A}$-algebras*

$$\begin{tikzcd}
{[\mathbb{S}Bim_{W'}]} \arrow[r] \arrow[d, "ch_\Delta"] & {[\mathbb{S}Bim^L]} \arrow[d, "ch_\Delta"] \\
\mathcal{H}' \arrow[r, "\rho"] & \mathcal{H}
\end{tikzcd}$$

1.3.5. We now assume that $(\mathfrak{h}, \{\alpha_s\}, \{\alpha_s^\vee\})$ $(s \in S)$ is Soergel's realization [10, Proposition 2.1] of (W, S) over $\mathbb{R}$. In this setting we can apply results from [6].

Recall that $(\mathfrak{h}, \{\alpha_r\}, \{\alpha_r^\vee\})$ $(r \in S')$ is also a reflection faithful realization of (W', S') over $\mathbb{R}$. Note that $\mathfrak{h}$ is not of minimal dimension here (cf. [6, §3.1]). But thanks to [5, Remark 3.19], results in [6] still apply.

Proposition 1.17. *The map $ch_\Delta : [\mathbb{S}Bim^L] \to \mathcal{H}$ sends $[B^L_w]$ to c_w for any $w \in W$.*

Proof. Thanks to [6], we know the character map $ch_\Delta : \mathbb{S}\mathrm{Bim}_{W'} \to \mathcal{H}'$ sends B^L_w to c_w for $w \in W'$. Then thanks to Theorem 1.7, we know that $\rho \circ ch_\Delta(B^L_w) = c_w$. Then by the commutative diagram in Corollary 1.16, we see that $ch_\Delta(B^L_w) = c_w$ for any $w \in W' \subset W$.

Hence thanks to the following commutative diagram $(s \in S_0)$

$$\begin{tikzcd}
{[\mathbb{S}\mathrm{Bim}^L]} \arrow[r, "{[R_s \otimes -]}"] \arrow[d, "ch_\Delta"] & {[\mathbb{S}\mathrm{Bim}^L]} \arrow[d, "ch_\Delta"] \\
\mathcal{H} \arrow[r, "T_s \cdot"] & \mathcal{H}
\end{tikzcd},$$

and Corollary 1.8 & 1.15, we have $ch_\Delta(B^L_w) = c_w$ for any $w \in W$. $\square$

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