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Singular equivalences of functor categories via Auslander-Buchweitz approximations

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The aim of this paper is to construct singular equivalences between functor categories. As a special case, we show that there exists a singular equivalence arising from a cotilting module T, namely, the singularity category of $({}^{\perp}T)/[T]$ and that of $(\mathsf{mod}\,\Lambda)/[T]$ are triangle equivalent. In particular, the canonical module ω over a commutative Noetherian ring R induces a singular equivalence between $(\mathsf{CM}R)/[\omega]$ and $(\mathsf{mod}\,R)/[\omega]$, which generalizes Matsui-Takahashi's theorem. Our result is based on a sufficient condition for an additive category $\mathcal{A}$ and its subcategory $\mathcal{X}$ so that the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a singular equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \simeq \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$, which is a functor category version of Xiao-Wu Chen's theorem.

1. Introduction

Let $\mathcal{A}$ be an additive category with weak-kernels. Then the functor category $\mathsf{mod}\,\mathcal{A}$, the category of finitely presented contravariant functors from $\mathcal{A}$ to the category of abelian groups, is abelian. The notion of *singularity category of* $\mathcal{A}$ is defined to be the Verdier quotient

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$$\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) := \frac{\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\mathcal{A})}{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\mathcal{A})},$$

where we denote by $\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\mathcal{A})$ the bounded derived category, and by $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\mathcal{A})$ the homotopy category of bounded complexes whose terms are projective. This concept was introduced as a homological invariant of rings by Buchweitz [6]. Recently it was applied by Orlov to study Landau-Ginzburg models [21]. A lot of studies on singularity categories has been done in various approaches (e.g. [13,17,22,24,25]).

For additive categories $\mathcal{A}$ and $\mathcal{A}'$ with weak-kernels, we say that $\mathcal{A}$ is *singularly equivalent* to $\mathcal{A}'$ if there exists a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \simeq \mathsf{D}_{\mathsf{sg}}(\mathcal{A}')$ [26]. If Λ is an Iwanaga-Gorenstein ring, then the singularity category of Λ is triangle equivalent to the stable category of Cohen-Macaulay Λ-modules. Thus the singular equivalence is a generalization of the stable equivalence for Iwanaga-Gorenstein rings.

It is a basic problem to compare homological properties of a ring Λ with its subalgebra $e\Lambda e$ given by an idempotent $e \in \Lambda$ (e.g. [2,9,10]). In this context, Xiao-Wu Chen investigated a sufficient condition for a ring Λ and its idempotent subalgebra $e\Lambda e$ so that they induce a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(e\Lambda e)$ [8, Thm. 1.3]. The first aim of this article is to provide its functor category version by using the following observations on Serre and Verdier quotients: Let $\mathcal{X}$ be a contravariantly finite subcategory of an additive category $\mathcal{A}$ with weak-kernels. Then $\mathcal{X}$ also admits weak-kernels, hence the canonical functor $Q : \operatorname{mod}\mathcal{A} \to \operatorname{mod}\mathcal{X}$ induces an equivalence

$$\frac{\operatorname{mod}\mathcal{A}}{\operatorname{mod}(\mathcal{A}/[\mathcal{X}])} \xrightarrow{\sim} \operatorname{mod}\mathcal{X}, \tag{1.0.1}$$

where the fraction denotes the Serre quotient (e.g. [7, Prop. 3.9]). Moreover, the equivalence (1.0.1) induces a triangle equivalence

$$\frac{\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\mathcal{A})}{\mathsf{D}^{\mathrm{b}}_{\mathcal{A}/[\mathcal{X}]}(\operatorname{mod}\mathcal{A})} \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod}\mathcal{X}), \tag{1.0.2}$$

where $\mathsf{D}^{\mathrm{b}}_{\mathcal{A}/[\mathcal{X}]}(\operatorname{mod}\mathcal{A})$ is a thick subcategory consisting of objects whose cohomologies belong to $\operatorname{mod}(\mathcal{A}/[\mathcal{X}])$ (see [20, Thm. 3.2] and [9, Thm. 2.3]). The equivalence (1.0.2) gives the following first result of this paper.

Theorem A *(Lemma 2.1, Theorem 2.2). Let $\mathcal{A}$ be an additive category with weak-kernels and $\mathcal{X}$ its contravariantly finite full subcategory. Suppose that $\mathsf{pd}_{\mathcal{X}}(\mathcal{A}(-, M)|_{\mathcal{X}}) < \infty$ for any $M \in \mathcal{A}$ and $\mathsf{pd}_{\mathcal{A}}(F) < \infty$ for any $F \in \operatorname{mod}(\mathcal{A}/[\mathcal{X}])$. Then the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$.*

Our second result is an application of Theorem A, which provides examples of singularly equivalent categories. We denote by $\widehat{\mathcal{X}}$ the full subcategory of $\mathcal{C}$ consisting of objects M which admit an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

with $X_n, \ldots, X_0 \in \mathcal{X}$ for some $n \in \mathbb{Z}_{\geq 0}$. Our result will be stated under the following condition which is a generalization of the setting appearing in Auslander-Buchweitz theory (see Condition 4.3 for details). A map $f : N \to M$ in $\mathcal{C}$ is called an $\mathcal{X}$-*epimorphism* if the induced map $\mathcal{C}(-, N)|_{\mathcal{X}} \xrightarrow{f \circ -} \mathcal{C}(-, M)|_{\mathcal{X}}$ is surjective.

Condition 1.1. Let $\mathcal{C}$ be an abelian category with enough projectives and let $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ be a sequence of full subcategories in $\mathcal{C}$ such that $\mathcal{X}$ and $\boldsymbol{\omega}$ are contravariantly finite in $\mathcal{A}$. We consider the following conditions:

(AB1) If a morphism $f : N \to M$ in $\mathcal{A}$ is an $\boldsymbol{\omega}$-epimorphism, then the kernel of f belongs to $\mathcal{A}$.
(AB2) $\mathrm{Ext}^i_{\mathcal{C}}(X, I) = 0$ for any $X \in \mathcal{X}, I \in \boldsymbol{\omega}$ and $i > 0$.
(AB3) For any $M \in \mathcal{A}$, there exists an exact sequence $0 \to Y_M \to X_M \xrightarrow{f} M$ in $\mathcal{A}$ such that f is a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$.

For example, the classical Auslander-Buchweitz theory (Condition 4.3) provides us with the triple $(\mathcal{C} = \mathcal{A}, \mathcal{X}, \boldsymbol{\omega})$ satisfies the Condition 1.1. Note that, in contrary to Condition 4.3, they are not required that: $\boldsymbol{\omega}$ is a *cogenerator of* $\mathcal{X}$; each morphism f appearing in $0 \to Y_M \to X_M \xrightarrow{f} M$ of (AB3) is surjective.

Since $\mathcal{X}/[\boldsymbol{\omega}]$ can be regarded as an analog of the costable category, we denote by

$$\overline{\mathcal{A}} := \mathcal{A}/[\boldsymbol{\omega}] \quad \text{and} \quad \overline{\mathcal{X}} := \mathcal{X}/[\boldsymbol{\omega}].$$

Our main result is the following:

Theorem B *(Theorem 3.1). Under Condition 1.1, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$.*

Typical examples satisfying Condition 1.1 come from cotilting theory. Let us recall the notion of cotilting subcategories of $\mathcal{C}$. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we denote by ${}^{\perp}\mathcal{X}$ the full subcategory of $\mathcal{C}$ of objects M with $\mathrm{Ext}^i_{\mathcal{C}}(M, X) = 0$ for any $i > 0$ and $X \in \mathcal{X}$.

Definition 1.2. Let $\mathcal{C}$ be an abelian category with enough projectives. A full subcategory $\mathcal{T}$ of $\mathcal{C}$ is called a *cotilting subcategory of* $\mathcal{C}$, if it satisfies the following conditions:

- There exists an integer $n \in \mathbb{Z}_{\geq 0}$ such that $\mathsf{id} I \leq n$ for any $I \in \mathcal{T}$;
- $\mathrm{Ext}^i_{\mathcal{C}}(I, J) = 0$ for any $I, J \in \mathcal{T}$ and $i > 0$;

- For each $M \in {}^{\perp}\mathcal{T}$, there exists an exact sequence

$$0 \to M \to I \to M' \to 0$$

with $I \in \mathcal{T}$ and $M' \in {}^{\perp}\mathcal{T}$.

We call an object $T \in \mathcal{C}$ a *cotilting object* if $\mathsf{add}\, T$ is a cotilting subcategory of $\mathcal{C}$.

The following result is immediate from Theorem B.

Corollary C *(Corollary 3.9). Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{T}$ its contravariantly finite cotilting subcategory. Then the canonical inclusion $\overline{{}^{\perp}\mathcal{T}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{{}^{\perp}\mathcal{T}})$.*

As examples of Corollary C, we have the followings:

Example 1.3.

(a) Let Λ be a finite dimensional k-algebra over a field k and T a cotilting Λ-module. Then the canonical inclusion $\overline{{}^{\perp}T} \hookrightarrow \overline{\mathsf{mod}}\Lambda$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathsf{mod}}\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{{}^{\perp}T})$.

(b) Let R be a commutative Cohen-Macaulay ring with a canonical R-module ω and $\mathsf{CM}R$ the full subcategory of maximal Cohen-Macaulay R-modules. Then the canonical inclusion $\overline{\mathsf{CM}}R \hookrightarrow \overline{\mathsf{mod}}R$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathsf{mod}}R) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{\mathsf{CM}}R)$.

Theorem B also provides an alternative proof for Matsui-Takahashi's theorem [19, Thm. 5.4(3)] (Corollary 3.11): For an Iwanaga-Gorenstein ring Λ, the canonical inclusion $\underline{\mathsf{CM}}\Lambda \hookrightarrow \underline{\mathsf{mod}}\Lambda$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{mod}}\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{CM}}\Lambda)$.

Notation and convention. Throughout the paper all categories and functors are assumed to be additive. The set of morphisms $M \to N$ in a category $\mathcal{A}$ is denoted by $\mathcal{A}(M, N)$. Morphisms are composed from right-to-left. Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We denote by $\mathcal{A}/[\mathcal{X}]$ the ideal quotient category of $\mathcal{A}$ modulo the ideal $[\mathcal{X}]$ of $\mathcal{A}$ consisting of all morphisms which factor through an object in $\mathcal{X}$. For each $M \in \mathcal{A}$, we denote by $\mathsf{add}\, M$ the full subcategory consisting of direct summands of a finite direct sum of M and we abbreviate $\mathcal{A}/[M]$ to indicate $\mathcal{A}/[\mathsf{add}\, M]$.

The word ring and algebra always mean ring with a unit and finite dimensional algebra over a field k, respectively. Let A be a ring. The symbol $\mathsf{mod}\, A$ denotes the category of finitely presented right A-modules. We denote by $\mathrm{Hom}_A(M, N)$ the morphism-set from M to N instead of $(\mathsf{mod}\, A)(M, N)$. The full subcategory of projective (resp. injective) modules in $\mathsf{mod}\, A$ will be denoted by $\mathsf{proj}\, A$ (resp. $\mathsf{inj}\, A$). The projective (resp. injective) dimension of right A-module M will be denoted by $\mathsf{pd}_A(M)$ (resp. $\mathsf{id}_A(M)$).

2. A functor category version of Chen's theorem

The aim of this section to provide a sufficient condition for an additive category $\mathcal{A}$ and its subcategory $\mathcal{X}$ so that the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \simeq \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$, which generalizes Xiao-Wu Chen's theorem.

The category $\mathsf{mod}\,\mathcal{A}$ is not necessarily abelian, however, if every morphism in $\mathcal{A}$ has weak-kernels, then $\mathsf{mod}\,\mathcal{A}$ is abelian ([12, Thm. 1.4]). Since we are interested in the case that $\mathsf{mod}\,\mathcal{A}$ is abelian. *Throughout this section, let $\mathcal{A}$ be an additive category with weak-kernels and $\mathcal{X}$ its contravariantly finite full subcategory.* Then, the canonical functor $Q : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{X}$ induces an equivalence

$$\frac{\mathsf{mod}\,\mathcal{A}}{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])} \xrightarrow{\sim} \mathsf{mod}\,\mathcal{X}. \tag{2.0.1}$$

Moreover, by [20, Thm. 3.2], it induces a triangle equivalence

$$\frac{\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A})}{\mathsf{D}^{\mathsf{b}}_{\mathcal{A}/[\mathcal{X}]}(\mathsf{mod}\,\mathcal{A})} \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X}).$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{mod}(\mathcal{A}/[\mathcal{X}]) \arrow[r, hook] \arrow[d, hook] & \mathsf{mod}\,\mathcal{A} \arrow[r, "Q"] \arrow[d, hook] & \mathsf{mod}\,\mathcal{X} \arrow[d] \\
\mathsf{D}^{\mathsf{b}}_{\mathcal{A}/[\mathcal{X}]}(\mathsf{mod}\,\mathcal{A}) \arrow[r, hook] & \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A}) \arrow[r, "Q'"] & \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X})
\end{tikzcd}$$

where the arrows of the shape $\hookrightarrow$ denote canonical inclusions, and Q' is the functor induced from Q. Note that $\mathsf{D}^{\mathsf{b}}_{\mathcal{A}/[\mathcal{X}]}(\mathsf{mod}\,\mathcal{A})$ is the thick subcategory of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A})$ containing $\mathsf{mod}(\mathcal{A}/[\mathcal{X}])$. The following lemma gives a natural sufficient condition so that the canonical functor $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X})$ induces a triangle functor $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$.

Lemma 2.1. *The following conditions are equivalent:*

(i) $\mathsf{pd}_{\mathcal{X}}(\mathcal{A}(-, M)|_{\mathcal{X}}) < \infty$ *for any* $M \in \mathcal{A}$;
(ii) *The canonical functor* $Q' : \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X})$ *restricts to* $Q' : \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{X})$.

If this is the case, we have an induced triangle functor $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$.

Proof. (i) $\Leftrightarrow$ (ii): Since the functor $Q' : \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X})$ restricts to $Q'|_{\mathsf{mod}\,\mathcal{A}} = Q : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{X}$, the condition (i) holds if and only if $Q'(\mathsf{proj}\,\mathcal{A}) \subseteq \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{X})$ if and only if the condition (ii) holds.

The latter statement follows from the universality of the Verdier quotient. $\square$

Since our aim is to compare the singularity categories $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$ and $\mathsf{D}_{\mathsf{sg}}(\mathcal{X})$, it is natural to assume that the equivalent conditions in Lemma 2.1 are satisfied. Our main result gives a necessary and sufficient condition so that the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$.

Theorem 2.2. *We assume that $\mathsf{pd}_{\mathcal{X}}(\mathcal{A}(-,M)|_{\mathcal{X}}) < \infty$ for any $M \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) $\mathsf{pd}_{\mathcal{A}}(F) < \infty$ *for any* $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$*;*
(ii) *The induced functor* $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$ *is a triangle equivalence.*

To prove Theorem 2.2, we firstly show Proposition 2.3 in a more general framework: Let $\mathcal{T}$ be a triangulated category with a translation [1]. For a class $\mathcal{S}$ of objects in $\mathcal{T}$, we denote by $\mathsf{tri}\,\mathcal{S}$ the smallest triangulated full subcategory of $\mathcal{T}$ containing $\mathcal{S}$. For two classes $\mathcal{U}$ and $\mathcal{V}$ of objects in $\mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the class of objects X occurring in a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Note that the operation $*$ is associative by the octahedral axiom.

Proposition 2.3. *Let $\mathcal{U}$ and $\mathcal{V}$ be triangulated full subcategories of $\mathcal{T}$ and consider the Verdier quotients with respect to them:*

$$\mathcal{U} \to \mathcal{T} \xrightarrow{Q_1} \mathcal{T}/\mathcal{U} \quad \text{and} \quad \mathcal{V} \to \mathcal{T} \xrightarrow{Q_2} \mathcal{T}/\mathcal{V}.$$

Then, there exist natural triangle equivalences

$$\frac{\mathcal{T}/\mathcal{U}}{\mathsf{tri}(Q_1\mathcal{V})} \simeq \frac{\mathcal{T}}{\mathsf{tri}(\mathcal{U},\mathcal{V})} \simeq \frac{\mathcal{T}/\mathcal{V}}{\mathsf{tri}(Q_2\mathcal{U})},$$

where $Q_1\mathcal{V}$ is the full subcategory of $\mathcal{T}/\mathcal{U}$ consisting of objects isomorphic to Q_1V for some $V \in \mathcal{V}$, and the symbol $Q_2\mathcal{U}$ is used in a similar meaning.

Proof. We shall show an equality $\mathsf{tri}(Q_1\mathcal{V}) = \mathsf{tri}(\mathcal{U},\mathcal{V})/\mathcal{U}$, where $\mathsf{tri}(\mathcal{U},\mathcal{V})$ denotes the smallest triangulated full subcategory of $\mathcal{T}$ containing $\mathcal{U}$ and $\mathcal{V}$. We set $\mathcal{S} := \mathcal{U} \cup \mathcal{V}$. Obviously we have $Q_1\mathcal{S} = Q_1\mathcal{V}$. Since $\mathsf{tri}(\mathcal{U},\mathcal{V}) = \bigcup_{n\geq 0} \mathcal{S}^{*n}$, we have the following equalities:

$$\mathsf{tri}(\mathcal{U},\mathcal{V})/\mathcal{U} = Q_1\left(\bigcup_{n\geq 0} \mathcal{S}^{*n}\right) = \bigcup_{n\geq 0} (Q_1\mathcal{S})^{*n} = \bigcup_{n\geq 0} (Q_1\mathcal{V})^{*n} = \mathsf{tri}(Q_1\mathcal{V}).$$

Hence we have a desired triangle equivalence $\dfrac{\mathcal{T}/\mathcal{U}}{\mathsf{tri}(Q_1\mathcal{V})} = \dfrac{\mathcal{T}/\mathcal{U}}{\mathsf{tri}(\mathcal{U},\mathcal{V})/\mathcal{U}} \xrightarrow{\sim} \dfrac{\mathcal{T}}{\mathsf{tri}(\mathcal{U},\mathcal{V})}$. $\square$

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Apply Proposition 2.3 for $\mathcal{T} = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A})$, $\mathcal{U} = \mathsf{D}^{\mathsf{b}}{}_{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])}(\mathsf{mod}\,\mathcal{A})$ and $\mathcal{V} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{A})$. Then $\mathcal{T}/\mathcal{U} = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X})$ and $\mathcal{T}/\mathcal{V} = \mathsf{D}_{\mathsf{sg}}(\mathcal{A})$. The assumption gives $Q_1\mathcal{V} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{X})$. Hence $\frac{\mathcal{T}/\mathcal{U}}{Q_1\mathcal{V}} = \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$. Thus we have a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{X}) \simeq \frac{\mathsf{D}_{\mathsf{sg}}(\mathcal{A})}{\mathsf{tri}(Q_2\mathcal{U})}$. This shows the condition (i) is equivalent to $Q_2\mathcal{U} = 0$, namely $\mathcal{U} \subset \mathcal{V}$, which is nothing but the condition (ii). □

We end this section with recovering the following Chen's theorem as a special case of Theorem 2.2 and Lemma 2.1.

Example 2.4. [8, Thm. 1.3] (see also [23, Thm. 5.2], [16, Prop. 3.3]). Let Λ be a Noetherian ring and e its idempotent. Assume that $\mathsf{pd}_{e\Lambda e}(\Lambda e) < \infty$. Then the canonical inclusion $e\Lambda e \hookrightarrow \Lambda$ induces a triangle functor $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\Lambda) \to \mathsf{D}_{\mathsf{sg}}(e\Lambda e)$, and the following are equivalent:

(i) $\mathsf{pd}_{\Lambda}(M) < \infty$ for any $M \in \mathsf{mod}(\Lambda/\Lambda e\Lambda)$;
(ii) The induced functor $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(e\Lambda e)$ is a triangle equivalence.

3. Sufficient conditions for singular equivalence

The aim of this section is to construct a singular equivalence from our generalized Auslander-Buchweitz condition (Condition 1.1). First we introduce some terminology. Let $\mathcal{C}$ denote an abelian category and let $\mathcal{C} \supseteq \mathcal{A} \supseteq \mathcal{B}$ be a sequence of full subcategories of $\mathcal{C}$. We call the kernel of a $\mathcal{B}$-epimorphism the *$\mathcal{B}$-epikernel*, for short. *We assume that $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels and $\mathcal{B}$ is contravariantly finite in $\mathcal{A}$.* Then the ideal-quotient category $\mathcal{A}/[\mathcal{B}]$ admits weak-kernels. In fact, for a morphism $\alpha : M \to L$ of $\mathcal{A}$, we obtain its weak-kernel as follows: We take a right $\mathcal{B}$-approximation $\beta : B_L \to L$ of L, and consider an induced exact sequence

$$0 \to N \xrightarrow{(\gamma\ \delta)} M \oplus B_L \xrightarrow{\binom{\alpha}{\beta}} L$$

in $\mathcal{C}$. Since $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels and the morphism $\binom{\alpha}{\beta}$ is an $\mathcal{B}$-epimorphism, we have $N \in \mathcal{A}$. It is basic that the morphism γ is a weak-kernel of α in $\mathcal{A}/[\mathcal{B}]$.

3.1. Singular equivalences from Auslander-Buchweitz approximation

In this subsection, we give a proof of the following main theorem.

Theorem 3.1. *Under Condition 1.1, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$.*

Let $\mathcal{C}$ be an abelian category with enough projectives and consider a sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ of full subcategories in $\mathcal{C}$ such that $\mathcal{X}$ and $\boldsymbol{\omega}$ are contravariantly finite in $\mathcal{A}$. We always assume (AB1) in Condition 1.1.

Proposition 3.2. *The ideal-quotient $\overline{\mathcal{A}}$ admits weak-kernels and $\overline{\mathcal{X}}$ is its contravariantly finite full subcategory. Moreover, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces the following equivalence*

$$\frac{\mathsf{mod}\,\overline{\mathcal{A}}}{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])} \xrightarrow{\sim} \mathsf{mod}\,\overline{\mathcal{X}}.$$

Proof. Since $\mathcal{A}$ is closed under $\boldsymbol{\omega}$-epikernels, $\overline{\mathcal{A}}$ admits weak-kernels. Since $\mathcal{X}$ is contravariantly finite in $\mathcal{A}$, so is $\overline{\mathcal{X}}$ in $\overline{\mathcal{A}}$. Note that there exists an equivalence $\mathcal{A}/[\mathcal{X}] \simeq \overline{\mathcal{A}}/[\overline{\mathcal{X}}]$. By (2.0.1), we have a desired equivalence. $\square$

To prove that the inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle functor $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \to \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$, we shall check a sufficient condition given in Lemma 2.1.

Lemma 3.3. *Assume (AB2) and (AB3). Let $X \in \mathcal{X}$ be given. Then:*

(a) *One has $\mathrm{Ext}^i_{\mathcal{C}}(X, I) = 0$ for any $I \in \widehat{\boldsymbol{\omega}}$ and $i > 0$.*
(b) *Every morphism $f : X \to I$ with $I \in \widehat{\boldsymbol{\omega}}$ factors through an object in $\boldsymbol{\omega}$.*

Proof. We only show the assertion (b). Since $I \in \widehat{\boldsymbol{\omega}}$, there exists an exact sequence $0 \to I' \to W \to I \to 0$ with $W \in \boldsymbol{\omega}$ and $I' \in \widehat{\boldsymbol{\omega}}$. Applying $\mathcal{C}(X, -)$, by (a), we conclude that f factors through W. $\square$

Proposition 3.4. *Assume (AB2) and (AB3). Then the canonical inclusion* $\mathrm{inc} : \overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ *admits a right adjoint R. Moreover, we have $\mathsf{pd}_{\overline{\mathcal{X}}}(\overline{\mathcal{A}}(-, M)|_{\overline{\mathcal{X}}}) = 0$ for any $M \in \mathcal{A}$.*

Proof. The proof is similar to one given in [5, Ch. V, Prop. 1.2], but we are in a slightly different situation. So we include a detailed proof. By (AB3), for each $M \in \mathcal{A}$, there exists an exact sequence in $\mathcal{A}$

$$0 \to Y_M \to X_M \xrightarrow{\alpha} M$$

with α a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$. We shall show that the morphism $\overline{\mathcal{X}}(X, X_M) \xrightarrow{\alpha\circ-} \overline{\mathcal{A}}(X, M)$ is a functorial isomorphism in $X \in \mathcal{X}$. Its surjectivity is clear, since α is a right $\mathcal{X}$-approximation. To show its injectivity, take a morphism $h \in \mathcal{X}(X, X_M)$ such that $\alpha \circ h$ factors through an object I of $\boldsymbol{\omega}$. Thus we have the following commutative diagram:

$$\begin{tikzcd} & & X \arrow[r, "h'"] \arrow[d, "h"'] & I \arrow[d, "h''"] \\ 0 \arrow[r] & Y_M \arrow[r] & X_M \arrow[r, "\alpha"] & M \end{tikzcd}$$

Since α is a right $\mathcal{X}$-approximation, there exists a morphism $\alpha' : I \to X_M$ such that $\alpha\alpha' = h''$. The morphism $h - \alpha' h'$ factors through $Y_M \in \widehat{\boldsymbol{\omega}}$. By Lemma 3.3(ii), this implies that $h - \alpha' h'$ factors through $\boldsymbol{\omega}$. Hence h factors through $\boldsymbol{\omega}$. By the Yoneda lemma, the assignment $M \mapsto X_M$ gives rise to a functor $R : \mathcal{A} \to \mathcal{X}$. The bifunctorial isomorphism $\overline{\mathcal{X}}(X, R(M)) \xrightarrow{\alpha\circ -} \overline{\mathcal{A}}(X, M)$ says the pair of functors (inc, R) forms an adjoint pair.

The latter statement is obvious. $\square$

Proposition 3.5. *Let $\mathcal{B}$ be a contravariantly finite full subcategory of $\mathcal{A}$ and assume that $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels. Let $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$ be given. Then there exists an exact sequence*

$$0 \to N \xrightarrow{g} M \xrightarrow{f} L \tag{3.5.1}$$

in $\mathcal{A}$ which satisfies the following conditions:

(a) *The morphism f is a $\mathcal{B}$-epimorphism;*
(b) *The induces sequence*

$$0 \to \mathcal{A}(-, N) \xrightarrow{g\circ -} \mathcal{A}(-, M) \xrightarrow{f\circ -} \mathcal{A}(-, L) \to F \to 0$$

is exact.

In particular, $\mathsf{pd}_{\mathcal{A}}(F) \leq 2$.

Proof. First F is a finitely presented $\mathcal{A}$-module. Indeed, a right $\mathcal{B}$-approximation $B_Y \to Y$ of any $Y \in \mathcal{A}$ induces a projective presentation

$$\mathcal{A}(-, B_Y) \to \mathcal{A}(-, Y) \to \mathcal{A}/[\mathcal{B}](-, Y) \to 0$$

of the $\mathcal{A}$-module $\mathcal{A}/[\mathcal{B}](-, Y)$. This shows that $\mathcal{A}/[\mathcal{B}](-, Y)$ belongs to $\mathsf{mod}\,\mathcal{A}$, hence so does F.

Thus we have a projective presentation $\mathcal{A}(-, M) \xrightarrow{f\circ -} \mathcal{A}(-, L) \to F \to 0$ of the $\mathcal{A}$-module F. Since F vanishes on $\mathcal{B}$, the induced morphism f is a $\mathcal{B}$-epimorphism. Thus we have an exact sequence $0 \to N \xrightarrow{g} M \xrightarrow{f} L$ in $\mathcal{A}$. Applying the Yoneda embedding, we have a projective resolution $0 \to \mathcal{A}(-, N) \to \mathcal{A}(-, M) \to \mathcal{A}(-, L) \to F \to 0$ of the $\mathcal{A}$-module F. $\square$

Let $M \in \mathcal{A}$ and $f : B_M \to M$ be a right $\mathcal{B}$-approximation of M. Then we write $\Omega_{\mathcal{B}}(M) := \operatorname{Ker} f$. We define $\Omega^n_{\mathcal{B}}(M)$ inductively for $n \geq 1$. We prove the following key-proposition which generalizes the well-known result given in [3, Prop. 4.1, 4.2] and [4, Prop. 1.2]. The proof is similar but a bit different from the original ones.

Proposition 3.6. *For $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$, the exact sequence (3.5.1) in Proposition 3.5 induces a projective resolution*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A}/[\mathcal{B}](-,\Omega_{\mathcal{B}}^2(N)) \arrow[r, "\Omega_{\mathcal{B}}^2 g\circ -"] & \mathcal{A}/[\mathcal{B}](-,\Omega_{\mathcal{B}}^2(M)) \arrow[r, "\Omega_{\mathcal{B}}^2 f\circ -"] & \mathcal{A}/[\mathcal{B}](-,\Omega_{\mathcal{B}}^2(L)) \\
\arrow[r] & \mathcal{A}/[\mathcal{B}](-,\Omega_{\mathcal{B}}(N)) \arrow[r, "\Omega_{\mathcal{B}} g\circ -"] & \mathcal{A}/[\mathcal{B}](-,\Omega_{\mathcal{B}}(M)) \arrow[r, "\Omega_{\mathcal{B}} f\circ -"] & \mathcal{A}/[\mathcal{B}](-,\Omega_{\mathcal{B}}(L)) \\
\arrow[r] & \mathcal{A}/[\mathcal{B}](-,N) \arrow[r, "g\circ -"] & \mathcal{A}/[\mathcal{B}](-,M) \arrow[r, "f\circ -"] & \mathcal{A}/[\mathcal{B}](-,L) \arrow[r] & F \arrow[r] & 0
\end{tikzcd} \tag{3.6.1}$$

of the $\mathcal{A}/[\mathcal{B}]$-module F.

Proof. For the sequence (3.5.1), we take right $\mathcal{B}$-approximations $\alpha_L : B_L \to L$ and $\alpha_N : B_N \to N$. Since the morphism f is $\mathcal{B}$-epimorphism, we have a morphism $\beta : B_L \to M$ such that $\alpha_L = f \circ \beta$. The induced morphism $\alpha_M := \left(\begin{smallmatrix}\beta \\ g\alpha_N\end{smallmatrix}\right) : B_M := B_L \oplus B_N \to M$ is a right $\mathcal{B}$-approximation of M. Since $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels, we have the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_{\mathcal{B}}(N) \arrow[r, "\Omega_{\mathcal{B}} g"] \arrow[d] & \Omega_{\mathcal{B}}(M) \arrow[r, "\Omega_{\mathcal{B}} f"] \arrow[d] & \Omega_{\mathcal{B}}(L) \arrow[d] & \\
0 \arrow[r] & B_N \arrow[r] \arrow[d, "\alpha_N"] & B_M \arrow[r] \arrow[d, "\alpha_M"] & B_L \arrow[r] \arrow[d, "\alpha_L"] & 0 \\
0 \arrow[r] & N \arrow[r, "g"] & M \arrow[r, "f"] & L &
\end{tikzcd}$$

where all columns and rows are exact, and the middle row splits. Applying the Yoneda embedding and the Snake Lemma, we have the following commutative diagram in $\mathsf{mod}\,\mathcal{A}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{A}(-,\Omega_{\mathcal{B}}(N)) \arrow[r] \arrow[d] & \mathcal{A}(-,\Omega_{\mathcal{B}}(M)) \arrow[r] \arrow[d] & \mathcal{A}(-,\Omega_{\mathcal{B}}(L)) \arrow[d] & & \\
0 \arrow[r] & \mathcal{A}(-,B_N) \arrow[r] \arrow[d, "\alpha_N\circ -"] & \mathcal{A}(-,B_M) \arrow[r] \arrow[d, "\alpha_M\circ -"] & \mathcal{A}(-,B_L) \arrow[r] \arrow[d, "\alpha_L\circ -"] & 0 & \\
0 \arrow[r] & \mathcal{A}(-,N) \arrow[r, "g\circ -"] \arrow[d] & \mathcal{A}(-,M) \arrow[r, "f\circ -"] \arrow[d] & \mathcal{A}(-,L) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
 & \mathcal{A}/[\mathcal{B}](-,N) \arrow[r] \arrow[d] & \mathcal{A}/[\mathcal{B}](-,M) \arrow[r] \arrow[d] & \mathcal{A}/[\mathcal{B}](-,L) \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

In particular, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{A}(-,\Omega_{\mathcal{B}}(N)) \longrightarrow \mathcal{A}(-,\Omega_{\mathcal{B}}(M)) \longrightarrow \mathcal{A}(-,\Omega_{\mathcal{B}}(L)) \\ \xrightarrow{\delta} \mathcal{A}/[\mathcal{B}](-,N) \longrightarrow \mathcal{A}/[\mathcal{B}](-,M) \longrightarrow \mathcal{A}/[\mathcal{B}](-,L) \longrightarrow F \longrightarrow 0 \end{aligned}$$

in $\mathsf{mod}\,\mathcal{A}$. We have an exact sequence $0 \to \Omega_{\mathcal{B}}(N) \to \Omega_{\mathcal{B}}(M) \xrightarrow{\Omega_{\mathcal{B}} f} \Omega_{\mathcal{B}}(L)$ such that $\Omega_{\mathcal{B}} f$ is an $\mathcal{B}$-epimorphism. Inductively, we have a desired projective resolution of the $\mathcal{A}/[\mathcal{B}]$-module F. $\square$

Lemma 3.7. *Under Condition 1.1,*

(a) *For any $L \in \mathcal{A}$, there exists $n \geq 0$ such that $\Omega^n_{\mathcal{X}}(L) \in \mathcal{X}$.*
(b) *For each $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$, we have $\mathsf{pd}_{\mathcal{A}/[\mathcal{X}]}(F) < \infty$.*

Proof. (a) For an object $L \in \mathcal{A}$, due to (AB3), we get an exact sequence

$$0 \to Y \to X_0 \xrightarrow{f_0} L$$

such that f_0 is a right $\mathcal{X}$-approximation of L and $Y \in \widehat{\boldsymbol{\omega}}$. Since $Y \in \widehat{\boldsymbol{\omega}}$, we get an exact sequence

$$0 \to I_n \to I_{n-1} \xrightarrow{f_{n-1}} \cdots \to I_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} L$$

with $I_i \in \boldsymbol{\omega}$ for $1 \leq i \leq n$. By Lemma 3.3, each morphism $f_i : I_i \to \operatorname{Im} f_i$ is a right $\mathcal{X}$-approximation of $\operatorname{Im} f_i$ for each $1 \leq i \leq n$ hence $I_n = \Omega^n_{\mathcal{X}}(L) \in \mathcal{X}$.

(b) We consider the projective resolution (3.6.1) of the $\mathcal{A}/[\mathcal{X}]$-module F given in Proposition 3.6 by setting $\mathcal{B} := \mathcal{X}$. Then the assertion follows from (a), since $\mathcal{A}/[\mathcal{X}](-,\Omega^n_{\mathcal{X}}(L)) = 0$. $\square$

Proposition 3.8. *Under Condition 1.1, for each $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$, one has $\mathsf{pd}_{\overline{\mathcal{A}}}(F) < \infty$.*

Proof. Since $\mathsf{pd}_{\mathcal{A}/[\mathcal{X}]}(F) < \infty$ by Lemma 3.7 and the canonical inclusion ι : $\mathsf{mod}(\mathcal{A}/[\mathcal{X}]) \hookrightarrow \mathsf{mod}(\overline{\mathcal{A}})$ is exact, it is enough to check the case of $F = \mathcal{A}/[\mathcal{X}](-,M)$ for some $M \in \mathcal{A}$. By (AB3), there exists an exact sequence $0 \to Y_M \xrightarrow{g} X_M \xrightarrow{f} M$ in $\mathcal{A}$ with f a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$. Applying the Yoneda embedding yields a projective resolution

$$0 \to \mathcal{A}(-,Y_M) \xrightarrow{g\circ-} \mathcal{A}(-,X_M) \xrightarrow{f\circ-} \mathcal{A}(-,M) \to \mathcal{A}/[\mathcal{X}](-,M) \to 0$$

of the $\mathcal{A}$-module $\mathcal{A}/[\mathcal{X}](-,M)$. Applying Proposition 3.6 to $\mathcal{B} := \boldsymbol{\omega}$, we have a projective resolution of the $\overline{\mathcal{A}}$-module $\mathcal{A}/[\mathcal{X}](-,M)$:

$$\cdots \longrightarrow \overline{\mathcal{A}}(-, \Omega_{\boldsymbol{\omega}}(Y_M)) \xrightarrow{\Omega_\omega g \circ -} \overline{\mathcal{A}}(-, \Omega_{\boldsymbol{\omega}}(X_M)) \xrightarrow{\Omega_\omega f \circ -} \overline{\mathcal{A}}(-, \Omega_{\boldsymbol{\omega}}(M))$$
$$\longrightarrow \overline{\mathcal{A}}(-, Y_M) \xrightarrow{g \circ -} \overline{\mathcal{A}}(-, X_M) \xrightarrow{f \circ -} \overline{\mathcal{A}}(-, M) \longrightarrow A/[\mathcal{X}](-, M) \longrightarrow 0.$$

Since $Y_M \in \widehat{\boldsymbol{\omega}}$, one has $\Omega^n_{\boldsymbol{\omega}}(Y_M) \in \boldsymbol{\omega}$ for some $n \geq 0$. Thus $\overline{\mathcal{A}}(-, \Omega^n_{\mathcal{X}}(Y_M)) = 0$ and hence $\mathsf{pd}_{\overline{\mathcal{A}}}(\mathcal{A}/[\mathcal{X}](-, M)) < \infty$. □

We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 2.1 and Proposition 3.4, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle functor $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \to \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$. By Theorem 2.2 and Proposition 3.8, the triangle functor $\bar{Q}$ is an equivalence. □

3.2. Singular equivalences from cotilting objects

In this subsection we construct a singular equivalence from a given cotilting subcategory, using Theorem 3.1. We denote by $\mathsf{P}(\mathcal{C})$ (resp. $\mathsf{GP}(\mathcal{C})$) the full subcategory of $\mathcal{C}$ consisting of projective (resp. Gorenstein projective) objects. We abbreviate $\Omega M := \Omega_{\mathsf{P}(\mathcal{C})} M$ for each $M \in \mathcal{C}$ and denote by $\Omega^n \mathcal{A}$ the full subcategory of $\mathcal{C}$ consisting of objects isomorphic to $\Omega^n M$ for some $M \in \mathcal{A}$. Moreover we define $\Omega^- M$ to be the kernel of a left $\mathsf{P}(\mathcal{C})$-approximation of M. Inductively we define $\Omega^{-n} M$ for any $n \geq 1$.

Corollary 3.9. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{T}$ its contravariantly finite cotilting subcategory. Then the canonical inclusion $\overline{{}^\perp\mathcal{T}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{{}^\perp\mathcal{T}})$.*

Proof. Setting $\mathcal{X} := {}^\perp\mathcal{T}$ and $\boldsymbol{\omega} := \mathcal{T}$, we shall show that the sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ satisfies conditions (AB1)-(AB3). The condition (AB1) is obvious, because $\mathcal{A} = \mathcal{C}$. The condition (AB2) holds by definition.

(AB3): By [1, Thm. 1.1], for any $M \in \widehat{\mathcal{X}}$, there exists an exact sequence

$$0 \to Y_M \to X_M \to M \to 0$$

with $Y_M \in \widehat{\omega}$ and $X_M \in \mathcal{X}$. It remains to show $\widehat{\mathcal{X}} = \mathcal{A}$. Since there exists an integer $n \geq 0$ such that $\mathsf{id} I \leq n$ for all $I \in \boldsymbol{\omega}$, it follows that $\Omega^n M \in \mathcal{X}$ holds for all $M \in \mathcal{A}$. This shows $\widehat{\mathcal{X}} = \mathcal{A}$. Thanks to Theorem 3.1, we have a desired triangle equivalence. □

3.3. Matsui-Takahashi's singular equivalence

We provide an alternative proof for Matsui-Takahashi's singular equivalence.

Definition 3.10. Let $\mathcal{C}$ be an abelian category with enough projectives. A full subcategory $\mathcal{A}$ of $\mathcal{C}$ is called *quasi-resolving* if it is closed under kernels of epimorphisms and contains all projectives. A quasi-resolving subcategory is called *resolving* if it is closed under extensions and direct summands.

Corollary 3.11. [19, Thm. 5.4(3)] *Let $\mathcal{A}$ be a quasi-resolving subcategory of an abelian category $\mathcal{C}$ with enough projectives. Assume that $\mathcal{A}$ together with an integer $n \in \mathbb{Z}_{\geq 0}$ satisfies the condition*

$$\Omega^n\mathcal{A} \text{ is contained in } \mathsf{GP}(\mathcal{C}) \text{ and closed under cosyzygies} \tag{$*$}$$

and set $\mathcal{X} := \Omega^n\mathcal{A}$. Then the canonical inclusion $\underline{\mathcal{X}} \hookrightarrow \underline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}})$.

Proof. Setting $\mathcal{X} := \Omega^n\mathcal{A}$ and $\boldsymbol{\omega} := \mathsf{P}(\mathcal{C})$, we shall show that the sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ of subcategories in $\mathcal{C}$ satisfies the conditions (AB1)-(AB3). (AB1): Since $\mathsf{P}(\mathcal{C})$-epikernels are epimorphisms, the condition (AB1) follows from the definition of quasi-resolving subcategories.

(AB2): Since $\mathcal{X} \subseteq \mathsf{GP}(\mathcal{C})$, we have $\mathcal{X} \subseteq {}^{\perp}\boldsymbol{\omega}$.

(AB3): Let $M \in \mathcal{A}$. By the condition ($*$), we have an exact sequence

$$0 \to G \to P_{n-1} \to \cdots P_0 \to M \to 0$$

with $G \in \mathcal{X}$ and $P_{n-1}, \cdots, P_0 \in \mathsf{P}(\mathcal{C})$. Since $G \in \mathsf{GP}(\mathcal{C})$, we have an exact sequence

$$0 \to G \xrightarrow{g_n} Q_{n-1} \xrightarrow{g_{n-1}} \cdots \to Q_0 \xrightarrow{g_0} \Omega^{-n}(G) \to 0$$

with the canonical morphisms $\operatorname{Im} g_i \to Q_i$ being left $\mathsf{P}(\mathcal{C})$-approximations for each $1 \leq i \leq n$. Thus we have the following chain map, where $\Omega^{-n}(G) \in \Omega^n\mathcal{A} = \mathcal{X}$ by the condition ($*$).

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & Q_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_0 \arrow[r] \arrow[d] & \Omega^{-n}(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

By taking the mapping cone of the above chain map, we have an exact sequence

$$0 \to G \to Q_{n-1} \oplus G \to Q_{n-2} \oplus P_{n-1} \to \cdots \to Q_0 \oplus P_1 \to \Omega^{-n}(G) \oplus P_0 \to M \to 0.$$

Since the left-most morphism $G \to Q_{n-1} \oplus G$ is a split-monomorphism, we have the following exact sequence

$$0 \to Q_{n-1} \to Q_{n-2} \oplus P_{n-1} \to \cdots \to Q_0 \oplus P_1 \to \Omega^{-n}(G) \oplus P_0 \xrightarrow{f} M \to 0. \tag{3.11.1}$$

Obviously $\operatorname{Ker} f \in \widehat{\omega}$ holds. The exact sequence $0 \to \operatorname{Ker} f \to \Omega^{-n}(G) \oplus P_0 \xrightarrow{f} M \to 0$ is a desired one. Indeed, f is a right $\mathcal{X}$-approximation by Lemma 3.3. $\square$

Recall that an additive category $\mathcal{A}$ with weak-kernels is said to be *Iwanaga-Gorenstein* if $\mathsf{id}_{\mathcal{A}}(\mathcal{A}(-,M)), \mathsf{id}_{\mathcal{A}^{\mathsf{op}}}(\mathcal{A}(M,-)) < \infty$ for any $M \in \mathcal{A}$. Typical examples of Iwanaga-Gorenstein rings are finite dimensional selfinjective algebras over a field k and commutative Gorenstein rings of finite Krull dimension. As an obvious consequence of Corollary 3.9 or 3.11, we have:

Example 3.12. Let Λ be an Iwanaga-Gorenstein ring with $\mathsf{id}_{\Lambda}(\Lambda) = n$ and $\mathsf{CM}\Lambda := {}^{\perp}\Lambda$. Then the canonical inclusion $\underline{\mathsf{CM}}\Lambda \hookrightarrow \underline{\mathsf{mod}}\Lambda$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{mod}}\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{CM}}\Lambda)$.

4. More results and examples

In this section, we provide further investigations on Condition 1.1. First we give sufficient conditions so that $\mathcal{X}/[\omega]$ is Iwanaga-Gorenstein and of finite global dimension, respectively.

Theorem 4.1. *Let Λ be a finite dimensional algebra and $T \in \mathsf{mod}\,\Lambda$ a cotilting module. We set $\underline{{}^{\perp}T} := {}^{\perp}T/[\Lambda]$ and $\overline{{}^{\perp}T} := {}^{\perp}T/[T]$. Then the followings hold:*

(a) *If Λ is Iwanaga-Gorenstein, then so is $\overline{{}^{\perp}T}$. Moreover, one has $\mathsf{id}_{(\overline{{}^{\perp}T})}F \leq 3\max\{\mathsf{pd}_{\Lambda}T, \mathsf{id}_{\Lambda}\Lambda\}$ for any projective $(\overline{{}^{\perp}T})$-module F.*
(b) *If $\mathsf{gl.dim}\Lambda = n$, then we have $\mathsf{gl.dim}(\overline{{}^{\perp}T}) \leq 3n - 1$.*

The assertion (b) can be found in [18, Thm. 6.1]. Let us recall from [14, Thm. 3.4] (see also [11,15]), there exist Auslander-Reiten translations on ${}^{\perp}T$, that is, mutually equivalences

$$\tau : \underline{{}^{\perp}T} \xrightarrow{\sim} \overline{{}^{\perp}T} \quad \text{and} \quad \tau^{-} : \overline{{}^{\perp}T} \xrightarrow{\sim} \underline{{}^{\perp}T}.$$

Moreover, they induce functorial isomorphisms

$$D\operatorname{Ext}^1_A(M,N) \cong \underline{{}^{\perp}T}(\tau^{-}N, M) \cong \overline{{}^{\perp}T}(N, \tau M)$$

in $M, N \in {}^{\perp}T$ which are known as Auslander-Reiten dualities, where $D := \operatorname{Hom}_k(-.k)$.

Proof of Theorem 4.1. (a) Since there exists an equivalence $\overline{{}^{\perp}T} \xrightarrow{\sim} \underline{{}^{\perp}T}$, we shall show that $\underline{{}^{\perp}T}$ is Iwanaga-Gorenstein. Thanks to Auslander-Reiten duality, every injective $(\underline{{}^{\perp}T})$-module is of the form $\operatorname{Ext}^1_{\Lambda}(-,M)$ for some $M \in {}^{\perp}T$. Since T is a cotilting module, we get an exact sequence $0 \to M \to T' \to N \to 0$ with $T' \in \mathsf{add}\,T$ and $N \in {}^{\perp}T$. The induced sequence

$$0 \to \mathrm{Hom}_\Lambda(-, M) \to \mathrm{Hom}_\Lambda(-, T') \to \mathrm{Hom}_\Lambda(-, N) \to \mathrm{Ext}^1_\Lambda(-, M) \to 0$$

gives a projective resolution of $({}^\perp T)$-module $\mathrm{Ext}^1_\Lambda(-, M)$. By Proposition 3.6, we have a projective resolution

$$\begin{aligned} \cdots \longrightarrow \underline{{}^\perp T}(-, \Omega_\Lambda(M)) \longrightarrow \underline{{}^\perp T}(-, \Omega_\Lambda(T')) \longrightarrow \underline{{}^\perp T}(-, \Omega_\Lambda(N)) \\ \xrightarrow{\delta} \underline{{}^\perp T}(-, M) \longrightarrow \underline{{}^\perp T}(-, T') \longrightarrow \underline{{}^\perp T}(-, N) \longrightarrow \mathrm{Ext}^1_\Lambda(-, M) \longrightarrow 0 \end{aligned} \tag{4.1.1}$$

of the $(\underline{{}^\perp T})$-module $\mathrm{Ext}^1_\Lambda(-, M)$. Since Λ is Iwanaga-Gorenstein, T is a tilting module, in particular $\mathsf{pd}_\Lambda(T) < \infty$. Thus there exists an integer $n \geq 0$ such that $\Omega^n_\Lambda(T') \in \mathsf{proj}\,\Lambda$. Hence every injective $(\underline{{}^\perp T})$-module $\mathrm{Ext}^1_\Lambda(-, M)$ is of finite projective dimension. Next we shall show that every projective $(\underline{{}^\perp T})$-module $\underline{{}^\perp T}(-, M)$ is of finite injective dimension. Considering the first syzygy of M, namely an exact sequence $0 \to \Omega_\Lambda M \to P \to M \to 0$ with $P \in \mathsf{proj}\,\Lambda$, we get an injective resolution

$$0 \to \underline{{}^\perp T}(-, M) \to \mathrm{Ext}^1_\Lambda(-, \Omega_\Lambda M) \to \mathrm{Ext}^1_\Lambda(-, P) \to \mathrm{Ext}^1_\Lambda(-, M) \to \cdots \tag{4.1.2}$$

of the $(\underline{{}^\perp T})$-module $\underline{{}^\perp T}(-, M)$. Since Λ is Iwanaga-Gorenstein, we have $\mathsf{id}_\Lambda P < \infty$. We have thus concluded that $\overline{{}^\perp T}$ is Iwanaga-Gorenstein. The latter formula follows from the sequence (4.1.1) and (4.1.2).

(b) We shall show that $\mathsf{gl.dim}(\underline{{}^\perp T}) \leq 3n - 1$. Let $F \in \mathsf{mod}(\underline{{}^\perp T})$ with a projective presentation $\underline{{}^\perp T}(-, M) \to \underline{{}^\perp T}(-, L) \to F \to 0$. Since F vanishes on $\mathsf{proj}\,\Lambda$, the corresponding morphism $f : M \to L$ is an epimorphism in $\mathsf{mod}\,\Lambda$. Since ${}^\perp T$ is closed under epimorphisms, we have an exact sequence $0 \to N \to M \to L \to 0$ in ${}^\perp T$ which induces a projective resolution

$$\begin{aligned} \cdots \longrightarrow \underline{{}^\perp T}(-, \Omega_\Lambda(N)) \longrightarrow \underline{{}^\perp T}(-, \Omega_\Lambda(M)) \longrightarrow \underline{{}^\perp T}(-, \Omega_\Lambda(L)) \\ \longrightarrow \underline{{}^\perp T}(-, N) \longrightarrow \underline{{}^\perp T}(-, M) \longrightarrow \underline{{}^\perp T}(-, L) \longrightarrow F \longrightarrow 0 \end{aligned}$$

of the $(\underline{{}^\perp T})$-module F. The assumption $\mathsf{gl.dim}\Lambda = n$ implies $\Omega^n_\Lambda(L) \in \mathsf{proj}\,\Lambda$. Hence $\mathsf{pd}_{(\underline{{}^\perp T})} F \leq 3n - 1$. $\square$

Theorem 4.1 contains the following well-known result.

Example 4.2. [3, Prop. 10.2] Let Λ be a finite dimensional algebra with $\mathsf{gl.dim}\Lambda = n$. Then we have $\mathsf{gl.dim}(\underline{\mathsf{mod}}\Lambda) \leq 3n - 1$.

Next we explain that (AB1)-(AB3) in Condition 1.1 are satisfied in the classical Auslander-Buchweitz theory: Let $\mathcal{C}$ be an abelian category with enough projectives and $\mathcal{X} \supseteq \boldsymbol{\omega}$ a sequence of full subcategories in $\mathcal{C}$. We say that $\boldsymbol{\omega}$ is a *cogenerator of* $\mathcal{X}$ if, for each $X \in \mathcal{X}$, there exists an exact sequence $0 \to X \to I \to X' \to 0$ with $I \in \boldsymbol{\omega}$, $X' \in \mathcal{X}$.

Condition 4.3. [1, p. 9, 17] For a sequence $\mathcal{X} \supseteq \boldsymbol{\omega}$ of full subcategories in $\mathcal{C}$, we consider the following conditions:

- $\widehat{\mathcal{X}} = \mathcal{C}$;
- $\mathcal{X}$ is closed under direct summands and extension;
- $\mathrm{Ext}^i_{\mathcal{C}}(X, I) = 0$ for any $X \in \mathcal{X}, I \in \boldsymbol{\omega}$ and $i > 0$;
- $\boldsymbol{\omega}$ is a cogenerator of $\mathcal{X}$ which is closed under direct summands.

Under these conditions, it is known that, for each $M \in \mathcal{C}$, there exists an exact sequence

$$0 \to Y_M \to X_M \to M \to 0 \tag{4.3.1}$$

with $X_M \in \mathcal{X}, Y_M \in \widehat{\boldsymbol{\omega}}$ [1, Thm. 1.1]. The sequence (4.3.1) is called the *Auslander-Buchweitz approximation of* M. As a benefit of our *generalized Auslander-Buchweitz approximation* in (AB3), we shall show Proposition 4.4. Notice that, in the proposition, the subcategory ω is not necessarily a cogenerator of $\mathcal{X}$, and right $\mathcal{X}$-approximations of objects of $\mathcal{A}$ appearing in (AB3) are not necessarily surjective.

Proposition 4.4. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{X} \supseteq \boldsymbol{\omega}$ a sequence of full subcategories of $\mathcal{A}$. Suppose that $\mathcal{X}$ is a torsion class of $\mathcal{A}$ and $\boldsymbol{\omega}$ is contravariantly finite in $\mathcal{A}$ and satisfies $\mathrm{Ext}^i_{\mathcal{A}}(X, I) = 0$ for any $X \in \mathcal{X}, I \in \boldsymbol{\omega}$ and $i > 0$. Then the sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ satisfies (AB1)-(AB3).*

Proof. The conditions (AB1) and (AB2) are obvious. Since $\mathcal{X}$ is a torsion class, for any $M \in \mathcal{A}$ there exists an exact sequence $0 \to X \to M$ with $X \in \mathcal{X}$, hence (AB3) holds. $\square$

We end this section by giving examples of singularly equivalent categories using Corollary 3.9.

Example 4.5. Fix an integer $n \in \mathbb{Z}_{>0}$. Let Λ be the algebra defined by the following quiver with relations.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2, \arrow[l, bend left, "\beta"] \end{tikzcd} \qquad \langle (\alpha\beta)^n \alpha \rangle$$

We describe the Auslander-Reiten quiver of Λ. Since Λ is a Nakayama algebra, an indecomposable module is determined by the pair (m, l) of the socle m and the Loewy length l. We shall denote the module by $[m]_l$.

$$\begin{tikzcd}[column sep=small]
 & {[2]_2} \arrow[dl] \arrow[dr] & & {[1]_4} \arrow[dl] & & {[1]_{2n}} \arrow[dr] & & \\
{[1]_1} \arrow[dr] & & {[2]_2} \arrow[dl] \arrow[dr] & & \cdots & & {[1]_{2n+1}} \arrow[dl] \arrow[dr] & \\
 & {[1]_2} \arrow[dl] \arrow[dr] & & {[2]_4} \arrow[dl] & & {[2]_{2n}} \arrow[dr] & & {[1]_{2n+2}} \arrow[dl] \\
{[2]_1} \arrow[dr] & & {[1]_3} \arrow[dl] \arrow[dr] & & \cdots & & {[2]_{2n+1}} \arrow[dl] & \\
 & {[2]_2} & & {[1]_4} & & {[1]_{2n}} & &
\end{tikzcd}$$

We can easily check that the module $T := [1]_1 \oplus [1]_{2n+2}$ is a cotilting module of $\mathsf{id}_\Lambda(T) = 1$. Due to Corollary 3.9, we conclude that $\overline{\mathsf{mod}}\Lambda := (\mathsf{mod}\,\Lambda)/[T]$ is singularly equivalent to $\overline{{}^\perp T} := ({}^\perp T)/[T]$. Their Auslander-Reiten quivers are described as follows:

$\overline{\mathsf{mod}}A$

$$\begin{tikzcd}[column sep=small]
 & {[2]_2} \arrow[dr] & & {[1]_4} \arrow[dl] & & & {[1]_{2n}} \arrow[dl] \arrow[dr] & \\
 & & {[2]_2} \arrow[dl] \arrow[dr] & & \cdots & {[2]_{2n-1}} \arrow[dr] & & {[1]_{2n+1}} \arrow[dl] \\
 & {[1]_2} \arrow[dl] \arrow[dr] & & {[2]_4} \arrow[dl] & & & {[2]_{2n}} \arrow[dl] \arrow[dr] & \\
{[2]_1} \arrow[dr] & & {[1]_3} \arrow[dl] \arrow[dr] & & \cdots & {[1]_{2n-1}} \arrow[dr] & & {[2]_{2n+1}} \arrow[dl] \\
 & {[2]_2} & & {[1]_4} & & & {[1]_{2n}} &
\end{tikzcd}$$

$\overline{{}^\perp T}$

$$\begin{tikzcd}[column sep=small]
 & {[1]_4} \arrow[dl] \arrow[dr] & & {[1]_7} \arrow[dl] & & & {[1]_{2n}} \arrow[dl] \arrow[dr] & \\
{[1]_2} \arrow[dr] & & {[1]_5} \arrow[dl] \arrow[dr] & & \cdots & {[1]_{2n-2}} \arrow[dr] & & {[1]_{2n+1}} \arrow[dl] \\
 & {[1]_3} \arrow[dr] & & {[1]_6} \arrow[dl] & & & {[1]_{2n-1}} \arrow[dl] \arrow[dr] & \\
 & & {[1]_4} & & \cdots & {[1]_{2n-3}} & & {[1]_{2n}}
\end{tikzcd}$$

Claim. *If $n = 1$, both $\overline{\mathsf{mod}}\Lambda$ and $\overline{{}^\perp T}$ are of finite global dimension, otherwise they are non Iwanaga-Gorenstein.*

Proof. We only check the case of $n \geq 2$. By calculations, the injective $(\overline{{}^\perp T})$-module $D\overline{{}^\perp T}([1]_3, -)$ has the following projective resolution:

$$\cdots \to P_5 \to P_3 \to P_{2n+1} \to P_{2n-1} \to P_{2n+1} \to P_3 \to P_4 \to P_{2n+1} \to I_3 \to 0,$$

where we set $I_3 := D\overline{{}^\perp T}([1]_3, -)$ and $P_l := \overline{{}^\perp T}(-, [1]_l)$ for each $1 \leq l \leq 2n+1$. We notice that $\Omega^2 I_3 \cong \Omega^8 I_3$. Hence $\overline{{}^\perp T}$ is non Iwanaga-Gorenstein. It remains to check the assertion for $\overline{\mathsf{mod}}\Lambda$. We denote by $Q : \mathsf{mod}(\overline{\mathsf{mod}}\Lambda) \to \mathsf{mod}(\overline{{}^\perp T})$ the canonical functor. There exists an injective object $J \in \mathsf{inj}(\overline{\mathsf{mod}}\Lambda)$ such that $QJ \cong I_3$. If $\overline{\mathsf{mod}}\Lambda$ is Iwanaga-Gorenstein, then J is of finite projective dimension. Moreover, since Q is exact and preserves projectives, it turns out that I_3 is of finite projective dimension. This is a contradiction. $\square$

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