



# $R$ matrix for generalized quantum group of type $A$

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## ABSTRACT

The generalized quantum group  $\mathcal{U}(\epsilon)$  of type  $A$  is an affine analogue of quantum group associated to a general linear Lie superalgebra  $\mathfrak{gl}_{M|N}$ . We prove that there exists a unique  $R$  matrix on the tensor product of fundamental type representations of  $\mathcal{U}(\epsilon)$  for arbitrary parameter sequence  $\epsilon$  corresponding to a non-conjugate Borel subalgebra of  $\mathfrak{gl}_{M|N}$ . We give an explicit description of its spectral decomposition, and then as an application, construct a family of finite-dimensional irreducible  $\mathcal{U}(\epsilon)$ -modules which have subspaces isomorphic to the Kirillov-Reshetikhin modules of usual affine type  $A_{M-1}^{(1)}$  or  $A_{N-1}^{(1)}$ .

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## 1. Introduction

A generalized quantum group  $\mathcal{U}(\epsilon)$  associated to  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  with  $\epsilon_i \in \{0, 1\}$  is a Hopf algebra introduced in [18], which appears in studying solutions of the tetrahedron equation or the three-dimensional Yang-Baxter equation.

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The generalized quantum group  $\mathcal{U}(\epsilon)$  of type  $A$  is isomorphic to the usual quantum affine algebra of type  $A_{n-1}^{(1)}$ , when  $\epsilon$  is homogeneous, that is,  $\epsilon_i = \epsilon_j$  for all  $i \neq j$ . But it becomes a more interesting object when  $\epsilon$  is non-homogeneous, which is closely related to the quantized enveloping algebra associated to an affine Lie superalgebra [23], or which can be viewed as an affine analogue of the quantized enveloping algebra of the general linear Lie superalgebra  $\mathfrak{gl}_{M|N}$  [22], where  $M$  and  $N$  are the numbers of 0 and 1 in  $\epsilon$ , respectively. We remark that the subalgebra  $\mathring{\mathcal{U}}(\epsilon)$  of  $\mathcal{U}(\epsilon)$  associated to the Lie superalgebra  $\mathfrak{gl}_{M|N}$  was also introduced in [7] independently, as symmetries appearing in the study of wave functions of quantum mechanical systems [24].

When the parameter  $\epsilon$  is standard, that is,  $\epsilon_{M|N} = (0^M, 1^N)$ , it is shown in [18] that there exists a unique  $R$  matrix on the tensor product of finite-dimensional  $\mathcal{U}(\epsilon_{M|N})$ -modules  $\mathcal{W}_{s,\epsilon}(x)$ , which correspond to fundamental representations of type  $A_{N-1}^{(1)}$  with spectral parameter  $x$  when  $N \geq 3$ . Indeed, the  $R$  matrix is obtained by reducing the solution of the tetrahedron equation, and the uniqueness follows from the irreducibility of tensor product  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  for generic  $x$  and  $y$ . An explicit spectral decomposition associated to the  $R$  matrix is obtained by analyzing the maximal vectors with respect to  $\mathring{\mathcal{U}}(\epsilon_{M|N})$ .

By applying fusion construction using the  $R$  matrix in [18], a family of irreducible  $\mathcal{U}(\epsilon_{M|N})$ -modules is constructed in [17], which are parametrized by rectangular partitions inside an  $(M|N)$ -hook. Moreover the existence of their crystal base is proved together with a combinatorial description of the associated crystal graphs. It can be viewed as a natural super-analogue of Kirillov-Reshetikhin modules (simply KR modules) of type  $A_\ell^{(1)}$ , which is an important family of finite-dimensional irreducible modules of quantum affine algebras (cf. [5,15]).

The results in [18] and [17] suggests that there is a close connection between finite-dimensional representations of  $\mathcal{U}(\epsilon_{M|N})$  and the quantum affine algebra  $U_q(A_{n-1}^{(1)})$  of type  $A_{n-1}^{(1)}$ . The purpose of this paper is to extend the results in [18] and [17] to arbitrary parameter sequence  $\epsilon$ , and find a more concrete connection between the finite-dimensional representations of  $\mathcal{U}(\epsilon)$  and  $U_q(A_\ell^{(1)})$ . From a viewpoint of representations of  $\mathfrak{gl}_{M|N}$ , the sequence  $\epsilon$  represents the type of Borel subalgebras of  $\mathfrak{gl}_{M|N}$ , which are not conjugate to each other under the Weyl group. It is not obvious whether the representation theory of  $\mathcal{U}(\epsilon)$  is the same under a different choice of permutations of  $\epsilon_{M|N}$ . For example, if we change the Borel in the generalized quantum group, then the defining relations and the crystal structure associated to  $\mathcal{U}(\epsilon)$ -modules become much different from the ones with respect to  $\epsilon_{M|N}$  as  $\epsilon$  gets far from  $\epsilon_{M|N}$  (cf. [2,16]).

We first show that there exists a unique  $R$  matrix on the tensor product of finite-dimensional  $\mathcal{U}(\epsilon)$ -modules  $\mathcal{W}_{s,\epsilon}(x)$  of fundamental type (Theorem 4.9). Since the existence of  $R$  matrix for arbitrary  $\epsilon$  was shown in [18], it suffices to prove the irreducibility of the tensor product  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  for generic  $x$  and  $y$ . We use a method completely different from [18]. Indeed, motivated by the work [6], we introduce a functor called truncation, and show that it sends any  $\mathcal{U}(\epsilon)$ -module with polynomial weights to

a  $\mathcal{U}(\epsilon')$ -module, preserving the comultiplications in tensor product, where  $\epsilon'$  is a subsequence of  $\epsilon$ . This in particular enables us to define a connected oriented graph structure on  $\mathcal{W}_{l,\epsilon}(1) \otimes \mathcal{W}_{m,\epsilon}(1)$  at  $q = 0$ , which has additional arrows other than the ones associated to  $\mathcal{U}(\epsilon)$ , and hence the irreducibility of  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  (Theorem 4.7 and Corollary 4.8).

Next, we prove that the truncation functor is compatible with the  $R$  matrix. This immediately implies that the spectral decomposition associated to the  $R$  matrix for  $\mathcal{U}(\epsilon)$  is the same as that of type  $A_\ell^{(1)}$  (Theorem 5.2) and hence does not depend on the choice of  $\epsilon$ . As an application, we construct a family of irreducible  $\mathcal{U}(\epsilon)$ -modules  $\mathcal{W}_{s,\epsilon}^{(r)}$  which yields the usual KR modules under truncation (Theorem 5.4). We conjecture that  $\mathcal{W}_{s,\epsilon}^{(r)}$  has a crystal base as in the case of  $\epsilon = \epsilon_{M|N}$ . The compatibility of truncation functor with the  $R$  matrix also plays a crucial role in understanding arbitrary finite-dimensional  $\mathcal{U}(\epsilon)$ -modules in connection with those of type  $A_\ell^{(1)}$ , which will be discussed with full details in a forthcoming paper.

We also remark that there are other recent works on the finite-dimensional representations of quantum affine *superalgebra* associated to  $\mathfrak{gl}_{M|N}$  [25–27]. It would be interesting to compare with these results.

The paper is organized as follows. In Section 2, we review basic materials for a generalized quantum group and its crystal base. In Section 3, we present the classical Schur–Weyl duality for  $\mathring{\mathcal{U}}(\epsilon)$  and then realize the irreducible polynomial representation of  $\mathring{\mathcal{U}}(\epsilon)$ . In Section 4, we prove the main theorem on the existence of the  $R$  matrix. In Section 5, we construct KR type modules of  $\mathcal{U}(\epsilon)$  using the  $R$  matrix.

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## 2. Generalized quantum group $\mathcal{U}(\epsilon)$ of type $A$

### 2.1. Generalized quantum group

Let  $\mathbb{Z}_+$  be the set of non-negative integers. Throughout the paper, let  $q$  be an indeterminate, and put

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}} \quad (m \in \mathbb{Z}_+),$$

$$[m]! = [m][m-1] \cdots [1] \quad (m \geq 1), \quad [0]! = 1,$$

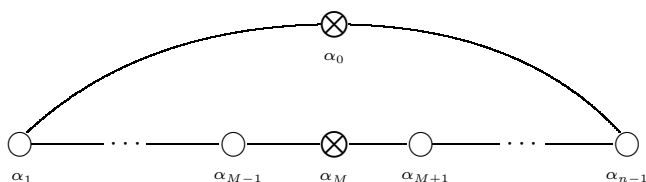
We fix a positive integer  $n \geq 4$ . Let  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  be a sequence with  $\epsilon_i \in \{0, 1\}$  for  $1 \leq i \leq n$ . Let  $\mathbb{I} = \{1, 2, \dots, n\}$  with usual linear ordering. We assume that  $M$  is the number of  $i$  with  $\epsilon_i = 0$  and  $N$  is the number of  $i$  with  $\epsilon_i = 1$  in  $\epsilon$ . We denote by  $\epsilon_{M|N}$  the sequence when  $\epsilon_1 = \dots = \epsilon_M = 0$  and  $\epsilon_{M+1} = \dots = \epsilon_n = 1$ .

Let  $P = \bigoplus_{i \in \mathbb{I}} \mathbb{Z}\delta_i$  be the free abelian group generated by  $\delta_i$  with a symmetric bilinear form  $(\cdot | \cdot)$  given by  $(\delta_i | \delta_j) = (-1)^{\epsilon_i} \delta_{ij}$  for  $i, j \in \mathbb{I}$ . Let  $\{\delta_i^\vee | i \in \mathbb{I}\} \subset P^\vee := \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$  be the dual basis such that  $\langle \delta_i, \delta_j^\vee \rangle = \delta_{ij}$  for  $i, j \in \mathbb{I}$ .

Let  $I = \{0, 1, \dots, n-1\}$  and

$$\alpha_i = \delta_i - \delta_{i+1}, \quad \alpha_i^\vee = \delta_i^\vee - (-1)^{\epsilon_i + \epsilon_{i+1}} \delta_{i+1}^\vee \quad (i \in I),$$

where  $\delta_0 = \delta_n$ . We put  $I_{\text{even}} = \{i \in I | (\alpha_i | \alpha_i) = \pm 2\}$  and  $I_{\text{odd}} = \{i \in I | (\alpha_i | \alpha_i) = 0\}$ . When  $\epsilon = \epsilon_{M|N}$ , the Dynkin diagram associated to the Cartan matrix  $(\langle \alpha_j, \alpha_i^\vee \rangle)_{i,j \in I}$  is



where  $\otimes$  denotes an isotropic simple root  $\alpha_i$  ( $i \in I_{\text{odd}}$ ).

Set

$$q_i = (-1)^{\epsilon_i} q^{(-1)^{\epsilon_i}} = \begin{cases} q, & \text{if } \epsilon_i = 0, \\ -q^{-1}, & \text{if } \epsilon_i = 1, \end{cases} \quad (i \in \mathbb{I}).$$

**Definition 2.1.** We define  $\mathcal{U}(\epsilon)$  to be the associative  $\mathbb{Q}(q)$ -algebra with 1 generated by  $q^h, e_i, f_i$  for  $h \in P^\vee$  and  $i \in I$  satisfying

$$q^0 = 1, \quad q^{h+h'} = q^h q^{h'} \quad (h, h' \in P^\vee), \quad (2.1)$$

$$\omega_j e_i \omega_j^{-1} = q_j^{\langle \alpha_i, \delta_j^\vee \rangle} e_i, \quad \omega_j f_i \omega_j^{-1} = q_j^{-\langle \alpha_i, \delta_j^\vee \rangle} f_i, \quad (2.2)$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{\omega_i \omega_{i+1}^{-1} - \omega_i^{-1} \omega_{i+1}}{q - q^{-1}}, \quad (2.3)$$

$$e_i^2 = f_i^2 = 0 \quad (i \in I_{\text{odd}}), \quad (2.4)$$

where  $\omega_j = q^{(-1)^{\epsilon_j} \delta_j^\vee}$  ( $j \in \mathbb{I}$ ), and the Serre-type relations

$$\begin{aligned} e_i e_j - e_j e_i &= f_i f_j - f_j f_i = 0, & (|i - j| > 1), \\ e_i^2 e_j - (-1)^{\epsilon_i} [2] e_i e_j e_i + e_j e_i^2 &= 0, \\ f_i^2 f_j - (-1)^{\epsilon_i} [2] f_i f_j f_i + f_j f_i^2 &= 0, & (i \in I_{\text{even}} \text{ and } |i - j| = 1), \end{aligned} \quad (2.5)$$

and

$$\begin{aligned}
& e_i e_{i-1} e_i e_{i+1} - e_i e_{i+1} e_i e_{i-1} + e_{i+1} e_i e_{i-1} e_i \\
& \quad - e_{i-1} e_i e_{i+1} e_i + (-1)^{\epsilon_i} [2] e_i e_{i-1} e_{i+1} e_i = 0, \\
& f_i f_{i-1} f_i f_{i+1} - f_i f_{i+1} f_i f_{i-1} + f_{i+1} f_i f_{i-1} f_i \\
& \quad - f_{i-1} f_i f_{i+1} f_i + (-1)^{\epsilon_i} [2] f_i f_{i-1} f_{i+1} f_i = 0,
\end{aligned} \quad (i \in I_{\text{odd}}). \quad (2.6)$$

Throughout the paper, we understand the subscript  $i \in I$  modulo  $n$ . We call  $\mathcal{U}(\epsilon)$  the *generalized quantum group of affine type A associated to  $\epsilon$*  (see [18]).

Put  $k_i = \omega_i \omega_{i+1}^{-1}$  for  $i \in I$ . Then we have for  $i, j \in I$

$$k_i e_j k_i^{-1} = D_{ij} e_j, \quad k_i f_j k_i^{-1} = D_{ij}^{-1} f_j, \quad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \quad (2.7)$$

where  $D_{ij} = q_i^{\langle \alpha_j, \delta_i^\vee \rangle} q_{i+1}^{-\langle \alpha_j, \delta_{i+1}^\vee \rangle}$ . There is a Hopf algebra structure on  $\mathcal{U}(\epsilon)$ , where the comultiplication  $\Delta$ , the antipode  $S$ , and the counit  $\varepsilon$  are given by

$$\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_i) &= e_i \otimes 1 + k_i^{-1} \otimes e_i, \\
\Delta(f_i) &= f_i \otimes k_i + 1 \otimes f_i, \\
S(q^h) &= q^{-h}, \quad S(e_i) = -k_i e_i, \quad S(f_i) = -f_i k_i^{-1}, \\
\varepsilon(q^h) &= 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,
\end{aligned} \quad (2.8)$$

for  $h \in P^\vee$  and  $i \in I$ . Let  $\eta$  be the anti-involution on  $\mathcal{U}(\epsilon)$  defined by

$$\eta(q^h) = q^h, \quad \eta(e_i) = q_i f_i k_i^{-1}, \quad \eta(f_i) = q_i^{-1} k_i e_i,$$

for  $h \in P^\vee$  and  $i \in I$ . It satisfies

$$\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta.$$

We have an isomorphism between  $\mathcal{U}(\epsilon)$  and  $\mathcal{U}(\tilde{\epsilon})$  where  $\tilde{\epsilon}$  is obtained from  $\epsilon$  by permutation of  $\epsilon_i$ 's, which is not an isomorphism of Hopf algebras [21, Theorem 2.7] (cf. [20, 37.1]).

**Theorem 2.2.** For  $1 \leq i \leq n-1$ , let  $\tilde{\epsilon} = (\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_n)$  be the sequence given by exchanging  $\epsilon_i$  and  $\epsilon_{i+1}$  in  $\epsilon$ . Then there exists an isomorphism of algebras  $\tau_i : \mathcal{U}(\epsilon) \longrightarrow \mathcal{U}(\tilde{\epsilon})$  given by

$$\begin{aligned}
\tau_i(\omega_i) &= \omega_{i+1}, \quad \tau_i(\omega_{i+1}) = \omega_i, \quad \tau_i(\omega_k) = \omega_k \quad (k \neq i, i+1), \\
\tau_i(e_i) &= -f_i k_i, \quad \tau_i(f_i) = -k_i^{-1} e_i, \\
\tau_i(e_j) &= [e_i, e_j]_{D_{ij}}, \quad \tau_i(f_j) = [f_j, f_i]_{D_{ij}^{-1}} \quad (|i-j|=1), \\
\tau_i(e_j) &= e_j, \quad \tau_i(f_j) = f_j \quad (|i-j|>1),
\end{aligned}$$

where the inverse map is given by

$$\begin{aligned}\tau_i^{-1}(\omega_i) &= \omega_{i+1}, & \tau_i^{-1}(\omega_{i+1}) &= \omega_i, & \tau_i^{-1}(\omega_k) &= \omega_k \quad (k \neq i, i+1), \\ \tau_i^{-1}(e_i) &= -k_i^{-1}f_i, & \tau_i^{-1}(f_i) &= -e_i k_i, \\ \tau_i^{-1}(e_j) &= [e_j, e_i]_{D_{ij}}, & \tau_i^{-1}(f_j) &= [f_i, f_j]_{D_{ij}^{-1}} \quad (|i-j|=1), \\ \tau_i^{-1}(e_j) &= e_j, & \tau_i^{-1}(f_j) &= f_j \quad (|i-j|>1),\end{aligned}$$

for  $j \in I$  and  $k \in \mathbb{I}$ . Here  $[X, Y]_t = XY - tYX$  for  $t \in \mathbb{Q}(q)$ .  $\square$

## 2.2. Crystal base of $\mathcal{U}(\epsilon)$ -modules

For a  $\mathcal{U}(\epsilon)$ -module  $V$  and  $\mu = \sum_i \mu_i \delta_i \in P$ , let

$$V_\mu = \{ u \in V \mid \omega_i u = q_i^{\mu_i} u \quad (i \in \mathbb{I}) \}$$

be the  $\mu$ -weight space of  $V$ . For a non-zero vector  $u \in V_\mu$ , we denote by  $\text{wt}(u) = \mu$  the weight of  $u$ . Let  $P_{\geq 0} = \sum_{i \in \mathbb{I}} \mathbb{Z}_+ \delta_i$  and let  $\mathcal{O}_{\geq 0}$  be the category of  $\mathcal{U}(\epsilon)$ -modules with objects  $V$  such that

$$V = \bigoplus_{\mu \in P_{\geq 0}} V_\mu \quad \text{with } \dim V_\mu < \infty. \quad (2.9)$$

By definition it is clear that  $\mathcal{O}_{\geq 0}$  is closed under taking submodules, quotients and tensor products. Note that the category  $\mathcal{O}_{\geq 0}$  satisfies the conditions (i)–(iv) of category  $\mathcal{O}_{int}$  in [2, Definition 2.2] by viewing  $U_q(\mathfrak{g})_i$  for  $i \in I$  there as the subalgebra of  $\mathcal{U}(\epsilon)$  generated by  $e_i, f_i, k_i, k_i^{-1}$ .

**Remark 2.3.** There is another comultiplication on  $\mathcal{U}(\epsilon)$  given by

$$\begin{aligned}\Delta_+(q^h) &= q^h \otimes q^h, \\ \Delta_+(e_i) &= 1 \otimes e_i + e_i \otimes k_i, \\ \Delta_+(f_i) &= k_i^{-1} \otimes f_i + f_i \otimes 1,\end{aligned} \quad (2.10)$$

(while  $\Delta_+^{\text{op}}$  is used in [17]). Let  $\otimes$  and  $\otimes_+$  denote the tensor product with respect to  $\Delta$  and  $\Delta_+$ , respectively. For  $\mathcal{U}(\epsilon)$ -modules  $M$  and  $N$ , we have a  $\mathcal{U}(\epsilon)$ -linear isomorphism  $\psi : M \otimes N \longrightarrow M \otimes_+ N$  given by

$$\psi(u \otimes v) = \left( \prod_{i \in \mathbb{I}} q_i^{\mu_i \nu_i} \right) u \otimes v, \quad (2.11)$$

for  $u \in M_\mu$  and  $v \in N_\nu$  with  $\mu = \sum_i \mu_i \delta_i$  and  $\nu = \sum_i \nu_i \delta_i$ .

Let us recall the notion of crystal base for  $V \in \mathcal{O}_{\geq 0}$  [17] (cf. [2]). The Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $V$  for  $i \in I$  are defined as follows. Suppose that  $u \in V_\mu$  is given.

*Case 1.* Suppose that  $i \in I_{\text{odd}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (0, 1)$ . We define

$$\tilde{e}_i u = \eta(f_i)u = q_i^{-1} k_i e_i u, \quad \tilde{f}_i u = f_i u.$$

*Case 2.* Suppose that  $i \in I_{\text{odd}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ . We define

$$\tilde{e}_i u = e_i u, \quad \tilde{f}_i u = \eta(e_i)u = q_i f_i k_i^{-1} u.$$

*Case 3.* Suppose that  $i \in I_{\text{even}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (0, 0)$ . Let  $\zeta : U_q(\mathfrak{sl}_2) \rightarrow \mathcal{U}(\epsilon)_i$  be the  $\mathbb{Q}(q)$ -algebra isomorphism given by  $\zeta(e) = e_i$ ,  $\zeta(f) = f_i$  and  $\zeta(k) = k_i$ , where  $U_q(\mathfrak{sl}_2) = \langle e, f, k^{\pm 1} \rangle$  is the usual quantum group for  $\mathfrak{sl}_2$  with relation  $kek^{-1} = q^2 e$ ,  $kfk^{-1} = q^{-2} f$ ,  $ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$ . The induced comultiplication  $\Delta^\zeta := (\zeta^{-1} \otimes \zeta^{-1}) \circ \Delta \circ \zeta$  on  $U_q(\mathfrak{sl}_2)$  is

$$\begin{aligned} \Delta^\zeta(k^{\pm 1}) &= k^{\pm 1} \otimes k^{\pm 1}, \\ \Delta^\zeta(e) &= k^{-1} \otimes e + e \otimes 1, \\ \Delta^\zeta(f) &= 1 \otimes f + f \otimes k. \end{aligned}$$

So we define  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $V$  to be the usual Kashiwara operators on the lower crystal base of  $U_q(\mathfrak{sl}_2)$ -module induced from  $\zeta$ . In other words, if  $u = \sum_{k \geq 0} f_i^{(k)} u_k$ , where  $f_i^{(k)} = f_i^k / [k]!$  and  $e_i u_k = 0$  for  $k \geq 0$ , then we define

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

*Case 4.* Suppose that  $i \in I_{\text{even}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (1, 1)$ . Let  $\xi : U_q(\mathfrak{sl}_2) \rightarrow \mathcal{U}(\epsilon)_i$  be the  $\mathbb{Q}(q)$ -algebra homomorphism given by  $\xi(e) = -e_i$ ,  $\xi(f) = f_i$  and  $\xi(k) = k_i^{-1}$ . Then the induced comultiplication  $\Delta^\xi$  on  $U_q(\mathfrak{sl}_2)$  is

$$\begin{aligned} \Delta^\xi(k^{\pm 1}) &= k^{\pm 1} \otimes k^{\pm 1}, \\ \Delta^\xi(e) &= k \otimes e + e \otimes 1, \\ \Delta^\xi(f) &= 1 \otimes f + f \otimes k^{-1}. \end{aligned}$$

So we define  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $V$  to be the Kashiwara operators on the upper crystal base of  $U_q(\mathfrak{sl}_2)$ -module induced from  $\xi$ . In other words, if  $u = \sum_{k \geq 0} f_i^{(k)} u_k$ , where  $e_i u_k = 0$  for  $k \geq 0$  and  $l_k = \langle \text{wt}(u_k), \alpha_i^\vee \rangle$ , then we define

$$\tilde{e}_i u = \sum_{k \geq 1} q^{-l_k + 2k - 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} q^{l_k - 2k - 1} f_i^{(k+1)} u_k.$$

Let  $A_0$  be the subring of  $\mathbb{Q}(q)$  consisting of  $f(q)/g(q)$  with  $f(q), g(q) \in \mathbb{Q}[q]$  and  $g(0) \neq 0$ .

**Definition 2.4.** Let  $V \in \mathcal{O}_{\geq 0}$  be given. A pair  $(L, B)$  is a crystal base of  $V$  if it satisfies the following conditions:

- (1)  $L$  is an  $A_0$ -lattice of  $V$  and  $L = \bigoplus_{\mu \in P_{\geq 0}} L_{\mu}$ , where  $L_{\mu} = L \cap V_{\mu}$ ,
- (2)  $B$  is a signed basis of  $L/qL$ , that is  $B = \mathbf{B} \cup -\mathbf{B}$  where  $\mathbf{B}$  is a  $\mathbb{Q}$ -basis of  $L/qL$ ,
- (3)  $B = \bigsqcup_{\mu \in P_{\geq 0}} B_{\mu}$  where  $B_{\mu} \subset (L/qL)_{\mu}$ ,
- (4)  $\tilde{e}_i L \subset L, \tilde{f}_i L \subset L$  and  $\tilde{e}_i B \subset B \cup \{0\}, \tilde{f}_i B \subset B \cup \{0\}$  for  $i \in I$ ,
- (5)  $\tilde{f}_i b = b'$  if and only if  $\tilde{e}_i b' = \pm b$  for  $i \in I$  and  $b, b' \in B$ .

Let us call  $B/\{\pm 1\}$  a crystal of  $V$ , which is an  $I$ -colored oriented graph. For  $i \in I$  and  $b \in B/\{\pm 1\}$ , we define

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k(b) \in B/\{\pm 1\}\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k(b) \in B/\{\pm 1\}\}.$$

We have a tensor product rule for crystals (see [2] and [17, Proposition 3.4]).

**Proposition 2.5.** Let  $V_1, V_2 \in \mathcal{O}_{\geq 0}$  be given. Suppose that  $(L_i, B_i)$  is a crystal base of  $V_i$  for  $i = 1, 2$ . Then  $(L_1 \otimes L_2, B_1 \otimes B_2)$  is a crystal base of  $V_1 \otimes V_2$ , where  $B_1 \otimes B_2 \subset (L_1/qL_1) \otimes (L_2/qL_2) = (L_1 \otimes L_2)/(qL_1 \otimes L_2)$ . Moreover,  $\tilde{e}_i$  and  $\tilde{f}_i$  act on  $B_1 \otimes B_2$  as follows:

- (1) if  $i \in I_{\text{odd}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (0, 1)$ , then

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i b_2, & \text{if } \langle \text{wt}(b_2), \alpha_i^{\vee} \rangle > 0, \\ \tilde{e}_i b_1 \otimes b_2, & \text{if } \langle \text{wt}(b_2), \alpha_i^{\vee} \rangle = 0, \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{f}_i b_2, & \text{if } \langle \text{wt}(b_2), \alpha_i^{\vee} \rangle > 0, \\ \tilde{f}_i b_1 \otimes b_2, & \text{if } \langle \text{wt}(b_2), \alpha_i^{\vee} \rangle = 0, \end{cases} \end{aligned}$$

- (2) if  $i \in I_{\text{odd}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ , then

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i b_2, & \text{if } \langle \text{wt}(b_1), \alpha_i^{\vee} \rangle = 0, \\ \tilde{e}_i b_1 \otimes b_2, & \text{if } \langle \text{wt}(b_1), \alpha_i^{\vee} \rangle > 0, \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{f}_i b_2, & \text{if } \langle \text{wt}(b_1), \alpha_i^{\vee} \rangle = 0, \\ \tilde{f}_i b_1 \otimes b_2, & \text{if } \langle \text{wt}(b_1), \alpha_i^{\vee} \rangle > 0, \end{cases} \end{aligned}$$



(3) if  $i \in I_{\text{even}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (0, 0)$ , then

$$\begin{aligned}\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1), \end{cases}\end{aligned}$$

(4) if  $i \in I_{\text{even}}$  and  $(\epsilon_i, \epsilon_{i+1}) = (1, 1)$ , then

$$\begin{aligned}\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \sigma_i \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \sigma_i \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}\end{aligned}$$

where  $\sigma_i = (-1)^{(\text{wt}(b_1), \alpha_i)}$ .

**Proof.** The proof is almost the same as in [17, Proposition 3.4], where the order of tensor product is reversed due to a different convention of comultiplication.  $\square$

**Remark 2.6.** Let  $\mathcal{V} = \bigoplus_{i \in \mathbb{I}} \mathbb{Q}(q)v_i$  denote the  $\mathcal{U}(\epsilon)$ -module, where

$$\omega_i v_j = q_i^{\delta_{ij}} v_j, \quad e_k v_j = \delta_{j, k+1} v_k, \quad f_k v_j = \delta_{j, k} v_{k+1}, \quad (2.12)$$

for  $i, j \in \mathbb{I}$  and  $k \in I$ . It is clear that the pair  $\mathcal{L} = \bigoplus_{i \in \mathbb{I}} A_0 v_i$  and  $\mathcal{B} = \{\pm v_i \pmod{q\mathcal{L}} \mid i \in \mathbb{I}\}$  is a crystal base of  $\mathcal{V}$ . The crystal structure on  $\mathcal{B}^{\otimes \ell} / \{\pm 1\}$  for  $\ell \geq 1$  can be described explicitly by Proposition 2.5, which is the same as in [2] or [17] except that the tensor product order is reversed.

### 3. Schur-Weyl duality and polynomial representations of $\mathring{\mathcal{U}}(\epsilon)$

#### 3.1. Schur-Weyl duality

Put  $\mathring{I} = I \setminus \{0\}$ . Let  $\mathring{\mathcal{U}}(\epsilon)$  be the  $\mathbb{Q}(q)$ -subalgebra of  $\mathcal{U}(\epsilon)$  generated by  $q^h$  and  $e_i, f_i$  for  $h \in P^\vee$  and  $i \in \mathring{I}$ .

Let us consider  $\mathcal{V} = \bigoplus_{i \in \mathbb{I}} \mathbb{Q}(q)v_i$  in (2.12) as a  $\mathring{\mathcal{U}}(\epsilon)$ -module. Fix  $\ell \geq 2$ . Let  $\Phi_\ell : \mathring{\mathcal{U}}(\epsilon) \rightarrow \text{End}_{\mathbb{Q}(q)}(\mathcal{V}^{\otimes \ell})$  denote the action of  $\mathring{\mathcal{U}}(\epsilon)$  on  $\mathcal{V}^{\otimes \ell}$  with respect to (2.8). Note that  $\mathcal{V}^{\otimes \ell}$  is semisimple (see [17, Corollary 4.1]).

We have a  $\mathcal{U}(\epsilon)$ -linear map  $\mathcal{R} : \mathcal{V}^{\otimes 2} \longrightarrow \mathcal{V}^{\otimes 2}$  given by

$$\mathcal{R}(v_i \otimes v_j) = \begin{cases} q^{-1}q_i^{-1}v_i \otimes v_j, & \text{if } i = j, \\ q^{-1}v_j \otimes v_i, & \text{if } i > j, \\ (q^{-2} - 1)v_i \otimes v_j + q^{-1}v_j \otimes v_i, & \text{if } i < j, \end{cases} \quad (3.1)$$

satisfying the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23},$$

where  $\mathcal{R}_{ij}$  denotes the map acting as  $\mathcal{R}$  on the  $i$ -th and the  $j$ -th components and the identity elsewhere on  $\mathcal{V}^{\otimes 3}$  (cf. [11]).

Let  $\mathcal{H}_\ell(q^{-2})$  be the Iwahori-Hecke algebra of type  $A$  over  $\mathbb{Q}(q)$  generated by  $h_i$  for  $i \in \{1, \dots, \ell - 1\}$  subject to the relations

$$\begin{aligned} (h_i - q^{-2})(h_i + 1) &= 0, \\ h_i h_j &= h_j h_i, \quad (|i - j| > 1), \\ h_i h_j h_i &= h_j h_i h_j \quad (|i - j| = 1), \end{aligned}$$

for  $i, j \in \{1, \dots, \ell - 1\}$ . Let  $W$  be the symmetric group on  $\{1, \dots, \ell\}$  and  $s_i = (i \ i + 1)$  be the transposition for  $1 \leq i \leq \ell - 1$ . For  $w \in W$ , let  $\ell(w)$  denote the length of  $w$  and let  $h(w)$  be the element in  $\mathcal{H}_\ell(q^{-2})$  associated to  $w$  such that  $h(s_i) = h_i$  for  $1 \leq i \leq \ell - 1$ .

We can check that there exists a well-defined action of  $\mathcal{H}_\ell(q^{-2})$  on  $\mathcal{V}^{\otimes \ell}$ , say,  $\Psi_\ell : \mathcal{H}_\ell(q^{-2}) \longrightarrow \text{End}_{\mathbb{Q}(q)}(\mathcal{V}^{\otimes \ell})$ , where  $\Psi_\ell(h_i)$  acts as  $\mathcal{R}$  on the  $i$ -th and  $(i+1)$ -th components and the identity elsewhere. Then we have an analogue of Schur-Weyl duality for  $\mathcal{U}(\epsilon)$  (cf. [11]) as follows. The proof is similar to the case when  $\epsilon_i = 0$  for all  $i$ .

**Theorem 3.1.** *We have*

$$\text{End}_{\mathcal{H}_\ell(q^{-2})}(\mathcal{V}^{\otimes \ell}) = \Phi_\ell(\mathcal{U}(\epsilon)), \quad \text{End}_{\mathcal{U}(\epsilon)}(\mathcal{V}^{\otimes \ell}) = \Psi_\ell(\mathcal{H}_\ell(q^{-2})). \quad \square$$

### 3.2. Polynomial representations of $\mathcal{U}(\epsilon)$

Recall that  $M$  is the number of  $i$ 's with  $\epsilon_i = 0$  and  $N$  is the number of  $i$ 's with  $\epsilon_i = 1$  in  $\epsilon$ .

Let  $\mathcal{P}$  be the set of all partitions. A partition  $\lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P}$  is called an  $(M|N)$ -hook partition if  $\lambda_{M+1} \leq N$  (cf. [4]). We denote the set of all  $(M|N)$ -hook partitions by  $\mathcal{P}_{M|N}$ . For a Young diagram  $\lambda$ , a tableau  $T$  obtained by filling  $\lambda$  with letters in  $\mathbb{I}$  is called semistandard if (1) the letters in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the letters  $i$  with  $\epsilon_i = 0$  (resp.  $\epsilon_i = 1$ ) are strictly increasing in each column (resp. row). Let  $SST_\epsilon(\lambda)$  be the set of all

semistandard tableaux of shape  $\lambda$ . Then  $SST_\epsilon(\lambda)$  is non-empty if and only if  $\lambda \in \mathcal{P}_{M|N}$ . For  $T \in SST_\epsilon(\lambda)$ , let  $w(T)$  be the word given by reading the entries in  $T$  column by column from left to right, and from bottom to top in each column.

For  $T \in SST_\epsilon(1^r)$  with  $r \geq 1$ , let  $d(T) = \sum_{u < v} d_u d_v$ , where  $d_u$  is the number of occurrences of  $u$  in  $T$  for  $u \in \mathbb{I}$ . In general, for a column-semistandard tableau  $T$ , that is, each column of  $T$  is semistandard, we define  $d(T) = \sum_{k \geq 1} d(T_k)$ , where  $T_k$  is the  $k$ -th column from the left.

We fix  $\ell \geq 2$ , and let  $W$  denote the symmetric group on  $\{1, \dots, \ell\}$ . Suppose that  $\lambda \in \mathcal{P}$  is given with  $\sum_{i \geq 1} \lambda_i = \ell$ . Let  $T_+^\lambda$  be the standard tableau obtained by filling  $\lambda$  with  $\{1, \dots, \ell\}$  row by row from top to bottom and from left to right in each row, and let  $T_-^\lambda$  be the tableau obtained by filling  $\lambda$  with  $\{1, \dots, \ell\}$  column by column from left to right and from top to bottom in each column.

Let  $w_\lambda \in W$  be such that  $w_\lambda(T_-^\lambda) = T_+^\lambda$ , where  $w_\lambda(T_-^\lambda)$  is the tableau obtained by acting  $w_\lambda$  on the letters in  $T_-^\lambda$ . Let  $W_+^\lambda$  (resp.  $W_-^\lambda$ ) be the Young subgroup of  $W$  stabilizing the rows (resp. columns) of  $T_+^\lambda$  (resp.  $T_-^\lambda$ ). Then the  $q$ -deformed Young symmetrizer is given by

$$Y^\lambda(q) = h(w_\lambda^{-1})e_+^\lambda h(w_\lambda)e_-^\lambda,$$

[10] where

$$e_+^\lambda = \sum_{w \in W_+^\lambda} h(w), \quad e_-^\lambda = \sum_{w \in W_-^\lambda} (-q^2)^{\ell(w)} h(w).$$

For  $1 \leq u < v \leq \ell$ , let  $W_{uv} = \langle s_i \mid u \leq i \leq v-1 \rangle$ . Suppose that  $a$  is a letter in  $T_-^\lambda$  such that  $a+1$  is located in the same column. We put  $C_a = 1 + h_a$ . Then we have

$$Y^\lambda(q)C_a = 0, \tag{3.2}$$

Next, suppose that  $a$  is a letter in  $T_-^\lambda$ , where there is another letter  $d$  to the right. Let  $b$  be the letter at the bottom of column where  $a$  is placed, and  $c = b+1$  the letter at the top of the column where  $d$  is placed. Let  $\mathcal{G}_a^\lambda$  be the set of minimal length right coset representatives of  $W_{ab} \times W_{cd}$  in  $W_{ad}$ . We define the Garnir element at  $a$  to be

$$G_a^\lambda = \sum_{w \in \mathcal{G}_a^\lambda} (-q^2)^{\ell(w)} h(w).$$

The collection of boxes in the Young diagram  $\lambda$  corresponding to the letters from  $a$  to  $d$  in  $T_-^\lambda$  is called a Garnir belt at  $a$ . Then we have the following relations [3, (15)]

$$Y^\lambda(q)G_a^\lambda = 0. \tag{3.3}$$

Let  $T$  be a tableau of shape  $\lambda$  with letters in  $\mathbb{I}$ , and let  $T(i)$  be the letter in  $T$  at the position corresponding to  $i$  in  $T_-^\lambda$  for  $1 \leq i \leq \ell$ . Let

$$v_T = Y^\lambda(q) (v_{T(1)} \otimes \cdots \otimes v_{T(\ell)}).$$

For  $\sigma \in W$ , let  $T^\sigma$  be the tableau given by replacing  $T(i)$  with  $T(\sigma(i))$  for  $1 \leq i \leq \ell$ .

Let  $a$  be a letter in  $T_-^\lambda$  with  $d$  to the right in the same row and with  $b, c$  as above. Let  $w_0$  be the longest element in  $W_{ab} \times W_{cd}$ , and let  $\tilde{\mathcal{G}}_a^\lambda = w_0 \mathcal{G}_a^\lambda w_0$ . Let  $u_1, \dots, u_s$  and  $u_{s+1}, \dots, u_{r+s}$  be the letters in  $T$  corresponding to  $c, \dots, d$  and  $a, \dots, b$  in  $T_-^\lambda$ , respectively. Then we may identify  $\sigma \in \tilde{\mathcal{G}}_a^\lambda$  with a permutation on  $\{1, \dots, r+s\}$  satisfying  $\sigma(1) < \cdots < \sigma(s)$  and  $\sigma(s+1) < \cdots < \sigma(s+r)$  so that  $T^\sigma$  is the tableau obtained from  $T$  by replacing  $u_i$ 's with  $u_{\sigma(i)}$ 's for  $1 \leq i \leq r+s$ . With this identification, we let  $\tilde{\ell}(\sigma)$  be the length of  $\sigma$  as a permutation on  $\{1, \dots, r+s\}$ , and put

$$\begin{aligned} X_\sigma &= \{i \mid 1 \leq i \leq s, s+1 \leq \sigma^{-1}(i) \leq s+r\}, \\ Y_\sigma &= \{j \mid s+1 \leq j \leq s+r, 1 \leq \sigma^{-1}(j) \leq r\}. \end{aligned}$$

**Lemma 3.2.** *Suppose that  $T$  is column-semistandard such that either  $T(a) = T(d) \in \mathbb{I}_1$  or  $T(a) > T(d)$ . Then under the above hypothesis, we have*

$$v_T = - \sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda, \sigma \neq 1} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T)} v_{T^\sigma},$$

where

$$\begin{aligned} m(\sigma, T) &= - \left| \{(i, j) \mid 1 \leq i < j \leq s, i \notin X_\sigma, j \in X_\sigma, u_i = u_j\} \right| \\ &\quad - \left| \{(k, l) \mid s+1 \leq k < l \leq s+r, k \in Y_\sigma, l \notin Y_\sigma, u_k = u_l\} \right| \\ &\quad + \left| \{(x, y) \mid 1 \leq x \leq s, s+1 \leq y \leq s+r, x \in X_\sigma \text{ or } y \in Y_\sigma, u_x = u_y\} \right|. \end{aligned}$$

**Proof.** We have  $v_T = Y^\lambda(q)v$ , where  $v = (v_{T(1)} \otimes \cdots \otimes v_{T(\ell)})$ . Following the above notations, we have  $v = v' \otimes v_{u_{1+s}} \otimes \cdots \otimes v_{u_{r+s}} \otimes v_{u_1} \otimes \cdots \otimes v_{u_s} \otimes v''$ . Note that

$$v_{Tw_0} = Y^\lambda(q) (v' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{1+s}} \otimes v_{u_s} \otimes \cdots \otimes v_{u_1} \otimes v''),$$

where  $u_{r+s} \geq \cdots \geq u_{1+s} = T(a) \geq u_s = T(d) \geq \cdots \geq u_1$ .

For  $w \in \mathcal{G}_a^\lambda$ , we have by (3.1) and (3.2)

$$\begin{aligned} &h(w) (v' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{1+s}} \otimes v_{u_s} \otimes \cdots \otimes v_{u_1} \otimes v'') \\ &= q^{-\ell(w)} (-q)^{m(\sigma, T^{w_0})} (v' \otimes v_{u_{\sigma(r+s)}} \otimes \cdots \otimes v_{u_{\sigma(1+s)}} \otimes v_{u_{\sigma(s)}} \otimes \cdots \otimes v_{u_{\sigma(1)}} \otimes v''), \end{aligned} \quad (3.4)$$

where  $\sigma$  is the permutation on  $\{1, \dots, r+s\}$  corresponding to  $w_0 w w_0$  and

$$m(\sigma, T^{w_0}) = \left| \{(i, j) \mid i < j, \sigma^{-1}(i) < \sigma^{-1}(j), u_i = u_j\} \right|.$$

Hence it follows from (3.3) and (3.4) that

$$\begin{aligned}
0 &= Y^\lambda(q) G_a^\lambda(q) (v' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{1+s}} \otimes v_{u_s} \otimes \cdots \otimes v_{u_1} \otimes v'') \\
&= Y^\lambda(q) \sum_{w \in \mathcal{G}_a^\lambda} (-q^2)^{\ell(w)} h(w) (v' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{1+s}} \otimes v_{u_s} \otimes \cdots \otimes v_{u_1} \otimes v'') \\
&= Y^\lambda(q) \sum_{w \in \mathcal{G}_a^\lambda} (-q)^{\ell(w) + m(\sigma, T^{w_0})} (v' \otimes v_{u_{\sigma(r+s)}} \otimes \cdots \otimes v_{u_{\sigma(1+s)}} \otimes v_{u_{\sigma(s)}} \otimes \cdots \otimes v_{u_{\sigma(1)}} \otimes v'') \\
&= \sum_{w \in \mathcal{G}_a^\lambda} (-q)^{\ell(w) + m(\sigma, T^{w_0})} v_{T^{w_0}w} = \sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T^{w_0})} v_{T^\sigma w_0}.
\end{aligned}$$

We have

$$\sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T^{w_0})} v_{(T^\sigma)w_0} = 0. \quad (3.5)$$

For  $\sigma \in \tilde{\mathcal{G}}_a^\lambda$ , let  $U^\sigma$  be the subtableau of  $T^\sigma$  corresponding to the Garnir belt at  $a$ , where  $U = U^{\text{id}}$ . We define  $d_a(T^\sigma)$  in the same way as in  $d(T)$  only by using the letters in  $U^\sigma$ . Let  $l_p > \cdots > l_1 \geq r_q > \cdots > r_1$  be the distinct letters appearing in  $U$ , where  $l_i$  and  $r_j$  are located in the left and right columns of  $U$ , respectively.

Let  $m_i$  (resp.  $n_j$ ) be the number of occurrences of  $l_i$ 's (resp.  $r_j$ 's) in  $U$ , which remain in the same column after applying  $\sigma$ . Let  $m'_i$  (resp.  $n'_i$ ) be the number of  $l_i$ 's (resp.  $r_j$ 's) which are placed on the right (resp. left) column of  $U^\sigma$  after applying  $\sigma$  to  $U$ . Note that  $\sum_i m'_i = \sum_j n'_j$ .

Case 1. Suppose that  $l_1 \neq r_q$ . We have

$$d_a(T) = \sum_{1 \leq i < j \leq p} (m_i + m'_i)(m_j + m'_j) + \sum_{1 \leq k < l \leq q} (n_k + n'_k)(n_l + n'_l), \quad (3.6)$$

while

$$\begin{aligned}
d_a(T^\sigma) &= \sum_{1 \leq i < j \leq p} (m_i m_j + m'_i m'_j) + \sum_{i,k} m_i n'_k + \sum_{1 \leq k < l \leq q} (n_k n_l + n'_k n'_l) + \sum_{j,l} m'_j n_l \\
&= \sum_{1 \leq i < j \leq p} (m_i m_j + m'_i m'_j) + \sum_i m_i \sum_k n'_k + \sum_{1 \leq k < l \leq q} (n_k n_l + n'_k n'_l) + \sum_j m'_j \sum_l n_l.
\end{aligned} \quad (3.7)$$

Since we have

$$\begin{aligned}
m(\sigma, T^{w_0}) &= |\{(i, j) \mid 1 \leq i < j \leq s, i \in X_\sigma, j \notin X_\sigma, u_i = u_j\}| \\
&\quad + |\{(k, l) \mid s+1 \leq k < l \leq s+r, k \notin Y_\sigma, l \in Y_\sigma, u_k = u_l\}|, \\
m(\sigma, T) &= -|\{(i, j) \mid 1 \leq i < j \leq s, i \notin X_\sigma, j \in X_\sigma, u_i = u_j\}| \\
&\quad - |\{(k, l) \mid s+1 \leq k < l \leq s+r, k \in Y_\sigma, l \notin Y_\sigma, u_k = u_l\}|,
\end{aligned}$$

one can check easily that

$$m(\sigma, T^{w_0}) - m(\sigma, T) = \sum_{1 \leq i \leq p} m_i m'_i + \sum_{1 \leq j \leq q} n_j n'_j. \quad (3.8)$$

By (3.6), (3.7), and (3.8), we have

$$d_a(T) - d_a(T^\sigma) = m(\sigma, T) - m(\sigma, T^{w_0}). \quad (3.9)$$

By (3.2), (3.5) and (3.9), we have

$$\begin{aligned} 0 &= \sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T^{w_0})} v_{(T^\sigma)^{w_0}} = \sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T^{w_0}) - d_a(T^\sigma)} v_{T^\sigma} \\ &= \sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T) - d_a(T)} v_{T^\sigma} = (-q)^{-d_a(T)} \sum_{\sigma \in \tilde{\mathcal{G}}_a^\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma, T)} v_{T^\sigma}. \end{aligned}$$

This proves the identity in the lemma.

*Case 2.* Suppose that  $l_1 = r_q$ . In this case,  $d_a(T)$  is the same as in *Case 1*, and

$$d_a(T) - d_a(T^\sigma) = - \sum_{1 \leq i \leq p} m_i m'_i - \sum_{1 \leq j \leq q} n_j n'_j + m_p n'_1 + m'_p n_1.$$

Note that

$$\begin{aligned} m(\sigma, T^{w_0}) &= |\{(i, j) \mid 1 \leq i < j \leq s, i \in X_\sigma, j \notin X_\sigma, u_i = u_j\}| \\ &\quad + |\{(k, l) \mid s+1 \leq k < l \leq s+r, k \notin Y_\sigma, l \in Y_\sigma, u_k = u_l\}| \\ &\quad + |\{(x, y) \mid 1 \leq x \leq s, s+1 \leq y \leq s+r, x \in X_\sigma, y \in Y_\sigma, u_x = u_y\}|, \end{aligned}$$

where the last summand is equal to  $m'_p n'_1$ . By similar arguments as in (3.8), we have

$$d_a(T) - d_a(T^\sigma) = m(\sigma, T) - m(\sigma, T^{w_0}).$$

This also proves the identity in the lemma as in (3.9).  $\square$

For  $\lambda \in \mathcal{P}_{M|N}$  with  $\sum_i \lambda_i = \ell$ , let

$$V_\epsilon(\lambda) = \sum_{T \in SST_\epsilon(\lambda)} \mathbb{Q}(q) v_T. \quad (3.10)$$

Let  $H_\lambda$  be the tableau in  $SST_\epsilon(\lambda)$ , which is defined inductively as follows:

- (1) Fill the first row (resp. column) of  $\lambda$  with 1 if  $\epsilon_1 = 0$  (resp.  $\epsilon_1 = 1$ ).
- (2) Suppose that we have filled a subdiagram of  $\lambda$  from 1 to  $i$ . Then fill the first row (resp. column) of the remaining diagram with  $i+1$  if  $\epsilon_{i+1} = 0$  (resp.  $\epsilon_{i+1} = 1$ ).

**Example 3.3.** Suppose that  $n = 5$ ,  $\epsilon = (0, 1, 1, 0, 0)$  and  $\lambda = (6, 5, 4, 2, 1)$ . In this case, we have

$$H_\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 3 & 4 & 4 & 4 & \\ \hline 2 & 3 & 5 & 5 & & \\ \hline 2 & 3 & & & & \\ \hline 2 & & & & & \\ \hline \end{array}$$

**Proposition 3.4.** *We have the following.*

- (1)  $V_\epsilon(\lambda)$  is a  $\dot{\mathcal{U}}(\epsilon)$ -submodule of  $\mathcal{V}^{\otimes \ell}$ .
- (2)  $V_\epsilon(\lambda)$  is an irreducible  $\dot{\mathcal{U}}(\epsilon)$ -module with basis  $\{v_T \mid T \in SST_\epsilon(\lambda)\}$ .
- (3)  $v_{H_\lambda}$  is a highest weight vector in  $V_\epsilon(\lambda)$ .

**Proof.** (1) It is clear that  $V_\epsilon(\lambda)$  is invariant under  $q^h$  for  $h \in P^\vee$ . It suffices to check  $f_i V_\epsilon(\lambda) \subset V_\epsilon(\lambda)$  for  $i \in \dot{I}$  since the proof for  $e_i$  is the same. The proof is similar to the case when  $\epsilon = (0, \dots, 0)$  (cf. [9]). For column-semistandard tableaux  $U$  and  $V$  of shape  $\lambda$ , we define  $U < V$  if there exists  $1 \leq k \leq \ell$  such that  $U(k) < V(k)$  and  $U(k') = V(k')$  for  $k < k' \leq \ell$ .

Suppose that  $T \in SST_\epsilon(\lambda)$  is given. By (2.8),  $f_i v_T$  is a  $\mathbb{Q}(q)$ -linear combination of  $v_{T'}$ 's, where we may assume that each of the  $T'$  is column-semistandard by (3.2). If such  $T'$  is not semistandard, then we may apply Lemma 3.2 to  $T'$  so that  $v_{T'}$  is a linear combination of  $T''$ 's which is column-semistandard and  $T' < T''$ . Repeating this process finitely many times, we conclude that  $f_i T$  is a linear combination of  $v_S$ 's for some  $S \in SST_\epsilon(\lambda)$ . Therefore, have  $f_i V_\epsilon(\lambda) \subset V_\epsilon(\lambda)$ .

(2) Since  $V_\epsilon(\lambda) = Y^\lambda(q)\mathcal{V}^{\otimes \ell}$  and  $Y^\lambda(q)$  is a primitive idempotent up to scalar multiplication [10], it follows from Theorem 3.1 that  $V_\epsilon(\lambda)$  is an irreducible  $\dot{\mathcal{U}}(\epsilon)$ -module. Recall that the dimension of the irreducible  $\mathcal{H}_\ell(q^{-2})$ -module  $S^\lambda$  generated by  $Y^\lambda(q)$  is the number of standard tableaux of shape  $\lambda$ . We may have an analogue of the Robinson-Schensted type correspondence, which is a bijection from the set of words of length  $\ell$  with letters in  $\mathbb{I}$  to the set of pair of standard tableau and semistandard tableau of shape  $\lambda$  (cf. [4]). Comparing the dimensions of  $\mathcal{V}^{\otimes \ell}$  and its decomposition into  $\mathcal{H}(q^{-2}) \otimes \dot{\mathcal{U}}(\epsilon)$ -module  $S^\lambda \otimes V_\epsilon(\lambda)$ , we conclude that  $\dim_{\mathbb{Q}(q)} V_\epsilon(\lambda)$  is equal to  $|SST_\epsilon(\lambda)|$ , and hence  $\{v_T \mid T \in SST_\epsilon(\lambda)\}$  is a linear basis of  $V_\epsilon(\lambda)$ .

(3) The character of  $V_\epsilon(\lambda)$  is equal to that of polynomial representations of the general linear Lie superalgebra  $\mathfrak{gl}_{M|N}$  corresponding to  $\lambda \in \mathcal{P}_{M|N}$ , and  $\text{wt}(v_{H_\lambda})$  is maximal [8, Theorem 2.55]. This implies that  $e_i v_{H_\lambda} = 0$  for all  $i \in \dot{I}$  and hence  $v_{H_\lambda}$  is a highest weight vector.  $\square$

**Remark 3.5.** Let us call  $V_\epsilon(\lambda)$  an irreducible polynomial representation of  $\dot{\mathcal{U}}(\epsilon)$  corresponding to  $\lambda \in \mathcal{P}_{M|N}$ . The character of  $V_\epsilon(\lambda)$  is called a hook Schur polynomial [4], which depends only on  $\epsilon$  up to permutations. A tensor product of two irreducible polynomial representations is completely reducible and the multiplicity of each irreducible

component is given by usual Littlewood-Richardson coefficient (see [14, Theorem 4.18] and [17, Corollary 4.14]).

### 3.3. Crystal base of $V_\epsilon(\lambda)$

Let  $\lambda \in \mathcal{P}_{M|N}$  be given. We may define an  $\check{I}$ -colored oriented graph structure on  $SST_\epsilon(\lambda)$  by identifying  $T$  with  $w(T)^{\text{rev}}$ , the reverse word of  $w(T)$  and then applying Proposition 2.5 (cf. [2]).

Let

$$\begin{aligned} L_\epsilon(\lambda) &= \bigoplus_{T \in SST_\epsilon(\lambda)} A_0 v_T^*, \\ B_\epsilon(\lambda) &= \{ \pm v_T^* \pmod{qL_\epsilon(\lambda)} \mid T \in SST_\epsilon(\lambda) \}, \end{aligned} \quad (3.11)$$

where  $v_T^* = q^{-d(T)} v_T$  for  $T \in SST_\epsilon(\lambda)$ .

**Lemma 3.6.** *When  $\lambda = (1^r)$  or  $(r)$  for  $r \geq 1$ ,  $(L_\epsilon(\lambda), B_\epsilon(\lambda))$  is a  $\check{\mathcal{U}}(\epsilon)$ -crystal base of  $V_\epsilon(\lambda)$ , and the crystal  $B_\epsilon(\lambda)/\{\pm 1\}$  is isomorphic to  $SST_\epsilon(\lambda)$ .*

**Proof.** The proof is similar to that of [17, Proposition 3.3].  $\square$

**Proposition 3.7.** *Suppose that  $\epsilon = \epsilon_{M|N}$ . For  $\lambda \in \mathcal{P}_{M|N}$ ,  $(L_\epsilon(\lambda), B_\epsilon(\lambda))$  is a  $\check{\mathcal{U}}(\epsilon)$ -crystal base of  $V_\epsilon(\lambda)$ .*

**Proof.** The proof is similar to that of [19, Theorem 4.4]. Let  $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$  be given by

$$\begin{aligned} \mathcal{L}(\lambda) &= \sum_{r \geq 0, i_1, \dots, i_r \in \check{I}} A_0 \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda, \\ \mathcal{B}(\lambda) &= \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda \pmod{qL(\lambda)} \mid r \geq 0, i_1, \dots, i_r \in \check{I} \} \setminus \{0\}, \end{aligned}$$

where  $v_\lambda = v_{H_\lambda}$  is a highest weight vector in  $V_\epsilon(\lambda)$  and  $x = e, f$  for each  $i_k$ . Following the same arguments in [2], it is shown in [17] that  $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$  is a crystal base of  $V_\epsilon(\lambda)$ . The crystal  $\mathcal{B}(\lambda)/\{\pm 1\}$  is equal to  $SST_\epsilon(\lambda)$  which is connected.

Let  $\mu = (\mu_1, \dots, \mu_r) = \lambda'$  be the conjugate partition of  $\lambda$ , and

$$V_\epsilon^\mu = V_\epsilon((1^{\mu_1})) \otimes \cdots \otimes V_\epsilon((1^{\mu_r})).$$

Let  $I_\epsilon^\mu$  be the subspace of  $V_\epsilon^\mu$  spanned by the vectors induced from the relation (3.3), which includes the relations in Lemma 3.2. Since  $I_\epsilon^\mu$  is a  $\check{\mathcal{U}}(\epsilon)$ -submodule, the quotient  $V_\epsilon^\mu / I_\epsilon^\mu$  is isomorphic to  $V_\epsilon(\lambda)$  by Proposition 3.4. So we have a well-defined  $\check{\mathcal{U}}(\epsilon)$ -linear map

$$\pi^\mu : V_\epsilon^\mu \longrightarrow V_\epsilon(\lambda)$$



given by  $\pi^\mu(v_{T_1} \otimes \cdots \otimes v_{T_r}) = v_T$  where  $T$  is the column semistandard tableau whose  $i$ -th column (from the left) is  $T_i$  for  $1 \leq i \leq r$ . Since the decomposition of  $V^\mu$  is equal to the usual Pieri rule of Schur functions, it has exactly one component isomorphic to  $V_\epsilon(\lambda)$ . Hence  $\pi^\mu$  is equal to the projection onto  $V_\epsilon(\lambda)$  up to scalar multiplication.

Let  $L_\epsilon^\mu = L_\epsilon((1^{\mu_1})) \otimes \cdots \otimes L_\epsilon((1^{\mu_r}))$  be the crystal lattice of  $V_\epsilon^\mu$ . By [17, Theorem 4.14],  $\pi^\mu(L_\epsilon^\mu)$  is a crystal lattice of  $V_\epsilon(\lambda)$  whose  $\text{wt}(H_\lambda)$ -weight space is equal to  $A_0 v_{H_\lambda}^*$ . Since the crystal of  $V_\epsilon(\lambda)$  is connected, we conclude that  $\{v_T^* | T \in SST_\epsilon(\lambda)\}$  is an  $A_0$ -basis of  $\pi^\mu(L_\epsilon^\mu)$  which is equal to  $L_\epsilon(\lambda)$ .  $\square$

**Remark 3.8.** For arbitrary  $\epsilon$ , the  $\mathring{I}$ -colored oriented graph  $SST_\epsilon(\lambda)$  is not in general connected (see [16] for more details). Furthermore, it is not known yet whether  $V_\epsilon(\lambda)$  has a crystal base for any  $\lambda \in \mathcal{P}_{M|N}$ . We expect that  $(L_\epsilon(\lambda), B_\epsilon(\lambda))$  in (3.11) is a crystal base of  $V_\epsilon(\lambda)$ .

#### 4. $R$ matrix for finite-dimensional $\mathcal{U}(\epsilon)$ -modules

##### 4.1. Finite-dimensional $\mathcal{U}(\epsilon)$ -modules of fundamental type

Let

$$\mathbb{Z}_+^n(\epsilon) = \{\mathbf{m} = (m_1, \dots, m_n) \mid m_i \in \mathbb{Z}_+ \text{ if } \epsilon_i = 0, m_i \in \{0, 1\} \text{ if } \epsilon_i = 1, (i \in \mathbb{I})\}.$$

For  $\mathbf{m} \in \mathbb{Z}_+^n(\epsilon)$ , let  $|\mathbf{m}| = m_1 + \cdots + m_n$ . For  $i \in \mathbb{I}$ , put  $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$  where 1 appears only in the  $i$ -th component.

For  $s \in \mathbb{Z}_+$ , let

$$\mathcal{W}_{s,\epsilon} = \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_+^n(\epsilon) \\ |\mathbf{m}|=s}} \mathbb{Q}(q)|\mathbf{m}\rangle$$

be the  $\mathbb{Q}(q)$ -vector space spanned by  $|\mathbf{m}\rangle$  for  $\mathbf{m} \in \mathbb{Z}_+^n(\epsilon)$  with  $|\mathbf{m}| = s$ .

For a non-zero parameter  $x \in \mathbb{Q}(q)$ , we denote by  $\mathcal{W}_{s,\epsilon}(x)$  a  $\mathcal{U}(\epsilon)$ -module  $V$ , where  $V = \mathcal{W}_{s,\epsilon}$  as a  $\mathbb{Q}(q)$ -vector space and the actions of  $e_i, f_i, \omega_j$  for  $i \in I, j \in \mathbb{I}$  are given by

$$\begin{aligned} e_i |\mathbf{m}\rangle &= \begin{cases} x^{\delta_{i,0}} q^{m_{i+1}-m_i-1} [m_{i+1}] |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, & \text{if } \mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon), \\ 0, & \text{otherwise,} \end{cases} \\ f_i |\mathbf{m}\rangle &= \begin{cases} x^{-\delta_{i,0}} q^{m_i-m_{i+1}-1} [m_i] |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, & \text{if } \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon), \\ 0, & \text{otherwise,} \end{cases} \\ \omega_j |\mathbf{m}\rangle &= q_j^{m_j} |\mathbf{m}\rangle, \end{aligned}$$

for  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_+^n(\epsilon)$ . Here we understand  $\mathbf{e}_0 = \mathbf{e}_n$ .

**Remark 4.1.** We may identify  $\mathcal{W}_{s,\epsilon}(x)$  with  $V_\epsilon((s))$  (3.10) as a  $\mathcal{U}(\epsilon)$ -module, where  $|\mathbf{m}\rangle$  corresponds to  $v_T$ , where  $T$  is the tableau of shape  $(s)$  with  $m_i$  the number of occurrences of  $i$  in  $T$  ( $i \in \mathbb{I}$ ). Also the map

$$\phi(|\mathbf{m}\rangle) = q^{-\sum_{i < j} m_i m_j} |\mathbf{m}\rangle \quad (4.1)$$

gives an isomorphism of  $\mathcal{U}(\epsilon)$ -modules from  $\mathcal{W}_{s,\epsilon}(x)$  to itself with another  $\mathcal{U}(\epsilon)$ -action defined in [17, (2.15)].

Let us regard  $\mathcal{W}_{s,\epsilon} = \mathcal{W}_{s,\epsilon}(1)$  and set

$$\mathcal{L}_{s,\epsilon} = \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_+^n(\epsilon) \\ |\mathbf{m}|=s}} A_0 |\mathbf{m}\rangle, \quad \mathcal{B}_{s,\epsilon} = \{ \pm |\mathbf{m}\rangle \pmod{q\mathcal{L}_{s,\epsilon}} \mid \mathbf{m} \in \mathbb{Z}_+^n(\epsilon), |\mathbf{m}| = s \}. \quad (4.2)$$

**Proposition 4.2.** For  $s \in \mathbb{Z}_+$ , the pair  $(\mathcal{L}_{s,\epsilon}, \mathcal{B}_{s,\epsilon})$  is a crystal base of  $\mathcal{W}_{s,\epsilon}$ , where

$$\begin{aligned} \tilde{e}_i |\mathbf{m}\rangle &= \begin{cases} |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, & \text{if } \mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon), \\ 0, & \text{otherwise,} \end{cases} \pmod{q\mathcal{L}_{s,\epsilon}}, \\ \tilde{f}_i |\mathbf{m}\rangle &= \begin{cases} |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, & \text{if } \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon), \\ 0, & \text{otherwise,} \end{cases} \pmod{q\mathcal{L}_{s,\epsilon}}, \end{aligned}$$

for  $i \in I$  and  $|\mathbf{m}\rangle \in \mathbb{Z}_+^n(\epsilon)$ , and the crystal  $\mathcal{B}_{s,\epsilon}/\{\pm 1\}$  is connected.

**Proof.** It follows from the same arguments as in Lemma 3.6 that  $(\mathcal{L}_{s,\epsilon}, \mathcal{B}_{s,\epsilon})$  is a crystal base of  $\mathcal{W}_{s,\epsilon}$ . The crystal  $SST_\epsilon((s))$  is connected with highest element  $H_{(s)}$ . Since the crystal  $\mathcal{B}_{s,\epsilon}/\{\pm 1\}$  of  $\mathcal{W}_{s,\epsilon}$  is equal to  $SST_\epsilon((s))$  as an  $\overset{\circ}{I}$ -colored graph,  $\mathcal{B}_{s,\epsilon}/\{\pm 1\}$  is connected as an  $I$ -colored oriented graph.  $\square$

#### 4.2. Subalgebra $\mathcal{U}(\epsilon')$

Suppose that  $n \geq 4$  and let  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  be given. Let  $\epsilon' = (\epsilon'_1, \dots, \epsilon'_{n-1})$  be the sequence obtained from  $\epsilon$  by removing  $\epsilon_i$  for some  $i \in \mathbb{I}$ . We further assume that  $\epsilon'$  is homogeneous when  $n = 4$ , that is,  $\epsilon' = (000)$  or  $(111)$ .

Put  $I' = \{0, 1, \dots, n-2\}$  and  $\mathbb{I}' = \{1, 2, \dots, n-1\}$ . Let us denote by  $\omega'_l$ ,  $e'_j$ , and  $f'_j$  the generators of  $\mathcal{U}(\epsilon')$  for  $l \in \mathbb{I}'$  and  $j \in I'$ . Let us define  $\Omega_l, E_j, F_j$  for  $l \in \mathbb{I}'$  and  $j \in I'$  as follows:

$$\Omega_l = \begin{cases} \omega_l, & \text{if } 1 \leq l \leq i-1, \\ \omega_{l+1}, & \text{if } i \leq l \leq n-1. \end{cases} \quad (4.3)$$

Case 1. If  $2 \leq i \leq n-1$ , then

$$E_j = \begin{cases} e_j, & \text{if } j \leq i-2, \\ [e_{i-1}, e_i]_{D_{i-1}i}, & \text{if } j = i-1, \\ e_{j+1}, & \text{if } j \geq i, \end{cases} \quad F_j = \begin{cases} f_j, & \text{if } j \leq i-2, \\ [f_i, f_{i-1}]_{D_{i-1}^{-1}i}, & \text{if } j = i-1, \\ f_{j+1}, & \text{if } j \geq i. \end{cases} \quad (4.4)$$

Case 2. If  $i = n$ , then

$$E_j = \begin{cases} e_j, & \text{if } j \neq 0, \\ [e_{n-1}, e_0]_{D_{n-1}0}, & \text{if } j = 0, \end{cases} \quad F_j = \begin{cases} f_j, & \text{if } j \neq 0, \\ [f_0, f_{n-1}]_{D_{n-1}^{-1}0}, & \text{if } j = 0. \end{cases} \quad (4.5)$$

Case 3. If  $i = 1$ , then

$$E_j = \begin{cases} [e_0, e_1]_{D_{01}}, & \text{if } j = 0, \\ e_{j+1}, & \text{if } j \neq 0, \end{cases} \quad F_j = \begin{cases} [f_1, f_0]_{D_{01}^{-1}}, & \text{if } j = 0, \\ f_{j+1}, & \text{if } j \neq 0. \end{cases} \quad (4.6)$$

**Theorem 4.3.** *There exists a homomorphism of  $\mathbb{Q}(q)$ -algebras  $\phi : \mathcal{U}(\epsilon') \rightarrow \mathcal{U}(\epsilon)$  such that*

$$\phi(\omega'_l) = \Omega_l, \quad \phi(e'_j) = E_j, \quad \phi(f'_j) = F_j,$$

for  $l \in \mathbb{I}'$  and  $j \in I'$ .

**Proof.** Let us prove Case 1 since the proof of the other cases are similar. Let  $\tilde{\epsilon} = (\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_n)$  be the sequence obtained from  $\epsilon$  by exchanging  $\epsilon_i$  and  $\epsilon_{i+1}$ , and let  $\tau_i : \mathcal{U}(\epsilon) \rightarrow \mathcal{U}(\tilde{\epsilon})$  be the isomorphism in Theorem 2.2. Let us check that  $\Omega_l, E_j, F_j$  satisfy the relations in Definition 2.1. Note that  $D_{i-1}i = q_i^{-1}$ .

First, the relations (2.1) and (2.2) are trivial. Let us check that (2.3) holds. Let  $E_j$  and  $F_l$  be given for  $j, l \in I'$ . If  $j \neq l$  or  $j = l \neq i-1$ , then it is clear. When  $j = l = i-1$ , we have  $\tau_i^{-1}(e_{i-1}) = [e_{i-1}, e_i]_{D_{i-1}i} = E_{i-1}$ ,  $\tau_i^{-1}(f_{i-1}) = F_{i-1}$ , and  $\tau_i^{-1}(k_{i-1}) = k_{i-1}k_i$ . Hence (2.3) holds. We can check the relation (2.4) by the same argument.

Next, consider the relations (2.5). The first one is immediate. So it is enough to show the second one. We may only consider four non-trivial cases when the pair of relevant indices in  $I'$  are  $(i-2, i-1), (i-1, i-2), (i-1, i), (i, i-1)$  with the first index in the pair in  $I'_{\text{even}}$ . In case of  $(i-2, i-1)$ , we have

$$\begin{aligned} & E_{i-2}^2 E_{i-1} - (-1)^{\epsilon_{i-2}} [2] E_{i-2} E_{i-1} E_{i-2} + E_{i-1} E_{i-2}^2 \\ &= e_{i-2}^2 e_{i-1} e_i - q_i^{-1} e_{i-2}^2 e_i e_{i-1} - (q_{i-1} + q_{i-1}^{-1}) e_{i-2} e_{i-1} e_i e_{i-2} \\ &\quad + (q_{i-1} + q_{i-1}^{-1}) q_i^{-1} e_{i-2} e_i e_{i-1} e_{i-2} + e_{i-1} e_i e_{i-2}^2 - q_i^{-1} e_i e_{i-1} e_{i-2}^2, \end{aligned}$$

which is zero, since  $e_{i-2}^2 e_{i-1} + e_{i-1} e_{i-2}^2 = (q_{i-1} + q_{i-1}^{-1}) e_{i-2} e_{i-1} e_{i-2}$  and hence

$$\begin{aligned}
& e_{i-2}^2 e_{i-1} e_i - (q_{i-1} + q_{i-1}^{-1}) e_{i-2} e_{i-1} e_i e_{i-2} + e_{i-1} e_i e_{i-2}^2 = 0, \\
& -q_i^{-1} e_{i-2}^2 e_i e_{i-1} + q_i^{-1} (q_{i-1} + q_{i-1}^{-1}) e_{i-2} e_i e_{i-1} e_{i-2} - q_i^{-1} e_i e_{i-1} e_{i-2}^2 = 0.
\end{aligned}$$

The proof for  $(i, i-1)$  is the same. In case of  $(i-1, i-2)$  and  $(i-1, i)$ , the proof reduces to the case of  $(i-2, i-1)$  or  $(i, i-1)$  by applying  $\tau_i$  to  $E_l$ 's for  $l = i-2, i-1, i$ .

Finally let us check the relation (2.6). We may only consider the cases when the relevant triple of indices in  $I'$  are  $(i-3, i-2, i-1)$ ,  $(i-2, i-1, i)$ ,  $(i-1, i, i+1)$  with the index in the middle in  $I'_{\text{odd}}$ . In case of  $(i-1, i, i+1)$  and  $i \in I'_{\text{odd}}$ , we have

$$\begin{aligned}
& E_i E_{i-1} E_i E_{i+1} - E_i E_{i+1} E_i E_{i-1} + E_{i+1} E_i E_{i-1} E_i \\
& \quad - E_{i-1} E_i E_{i+1} E_i + (-1)^{\epsilon_i} [2] E_i E_{i-1} E_{i+1} E_i \\
& = e_{i+1} (e_{i-1} e_i - q_i^{-1} e_i e_{i-1}) e_{i+1} e_{i+2} - e_{i+1} e_{i+2} e_{i+1} (e_{i-1} e_i - q_i^{-1} e_i e_{i-1}) \\
& \quad + e_{i+2} e_{i+1} (e_{i-1} e_i - q_i^{-1} e_i e_{i-1}) e_{i+1} - (e_{i-1} e_i - q_i^{-1} e_i e_{i-1}) e_{i+1} e_{i+2} e_{i+1} \\
& \quad + (-1)^{\epsilon_{i+1}} [2] e_{i+1} (e_{i-1} e_i - q_i^{-1} e_i e_{i-1}) e_{i+2} e_{i+1},
\end{aligned}$$

which is zero by (2.6) for  $\mathcal{U}(\epsilon)$  with respect to  $i+1 \in I_{\text{odd}}$ . The proof for  $(i-3, i-2, i-1)$  is the same. The proof for  $(i-2, i-1, i)$  reduces to the previous cases by applying  $\tau_i$  to  $E_l$  for  $l = i-2, i-1, i$ . We leave the proof for  $F_j$ 's to the reader.  $\square$

### 4.3. Truncation to $\mathcal{U}(\epsilon')$ -modules

Let  $\epsilon'$  be as in Section 4.2. Suppose that  $M'$  is the number of  $j$ 's with  $\epsilon'_j = 0$  and  $N'$  is the number of  $j$ 's with  $\epsilon'_j = 1$  in  $\epsilon'$ .

Let  $\mathcal{O}_{\geq 0}^\epsilon$  and  $\mathcal{O}_{\geq 0}^{\epsilon'}$  denote the categories  $\mathcal{O}_{\geq 0}$  for  $\mathcal{U}(\epsilon)$  and  $\mathcal{U}(\epsilon')$ , respectively. For  $V \in \mathcal{O}_{\geq 0}^\epsilon$ , we define

$$\text{tr}_{\epsilon'}^\epsilon(V) = \bigoplus_{\substack{\mu \in \text{wt}(V) \\ (\mu|\delta_i)=0}} V_\mu, \tag{4.7}$$

where  $\text{wt}(V)$  is the set of weights of  $V$ . For  $V, W \in \mathcal{O}_{\geq 0}^\epsilon$ , it is clear that

$$\text{tr}_{\epsilon'}^\epsilon(V \otimes W) = \text{tr}_{\epsilon'}^\epsilon(V) \otimes \text{tr}_{\epsilon'}^\epsilon(W), \tag{4.8}$$

as a  $\mathbb{Q}(q)$ -vector space.

**Proposition 4.4.** *For  $V, W \in \mathcal{O}_{\geq 0}^\epsilon$ , let  $V' = \text{tr}_{\epsilon'}^\epsilon(V)$  and  $W' = \text{tr}_{\epsilon'}^\epsilon(W)$ . Then*

- (1)  $V'$  is invariant under the action of  $\mathcal{U}(\epsilon')$  via  $\phi$ , and hence a  $\mathcal{U}(\epsilon')$ -module in  $\mathcal{O}_{\geq 0}^{\epsilon'}$ ,
- (2)  $\text{tr}_{\epsilon'}^\epsilon(V \otimes W)$  is isomorphic to the tensor product of  $V'$  and  $W'$  as a  $\mathcal{U}(\epsilon')$ -module.

**Proof.** (1) We may assume that  $2 \leq i \leq n-1$  since the proofs for the other cases are similar. Let  $\mu \in \text{wt}(V)$  be such that  $(\mu|\delta_i) = 0$  so that  $V_\mu \subset V'$ . We claim that  $\mathcal{U}(\epsilon')V_\mu \subset V'$ . Indeed, it suffices to show that  $e'_j V_\mu \subset V'$  and  $f'_j V_\mu \subset V'$  for  $j \in I'$ .

Suppose that  $j \neq i-1$ . By (4.4), we have

$$e'_j V_\mu = E_j V_\mu = \begin{cases} e_j V_\mu \subset V_{\mu+\epsilon_j-\epsilon_{j+1}} & (j \leq i-2), \\ e_{j+1} V_\mu \subset V_{\mu+\epsilon_{j+1}-\epsilon_{j+2}} & (j \geq i). \end{cases}$$

Hence, we have  $E_j V_\mu \subset V'$  by (2.9). Suppose that  $j = i-1$ . We have  $E_{i-1} = e_{i-1}e_i - q_i^{-1}e_i e_{i-1}$ . Since  $(\mu|\delta_i) = 0$ , we have  $e_{i-1}V_\mu = 0$  by (2.9), and hence

$$e'_{i-1} V_\mu = E_{i-1} V_\mu = e_{i-1}e_i V_\mu \subset V_{\mu+\epsilon_{i-1}-\epsilon_{i+1}} \subset V'.$$

Similarly, we can show that  $F_j V_\mu \subset V'$ .

(2) We claim that

$$\Delta\phi(x) = (\phi \otimes \phi)\Delta'(x) \quad \text{on } V' \otimes W' \text{ for } x \in \mathcal{U}(\epsilon'), \quad (4.9)$$

where  $\Delta'$  is the comultiplication on  $\mathcal{U}(\epsilon')$  as in (2.8). Clearly it is true for  $\omega'_l$  for  $l \in \mathbb{I}'$ . So it suffices to consider  $e'_j$  and  $f'_j$  for  $j \in I'$ . Let us first prove it in *Case 1*.

Consider  $e'_j$ . It is easy to check that  $\Delta\phi(e'_j) = (\phi \otimes \phi)\Delta'(e'_j)$  if  $j \neq i-1$ . Suppose that  $j = i-1$ . We have  $\phi(e'_{i-1}) = E_{i-1} = e_{i-1}e_i - q_i^{-1}e_i e_{i-1}$ , and then

$$\begin{aligned} \Delta\phi(e'_{i-1}) &= \Delta(e_{i-1})\Delta(e_i) - q_i^{-1}\Delta(e_i)\Delta(e_{i-1}) \\ &= k_{i-1}^{-1}k_i^{-1} \otimes e_{i-1}e_i + k_{i-1}^{-1}e_i \otimes e_{i-1} + e_{i-1}k_i^{-1} \otimes e_i + e_{i-1}e_i \otimes 1 \\ &\quad - q_i^{-1}(k_i^{-1}k_{i-1}^{-1} \otimes e_i e_{i-1} + k_i^{-1}e_{i-1} \otimes e_i + e_i k_{i-1}^{-1} \otimes e_{i-1} + e_i e_{i-1} \otimes 1) \\ &= k_{i-1}^{-1}k_i^{-1} \otimes e_{i-1}e_i + k_{i-1}^{-1}e_i \otimes e_{i-1} + q_i^{-1}k_i^{-1}e_{i-1} \otimes e_i + e_{i-1}e_i \otimes 1 \\ &\quad - q_i^{-1}(k_i^{-1}k_{i-1}^{-1} \otimes e_i e_{i-1} + k_i^{-1}e_{i-1} \otimes e_i + q_i^{-1}k_{i-1}^{-1}e_i \otimes e_{i-1} + e_i e_{i-1} \otimes 1), \end{aligned}$$

where the last equality follows from (2.7). On the other hand, we have

$$\begin{aligned} (\phi \otimes \phi)\Delta'(e'_{i-1}) &= K_{i-1}^{-1} \otimes E_{i-1} + E_{i-1} \otimes 1 \\ &= k_i^{-1}k_{i-1}^{-1} \otimes (e_{i-1}e_i - q_i^{-1}e_i e_{i-1}) + (e_{i-1}e_i - q_i^{-1}e_i e_{i-1}) \otimes 1. \end{aligned}$$

Summarizing, we conclude that

$$\Delta\phi(e'_j) - (\phi \otimes \phi)\Delta'(e'_j) = \delta_{j,i-1}(1 - q_i^{-2})k_{i-1}^{-1}e_i \otimes e_{i-1}. \quad (4.10)$$

Similarly, we obtain

$$\Delta\phi(f'_j) - (\phi \otimes \phi)\Delta'(f'_j) = \delta_{j,i-1}(1 - q_i^2)f_{i-1} \otimes k_{i-1}f_i. \quad (4.11)$$

As in the proof of (1), we see that  $e_{i-1}$  and  $f_i$  acts trivially on  $V'$  and  $W'$ . Hence, we obtain (4.9). In Case 2, one can check that

$$\begin{aligned}\Delta\phi(e'_j) - (\phi \otimes \phi)\Delta'(e'_j) &= \delta_{j,0}(1 - q_0^{-2})k_{n-1}^{-1}e_n \otimes e_{n-1}, \\ \Delta\phi(f'_j) - (\phi \otimes \phi)\Delta'(f'_j) &= \delta_{j,0}(1 - q_0^2)f_{n-1} \otimes k_{n-1}f_n.\end{aligned}$$

In Case 3, the formula (4.10) and (4.11) also hold. Therefore we conclude that  $\mathbf{tr}_{\epsilon'}^{\epsilon}(V \otimes W)$  is isomorphic to the tensor product of  $V'$  and  $W'$  as a  $\mathcal{U}(\epsilon')$ -module.  $\square$

For a  $\mathcal{U}(\epsilon)$ -linear map  $f : V \rightarrow W$ , we have a well-defined  $\mathcal{U}(\epsilon')$ -linear map  $\mathbf{tr}_{\epsilon'}^{\epsilon}(f) : V' \rightarrow W'$  by Proposition 4.4. Hence we have a functor  $\mathbf{tr}_{\epsilon'}^{\epsilon} : \mathcal{O}_{\geq 0}^{\epsilon} \rightarrow \mathcal{O}_{\geq 0}^{\epsilon'}$ , which is exact.

**Proposition 4.5.** *Let  $\lambda \in \mathcal{P}_{M|N}$  be given. Let  $\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda))$  be defined as in (4.7).*

- (1)  $\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda))$  is a  $\mathcal{U}(\epsilon')$ -submodule of  $V_{\epsilon}(\lambda)$  via  $\phi$ .
- (2)  $\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda))$  is non-zero if and only if  $\lambda \in \mathcal{P}_{M'|N'}$ . In this case, we have

$$\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda)) \cong V_{\epsilon'}(\lambda),$$

as a  $\mathcal{U}(\epsilon')$ -module.

**Proof.** (1) It follows from Proposition 4.4(1).

(2) Let  $\mathcal{V}' = \mathbf{tr}_{\epsilon'}^{\epsilon}(\mathcal{V})$ . We first claim that  $\mathcal{V}'$  is isomorphic to the natural representation of  $\mathcal{U}(\epsilon')$  given in (2.12). Let us assume that  $2 \leq i \leq n-1$  since the proof for the other cases is similar. Let  $j \in (I')$  and  $k \in \mathbb{I} \setminus \{i\}$  given. It is clear from (4.4) that

$$E_j v_k = \begin{cases} v_j, & \text{if } k = j+1, \\ 0, & \text{if } k \neq j+1, \end{cases} \quad (j \leq i-2), \quad E_j v_k = \begin{cases} v_{j+1}, & \text{if } k = j+2, \\ 0, & \text{if } k \neq j+2, \end{cases} \quad (j \geq i). \quad (4.12)$$

When  $j = i-1$ , we have  $E_{i-1} = e_{i-1}e_i - q_i^{-1}e_i e_{i-1}$ , and

$$E_{i-1} v_k = \begin{cases} v_{i-1}, & \text{if } k = i+1, \\ 0, & \text{if } k \neq i+1. \end{cases} \quad (4.13)$$

By (4.12) and (4.13),  $\mathcal{V}'$  is isomorphic to the natural representation of  $\mathcal{U}(\epsilon')$ .

Note that  $SST_{\epsilon'}(\lambda) \subset SST_{\epsilon}(\lambda)$ . Since

$$\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda)) = \mathbf{tr}_{\epsilon'}^{\epsilon}(Y^{\lambda}(q)\mathcal{V}^{\otimes \ell}) = Y^{\lambda}(q)\mathbf{tr}_{\epsilon'}^{\epsilon}(\mathcal{V}^{\otimes \ell}) = Y^{\lambda}(q)\mathbf{tr}_{\epsilon'}^{\epsilon}(\mathcal{V})^{\otimes \ell} = Y^{\lambda}(q)(\mathcal{V}')^{\otimes \ell},$$

we see that  $\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda))$  is a  $\mathbb{Q}(q)$ -span of  $\{v_{T'} \mid T' \in SST_{\epsilon'}(\lambda)\}$ , which in fact forms a basis, by Proposition 3.4. This implies that  $\mathbf{tr}_{\epsilon'}^{\epsilon}(V_{\epsilon}(\lambda))$  is non-zero if and only if  $\lambda \in$

$\mathcal{P}_{M-1|N}$  when  $\epsilon_i = 0$ , and  $\lambda \in \mathcal{P}_{M|N-1}$  when  $\epsilon_i = 1$ . Hence, if it is non-zero, then  $\mathrm{tr}_{\epsilon'}^\epsilon(V_\epsilon(\lambda))$  is isomorphic to  $V_{\epsilon'}(\lambda)$  by Proposition 4.4(2) and the fact that  $\mathcal{V}'$  is the natural representation of  $\mathcal{U}(\epsilon')$ .  $\square$

The following can be proved in a similar manner.

**Proposition 4.6.**

- (1) For  $s \in \mathbb{Z}_+$  and non-zero  $x \in \mathbb{Q}(q)$ ,  $\mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{s,\epsilon}(x))$  is a  $\mathcal{U}(\epsilon')$ -submodule of  $\mathcal{W}_{s,\epsilon}(x)$  via  $\phi$ , and

$$\mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{s,\epsilon}(x)) \cong \mathcal{W}_{s,\epsilon'}(x).$$

Moreover,  $(\mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{L}_{s,\epsilon}), \mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{B}_{s,\epsilon}))$  is a crystal base of  $\mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{s,\epsilon})$  isomorphic to  $(\mathcal{L}_{s,\epsilon'}, \mathcal{B}_{s,\epsilon'})$ .

- (2) For  $l, m \in \mathbb{Z}_+$  and non-zero  $x, y \in \mathbb{Q}(q)$ ,  $\mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y))$  is a  $\mathcal{U}(\epsilon')$ -module via  $\phi$ , and

$$\mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)) \cong \mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{l,\epsilon}(x)) \otimes \mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{m,\epsilon}(y)),$$

as a  $\mathcal{U}(\epsilon')$ -module.

**4.4. Irreducibility of  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$**

Let us show that  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  is irreducible for  $l, m \in \mathbb{Z}_+$  and generic  $x, y \in \mathbb{Q}(q)$ . When  $\epsilon = \epsilon_{M|N}$ , the irreducibility is shown in [18]. In this paper, we prove it in a different way for arbitrary  $\epsilon$ .

**Theorem 4.7.** For  $l, m \in \mathbb{Z}_+$ ,  $\mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon}$  is irreducible.

**Proof.** Let us assume without loss of generality that  $M, N \geq 1$  with  $\epsilon_1 = 0$ .

Let  $(\mathcal{L}_{s,\epsilon}, \mathcal{B}_{s,\epsilon})$  be the crystal base of  $\mathcal{W}_{s,\epsilon}$  in (4.2) for  $s = l, m$ . By Proposition 2.5,  $(\mathcal{L}_{l,\epsilon} \otimes \mathcal{L}_{m,\epsilon}, \mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon})$  is a crystal base of  $\mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon}$ . If  $M = 1$ , then it is proved in [17] that  $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon} / \{\pm 1\}$  is connected. Since  $\dim_{\mathbb{Q}(q)}(\mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon})_{(l+m)\delta_1} = 1$  and  $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon} / \{\pm 1\}$  is connected, it follows from [2, Lemma 2.7] that  $\mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon}$  is irreducible.

We assume that  $M \geq 2$ . Set  $\epsilon' = \epsilon_{M|0}$ , which is the subsequence of  $\epsilon$  obtained by removing all  $\epsilon_i = 1$ 's. Note that the length of  $\epsilon'$  may be less than 4 so that  $\mathcal{U}(\epsilon')$  is not well-defined, but  $\mathrm{tr}_{\epsilon'}^\epsilon$  can be defined in the same way as in (4.7). We put

$$\mathcal{W}_{s,\epsilon'} := \mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{W}_{s,\epsilon}), \quad \mathcal{L}_{s,\epsilon'} := \mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{L}_{s,\epsilon}) \subset \mathcal{L}_{s,\epsilon}, \quad \mathcal{B}_{s,\epsilon'} := \mathrm{tr}_{\epsilon'}^\epsilon(\mathcal{B}_{s,\epsilon}) \subset \mathcal{B}_{s,\epsilon}.$$

Let  $1 \leq j_1 < \dots < j_M \leq n$  be such that  $\epsilon_{j_k} = 0$  for  $1 \leq k \leq M$ . By Theorem 4.3, we have a  $U_q(\mathfrak{sl}_2)$ -action on  $\mathcal{W}_{l,\epsilon'} \otimes \mathcal{W}_{m,\epsilon'}$  corresponding to the pair  $(\epsilon_{j_k}, \epsilon_{j_{k+1}})$  or  $(\epsilon_{j_M}, \epsilon_{j_1})$ .

For  $0 \leq k \leq M-1$ , let us denote by  $\tilde{e}_{k'}$  and  $\tilde{f}_{k'}$  the Kashiwara operators corresponding to  $(\epsilon_{j_k}, \epsilon_{j_{k+1}})$  when  $k \neq 0$  and to  $(\epsilon_{j_M}, \epsilon_{j_1})$  when  $k = 0$ .

If we put  $I' = \{k' \mid k = 0, \dots, M-1\}$ , then  $\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}$  is invariant under  $\tilde{e}_{k'}$  and  $\tilde{f}_{k'}$  for  $k' \in I'$ , and hence  $\mathcal{B}_{l,\epsilon'} \otimes \mathcal{B}_{m,\epsilon'} / \{\pm 1\}$  is an  $I'$ -colored oriented graph. Since  $\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'} \subset \mathcal{L}_{l,\epsilon} \otimes \mathcal{L}_{m,\epsilon}$  and  $\mathcal{B}_{l,\epsilon'} \otimes \mathcal{B}_{m,\epsilon'} \subset \mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon}$ , we may regard  $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon} / \{\pm 1\}$  as an  $(I \sqcup I')$ -colored oriented graph.

Let  $\mathbf{b} = |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \in \mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon}$  be given. We will show that  $\mathbf{b}$  is connected to  $|l\mathbf{e}_1\rangle \otimes |m\mathbf{e}_1\rangle$ , which implies that  $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon} / \{\pm 1\}$  is connected as an  $(I \sqcup I')$ -colored oriented graph. Let us write  $\mathbf{m}_i = (m_{i1}, \dots, m_{in})$  for  $i = 1, 2$ .

We first claim that there exists a sequence  $i_1, \dots, i_r \in I$  such that  $(\epsilon_{i_k}, \epsilon_{i_{k+1}}) \neq (0, 0)$  for  $1 \leq k \leq r$  and

$$\mathbf{b}' := \tilde{x}_{i_1} \dots \tilde{x}_{i_r} \mathbf{b} \equiv |\mathbf{m}'_1\rangle \otimes |\mathbf{m}'_2\rangle \pmod{q\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}}, \quad (4.14)$$

for some  $|\mathbf{m}'_1\rangle \in \mathcal{W}_{l,\epsilon'}$  and  $|\mathbf{m}'_2\rangle \in \mathcal{W}_{m,\epsilon'}$ , where  $\tilde{x}_{i_s} = \tilde{e}_{i_s}$  or  $\tilde{f}_{i_s}$  for each  $1 \leq s \leq r$  (cf. Propositions 2.5 and 4.2).

Suppose that there exists  $k$  with  $\epsilon_k = 1$  such that  $m_{1k} = 1$  or  $m_{2k} = 1$ . Let  $i$  and  $j$  be the maximal and minimal indices respectively such that  $i < k < j$  and  $\epsilon_i = \epsilon_j = 0$ . If there is no such  $(i, j)$ , then we have  $\epsilon = \epsilon_{M|N}$  and identify this case with the one of  $\epsilon = (0^{M-1}, 1^N, 0)$ . Since we will choose  $i_1, \dots, i_r$  in  $\{i, i+1, \dots, j-1\}$ , we may assume for simplicity that  $m_{ab} = 0$  for  $a = 1, 2$  and  $b \notin \{i, \dots, j\}$ .

Let us use induction on  $L = |\mathbf{m}_1| + |\mathbf{m}_2|$ . Suppose that  $L = 1$ . If  $m_{1k} = 1$ , then  $\tilde{f}_{j-1}\tilde{f}_{j-2}\dots\tilde{f}_k\mathbf{b}$  satisfies (4.14). If  $m_{2k} = 1$ , then  $\tilde{e}_i\tilde{e}_{i+1}\dots\tilde{e}_{k-1}\mathbf{b}$  satisfies (4.14).

Suppose that  $L > 1$ . We may assume that  $\tilde{f}_{i+1}\mathbf{b} = \tilde{f}_{i+2}\mathbf{b} = \dots = \tilde{f}_{j-2}\mathbf{b} = 0$ , that is,  $\mathbf{b}$  is a lowest weight element as an element of  $\mathfrak{sl}_L$ -crystal with respect to  $\tilde{e}_l$  and  $\tilde{f}_l$  for  $i+1 \leq l \leq j-2$ , where  $L = j-i-1$ . Then by tensor product rule in Proposition 2.5(4), we have  $\tilde{f}_l|\mathbf{m}_2\rangle = 0$  and  $\tilde{f}_l(|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle) = 0$  for  $i+1 \leq l \leq j-2$ . By Proposition 2.5(4), it is straightforward to check that

$$\begin{aligned} \mathbf{m}_1 &= m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq y} \mathbf{e}_u + \sum_{z \leq v \leq j-1} \mathbf{e}_v + m_{1j}\mathbf{e}_j, \\ \mathbf{m}_2 &= m_{2i}\mathbf{e}_i + \sum_{y+1 \leq v \leq j-1} \mathbf{e}_v + m_{2j}\mathbf{e}_j, \end{aligned} \quad (4.15)$$

for some  $i < x < y < z < j$ . Here we assume that  $\sum_{z \leq v \leq j-1} \mathbf{e}_v$  in  $\mathbf{m}_1$  is empty if there is no such  $z$ . Now we take the following steps to construct  $\mathbf{b}'$  in (4.14).

*Step 1.* If there exists  $z$  such that  $y < z < j$  and  $m_{1z} = \dots = m_{1j-1} = 1$ , then by applying  $\tilde{f}_z\tilde{f}_{z+1}\dots\tilde{f}_{j-1}$  to  $\mathbf{b}$ ,  $\mathbf{m}_1$  in (4.15) is replaced by

$$m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq y} \mathbf{e}_u + \sum_{z+1 \leq v \leq j-1} \mathbf{e}_v + (m_{1j} + 1)\mathbf{e}_j. \quad (4.16)$$

Repeating this step, (4.16) is replaced by



$$m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq y} \mathbf{e}_u + (m_{1j} + j - z)\mathbf{e}_j.$$

Hence we may assume that  $\mathbf{m}_1$  in (4.15) is of the form  $m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq y} \mathbf{e}_u + m_{1j}\mathbf{e}_j$ .

Step 2. If  $m_{1j} = 0$ , then we have by Proposition 2.5(2)

$$\tilde{f}_{j-1}\mathbf{b} = |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2 - \mathbf{e}_{j-1} + \mathbf{e}_j\rangle.$$

Hence we may apply the induction hypothesis to conclude (4.14).

Step 3. If  $m_{1j} \neq 0$ , then by applying  $\tilde{e}_i\tilde{e}_{i+1}\dots\tilde{e}_{j-2}\tilde{e}_{j-1}$  to  $\mathbf{b}$ ,  $\mathbf{m}_1$  and  $\mathbf{m}_2$  are replaced by

$$\begin{aligned} m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq y+1} \mathbf{e}_u + (m_{1j} - 1)\mathbf{e}_j, \\ (m_{2i} + 1)\mathbf{e}_i + \sum_{y+2 \leq v \leq j-1} \mathbf{e}_v + m_{2j}\mathbf{e}_j, \end{aligned}$$

respectively. Repeating this step  $d$  times such that  $m_{1j} - d \geq 0$  and  $y + d + 1 \leq j$ ,  $\mathbf{m}_1$  and  $\mathbf{m}_2$  are replaced by

$$\begin{aligned} m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq y+d} \mathbf{e}_u + (m_{1j} - d)\mathbf{e}_j, \\ (m_{2i} + d)\mathbf{e}_i + \sum_{y+d+1 \leq v \leq j-1} \mathbf{e}_v + m_{2j}\mathbf{e}_j, \end{aligned}$$

respectively. We may keep this process until  $m_{1j} - d = 0$ , which belongs to the case in Step 2, or  $\sum_{y+d+1 \leq v \leq j-1} \mathbf{e}_v$  is empty. In the latter case,  $\mathbf{m}_1$  is replaced by  $m_{1i}\mathbf{e}_i + \sum_{x \leq u \leq j-1} \mathbf{e}_u + (m_{1j} - d)\mathbf{e}_j$  so that we may apply  $\tilde{f}_{j-1}$  and use induction hypothesis to have  $\mathbf{b}'$ . This proves the claim.

By construction of  $\mathbf{b}'$  and its weight, we have

$$\mathbf{b}' - |\mathbf{m}'_1\rangle \otimes |\mathbf{m}'_2\rangle \in (\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}) \cap (q\mathcal{L}_{l,\epsilon} \otimes \mathcal{L}_{m,\epsilon}) = q\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'},$$

and hence  $\mathbf{b}' \in (\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'} / q\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}) \subset (\mathcal{L}_{l,\epsilon} \otimes \mathcal{L}_{m,\epsilon} / q\mathcal{L}_{l,\epsilon} \otimes \mathcal{L}_{m,\epsilon})$ . If  $M = 2$ , then it is easy to show that  $\mathbf{b}' = |\mathbf{m}'_1\rangle \otimes |\mathbf{m}'_2\rangle \in \mathcal{B}_{l,\epsilon'} \otimes \mathcal{B}_{m,\epsilon'}$  is connected to  $|\mathbf{e}_1\rangle \otimes |\mathbf{m}\mathbf{e}_1\rangle$  under  $\tilde{e}_{k'}$  and  $\tilde{f}_{k'}$  for  $k = 0, 1$ . If  $M \geq 3$ , then we can also show that  $\mathbf{b}' = |\mathbf{m}'_1\rangle \otimes |\mathbf{m}'_2\rangle \in \mathcal{B}_{l,\epsilon'} \otimes \mathcal{B}_{m,\epsilon'}$  is connected to  $|\mathbf{e}_1\rangle \otimes |\mathbf{m}\mathbf{e}_1\rangle$  by using the fact that  $\mathcal{B}_{l,\epsilon'} \otimes \mathcal{B}_{m,\epsilon'} / \{\pm 1\}$  is a connected crystal of type  $A_{M-1}^{(1)}$  (cf. [1]).

Finally, note that  $\dim_{\mathbb{Q}(q)}(\mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon})_{(l+m)\delta_1} = 1$  and  $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon} / \{\pm 1\}$  is connected as an  $(I \sqcup I')$ -colored oriented graph. Thus we can apply the same argument as in the proof of [2, Lemma 2.7] using the connectedness with respect to  $I \sqcup I'$ . This implies that  $\mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon}$  is irreducible. The proof completes.  $\square$

**Corollary 4.8.** For  $l, m \in \mathbb{Z}_+$  and generic  $x, y \in \mathbb{Q}(q)$ ,  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  is irreducible.

**Proof.** It follows from [12, Lemma 3.4.2].  $\square$

#### 4.5. Existence of $R$ matrix

For  $l, m \in \mathbb{Z}_+$  and non-zero  $x, y \in \mathbb{Q}(q)$ , consider a non-zero  $\mathbb{Q}(q)$ -linear map  $R$  on  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  such that

$$\Delta^{\text{op}}(g) \circ R = R \circ \Delta(g), \quad (4.17)$$

for  $g \in \mathcal{U}(\epsilon)$ , where  $\Delta^{\text{op}}$  is the opposite coproduct of  $\Delta$  in (2.8), that is,  $\Delta^{\text{op}}(g) = P \circ \Delta(g) \circ P$  and  $P(a \otimes b) = b \otimes a$ . We denote it by  $R(z)$ , where  $z = x/y$ , since  $R$  depends only on  $z$ .

We say that  $R(z)$  satisfies the Yang-Baxter equation if we have

$$R_{12}(u)R_{13}(uv)R_{23}(v) = R_{23}(v)R_{13}(uv)R_{12}(u), \quad (4.18)$$

on  $\mathcal{W}_{s_1,\epsilon}(x_1) \otimes \mathcal{W}_{s_2,\epsilon}(x_2) \otimes \mathcal{W}_{s_3,\epsilon}(x_3)$  with  $u = x_1/x_2$  and  $v = x_2/x_3$  for  $(s_1), (s_2), (s_3) \in \mathcal{P}_{M|N}$ . Here  $R_{ij}(z)$  denotes the map which acts as  $R(z)$  on the  $i$ -th and the  $j$ -th component and the identity elsewhere. We call  $R(z)$  the  $R$  matrix. If  $M = 0$ , then the existence of  $R$  matrix is already known. Hence we may assume that  $M \geq 1$ .

**Theorem 4.9.** *Let  $l, m \in \mathbb{Z}_+$  given with  $(l), (m) \in \mathcal{P}_{M|N}$ . Suppose that  $M \geq 1$  and let  $i \in \mathbb{I}$  be such that  $\epsilon_i = 0$ . Then there exists a unique non-zero linear map  $R(z) \in \text{End}_{\mathbb{Q}(q)}(\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y))$  satisfying (4.17) and (4.18), and  $R(z)(|l\mathbf{e}_i\rangle \otimes |m\mathbf{e}_i\rangle) = |l\mathbf{e}_i\rangle \otimes |m\mathbf{e}_i\rangle$  for generic  $x, y \in \mathbb{Q}(q)$ .*

**Proof.** The existence of such a map for arbitrary  $\epsilon$  is proved in [18, Theorem 5.1] with respect to  $\Delta_+$  in (2.10), say  $R_+$ . Let

$$\chi = \psi \circ (\phi \otimes \phi), \quad (4.19)$$

where  $\psi$  and  $\phi$  are given in (2.11) and (4.1), respectively. Then

$$R := \chi^{-1} \circ R_+ \circ \chi$$

satisfies the conditions (4.17) and (4.18), and  $R(z)(|l\mathbf{e}_i\rangle \otimes |m\mathbf{e}_i\rangle) = |l\mathbf{e}_i\rangle \otimes |m\mathbf{e}_i\rangle$  with respect to  $\Delta$ . The uniqueness follows from the irreducibility in Corollary 4.8 and normalization by  $R(z)(|l\mathbf{e}_i\rangle \otimes |m\mathbf{e}_i\rangle) = |l\mathbf{e}_i\rangle \otimes |m\mathbf{e}_i\rangle$ .  $\square$

**Remark 4.10.** The  $R$  matrix in Theorem 4.9 does not depend on the choice of  $i$  such that  $\epsilon_i = 0$ .

## 5. Kirillov-Reshetikhin modules

### 5.1. Spectral decomposition

In this section, we assume that  $M \geq 1$ . Let  $l, m \in \mathbb{Z}_+$  be given. Let  $R_\epsilon(z)$  be the  $R$  matrix on  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  in Theorem 4.9. We have as a  $\dot{\mathcal{U}}(\epsilon)$ -module,

$$\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \cong \bigoplus_{t \in H(l,m)} V_\epsilon((l+m-t, t)),$$

where  $H(l, m) = \{t \mid 0 \leq t \leq \min\{l, m\}, (l+m-t, t) \in \mathcal{P}_{M|N}\}$ .

Let us take a sequence  $\epsilon'' = (\epsilon''_1, \dots, \epsilon''_{n''})$  of 0, 1's with  $n'' \gg n$  satisfying the following:

- (1)  $\epsilon$  is a subsequence of  $\epsilon''$ ,
- (2) we have as a  $\dot{\mathcal{U}}(\epsilon'')$ -module

$$\mathcal{W}_{l,\epsilon''}(x) \otimes \mathcal{W}_{m,\epsilon''}(y) \cong \bigoplus_{0 \leq t \leq \min\{l,m\}} V_{\epsilon''}((l+m-t, t)),$$

- (3) if  $\epsilon' = \epsilon_{M''|0}$  with  $M'' = |\{i \mid \epsilon''_i = 0\}| \gg 0$ , then we have as a  $\dot{\mathcal{U}}(\epsilon')$ -module

$$\mathcal{W}_{l,\epsilon'}(x) \otimes \mathcal{W}_{m,\epsilon'}(y) \cong \bigoplus_{0 \leq t \leq \min\{l,m\}} V_{\epsilon'}((l+m-t, t)).$$

Let  $R_{\epsilon''}(z)$  and  $R_{\epsilon'}(z)$  denote the  $R$  matrices on  $\mathcal{W}_{l,\epsilon''}(x) \otimes \mathcal{W}_{m,\epsilon''}(y)$  and  $\mathcal{W}_{l,\epsilon'}(x) \otimes \mathcal{W}_{m,\epsilon'}(y)$  in Theorem 4.9, respectively.

**Lemma 5.1.** *For  $\epsilon = \epsilon$  or  $\epsilon'$ , we have*

$$\text{tr}_\epsilon^{\epsilon''}(PR_{\epsilon''}(z)) = PR_\epsilon(z).$$

**Proof.** For  $\epsilon = \epsilon$  or  $\epsilon'$ , we have a well-defined  $\mathcal{U}(\epsilon)$ -linear map

$$\text{tr}_\epsilon^{\epsilon''}(PR_{\epsilon''}(z)) : \mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \longrightarrow \mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x).$$

By Proposition 4.6 and Theorem 4.9,  $P\text{tr}_\epsilon^{\epsilon''}(PR_{\epsilon''}(z))$  satisfies the conditions in Theorem 4.9 with respect to  $\epsilon$ , which implies that  $P\text{tr}_\epsilon^{\epsilon''}(PR_{\epsilon''}(z)) = R_\epsilon(z)$ .  $\square$

For  $0 \leq t \leq \min\{l, m\}$ , let  $v'(l, m, t)$  be the highest weight vector of  $V_{\epsilon'}((l+m-t, t))$  in  $\mathcal{W}_{l,\epsilon'}(x) \otimes \mathcal{W}_{m,\epsilon'}(y)$  such that

$$\begin{aligned} v'(l, m, t) &\in \mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}, \\ v'(l, m, t) &\equiv |(l-t)\mathbf{e}_1 + t\mathbf{e}_2\rangle \otimes |m\mathbf{e}_1\rangle \pmod{q\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}}. \end{aligned} \tag{5.1}$$

We also define  $v'(m, l, t)$  in the same manner. For  $0 \leq t \leq \min\{l, m\}$ , we may regard

$$V_\epsilon((l+m-t, t)) \subset V_{\epsilon''}((l+m-t, t)), \quad V_{\epsilon'}((l+m-t, t)) \subset V_{\epsilon''}((l+m-t, t))$$

as a  $\mathbb{Q}(q)$ -vector space, and let  $\mathcal{P}_t^{l,m} : \mathcal{W}_{l,\epsilon''}(x) \otimes \mathcal{W}_{m,\epsilon''}(y) \longrightarrow \mathcal{W}_{m,\epsilon''}(y) \otimes \mathcal{W}_{l,\epsilon''}(x)$  be a  $\mathcal{U}(\epsilon'')$ -linear map given by  $\mathcal{P}_t^{l,m}(v'(l, m, t')) = \delta_{tt'} v'(m, l, t')$ . Then we have the following spectral decomposition of  $PR_{\epsilon''}(z)$

$$PR_{\epsilon''}(z) = \sum_{0 \leq t \leq \min\{l, m\}} \rho_t(z) \mathcal{P}_t^{l,m},$$

for some  $\rho_t(z) \in \mathbb{Q}(q)$ . By Proposition 4.5 and Lemma 5.1, we have

$$\begin{aligned} PR_{\epsilon'}(z) &= \sum_{0 \leq t \leq \min\{l, m\}} \rho_t(z) \mathcal{P}_t^{l,m}, \\ PR_\epsilon(z) &= \sum_{t \in H(l, m)} \rho_t(z) \mathcal{P}_t^{l,m}, \end{aligned} \tag{5.2}$$

where we understand  $\mathcal{P}_t^{l,m}$  as defined on  $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$  for  $\epsilon = \epsilon'$  or  $\epsilon$ . Then we have the following, which is proved in case of  $\epsilon = \epsilon_{M|N}$  [18].

**Theorem 5.2.** *We have*

$$\begin{aligned} PR_\epsilon(z) &= \sum_{t=0}^{\min\{l, m, n-1\}} \left( \prod_{i=1}^t \frac{1 - q^{l+m-2i+2}z}{z - q^{l+m-2i+2}} \right) \mathcal{P}_t^{l,m} & (M=1), \\ PR_\epsilon(z) &= \sum_{t=0}^{\min\{l, m\}} \left( \prod_{i=1}^t \frac{1 - q^{l+m-2i+2}z}{z - q^{l+m-2i+2}} \right) \mathcal{P}_t^{l,m} & (2 \leq M \leq n), \end{aligned}$$

where we assume that  $\rho_0(z) = 1$ .

**Proof.** It is well-known that  $PR_{\epsilon'}(z)$  for  $\epsilon' = \epsilon_{M''|0}$  has the following spectral decomposition

$$PR_{\epsilon'}(z) = \sum_{0 \leq t \leq \min\{l, m\}} \rho'_t(z) \mathcal{P}_t^{l,m},$$

where

$$\rho'_0(z) = 1, \quad \rho'_t(z) = \prod_{i=1}^t \frac{1 - q^{l+m-2i+2}z}{z - q^{l+m-2i+2}} \quad (1 \leq t \leq \min\{l, m\}),$$

(see for example, [18, (6.16)]). We remark that  $\chi(v'(l, m, t))$  and  $\chi(v'(m, l, t))$  for  $0 \leq t \leq \min\{l, m\}$  are the same scalar multiplications of the highest weight vectors in [18, (6.14)], where  $\chi$  is as in (4.19). Hence it follows from (5.2) that

$$\rho_t(z) = \rho'_t(z) \quad (t \in H(l, m)),$$

which completes the proof.  $\square$

**Remark 5.3.** If  $l = m = 1$ ,  $PR_\epsilon(z)$  can be expressed in terms of  $\mathcal{R}$  in (3.1). For example, if  $2 \leq M \leq n$  and  $\epsilon_1 = 0$ , we have  $PR_\epsilon(z) = \mathcal{P}_0^{1,1} + (\frac{1-q^2z}{z-q^2})\mathcal{P}_1^{1,1}$ . Note that  $v_0 = |\mathbf{e}_1\rangle \otimes |\mathbf{e}_1\rangle$  and  $v_1 = |\mathbf{e}_2\rangle \otimes |\mathbf{e}_1\rangle - q|\mathbf{e}_1\rangle \otimes |\mathbf{e}_2\rangle$  are the highest weight vectors of  $V_\epsilon((2))$  and  $V_\epsilon((1, 1))$  respectively, satisfying (5.1). Since  $\mathcal{P}_0^{1,1}(v_0) = v_0$ ,  $\mathcal{P}_1^{1,1}(v_1) = v_1$  and  $\mathcal{R}$  is  $\mathcal{U}(\epsilon)$ -linear, we can check by comparing the eigenvalues of both sides that

$$PR_\epsilon(z) = \frac{zq\mathcal{R} - q^{-1}\mathcal{R}^{-1}}{zq^{-1} - q}.$$

## 5.2. Kirillov-Reshetikhin modules

As an application of Theorem 5.2, let us construct a family of irreducible  $\mathcal{U}(\epsilon)$ -modules in  $\mathcal{O}_{\geq 0}$  which corresponds to usual Kirillov-Reshetikhin modules [13] under truncation.

Fix  $s \geq 1$  and put  $V_x = \mathcal{W}_{s,\epsilon}(x)$  for non-zero  $x \in \mathbb{Q}(q)$ . We take a normalization

$$\check{R}(z) = \left( \prod_{i=1}^s \frac{z - q^{2s-2i+2}}{1 - q^{2s-2i+2}z} \right) PR_\epsilon(z),$$

where  $R_\epsilon(z)$  is the  $R$  matrix on  $V_x \otimes V_y$  in Theorem 4.9. Since  $(s^2) \notin \mathcal{P}_{M|N}$  if and only if  $M = 1$  and  $s > n - 1$ , we have

$$\check{R}(z) = \begin{cases} \sum_{t=0}^{n-1} \left( \prod_{i=t+1}^s \frac{z - q^{2s-2i+2}}{1 - q^{2s-2i+2}z} \right) \mathcal{P}_t^{s,s}, & \text{if } (s^2) \notin \mathcal{P}_{M|N}, \\ \mathcal{P}_s^{s,s} + \sum_{t=0}^{s-1} \left( \prod_{i=t+1}^s \frac{z - q^{2s-2i+2}}{1 - q^{2s-2i+2}z} \right) \mathcal{P}_t^{s,s}, & \text{if } (s^2) \in \mathcal{P}_{M|N}. \end{cases}$$

For  $r \geq 2$ , let  $W$  denote the symmetric group on  $\{1, \dots, r\}$  generated by  $s_i = (i \ i+1)$  for  $1 \leq i \leq r - 1$ . By Theorem 4.9, we have  $\mathcal{U}(\epsilon)$ -linear maps

$$\check{R}_w(x_1, \dots, x_r) : V_{x_1} \otimes \cdots \otimes V_{x_r} \longrightarrow V_{x_{w(1)}} \otimes \cdots \otimes V_{x_{w(r)}}, \quad (5.3)$$

for  $w \in W$  and generic  $x_1, \dots, x_r$  satisfying the following:

$$\begin{aligned} \check{R}_1(x_1, \dots, x_r) &= \text{id}_{V_{x_1} \otimes \cdots \otimes V_{x_r}}, \\ \check{R}_{s_i}(x_1, \dots, x_r) &= \left( \otimes_{j < i} \text{id}_{V_{x_j}} \right) \otimes \check{R}(x_i/x_{i+1}) \otimes \left( \otimes_{j > i+1} \text{id}_{V_{x_j}} \right), \\ \check{R}_{ww'}(x_1, \dots, x_r) &= \check{R}_{w'}(x_{w(1)}, \dots, x_{w(r)}) \check{R}_w(x_1, \dots, x_r), \end{aligned}$$

for  $w, w' \in W$  with  $\ell(ww') = \ell(w) + \ell(w')$ . Let  $w_0$  denote the longest element in  $W$ . By Theorem 5.2,  $\check{R}_{w_0}(x_1, \dots, x_r)$  does not have a pole at  $q^{2k}$  for  $k \in \mathbb{Z}_+$  as a function in  $x_1, \dots, x_r$ . Hence we have a  $\mathcal{U}(\epsilon)$ -linear map

$$\check{R}_r := \check{R}_{w_0}(q^{r-1}, q^{r-3}, \dots, q^{1-r}) : V_{q^{r-1}} \otimes \cdots \otimes V_{q^{1-r}} \longrightarrow V_{q^{1-r}} \otimes \cdots \otimes V_{q^{r-1}}.$$

Then we define a  $\mathcal{U}(\epsilon)$ -module

$$\mathcal{W}_{s,\epsilon}^{(r)} := \text{Im} \check{R}_r. \quad (5.4)$$

It is proved in [17] that  $\mathcal{W}_{s,\epsilon}^{(r)}$  is irreducible when  $\epsilon = \epsilon_{M|N}$ , where the proof uses the crystal base of polynomial representation of  $\mathring{\mathcal{U}}(\epsilon_{M|N})$ . Now we give another proof of the irreducibility of  $\mathcal{W}_{s,\epsilon}^{(r)}$ , which is available for arbitrary  $\epsilon$ .

**Theorem 5.4.** *Let  $r, s \geq 1$  be given. Then  $\mathcal{W}_{s,\epsilon}^{(r)}$  is non-zero if and only if  $(s^r) \in \mathcal{P}_{M|N}$ . In this case,  $\mathcal{W}_{s,\epsilon}^{(r)}$  is irreducible, and it is isomorphic to  $V_\epsilon((s^r))$  as a  $\mathring{\mathcal{U}}(\epsilon)$ -module.*

**Proof.** Let us take a sequence  $\epsilon'' = (\epsilon''_1, \dots, \epsilon''_{n''})$  of 0, 1's satisfying the following:

- (1)  $\epsilon$  is a subsequence of  $\epsilon''$ ,
- (2) we have as a  $\mathring{\mathcal{U}}(\epsilon'')$ -module

$$V_{\epsilon''}((s))^{\otimes r} \cong \bigoplus_{\lambda \in \mathcal{P}} V_{\epsilon''}(\lambda)^{\oplus K_{\lambda}(s^r)}, \quad (5.5)$$

where  $K_{\lambda}(s^r)$  is the Kostka number associated to  $\lambda$  and  $(s^r)$  (cf. Remark 3.5),

- (3) if  $\epsilon' = \epsilon_{M''|0}$  with  $M'' = |\{i \mid \epsilon''_i = 0\}|$ , then we have as a  $\mathring{\mathcal{U}}(\epsilon')$ -module

$$V_{\epsilon'}((s))^{\otimes r} \cong \bigoplus_{\lambda \in \mathcal{P}} V_{\epsilon'}(\lambda)^{\oplus K_{\lambda}(s^r)}. \quad (5.6)$$

Let us define a  $\mathcal{U}(\epsilon'')$ -module  $\mathcal{W}_{s,\epsilon''}^{(r)}$  by the same way as in (5.4), where  $\check{R}_r''$  and  $V_x''$  denote the corresponding ones. We define  $\mathcal{W}_{s,\epsilon'}^{(r)}$ ,  $\check{R}_r'$  and  $V_x'$  similarly.

By Lemma 5.1, we have

$$\text{tr}_{\epsilon'}^{\epsilon''}(\check{R}_r'') = \check{R}_r'.$$

Hence by (5.5), (5.6) and Proposition 4.5, the decomposition of  $\mathcal{W}_{s,\epsilon''}^{(r)}$  into polynomial  $\mathring{\mathcal{U}}(\epsilon'')$ -modules is the same as that of  $\mathcal{W}_{s,\epsilon'}^{(r)}$  into polynomial  $\mathring{\mathcal{U}}(\epsilon')$ -modules. It is well-known that  $\mathcal{W}_{s,\epsilon'}^{(r)}$  is irreducible and isomorphic to  $V_{\epsilon'}((s^r))$  as a  $\mathring{\mathcal{U}}(\epsilon')$ -module since  $\mathcal{U}(\epsilon'') \cong U_q(A_{M''-1}^{(1)})$ . Therefore,  $\mathcal{W}_{s,\epsilon''}^{(r)}$  is irreducible and isomorphic to  $V_{\epsilon''}((s^r))$  as a  $\mathring{\mathcal{U}}(\epsilon'')$ -module.

Again by Lemma 5.1, we have

$$\text{tr}_{\epsilon}^{\epsilon''}(\check{R}_r'') = \check{R}_r.$$

Since  $\mathbf{t}\epsilon^{\epsilon''}(V_{\epsilon''}((s^r)))$  is non-zero if and only if  $(s^r) \in \mathcal{P}_{M|N}$ , which is equal to  $V_{\epsilon}((s^r))$  in this case, it follows that  $\mathcal{W}_{s,\epsilon}^{(r)}$  is non-zero if and only if  $(s^r) \in \mathcal{P}_{M|N}$ . This implies in this case that  $\mathcal{W}_{s,\epsilon}^{(r)}$  is irreducible, and it is isomorphic to  $V_{\epsilon}((s^r))$  as a  $\mathcal{U}(\epsilon)$ -module.  $\square$

The following can be proved by similar arguments.

**Corollary 5.5.** *Suppose that  $(s^r) \in \mathcal{P}_{M|N}$  is given.*

- (1) *If  $r \leq M$  and  $M \geq 3$ , then  $\mathbf{t}\epsilon_{\epsilon'}(\mathcal{W}_{s,\epsilon}^{(r)})$  is the Kirillov-Reshetikhin module of type  $A_{M-1}^{(1)}$  corresponding to the partition  $(s^r)$ , where  $\epsilon' = \epsilon_{M|0}$ .*
- (2) *If  $s \leq N$  and  $N \geq 3$ , then  $\mathbf{t}\epsilon_{\epsilon'}(\mathcal{W}_{s,\epsilon}^{(r)})$  is the Kirillov-Reshetikhin module of type  $A_{N-1}^{(1)}$  corresponding to the partition  $(r^s)$ , where  $\epsilon' = \epsilon_{0|N}$ .*

**Remark 5.6.** As in case of  $\epsilon = \epsilon_{M|N}$  [17], we also expect that  $\mathcal{W}_{s,\epsilon}^{(r)}$  has a crystal base for arbitrary  $\epsilon$  (cf. Remark 3.8).

One may use a similar argument as in the proof of Theorem 5.4 to prove the irreducibility of a tensor product of  $\mathcal{W}_{l,\epsilon}(x)$ 's and its quotient in some special cases. Let  $l_1, \dots, l_r \in \mathbb{Z}_+$  and non-zero  $x_1, \dots, x_r \in \mathbb{Q}(q)$  be given and let  $\epsilon' = \epsilon_{M|0}$ . We further assume that  $M \geq 3$  so that  $\mathcal{U}(\epsilon')$  is of type  $A_{\ell}^{(1)}$  with  $\ell \geq 2$ .

**Proposition 5.7.** *If  $M \geq r$  and  $\mathcal{W}_{l_1,\epsilon'}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon'}(x_r)$  is irreducible, then  $\mathcal{W}_{l_1,\epsilon}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon}(x_r)$  is also irreducible.*

**Proof.** Suppose that  $\mathcal{W}_{l_1,\epsilon}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon}(x_r)$  is not irreducible and let  $W$  be a proper non-trivial submodule. By the assumption  $M \geq r$  and the Pieri rule of irreducible polynomial representations (see Remark 3.5), the multiplicity of  $V_{\epsilon}(\lambda)$  for  $\lambda \in \mathcal{P}$  in  $\mathcal{W}_{l_1,\epsilon}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon}(x_r)$  is equal to that of  $V_{\epsilon'}(\lambda)$  for  $\lambda \in \mathcal{P}$  in  $\mathcal{W}_{l_1,\epsilon'}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon'}(x_r)$ . This also holds for  $W$  and  $\mathbf{t}\epsilon_{\epsilon'}(W)$ , which implies that  $\mathbf{t}\epsilon_{\epsilon'}(W)$  is a proper non-zero subspace of  $\mathbf{t}\epsilon_{\epsilon'}(\mathcal{W}_{l_1,\epsilon}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon}(x_r))$ . Since  $\mathbf{t}\epsilon_{\epsilon'}(W) = W \cap \mathbf{t}\epsilon_{\epsilon'}(\mathcal{W}_{l_1,\epsilon}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon}(x_r))$ , it follows that  $\mathbf{t}\epsilon_{\epsilon'}(W)$  is a proper non-zero  $\mathcal{U}(\epsilon')$ -submodule, which is a contradiction.  $\square$

**Remark 5.8.** Proposition 5.7 together with the irreducibility of  $\mathcal{W}_{l,\epsilon'} \otimes \mathcal{W}_{m,\epsilon'}$  also implies Theorem 4.7 when  $M \geq 3$ . But we do not know whether it holds for  $M = 2$ . We also would like to point out that the proof of Theorem 4.7 has its own interest since it describes a new connected crystal graph structure on  $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon}/\{\pm 1\}$ .

**Proposition 5.9.** *Suppose that  $x_i/x_j \notin q^{\mathbb{Z}_+}$  for  $1 \leq i < j \leq r$ . If  $M \geq r$  and the image of*

$$R'_{w_0}(x_1, \dots, x_r) : \mathcal{W}_{l_1,\epsilon'}(x_1) \otimes \dots \otimes \mathcal{W}_{l_r,\epsilon'}(x_r) \longrightarrow \mathcal{W}_{l_r,\epsilon'}(x_r) \otimes \dots \otimes \mathcal{W}_{l_1,\epsilon'}(x_1)$$

is irreducible, then the image of

$$R_{w_0}(x_1, \dots, x_r) : \mathcal{W}_{l_1, \epsilon}(x_1) \otimes \cdots \otimes \mathcal{W}_{l_r, \epsilon}(x_r) \longrightarrow \mathcal{W}_{l_r, \epsilon}(x_r) \otimes \cdots \otimes \mathcal{W}_{l_1, \epsilon}(x_1)$$

is irreducible, where  $R_{w_0}(x_1, \dots, x_r)$  is defined as in (5.3) using  $PR_\epsilon(z)$  in Theorem 5.2, and  $R'_{w_0}(x_1, \dots, x_r) = \mathbf{tr}_{\epsilon'}(R_{w_0}(x_1, \dots, x_r))$ .

**Proof.** It follows from Lemma 5.1, Theorem 5.2, and the same argument as in Proposition 5.7.  $\square$

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