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# From endomorphisms to bi-skew braces, regular subgroups, the Yang–Baxter equation, and Hopf–Galois structures

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## ABSTRACT

The interplay between set-theoretic solutions of the Yang–Baxter equation of Mathematical Physics, skew braces, regular subgroups, and Hopf–Galois structures has spawned a considerable body of literature in recent years.

In a recent paper, Alan Koch generalised a construction of Lindsay N. Childs, showing how one can obtain bi-skew braces  $(G, \cdot, \circ)$  from an endomorphism of a group  $(G, \cdot)$  whose image is abelian.

In this paper, we characterise the endomorphisms of a group  $(G, \cdot)$  for which Koch's construction, and a variation on it, yield (bi-)skew braces. We show how the set-theoretic solutions of the Yang–Baxter equation derived by Koch's

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Yang–Baxter equation  
Hopf–Galois structures

construction carry over to our more general situation, and  
discuss the related Hopf–Galois structures.

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## 1. Introduction

The interplay between set-theoretic solutions of the Yang–Baxter equation of Mathematical Physics, (skew) braces, regular subgroups, and Hopf–Galois structures has spawned a considerable body of literature in recent years.

Childs gave in [5] the following construction. Let  $G = (G, \cdot)$  be a group, and let  $\text{Perm}(G)$  be the group of all permutations on the underlying set  $G$ . Let  $\varphi$  be an endomorphism of  $G$  which is fixed point free (meaning that  $\varphi g = g$  only for  $g = 1$ ; here where  $\varphi g$  is our notation for the value of an endomorphism  $\varphi$  of the group  $G$  on an element  $g \in G$ ), and abelian (meaning that the image of  $\varphi$  is an abelian subgroup of  $G$ , or equivalently  $\varphi[G, G] = 1$ ). Consider the subset

$$N = \{h \mapsto g \cdot h \cdot (\varphi g)^{-1} : g \in G\}$$

of  $\text{Perm}(G)$ . Then  $N$  is a regular subgroup of  $\text{Perm}(G)$  which is normalised by the image  $\lambda(G)$  of the left regular representation of  $G$ . To such an  $N$  is associated a certain group operation  $\circ$  on  $G$ , such that  $N$  is isomorphic to  $(G, \circ)$ , and  $(G, \circ, \cdot)$  is a skew left brace.

In the recent manuscript [12] Koch generalised this construction. Let  $\psi$  be an abelian endomorphism of the group  $G$ . Then the set

$$\{h \mapsto g \cdot (\psi g)^{-1} \cdot h \cdot \psi g : g \in G\} \quad (1.1)$$

is a regular subgroup of  $\text{Perm}(G)$  which both normalises, and is normalised by,  $\lambda(G)$ . The associated skew brace  $(G, \circ, \cdot)$  is thus a bi-skew brace ([6,3]) in the sense that  $(G, \cdot, \circ)$  is also a skew brace. In [12] Koch showed that his construction encompasses that of Childs.

The goal of this paper is to further extend Koch's construction to the following situations. Given an endomorphism  $\psi$  of  $G$  and  $\varepsilon = \pm 1$ , we consider the subset

$$N = \{h \mapsto g \cdot (\psi g)^\varepsilon \cdot h \cdot (\psi g)^{-\varepsilon} : g \in G\} \quad (1.2)$$

of  $\text{Perm}(G)$ . It is immediate to see that  $N$  is a regular subset of  $\text{Perm}(G)$  which normalises  $\lambda(G)$ . We determine the conditions on  $\psi$  so that  $N$  is a subgroup of  $\text{Perm}(G)$ , and then is normalised by  $\lambda(G)$ .

Our first main result is concerned with the case when  $\varepsilon = -1$  in (1.2), which is a direct extension of the case considered by Koch. In the statement, given an endomorphism  $\psi$  of  $G$  and an element  $g \in G$ , we use the notations  $[g, \psi] = g \cdot (\psi g)^{-1}$ , and  $[G, \psi] = \langle [g, \psi] : g \in G \rangle$ , the reason being that if  $\psi$  is an automorphism of  $G$ , then  $[g, \psi]$  is indeed the commutator  $g\psi g^{-1}\psi^{-1}$  in the abstract holomorph of  $G$ .

**Theorem 1.1.** *Let  $G = (G, \cdot)$  be a group, and let  $\psi$  be an endomorphism of  $G$ . Consider the subset*

$$N = \{h \mapsto g \cdot (\psi g)^{-1} \cdot h \cdot \psi g : g \in G\}$$

*of  $\text{Perm}(G)$ . The following are equivalent:*

- (a)  *$N$  is a subgroup of  $\text{Perm}(G)$ ;*
- (b)  *$N$  is normalised by  $\lambda(G)$ ;*
- (c)  *$N$  is a regular subgroup of  $\text{Perm}(G)$  which normalises, and is normalised by,  $\lambda(G)$ ;*
- (d)  *${}^\psi[[G, \psi], G] \leq Z(G, \cdot)$ ;*
- (e)  *$(G, \cdot, \circ)$  is a bi-skew brace, for  $g \circ h = g \cdot (\psi g)^{-1} \cdot h \cdot \psi g$ .*

In the situation considered by Koch we have  ${}^\psi[G, G] = 1$ , so the condition in Theorem 1.1 is clearly satisfied; we show with an example that this condition is distinct from Koch's condition  ${}^\psi[G, G] = 1$ .

We then consider the situation when  $\varepsilon = 1$  in (1.2).

**Theorem 1.2.** *Let  $G = (G, \cdot)$  be a group, and let  $\psi$  be an endomorphism of  $G$ . Consider the subset*

$$N = \{h \mapsto g \cdot \psi g \cdot h \cdot (\psi g)^{-1} : g \in G\}$$

*of  $\text{Perm}(G)$ .*

1. *The following are equivalent:*

- (a)  *$N$  is a subgroup of  $\text{Perm}(G)$ ;*
- (b)  *$N$  is a regular subgroup of  $\text{Perm}(G)$  which normalises  $\lambda(G)$ ;*
- (c)  *${}^\psi[\psi G, G] \leq Z(G)$ ;*
- (d)  *$(G, \cdot, \circ)$  is a skew brace, for  $g \circ h = g \cdot \psi g \cdot h \cdot (\psi g)^{-1}$ .*

2. *The following are equivalent:*

- (a)  *$N$  is a regular subgroup of  $\text{Perm}(G)$  which normalises, and is normalised by,  $\lambda(G)$ ;*
- (b)  *${}^\psi[G, G] \leq Z(G)$ ;*
- (c)  *$(G, \cdot, \circ)$  is a bi-skew brace, for  $g \circ h = g \cdot \psi g \cdot h \cdot (\psi g)^{-1}$ .*

We show with examples that conditions (1c) and (2b) of Theorem 1.2 are distinct, and distinct from Koch's condition  ${}^\psi[G, G] = 1$ .

In [12] Koch also applied his construction to exhibit solutions of the Yang–Baxter equation. We do the same in the context of our Theorems 1.1 and 1.2.

In [12] Koch derived results about the Hopf–Galois extensions related to his construction. We are able to obtain similar results in our setting, studying Hopf algebras associated to regular subgroups of  $\text{Perm}(G)$  which normalise  $\lambda(G)$ .

Section 2 deals with some preliminaries. Since the subject of skew braces is fairly recent, some basic results are scattered through the literature; we sum them up here for the convenience of the reader.

Section 3 contains the proofs of Theorems 1.1 and 1.2, plus some supplementary material on the structure of the involved regular subgroups. Section 4 contains the examples mentioned above.

Section 5 describes the set-theoretic solutions of the Yang–Baxter equation associated to the skew braces of Theorems 1.1 and 1.2.

In Section 6 we introduce the Hopf–Galois structures associated to the regular subgroups of Theorems 1.1 and 1.2, which we investigate in Section 7 by considering certain subgroups of the relevant regular subgroups, which correspond to sub-Hopf algebras.

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## 2. Preliminaries

In what follows, if  $(G, \cdot)$  is a group, we write  $\text{End}(G, \cdot)$  for the monoid of endomorphisms of  $(G, \cdot)$ . We denote the action of  $\psi \in \text{End}(G, \cdot)$  on  $g \in G$  by a left exponent  ${}^\psi g$ , and thus compose such endomorphisms right-to-left. If  $\varepsilon$  is an integer, we have clearly  ${}^\psi(g^\varepsilon) = ({}^\psi g)^\varepsilon$ ; we thus write simply  ${}^\psi g^\varepsilon$  for such an element. Similarly, if  $\eta$  an element of  $\text{Perm}(G)$ , the group of permutations on the underlying set  $G$ , we write  ${}^\eta g$  for the image of  $g$  under  $\eta$ .

Let  $(G, \cdot)$  be a group. Denote by  $\lambda$  the *left regular representation*:

$$\begin{aligned}\lambda: (G, \cdot) &\rightarrow \text{Perm}(G) \\ g &\mapsto (h \mapsto g \cdot h)\end{aligned}$$

**Definition 2.1.** The (*permutational*) *holomorph* of  $(G, \cdot)$  is defined as the normaliser of the image  $\lambda(G)$  of  $G$  in  $\text{Perm}(G)$ :

$$\text{Hol}(G, \cdot) = N_{\text{Perm}(G)}(\lambda(G)).$$

It is immediate to see that

$$\text{Hol}(G, \cdot) = \lambda(G) \text{Aut}(G, \cdot),$$

that is, the holomorph is the (inner) semidirect product of  $\lambda(G)$  by the group of automorphisms  $\text{Aut}(G, \cdot)$ , the latter being a bona fide subgroup of  $\text{Perm}(G)$ . The group  $\text{Hol}(G, \cdot)$  is therefore isomorphic to the *abstract holomorph* of  $(G, \cdot)$ , which is the natural (outer) semidirect product of  $(G, \cdot)$  by  $\text{Aut}(G, \cdot)$ .

Write 1 for the identity of  $(G, \cdot)$ .

**Definition 2.2.** A subset  $N$  of  $\text{Perm}(G)$  is *regular* if the map

$$\begin{aligned} N &\rightarrow G \\ \eta &\mapsto \eta 1 \end{aligned}$$

is bijective. We denote by  $\nu$  the inverse of this map, so that  $N = \{\nu(g) : g \in G\}$ , where  $\nu(g)$  is the unique element of  $N$  such that  $\nu(g)1 = g$ .

**Example 2.3.** Let  $\gamma : G \rightarrow \text{Aut}(G, \cdot)$  be a function. Then

$$N = \{\lambda(g)\gamma(g) : g \in G\}$$

is a regular subset of  $\text{Hol}(G, \cdot)$ , since for every  $g \in G$ ,  $\lambda(g)\gamma(g)$  is the unique element taking 1 to  $g$ .

Regular subgroups are strictly connected with skew braces.

**Definition 2.4.** A *skew (left) brace* is a triple  $(G, \cdot, \circ)$ , where  $(G, \cdot)$  and  $(G, \circ)$  are groups, and for every  $g, h, k \in G$ ,

$$g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k).$$

(Here  $g^{-1}$  denotes the inverse of  $g$  with respect to  $\cdot$ .)

It is immediate to see that  $(G, \cdot)$  and  $(G, \circ)$  share the same identity 1 ([10, Lemma 1.7]).

**Definition 2.5.** A *bi-skew brace* is a triple  $(G, \cdot, \circ)$ , where both  $(G, \cdot, \circ)$  and  $(G, \circ, \cdot)$  are skew braces.

We summarise here the main results relating skew braces and regular subgroups. The first one combines [10, Theorem 4.2] and [10, Proposition 4.3]. The second one can be obtained as the first one, reversing the role of the two operations (see also the proof of [16, Proposition 2.1]). Finally, the third one is part of [3, Theorem 3.1], translated from right to left.

**Theorem 2.6.** Let  $(G, \cdot)$  be a group. The following data are equivalent:

1. an operation  $\circ$  on  $G$  such that  $(G, \cdot, \circ)$  is a skew brace;
2. a regular subgroup  $N = \{\nu(g) : g \in G\} \leq \text{Perm}(G)$  which normalises  $\lambda(G)$ .

**Proposition 2.7.** *Let  $(G, \cdot)$  be a group. The following data are equivalent:*

1. *an operation  $\circ$  such that  $(G, \circ, \cdot)$  is a skew brace;*
2. *a regular subgroup  $N = \{\nu(g) : g \in G\} \leq \text{Perm}(G)$  which is normalised by  $\lambda(G)$ .*

**Theorem 2.8.** *Let  $(G, \cdot)$  be a group. The following data are equivalent:*

1. *an operation  $\circ$  on  $G$  such that  $(G, \cdot, \circ)$  is a bi-skew brace;*
2. *a regular subgroup  $N = \{\nu(g) : g \in G\} \leq \text{Perm}(G)$  which normalises, and is normalised by,  $\lambda(G)$ .*

**Remark 2.9.** In all the previous results, the correspondence is given by

$$g \circ h = \nu(g)h.$$

In particular, the map

$$\nu : (G, \circ) \rightarrow N$$

is an isomorphism.

### 3. Proofs of Theorems 1.1 and 1.2

Let  $N$  be a subset of  $\text{Perm}(G)$  as in (1.2):

$$N = \{h \mapsto g \cdot {}^\psi g^\varepsilon \cdot h \cdot {}^\psi g^{-\varepsilon} : g \in G\}.$$

Note that every element of  $N$  can be written as  $\lambda(g)\iota({}^\psi g^\varepsilon)$  for some  $g \in G$ , where

$$\begin{aligned} \iota : (G, \cdot) &\rightarrow \text{Aut}(G, \cdot) \\ x &\mapsto (y \mapsto x \cdot y \cdot x^{-1}) \end{aligned}$$

is the homomorphism mapping  $x \in G$  to the conjugation by  $x$ . In particular, as in Example 2.3,  $N$  is a regular subset of  $\text{Perm}(G)$  which normalises  $\lambda(G)$ , and we may write  $N = \{\nu(g) : g \in G\}$ , with

$$\nu(g) = \lambda(g)\iota({}^\psi g^\varepsilon).$$

#### 3.1. Proof of Theorem 1.1

Let  $N = \{\nu(g) : g \in G\}$  with

$$\nu(g) = \lambda(g)\iota({}^\psi g^{-1}).$$

Then  $N$  is a regular subset of  $\text{Hol}(G, \cdot)$ . Let  $g, h \in G$ . Since

$$\nu(h)^{-1} = (\lambda(h)\iota(\psi h^{-1}))^{-1} = \lambda(\psi h \cdot h^{-1} \cdot \psi h^{-1})\iota(\psi h)$$

and

$$\begin{aligned}\nu(g)\nu(h)^{-1} &= \lambda(g)\iota(\psi g^{-1})\lambda(\psi h \cdot h^{-1} \cdot \psi h^{-1})\iota(\psi h) \\ &= \lambda(g \cdot \psi g^{-1} \cdot \psi h \cdot h^{-1} \cdot \psi h^{-1} \cdot \psi g)\iota(\psi(g^{-1} \cdot h))\end{aligned}$$

we find that  $\nu(g)\nu(h)^{-1} \in N$  if and only if

$$\iota(\psi(g \cdot \psi g^{-1} \cdot \psi h \cdot h^{-1} \cdot \psi h^{-1} \cdot \psi g)^{-1}) = \iota(\psi(g^{-1} \cdot h)),$$

that is,

$$\psi[[h^{-1} \cdot g, \psi], h] = \psi(h^{-1} \cdot g \cdot \psi(g^{-1} \cdot h) \cdot h \cdot \psi(h^{-1} \cdot g) \cdot g^{-1} \cdot h \cdot h^{-1}) \in Z(G, \cdot).$$

This shows that (a) and (d) are equivalent.

Since

$$\lambda(h)\nu(g)\lambda(h)^{-1} = \lambda(h)\lambda(g)\iota(\psi g^{-1})\lambda(h^{-1}) = \lambda(h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g)\iota(\psi g^{-1}),$$

we find that  $\lambda(h)\nu(g)\lambda(h)^{-1} \in N$  if and only if

$$\iota(\psi(h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g)^{-1}) = \iota(\psi g^{-1}),$$

that is,

$$\psi[[g, \psi], h] = \psi(g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1} \cdot h^{-1}) \in Z(G, \cdot).$$

This shows that (b) and (d) are equivalent, and so also equivalent to (c).

Finally, (c) and (e) are equivalent by Theorem 2.8.

### 3.2. Proof of Theorem 1.2

Let  $N = \{\nu(g) : g \in G\}$  with

$$\nu(g) = \lambda(g)\iota(\psi g).$$

Then  $N$  is a regular subset of  $\text{Hol}(G, \cdot)$ . Let  $g, h \in G$ . Since

$$\nu(h)^{-1} = (\lambda(h)\iota(\psi h))^{-1} = \lambda(\psi h^{-1} \cdot h^{-1} \cdot \psi h)\iota(\psi h^{-1})$$

and

$$\begin{aligned}\nu(g)\nu(h)^{-1} &= \lambda(g)\iota(\psi g)\lambda(\psi h^{-1} \cdot h^{-1} \cdot \psi h)\iota(\psi h^{-1}) \\ &= \lambda(g \cdot \psi g \cdot \psi h^{-1} \cdot h^{-1} \cdot \psi h \cdot \psi g^{-1})\iota(\psi(g \cdot h^{-1}))\end{aligned}$$

we find that  $\nu(g)\nu(h)^{-1} \in N$  if and only if

$$\iota(\psi(g \cdot \psi g^{-1} \cdot \psi h^{-1} \cdot h^{-1} \cdot \psi h \cdot \psi g)) = \iota(\psi(g \cdot h^{-1})),$$

that is,

$$\psi[\psi(g^{-1} \cdot h^{-1}), h^{-1}] \in Z(G, \cdot).$$

This shows that (1a) and (1c) are equivalent, and so also equivalent to (1b). Moreover, (1b) and (1d) are equivalent by Theorem 2.6.

Since

$$\lambda(h)\nu(g)\lambda(h)^{-1} = \lambda(h)\lambda(g)\iota(\psi g)\lambda(h^{-1}) = \lambda(h \cdot g \cdot \psi g \cdot h^{-1} \cdot \psi g^{-1})\iota(\psi g),$$

we find that  $\lambda(h)\nu(g)\lambda(h)^{-1} \in N$  if and only if

$$\iota(\psi(h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g)) = \iota(\psi g),$$

that is,

$$\psi(h \cdot g \cdot [\psi g^{-1}, h^{-1}] \cdot h^{-1} \cdot g^{-1}) \in Z(G, \cdot).$$

If  $N$  is a subgroup, then  $\psi[\psi g^{-1}, h^{-1}] \in Z(G, \cdot)$ , and so we find that  $N$  is normalised by  $\lambda(G)$  if and only if condition (2b) holds. As condition (2b) is stronger than condition (1c), we conclude that (2a) and (2b) are equivalent.

Finally, (2a) and (2c) are equivalent by Theorem 2.8.

### 3.3. Endomorphisms yielding the same subgroup

In [12, Proposition 3.3] Koch characterised the pairs  $\psi_1, \psi_2 \in \text{End}(G, \cdot)$  of abelian endomorphisms which give rise to the same regular subgroup of  $\text{Perm}(G)$ . The very same proof applies also in our setting.

**Proposition 3.1.** *Let  $(G, \cdot)$  be a group, and let  $\psi_1, \psi_2 \in \text{End}(G, \cdot)$ . Suppose that  $\psi_1, \psi_2$  satisfy the condition of Theorem 1.1 or one of the conditions of Theorem 1.2. Then the regular subgroups  $N_1, N_2$  of  $\text{Perm}(G)$  associated respectively to  $\psi_1$  and  $\psi_2$  coincide if and only if, for every  $g \in G$ ,*

$$\psi_1 g \cdot \psi_2 g^{-1} \in Z(G, \cdot).$$



### 3.4. A semidirect decomposition of $N$

Proposition 3.3 below is an immediate consequence of Fitting's Lemma for groups, as stated in [1, Proof of Theorem 4.1] and [2, Theorem 4.2], and of the next lemma (see [13, Section 3]).

**Lemma 3.2.** *In the situation of Theorems 1.1 and 1.2, the endomorphism  $\psi$  of  $(G, \cdot)$  is also an endomorphism of  $(G, \circ)$ .*

**Proof.** For every  $g, h \in G$ , we have

$$\psi g \circ \psi h = \psi g \cdot \psi(\psi g)^\varepsilon \cdot \psi h \cdot \psi(\psi g)^{-\varepsilon} = \psi(g \cdot \psi g^\varepsilon \cdot h \cdot \psi g^{-\varepsilon}) = \psi(g \circ h). \quad \square$$

Fitting's Lemma for groups now yields the following result.

**Proposition 3.3.** *Let  $(G, \cdot)$  be a group satisfying the descending condition on subgroups, and the ascending condition on normal subgroups. (In particular, a finite group will do). Let  $N = \{\nu(g) : g \in G\}$  be one of the regular subgroups arising from Theorems 1.1 or 1.2, and let  $(G, \cdot, \circ)$  be the corresponding (bi-)skew brace. Then there is a natural number  $n$  such that the following hold:*

1.  $J = \ker(\psi^n)$  is a normal subgroup of both  $(G, \cdot)$  and  $(G, \circ)$ ;
2.  $I = \psi^n G$  is a subgroup of both  $(G, \cdot)$  and  $(G, \circ)$ ;
3.  $\psi$  restricts to a nilpotent endomorphism on  $J$ ;
4.  $\psi$  restricts to an automorphism on  $I$ ;
5. both  $(G, \cdot)$  and  $(G, \circ)$  are semidirect products of  $J$  by  $I$ .

Applying  $\nu$  to  $(G, \circ)$ , we obtain that

6.  $N$  is a semidirect product of the normal subgroup  $\nu(J)$  by  $\nu(I)$ .

## 4. Examples

### 4.1. Examples for Theorem 1.1

**Example 4.1.** If  $G$  is any group and  $\psi$  is the identity on  $G$ , then condition (d) is clearly satisfied, as  $[G, \psi] = 1$ . In this case, for every  $g, h \in G$ ,

$$g \circ h = g \cdot g^{-1} \cdot h \cdot g = h \cdot g,$$

therefore the associated skew brace is the *almost trivial* skew brace.

**Example 4.2.** Let  $S$  be a group of nilpotence class two, and let  $G = S \times S$ . Let  $\psi : G \rightarrow G$  be the projection on the second factor:

$$\psi(a, b) = (1, b).$$

The endomorphism  $\psi$  is clearly not abelian, as its image is isomorphic to  $S$ . For  $g = (a, b) \in G$  we have

$$[g, \psi] = (a, b) \cdot (1, b^{-1}) = (a, 1),$$

so that

$$\psi[[G, \psi], G] = 1 \leq Z(G).$$

Thus  $\psi$  satisfies condition (d) of Theorem 1.1, but not Koch's condition  $\psi[G, G] = 1$ .

We now exhibit an example of a group  $G$  with an endomorphism  $\psi$  such that  $1 \neq \psi[[G, \psi], G] \leq Z(G)$ , so that in particular  $\psi[G, G] \neq 1$ .

**Example 4.3.** Let  $S$  be a group of nilpotence class 3, and let  $T = S/[[S, S], S]$ , so that  $T$  has nilpotence class 2. Write  $\pi : S \rightarrow T$  for the natural epimorphism, and consider  $G = S \times S \times T$ . Define  $\psi \in \text{End}(G)$  by

$$\psi(a, b, c) = (a, a, \pi(b)).$$

Again,  $\psi$  is non-abelian, as  $G$  projects onto  $S$ . For  $g = (a, b, c) \in G$  we have

$$[g, \psi] = (a, b, c) \cdot (a^{-1}, a^{-1}, \pi(b^{-1})) = (1, b \cdot a^{-1}, c \cdot \pi(b^{-1})),$$

so that

$$[G, \psi] = 1 \times S \times T.$$

It follows that

$$\begin{aligned} \psi[[G, \psi], G] &= \psi(1 \times [S, S] \times [T, T]) \\ &= 1 \times 1 \times \pi([S, S]) \\ &= 1 \times 1 \times [T, T] \end{aligned}$$

is a non-trivial subgroup of  $Z(G)$ .

#### 4.2. Examples for Theorem 1.2

**Example 4.4.** If  $G$  is any group and  $\psi$  is the identity on  $G$ , then both conditions (1c) and (2b) become  $[G, G] \leq Z(G)$ , that is, they hold when  $G$  has nilpotence class at most two. Here we have, for every  $g, h \in G$ ,

$$g \circ h = g \cdot g \cdot h \cdot g^{-1} = g \cdot [g, h] \cdot h \cdot g \cdot g^{-1} = g \cdot [g, h] \cdot h = g \cdot h \cdot [g, h].$$

**Example 4.5.** If  $G$  is a group of nilpotence class two, then any endomorphism  $\psi$  of  $G$  satisfies condition (2b) of Theorem 1.2. If  $\psi$  is an automorphism, then it does not satisfy Koch's condition  $\psi[G, G] = 1$ .

**Example 4.6.** Let  $S$  be a group of nilpotence class 3. Let  $T = S/[S, S]$ , so that  $T$  has nilpotence class 2, and let  $\pi : S \twoheadrightarrow T$  be the natural projection. Let  $G = S \times T$ , and define  $\psi \in \text{End}(G)$  by

$$\psi(x, y) = (1, \pi(x)).$$

Then

$$\psi[G, G] = \psi([S, S] \times [T, T]) = 1 \times [T, T],$$

is a non-trivial subgroup of  $Z(G)$ . Thus  $G$  satisfies condition (2b) of Theorem 1.2, but not Koch's condition  $\psi[G, G] = 1$ .

**Example 4.7.** Let  $S$  be a group of nilpotence class 3, and  $T = S/[S, S]$ ,  $U = S/[S, S]$ , so that  $T$ , respectively  $U$  have nilpotence class 2, respectively 1. Let  $\pi : S \twoheadrightarrow T$  and  $\sigma : T \twoheadrightarrow U$  be the natural projections. Take  $G = S \times T \times U$ , and define  $\psi \in \text{End}(G)$  by

$$\psi(a, b, c) = (1, \pi(a), \sigma(b)).$$

Now

$$\psi[G, G] = \psi([S, S] \times [T, T] \times [U, U]) = 1 \times [T, T] \times [U, U] = 1 \times [T, T] \times 1$$

is contained in  $Z(G)$  and non-trivial, as  $T$  has nilpotence class 2. In particular,  $\psi$  is not abelian. However we have

$$\psi G = 1 \times T \times U,$$

so that

$$\psi[\psi G, G] = \psi([1, S] \times [T, T] \times [U, U]) = 1 \times 1 \times [U, U] = 1.$$

We now construct an example where condition (1c) of Theorem 1.2 holds, but not condition (2b).

**Example 4.8.** Let  $S$  be a group of nilpotence class 4, let  $T = S/[[[S, S], S], S]$ , and  $U = S/[[S, S], S]$ , so that  $T$ , respectively  $U$  have nilpotence class 3, respectively 2. Let  $\pi : S \twoheadrightarrow T$  and  $\sigma : T \twoheadrightarrow U$  be the natural projections. Take  $G = S \times T \times U$ , and define  $\psi \in \text{End}(G)$  by

$$\psi(a, b, c) = (1, \pi(a), \sigma(b)).$$

Now

$$\psi[G, G] = \psi([S, S] \times [T, T] \times [U, U]) = 1 \times [T, T] \times [U, U]$$

is not contained in  $Z(G)$ , as  $T$  has nilpotence class 3. However we have

$$\psi G = 1 \times T \times U,$$

so that

$$\psi[\psi G, G] = \psi([1, S] \times [T, T] \times [U, U]) = 1 \times 1 \times [U, U] \leq Z(G).$$

## 5. Set-theoretic solutions of the Yang–Baxter equation

We recall that a *set-theoretic solution of the Yang–Baxter equation*, defined in [8], is a couple  $(X, r)$ , where  $X \neq \emptyset$  is a set and

$$\begin{aligned} r : X \times X &\rightarrow X \times X \\ (x, y) &\mapsto (\sigma_x(y), \tau_y(x)) \end{aligned}$$

is a bijective map satisfying

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

We say that  $(X, r)$  is

- *non-degenerate* if, for every  $x \in X$ ,  $\sigma_x$  and  $\tau_x$  are bijective;
- *involution* if  $r^2 = \text{id}_{X \times X}$ .

In the sequel, we say that  $(X, r)$  is a solution if  $(X, r)$  is a set-theoretic non-degenerate solution of the Yang–Baxter equation.

In [17] Rump found a connection between (left) braces  $(G, \cdot, \circ)$  and involutive solutions. (Recall that a brace can be defined a posteriori as a skew brace where  $(G, \cdot)$  is

abelian.) This was generalised in [10, Theorem 3.1] as follows. (Recall that for  $g \in G$ , the inverse with respect to  $\circ$  is denoted by  $g^{-1}$ , while we write  $\overline{g}$  for the inverse with respect to  $\circ$ .)

**Theorem 5.1.** *Let  $(G, \cdot, \circ)$  be a skew brace. Then  $(G, r)$  is a solution, where*

$$\begin{aligned} r: G \times G &\rightarrow G \times G \\ (g, h) &\mapsto (g^{-1} \cdot (g \circ h), \overline{g^{-1} \cdot (g \circ h)} \circ g \circ h). \end{aligned}$$

*The solution  $(G, r)$  is involutive if and only if  $(G, \cdot)$  is abelian.*

If  $\mathcal{G} = (G, \cdot, \circ)$  is a skew brace, then its *opposite skew brace* (see [18,14]) is  $\mathcal{G}' = (G, \cdot', \circ)$ , where

$$g \cdot' h = h \cdot g.$$

It follows that we get two solutions from a single skew brace  $\mathcal{G} = (G, \cdot, \circ)$ , namely  $(\mathcal{G}, r_{\mathcal{G}})$  and  $(\mathcal{G}', r_{\mathcal{G}'})$ . These solutions are nicely related:  $r_{\mathcal{G}'}$  is a two-sided inverse of  $r_{\mathcal{G}}$ , and  $r_{\mathcal{G}'} = r_{\mathcal{G}}$  if and only if  $(G, \cdot)$  is abelian (see, for instance, [14, Theorem 4.1]).

In particular, if  $\mathcal{G} = (G, \cdot, \circ)$  is a bi-skew brace, then also  $\mathcal{G}_1 = (G, \circ, \cdot)$  is a skew brace, and we get four solutions.

We may summarise all of this as follows.

**Proposition 5.2.** *Let  $(G, \cdot, \circ)$  be a skew brace. Then we get two solutions:*

$$\begin{aligned} r_{\mathcal{G}}(g, h) &= (g^{-1} \cdot (g \circ h), \overline{g^{-1} \cdot (g \circ h)} \circ g \circ h); \\ r_{\mathcal{G}'}(g, h) &= ((g \circ h) \cdot g^{-1}, \overline{(g \circ h) \cdot g^{-1}} \circ g \circ h). \end{aligned}$$

*These solutions are one the inverse of the other and coincide if and only if  $(G, \cdot)$  is abelian.*

*If in addition  $(G, \cdot, \circ)$  is a bi-skew brace, then we get other two solutions:*

$$\begin{aligned} r_{\mathcal{G}_1}(g, h) &= (\overline{g} \circ (g \cdot h), (\overline{g} \circ (g \cdot h))^{-1} \cdot g \cdot h); \\ r_{\mathcal{G}'_1}(g, h) &= ((g \cdot h) \circ \overline{g}, ((g \cdot h) \circ \overline{g})^{-1} \cdot g \cdot h) \end{aligned}$$

*These solutions are one the inverse of the other and coincide if and only if  $(G, \circ)$  is abelian.*

### 5.1. The case $\varepsilon = -1$

This part extends Koch's work; our solutions coincide with his in the particular case when the endomorphism  $\psi$  is abelian.

**Theorem 5.3.** *Let  $(G, \cdot)$  be a group, and let  $\psi \in \text{End}(G, \cdot)$ . If  $\psi[[G, \psi], G] \leq Z(G, \cdot)$ , then we get four solutions:*

$$\begin{aligned} r_{\mathcal{G}}(g, h) &= (\psi g^{-1} \cdot h \cdot \psi g, \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi g \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi(h^{-1} \cdot g)); \\ r_{\mathcal{G}'}(g, h) &= (g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1}, \psi h \cdot g \cdot \psi h^{-1}); \\ r_{\mathcal{G}_1}(g, h) &= (\psi g \cdot h \cdot \psi g^{-1}, \psi g \cdot h^{-1} \cdot \psi g^{-1} \cdot g \cdot h); \\ r_{\mathcal{G}'_1}(g, h) &= (g \cdot h \cdot \psi h^{-1} \cdot g^{-1} \cdot \psi h, \psi h^{-1} \cdot g \cdot \psi h). \end{aligned}$$

The solutions  $(G, r_{\mathcal{G}})$  and  $(G, r_{\mathcal{G}'})$  are one the inverse of the other and coincide if and only if  $(G, \cdot)$  is abelian.

The solutions  $(G, r_{\mathcal{G}_1})$  and  $(G, r_{\mathcal{G}'_1})$  are one the inverse of the other and coincide if and only if, for every  $g, h \in G$ ,

$$g \cdot \psi g^{-1} \cdot h \cdot \psi g = h \cdot \psi h^{-1} \cdot g \cdot \psi h.$$

**Proof.** By Theorem 1.1,  $(G, \cdot, \circ)$  is a bi-skew brace, where for every  $g, h \in G$ ,

$$g \circ h = g \cdot \psi g^{-1} \cdot h \cdot \psi g.$$

We just need to apply Proposition 5.2 in this setting. The key observation is the following: if  $g, h \in G$ , then

$$\begin{aligned} \psi(\psi g \cdot h \cdot \psi g^{-1}) &= \psi g \cdot \psi[g^{-1} \cdot \psi g, h] \cdot \psi(h \cdot g^{-1}) \\ &= \psi g \cdot \psi[[g^{-1}, \psi], h] \cdot \psi(h \cdot g^{-1}), \end{aligned}$$

and since  $\psi[[g^{-1}, \psi], h] \in Z(G, \cdot)$ , we find that for every  $k \in G$ ,

$$\psi(\psi g \cdot h \cdot \psi g^{-1}) \cdot k \cdot \psi(\psi g \cdot h^{-1} \cdot \psi g^{-1}) = \psi(g \cdot h \cdot g^{-1}) \cdot k \cdot \psi(g \cdot h^{-1} \cdot g^{-1}). \quad (*)$$

In this proof, every time we employ  $(*)$ , the symbol  $\stackrel{*}{=}$  is used. Recall also that  $\bar{g} = \psi g \cdot g^{-1} \cdot \psi g^{-1}$ .

The first solution is

$$\begin{aligned} r_{\mathcal{G}}(g, h) &= (g^{-1} \cdot (g \circ h), \overline{g^{-1} \cdot (g \circ h)} \circ g \circ h) \\ &= (\psi g^{-1} \cdot h \cdot \psi g, \overline{(\psi g^{-1} \cdot h \cdot \psi g)} \circ (g \cdot \psi g^{-1} \cdot h \cdot \psi g)). \end{aligned}$$

Here

$$\begin{aligned} \overline{\psi g^{-1} \cdot h \cdot \psi g} &= \psi(\psi g^{-1} \cdot h \cdot \psi g) \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot \psi(\psi g^{-1} \cdot h^{-1} \cdot \psi g) \\ &\stackrel{*}{=} (g^{-1} \cdot h \cdot g) \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot \psi(g^{-1} \cdot h^{-1} \cdot g) \\ &= \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi(h^{-1} \cdot g), \end{aligned}$$

and so

$$\begin{aligned}
 & (\overline{\psi g^{-1} \cdot h \cdot \psi g}) \circ (g \cdot \psi g^{-1} \cdot h \cdot \psi g) = \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi(h^{-1} \cdot g) \circ (g \cdot \psi g^{-1} \cdot h \cdot \psi g) \\
 & = \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi(h^{-1} \cdot g) \cdot \psi(\psi(g^{-1} \cdot h) \cdot h \cdot \psi(h^{-1} \cdot g)) \\
 & \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi(\psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi(h^{-1} \cdot g)) \\
 & \stackrel{*}{=} \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi(h^{-1} \cdot g) \cdot \psi(g^{-1} \cdot h \cdot g) \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi(g^{-1} \cdot h^{-1} \cdot g) \\
 & = \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi g \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi(h^{-1} \cdot g).
 \end{aligned}$$

The second solution is

$$\begin{aligned}
 r_{\mathcal{G}'}(g, h) &= ((g \circ h) \cdot g^{-1}, \overline{(g \circ h) \cdot g^{-1}} \circ g \circ h) \\
 &= (g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1}, \overline{(g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1})} \circ (g \cdot \psi g^{-1} \cdot h \cdot \psi g)).
 \end{aligned}$$

Here

$$\begin{aligned}
 & \overline{g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1}} \\
 &= \psi(g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1}) \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi(g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1}) \\
 &= \psi g \cdot \psi(\psi g^{-1} \cdot h \cdot \psi g) \cdot \psi g^{-1} \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi g \cdot \psi(\psi g^{-1} \cdot h^{-1} \cdot \psi g) \cdot \psi g^{-1} \\
 &\stackrel{*}{=} \psi g \cdot \psi(g^{-1} \cdot h \cdot g) \cdot \psi g^{-1} \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi g \cdot \psi(g^{-1} \cdot h^{-1} \cdot g) \cdot \psi g^{-1} \\
 &= \psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1},
 \end{aligned}$$

and so

$$\begin{aligned}
 & (\overline{\psi g^{-1} \cdot h \cdot \psi g}) \circ (g \cdot \psi g^{-1} \cdot h \cdot \psi g) \\
 &= (\psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1}) \circ (g \cdot \psi g^{-1} \cdot h \cdot \psi g) \\
 &= \psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1} \cdot \psi(\psi h \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1}) \\
 &\cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi(\psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1}) \\
 &= \psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1} \cdot \psi(\psi h \cdot g) \cdot \psi(\psi g^{-1} \cdot h \cdot \psi g) \cdot \psi(g^{-1} \cdot \psi h^{-1}) \\
 &\cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi(\psi h \cdot g) \cdot \psi(\psi g^{-1} \cdot h^{-1} \cdot \psi g) \cdot \psi(g^{-1} \cdot \psi h^{-1}) \\
 &\stackrel{*}{=} \psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1} \cdot \psi(\psi h \cdot g) \cdot \psi(g^{-1} \cdot h \cdot g) \cdot \psi(g^{-1} \cdot \psi h^{-1}) \\
 &\cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi(\psi h \cdot g) \cdot \psi(g^{-1} \cdot h^{-1} \cdot g) \cdot \psi(g^{-1} \cdot \psi h^{-1}) \\
 &= \psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot \psi h^{-1} \cdot \psi(\psi h) \cdot \psi h \cdot \psi(\psi h^{-1}) \\
 &\cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi(\psi h) \cdot \psi h^{-1} \cdot \psi(\psi h^{-1}) \\
 &\stackrel{*}{=} \psi h \cdot g \cdot \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g^{-1} \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot \psi h^{-1} \\
 &= \psi h \cdot g \cdot \psi h^{-1}.
 \end{aligned}$$

The third solution is

$$r_{\mathcal{G}_1}(g, h) = (\bar{g} \circ (g \cdot h), (\bar{g} \circ (g \cdot h))^{-1} \cdot g \cdot h).$$

Here

$$\begin{aligned} \bar{g} \circ (g \cdot h) &= (\psi g \cdot g^{-1} \cdot \psi g^{-1}) \circ (g \cdot h) \\ &= \psi g \cdot g^{-1} \cdot \psi g^{-1} \cdot \psi(\psi g \cdot g \cdot \psi g^{-1}) \cdot g \cdot h \cdot \psi(\psi g \cdot g^{-1} \cdot \psi g^{-1}) \\ &\stackrel{*}{=} \psi g \cdot g^{-1} \cdot \psi g^{-1} \cdot \psi(g \cdot g \cdot g^{-1}) \cdot g \cdot h \cdot \psi(g \cdot g^{-1} \cdot g^{-1}) \\ &= \psi g \cdot h \cdot \psi g^{-1}, \end{aligned}$$

and so

$$(\bar{g} \circ (g \cdot h))^{-1} \cdot g \cdot h = \psi g \cdot h^{-1} \cdot \psi g^{-1} \cdot g \cdot h.$$

Finally, the fourth solution is

$$r_{\mathcal{G}'_1}(g, h) = ((g \cdot h) \circ \bar{g}, ((g \cdot h) \circ \bar{g})^{-1} \cdot g \cdot h).$$

Here

$$\begin{aligned} (g \cdot h) \circ \bar{g} &= g \cdot h \cdot \psi(h^{-1} \cdot g^{-1}) \cdot \psi g \cdot g^{-1} \cdot \psi g^{-1} \cdot \psi(g \cdot h) \\ &= g \cdot h \cdot \psi h^{-1} \cdot g^{-1} \cdot \psi h, \end{aligned}$$

and so

$$((g \cdot h) \circ \bar{g})^{-1} \cdot g \cdot h = \psi h^{-1} \cdot g \cdot \psi h. \quad \square$$

## 5.2. The case $\varepsilon = 1$

**Theorem 5.4.** *Let  $(G, \cdot)$  be a group, and let  $\psi \in \text{End}(G, \cdot)$ . If  $\psi[\psi G, G] \leq Z(G, \cdot)$ , then we get two solutions:*

$$\begin{aligned} r_{\mathcal{G}}(g, h) &= (\psi g \cdot h \cdot \psi g^{-1}, \psi(h^{-1} \cdot g) \cdot h^{-1} \cdot \psi g^{-1} \cdot g \cdot \psi g \cdot h \cdot \psi(g^{-1} \cdot h)); \\ r_{\mathcal{G}'}(g, h) &= (g \cdot \psi g \cdot h \cdot \psi g^{-1} \cdot g^{-1}, \psi(g \cdot h^{-1} \cdot g^{-1}) \cdot g \cdot \psi(g \cdot h \cdot g^{-1})). \end{aligned}$$

*These solutions are one the inverse of the other and coincide if and only if  $(G, \cdot)$  is abelian.*

*If in addition  $\psi[G, G] \leq Z(G, \cdot)$ , we get other two solutions:*

$$\begin{aligned} r_{\mathcal{G}_1}(g, h) &= (\psi g^{-1} \cdot h \cdot \psi g, \psi g^{-1} \cdot h^{-1} \cdot \psi g \cdot g \cdot h); \\ r_{\mathcal{G}'_1}(g, h) &= (g \cdot h \cdot \psi h \cdot g^{-1} \cdot \psi h^{-1}, \psi h \cdot g \cdot \psi h^{-1}). \end{aligned}$$



These solutions are one the inverse of the other and coincide if and only if, for every  $g, h \in G$ ,

$$g \cdot {}^\psi g \cdot h \cdot {}^\psi g^{-1} = h \cdot {}^\psi h \cdot g \cdot {}^\psi h^{-1}.$$

**Proof.** This follows from Theorem 1.2, Proposition 5.2, and reasonings similar to the ones in the proof of Theorem 5.3.  $\square$

## 6. Hopf–Galois theory

Let  $L/K$  be a finite field extension. A *Hopf–Galois structure* on  $L/K$  consists of a cocommutative  $K$ -Hopf algebra  $H$ , together with an  $H$ -action  $*$  on  $L$ , such that  $L$  is a left  $H$ -module algebra, and the  $K$ -linear map

$$\begin{aligned} j: L \otimes_K H &\rightarrow \text{End}_K(L) \\ x \otimes h &\mapsto (y \mapsto x(h * y)) \end{aligned}$$

is bijective. In this situation, we say that the  $K$ -Hopf algebra  $H$  gives a Hopf–Galois structure on  $L/K$ . We refer to [4] for a general treatment about Hopf–Galois theory.

If  $L/K$  is a finite Galois extension with Galois group  $G$ , then the following result, known as Greither–Pareigis correspondence ([9, Theorem 3.1]), allows us to make use of group theory in order to find Hopf–Galois structures on  $L/K$ .

**Theorem 6.1** (*Greither–Pareigis*). *Let  $L/K$  be a finite Galois extension with Galois group  $G$ . Then the Hopf–Galois structures on  $L/K$  correspond bijectively to the regular subgroups of  $\text{Perm}(G)$  normalised by  $\lambda(G)$ .*

Explicitly, if  $N$  is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$ , then the corresponding Hopf–Galois extension is given by the  $K$ -Hopf algebra

$$L[N]^G = \{x \in L[N] : g \cdot x = x, \forall g \in G\},$$

where  $G$  acts on  $L$  via Galois action and on  $N$  via conjugation (after the identification  $G \leftrightarrow \lambda(G)$ ). The isomorphism class of  $N$  is called the *type* of the Hopf–Galois structure given by  $L[N]^G$ .

**Example 6.2.** If  $L/K$  is a finite Galois extension with Galois group  $(G, \cdot)$  and

$$\begin{aligned} \rho: (G, \cdot) &\rightarrow \text{Perm}(G) \\ g &\mapsto (h \mapsto h \cdot g^{-1}) \end{aligned}$$

is the *right regular representation*, then  $\rho(G)$  is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$ , which corresponds to the classical Galois structure given by  $K[G]$  ([4,

Proposition 6.10]). Now  $\lambda(G)$  is regular and normalised by itself. Since  $\rho(G) = \lambda(G)$  if and only if  $(G, \cdot)$  is abelian, we get that a non-abelian Galois extension has at least two non-isomorphic Hopf–Galois structures. Both are clearly of type  $(G, \cdot)$ .

The next result allows us to translate information about  $N$  to information about  $L[N]^G$ . It is implicitly contained in the proof of [9, Theorem 5.2], and here it is presented in the formulation of [7, Proposition 2.2].

**Proposition 6.3.** *Let  $L/K$  be a finite Galois extension with Galois group  $G$ , and let  $N \leq \text{Perm}(G)$  be a regular subgroup normalised by  $\lambda(G)$ . Then the  $K$ -sub-Hopf algebras of  $L[N]^G$  correspond bijectively to the subgroups of  $N$  normalised by  $\lambda(G)$ .*

Explicitly, if  $M$  is a subgroup of  $N$  normalised by  $\lambda(G)$ , then  $L[M]^G$  is a  $K$ -sub-Hopf algebra of  $L[N]^G$ .

Let us now recall some facts about opposite subgroups. If  $G$  is a finite group and  $N = \{\nu(g) : g \in G\}$  is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$ , consider the centraliser

$$N' = C_{\text{Perm}(G)}(N) = \{n' \in \text{Perm}(G) : \eta' \eta = \eta \eta', \text{ for all } \eta \in N\}.$$

In [9, Lemma 2.4.2] Greither and Pareigis proved that

$$N' = \{\varphi(\eta) : \eta \in N\},$$

where

$$\varphi(\eta)h = \nu(h)\eta 1.$$

The subgroup  $N'$  is called the *opposite subgroup* of  $N$ . From this explicit description, one can easily get that  $N'$  is regular, and in the proof of [9, Theorem 2.5] it is shown that  $N'$  is normalised by  $\lambda(G)$ . In particular,  $L[N']^G$  gives a Hopf–Galois structure on  $L/K$ , which is called the *opposite structure*. Clearly,  $N = N'$  if and only if  $N$  is abelian.

**Example 6.4.** For a finite Galois extension of fields  $L/K$  with Galois group  $G$ , since  $\rho(G) = C_{\text{Perm}(G)}(\lambda(G))$ , the Hopf–Galois structure corresponding to  $\lambda(G)$  is the opposite structure of the classical Galois structure given by  $K[G]$ . It is called the *canonical nonclassical Hopf–Galois structure*, and it is studied in details in [19].

As one can expect, the opposite subgroup is strictly related to the opposite skew brace (recall the correspondence between regular subgroups and operations of Proposition 2.7).

**Proposition 6.5.** *Let  $(G, \cdot)$  be a finite group, and let  $N$  be a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$ . If  $\mathcal{G} = (G, \circ, \cdot)$  is the skew brace corresponding to  $N$ , then  $\mathcal{G}' = (G, \circ', \cdot)$  is the skew brace corresponding to the opposite subgroup  $N'$ .*

**Proof.** Recall that

$$N' = \{\varphi(\eta) : \eta \in N\},$$

where

$$\varphi(\eta)h = \nu(h)\eta 1.$$

If we now set  $\eta = \nu(g)$ , we obtain

$$\varphi(\eta)h = \nu(h)\nu(g)1 = \nu(h)g = h \circ g,$$

that is,

$$\varphi(n) = \nu_1(g),$$

with  $\nu_1(g)h = h \circ g = g \circ' h$ .  $\square$

**Remark 6.6.** The proof of Proposition 6.5 is similar to the proof of [14, Proposition 3.4], but slightly different, since in [14] Koch and Truman build the skew braces corresponding to  $N$  directly on  $N$ , and so their result is up to isomorphism.

**Example 6.7.** Let  $(G, \cdot)$  be a finite group, and let  $\psi \in \text{End}(G, \cdot)$  such that  $\psi$  satisfies one the following conditions:

1.  $\psi[[G, \psi], G] \leq Z(G, \cdot)$  (in this case set  $\varepsilon = -1$ );
2.  $\psi[G, G] \leq Z(G, \cdot)$  (in this case set  $\varepsilon = 1$ ).

Then

$$N = \{\nu(g) : g \in G\}$$

is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$  (Theorems 1.1 and 1.2), where, for every  $g, h \in G$ ,

$$\nu(g)h = g \circ h = g \cdot {}^\psi g^\varepsilon \cdot h \cdot {}^\psi g^{-\varepsilon}.$$

We deduce by Proposition 6.5 that the opposite subgroup is  $N' = \{\nu'(g) : g \in G\}$ , with

$$\nu'(g)h = g \circ' h = h \circ g = h \cdot {}^\psi h^\varepsilon \cdot g \cdot {}^\psi h^{-\varepsilon}.$$

In [15] Koch and Truman, given a finite group  $(G, \cdot)$  and a regular subgroup  $N = \{\nu(g) : g \in G\} \leq \text{Perm}(G)$  normalised by  $\lambda(G)$ , introduced a set  $\Lambda_N = \lambda(G) \cap N$  of  $\lambda$ -points and a set  $P_N = \rho(G) \cap N$  of  $\rho$ -points. The following immediate facts hold:

- $\Lambda_N$  and  $P_N$  are subgroups of  $N$ ;
- since  $N$ ,  $\lambda(G)$ ,  $\rho(G)$  are normalised by  $\lambda(G)$ , also  $\Lambda_N$  and  $P_N$  are normalised by  $\lambda(G)$ ;
- as  $C_{\text{Perm}(G)}(\rho(G)) = \lambda(G)$ , the action of  $\lambda(G)$  on  $P_N$  via conjugation is trivial;
- $\Lambda_N = \{\nu(g) : \lambda(g) = \nu(g)\}$  and  $P_N = \{\nu(g) : \nu(g) = \rho(g^{-1})\}$ .

## 7. Special subgroups of $N$ normalised by $\lambda(G)$ , and their associated sub-Hopf algebras

If  $L/K$  is a finite Galois extension with Galois group  $(G, \cdot)$  and  $\psi \in \text{End}(G, \cdot)$  satisfies  ${}^\psi[[G, \psi], G] \leq Z(G, \cdot)$  (in this case set  $\varepsilon = -1$ ) or  ${}^\psi[G, G] \leq Z(G, \cdot)$  (in this case set  $\varepsilon = 1$ ), then  $N = \{\nu(g) : g \in G\}$ , with

$$\nu(g)h = g \cdot {}^\psi g^\varepsilon \cdot h \cdot {}^\psi g^{-\varepsilon},$$

is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$  (Theorems 1.1 and 1.2), and so  $L[N]^G$  gives a Hopf–Galois structure on  $L/K$ .

The classification of the types of Hopf–Galois structures of a finite field extension, once the isomorphism type of its Galois group is given, has met great interest in recent years. In this line of reasoning, one would hope to be able to determine, given an endomorphism  $\psi \in \text{End}(G, \cdot)$  as above, the isomorphism class of the corresponding regular subgroup  $N \leq \text{Perm}(G)$ . As pointed out in [12], it seems there is no easy way to solve this question, even when  $\psi$  is an abelian endomorphism. Some information about  $N$  is given by Proposition 3.3: there exists an integer  $n$  such that

$$N = \nu(\ker(\psi^n)) \rtimes \nu({}^{\psi^n}G).$$

But this is not enough, in general, to pinpoint the type of  $N$ . In the following, we will thus proceed as in [12, Section 6] locating certain special subgroups of  $N$ , which are normalised by  $\lambda(G)$ , and thus determining, by Proposition 6.3,  $K$ -sub-Hopf algebras of  $L[N]^G$ .

Recall that, for the operation  $\circ$  determined by  $N$ , the map

$$\nu: (G, \circ) \rightarrow N$$

is an isomorphism, so that (normal) subgroups of  $N$  correspond to (normal) subgroups of  $(G, \circ)$ . Recall furthermore that  $\psi$  is also an endomorphism of  $(G, \circ)$  (Lemma 3.2).

### 7.1. The case $\varepsilon = -1$

This case is a direct generalisation of Koch’s work in [12], and the results we find coincide with his, in the particular situation when  ${}^\psi[G, G] = 1$ .

Let  $L/K$  be a finite Galois extension with Galois group  $(G, \cdot)$ , and let  $\psi \in \text{End}(G, \cdot)$  such that

$$\psi[[G, \psi], G] \leq Z(G, \cdot).$$

Then  $N = \{\nu(g) : g \in G\}$ , with

$$\nu(g)h = g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi g,$$

is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$  (Theorem 1.1), and so  $L[N]^G$  gives a Hopf–Galois structure on  $L/K$ .

Consider  $G_0 = \ker(\psi)$ . It is normal in  $(G, \cdot)$  and in  $(G, \circ)$ , therefore its image  $N_0 = \{\nu(g_0) : g_0 \in G_0\}$  is normal in  $N$ . Note that for every  $h \in G$ ,  $g_0 \in G_0$ ,

$$\nu(g_0)h = g_0 \cdot {}^\psi g_0^{-1} \cdot h \cdot {}^\psi g_0 = g_0 \cdot h = {}^{\lambda(g_0)}h,$$

that is,  $N_0 = \{\lambda(g_0) : g_0 \in G_0\} = \lambda(G_0)$ , and so  $N_0$  is also isomorphic to  $(G_0, \cdot)$ , and it is normalised by  $\lambda(G)$ .

Consider now the subgroup  $G_1 = \{g \in G : {}^\psi g = g\}$  of  $(G, \cdot)$  and  $(G, \circ)$  of fixed points under  $\psi$ . In general, this is not a normal subgroup of  $(G, \circ)$ . Define  $N_1 = \{\nu(g_1) : g_1 \in G_1\}$ . For every  $h \in G$ ,  $g_1 \in G_1$ , we have

$$\nu(g_1)h = g_1 \cdot g_1^{-1} \cdot h \cdot g_1 = h \cdot g_1 = {}^{\rho(g_1^{-1})}h,$$

that is,  $N_1 = \rho(G_1)$ , and so  $N_1$  is also isomorphic to  $(G_1, \cdot)$ , and it is normalised by  $\lambda(G)$ .

Clearly  $G_0 \circ G_1 = G_0 \cdot G_1$ . Since  $N_0$  is normal in  $N$ ,

$$N_{01} := N_0 N_1 = \nu(G_0)\nu(G_1) = \nu(G_0 \circ G_1) = \nu(G_0 \cdot G_1)$$

is a subgroup of  $N$ . Note that  $N_1$  is normal in  $N_{01}$ : as in [12, Proposition 6.3], for every  $g_0 \in G_0$ ,  $g_1, h_1 \in G_1$ , we have

$$\begin{aligned} \nu(g_0)\nu(g_1)\nu(h_1)(\nu(g_0)\nu(g_1))^{-1} &= \lambda(g_0)\rho(g_1^{-1})\rho(h_1^{-1})(\lambda(g_0)\rho(g_1^{-1}))^{-1} \\ &= \lambda(g_0)\rho(g_1^{-1})\rho(h_1^{-1})\rho(g_1)\lambda(g_0^{-1}) \\ &= \rho(g_1^{-1} \cdot h_1^{-1} \cdot g_1) \in \rho(G_1) = N_1. \end{aligned}$$

Since  $G_0 \cap G_1$  is trivial, we conclude that  $N_{01}$  is the direct product of  $N_0$  and  $N_1$ , it is isomorphic to  $(G_0, \cdot) \times (G_1, \cdot)$ , and it is normalised by  $\lambda(G)$ .

We can also use the  $\lambda$ -points and  $\rho$ -points to find other subgroups normalised by  $\lambda(G)$ :

$$\begin{aligned} \Lambda_N &= \{\nu(g) : \nu(g) = \lambda(g)\} = \{\nu(g) : \lambda(g)\iota({}^\psi g^{-1}) = \lambda(g)\} \\ &= \{\nu(g) : \iota({}^\psi g^{-1}) = \text{id}_G\} = \{\lambda(g) : {}^\psi g \in Z(G, \cdot)\}; \\ P_N &= \{\nu(g) : \nu(g) = \rho(g^{-1})\} = \{\nu(g) : \lambda(g)\iota({}^\psi g^{-1}) = \rho(g^{-1})\} \\ &= \{\nu(g) : \iota(g \cdot {}^\psi g^{-1}) = \text{id}_G\} = \{\rho(g) : g \cdot {}^\psi g^{-1} \in Z(G, \cdot)\}. \end{aligned}$$

Since  $\lambda(G)$  acts trivially on  $P_N$  and  $N_1 \subseteq P_N$ ,  $\lambda(G)$  acts trivially also on  $N_1$ .

Summarising, by Proposition 6.3, we get (up to) five  $K$ -sub-Hopf algebras of  $L[N]^G$ :

- $L[N_0]^G$ ;
- $L[N_1]^G = K[N_1]$ ;
- $L[N_{01}]^G \cong (L[N_0] \otimes_L L[N_1])^G \cong L[N_0]^G \otimes_K K[N_1]$  (this  $K$ -Hopf algebra isomorphism follows, for example, by Galois descent: see [4, 2.12]);
- $L[\Lambda_N]^G$ ;
- $L[P_N]^G = K[P_N]$ .

**Remark 7.1.** Some of these  $K$ -sub-Hopf algebras may coincide. For example, if  $Z(G, \cdot) = 1$ , then  $N_0 = \Lambda_N$  and  $N_1 = P_N$ . But they may also be all distinct. Consider Example 4.2: if  $S$  is a group of nilpotence class two and  $G = S \times S$ , we can define  $\psi : G \rightarrow G$  to be the projection on the second factor:

$$\psi(a, b) = (1, b).$$

Then  $1 \neq Z(S) \neq S$ ,  $\psi[[G, \psi], G] \leq Z(G)$ , and

- $N_0 = \nu(\{(a, b) \in G : b = 1\}) = \nu(S \times 1)$ ;
- $N_1 = \nu(\{(a, b) \in G : a = 1\}) = \nu(1 \times S)$ ;
- $N_{01} = \nu((S \times 1)(1 \times S)) = \nu(G)$ ;
- $\Lambda_N = \nu(\{(a, b) \in G : b \in Z(S)\}) = \nu(S \times Z(S))$ ;
- $P_N = \nu(\{(a, b) : a \in Z(S)\}) = \nu(Z(S) \times S)$ .

Since all these subgroups of  $N$  are distinct, by Proposition 6.3 they yield distinct  $K$ -sub-Hopf algebras of  $L[N]^G$ .

Finally, we discuss about situations in which the type of the structure given by  $L[N]^G$  is explicit.

If  $\psi$  is different from zero and idempotent, then, for every  $n \geq 1$ ,  $\psi^n = \psi$ . In particular, by Proposition 3.3,

$$N = \nu(\ker(\psi)) \rtimes \nu(\psi G).$$

As above,  $G_0 = \ker(\psi)$  and  $G_1 = \{g \in G : \psi g = g\}$ . Since  $\psi^2 = \psi$ , it immediately follows that  $G_1 = \psi G$ , that is,

$$N = \nu(G_0) \rtimes \nu(G_1) = N_0 \rtimes N_1.$$

We have seen that this product is actually direct, and  $N_0 \cong (G_0, \cdot)$ ,  $N_1 \cong (G_1, \cdot)$ , so we conclude that

$$N \cong (\ker(\psi), \cdot) \times ({}^\psi G, \cdot).$$

Now suppose that  $\psi$  is fixed point free. If  $\psi$  is also abelian, then by [5] and [12, Section 4],  $N \cong (G, \cdot)$ . Here instead of  $\psi[G, G] = 1$ , we may assume the weaker condition  $\psi[[G, \psi], G] = 1$  (see Example 4.2), and still find that  $N \cong (G, \cdot)$ . Indeed, since  $\psi$  is fixed point free, then

$$\begin{aligned} \alpha: G &\rightarrow G \\ g &\mapsto g \cdot {}^\psi g^{-1}, \end{aligned}$$

is bijective, and we claim that  $\alpha: (G, \circ) \rightarrow (G, \cdot)$  is an isomorphism. For every  $g, h \in G$ , we have

$$\begin{aligned} \alpha(g \circ h) &= \alpha(g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi g) \\ &= g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi g \cdot {}^\psi({}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g \cdot g^{-1}) \\ &= g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi(g \cdot {}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g \cdot g^{-1} \cdot h) \cdot {}^\psi h^{-1} \\ &= g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi[[g, \psi], h^{-1}] \cdot {}^\psi h^{-1} = g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi h^{-1} \\ &= {}^\alpha g \cdot {}^\alpha h. \end{aligned}$$

Since  $\nu: (G, \circ) \rightarrow N$  is an isomorphism, we derive our assertion. In particular,  $N$  and  $\lambda(G)$  are isomorphic:

$$\varphi: N \xrightarrow{\nu^{-1}} (G, \circ) \xrightarrow{\alpha} (G, \cdot) \xrightarrow{\lambda} \lambda(G).$$

Under this isomorphism, an element  $\nu(g)$  is sent to  $\lambda(g \cdot {}^\psi g^{-1})$ . An isomorphism of regular subgroups of  $\text{Perm}(G)$  normalised by  $\lambda(G)$  yields an isomorphism of the corresponding Hopf algebras if and only if it is  $G$ -equivariant (see, for instance, [11, Corollary 2.3]), where the  $G$ -action is via conjugation, after the identification  $G \leftrightarrow \lambda(G)$ .

We claim that  $\varphi$  yields an isomorphism  $L[N]^G \cong L[\lambda(G)]^G$  as  $K$ -Hopf algebras. We need to check whether, for every  $g, h \in G$ ,

$$\varphi(\lambda(h)\nu(g)\lambda(h)^{-1}) = \lambda(h)\varphi(\nu(g))\lambda(h^{-1}). \quad (7.1)$$

The right-hand side is

$$\lambda(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1}).$$

Since  $\lambda(h)\nu(g)\lambda(h)^{-1} = \nu(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g)$  (we have already performed this computation in the proof of Theorem 1.1), the left-hand side is

$$\begin{aligned}
& \varphi(\nu(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g)) \\
&= \lambda(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g) \lambda({}^\psi(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g)^{-1}) \\
&= \lambda(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1} \cdot {}^\psi g) \lambda({}^\psi({}^\psi g^{-1} \cdot h \cdot {}^\psi g \cdot g^{-1} \cdot h^{-1})) \\
&= \lambda(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1}) \lambda({}^\psi(g \cdot {}^\psi g^{-1} \cdot h \cdot {}^\psi g \cdot g^{-1} \cdot h^{-1})) \\
&= \lambda(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1}) \lambda({}^\psi[[g, \psi], h]) = \lambda(h \cdot g \cdot {}^\psi g^{-1} \cdot h^{-1}),
\end{aligned}$$

and so (7.1) holds.

## 7.2. The case $\varepsilon = 1$

Let  $L/K$  be a finite Galois extension with Galois group  $(G, \cdot)$ , and let  $\psi \in \text{End}(G, \cdot)$  such that

$${}^\psi[G, G] \leq Z(G, \cdot).$$

Then  $N = \{\nu(g) : g \in G\}$ , with

$$\nu(g)h = g \cdot {}^\psi g \cdot h \cdot {}^\psi g^{-1},$$

is a regular subgroup of  $\text{Perm}(G)$  normalised by  $\lambda(G)$  (Theorem 1.2), and so  $L[N]^G$  gives a Hopf-Galois structure on  $L/K$ .

As above,  $N_0 = \{\nu(g) : g \in \ker(\psi)\}$  equals  $\lambda(G_0)$ , and it is a normal subgroup of  $N$  which is normalised by  $\lambda(G)$ , yielding the  $K$ -sub-Hopf algebra  $L[N_0]^G$ .

However, in this case,  $N_1 = \{\nu(g) : {}^\psi g = g\} = \{\lambda(g^2)\rho(g) : {}^\psi g = g\}$  is not (in general) normalised by  $\lambda(G)$ : if  $g, h \in G$  with  ${}^\psi g = g$ , then

$$\lambda(h)\lambda(g^2)\rho(g)\lambda(h^{-1}) = \lambda(h \cdot g^2 \cdot h^{-1})\rho(g)$$

belongs to  $N_1$  if and only if

$$h \cdot g^2 \cdot h^{-1} = g^2.$$

This happens, for example, if  $(G, \cdot)$  is abelian, but it is false in general.

We can find once more explicitly the  $\lambda$ -points and  $\rho$ -points:

$$\begin{aligned}
\Lambda_N &= \{\nu(g) : \nu(g) = \lambda(g)\} = \{\nu(g) : \lambda(g)\iota({}^\psi g) = \lambda(g)\} \\
&= \{\nu(g) : \iota({}^\psi g) = \text{id}_G\} = \{\lambda(g) : {}^\psi g \in Z(G, \cdot)\}; \\
P_N &= \{\nu(g) : \nu(g) = \rho(g^{-1})\} = \{\nu(g) : \lambda(g)\iota({}^\psi g) = \rho(g^{-1})\} \\
&= \{\nu(g) : \iota(g \cdot {}^\psi g) = \text{id}_G\} = \{\rho(g) : g \cdot {}^\psi g \in Z(G, \cdot)\}.
\end{aligned}$$

Since the action of  $\lambda(G)$  via conjugation on  $P_N$  is trivial, we find (up to) three  $K$ -sub-Hopf algebras of  $L[N]^G$ :



- $L[N_0]^G$ ;
- $L[\Lambda_N]^G$ ;
- $L[P_N]^G = K[P_N]$ .

**Remark 7.2.** If  $Z(G, \cdot) = 1$ , then  $N_0 = \Lambda_N$ , hence  $L[N_0]^G = L[\Lambda_N]^G$ . But these sub-Hopf algebras may be all different, as the same example of Remark 7.1 immediately shows.

We conclude this subsection with a study of fixed point free endomorphisms. Suppose that  $\psi$  is fixed point free. We claim that if  ${}^\psi[\psi G, G] = 1$ , then  $N \cong (G, \cdot)$  (note that this condition is weaker than the condition  ${}^\psi[G, G] = 1$ , as Example 4.7 shows). Indeed, since  $\psi$  is fixed point free, the map

$$\begin{aligned}\alpha: G &\rightarrow G \\ g &\mapsto g \cdot {}^\psi g\end{aligned}$$

is bijective. As  $N \cong (G, \circ)$  via  $\nu$ , it is enough to show that

$$\alpha: (G, \circ) \rightarrow (G, \cdot)$$

is a homomorphism. For every  $g, h \in G$ , we have

$$\begin{aligned}\alpha(g \circ h) &= \alpha(g \cdot {}^\psi g \cdot h \cdot {}^\psi g^{-1}) \\ &= g \cdot {}^\psi g \cdot h \cdot {}^\psi g^{-1} \cdot {}^\psi(g \cdot {}^\psi g \cdot h \cdot {}^\psi g^{-1}) \\ &= g \cdot {}^\psi g \cdot h \cdot {}^\psi({}^\psi g \cdot h \cdot {}^\psi g^{-1} \cdot h^{-1}) \cdot {}^\psi h \\ &= g \cdot {}^\psi g \cdot h \cdot {}^\psi[{}^\psi g, h] \cdot {}^\psi h = g \cdot {}^\psi g \cdot h \cdot {}^\psi h \\ &= \alpha g \cdot \alpha h.\end{aligned}$$

In particular,  $N$  and  $\lambda(G)$  are isomorphic:

$$\varphi: N \xrightarrow{\nu^{-1}} (G, \circ) \xrightarrow{\alpha} (G, \cdot) \xrightarrow{\lambda} \lambda(G).$$

Under this isomorphism, an element  $\nu(g)$  is sent to  $\lambda(g \cdot {}^\psi g)$ . Finally, we claim that  $\varphi$  yields an isomorphism  $L[N]^G \cong L[\lambda(G)]^G$  as  $K$ -Hopf algebras if and only if  $\psi$  is abelian. We need to check whether, for every  $g, h \in G$ ,

$$\varphi(\lambda(h)\nu(g)\lambda(h)^{-1}) = \lambda(h)\varphi(\nu(g))\lambda(h^{-1}). \quad (7.2)$$

The right-hand side is

$$\lambda(h \cdot g \cdot {}^\psi g \cdot h^{-1}).$$

Since  $\lambda(h)\nu(g)\lambda(h)^{-1} = \nu(h \cdot g \cdot {}^\psi g \cdot h^{-1} \cdot {}^\psi g^{-1})$  (we have already performed this computation in the proof of Theorem 1.2), the left-hand side is

$$\begin{aligned} & \varphi(\nu(h \cdot g \cdot {}^\psi g \cdot h^{-1} \cdot {}^\psi g^{-1})) \\ &= \lambda(h \cdot g \cdot {}^\psi g \cdot h^{-1} \cdot {}^\psi g^{-1})\lambda({}^\psi(h \cdot g \cdot {}^\psi g \cdot h^{-1} \cdot {}^\psi g^{-1})) \\ &= \lambda(h \cdot g \cdot {}^\psi g \cdot h^{-1})\lambda({}^\psi(g^{-1} \cdot h \cdot g \cdot h^{-1} \cdot h \cdot {}^\psi g \cdot h^{-1} \cdot {}^\psi g^{-1})) \\ &= \lambda(h \cdot g \cdot {}^\psi g \cdot h^{-1})\lambda({}^\psi[g^{-1}, h] \cdot {}^\psi[h, {}^\psi g]) \\ &= \lambda(h \cdot g \cdot {}^\psi g \cdot h^{-1})\lambda({}^\psi[g^{-1}, h]). \end{aligned}$$

Therefore (7.2) holds if and only if  $\psi$  is abelian.

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