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A Probabilistic Approach to Conjugacy Classes in the Finite Symplectic and Orthogonal Groups

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Markov chains are used to give a purely probabilistic way of understanding the conjugacy classes of the finite symplectic and orthogonal groups in odd characteristic. As a corollary of these methods, one obtains a probabilistic proof of Steinberg's count of unipotent matrices and generalizations of formulas of Rudvalis and Shinoda. © 2000 Academic Press

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1. INTRODUCTION

For compact Lie groups, conjugacy classes are essentially eigenvalues up to the Weyl group [Ad]. Thus the enormous physics and mathematical literature on eigenvalues of random matrices (see [M] for a survey) is often a study of conjugacy classes. Although the field is rapidly evolving, perhaps the closest thing to a true probabilistic understanding of eigenvalues of such matrices are the papers of Dyson [Dy1, Dy2]. An equally rich theory exists for the symmetric groups. The cycles of random permutations have a probabilistic description using Poisson processes [LShe]. The small cycles of a randomly chosen permutation are asymptotically Poisson, the medium length cycles relate to Brownian motion [DeP], and the long cycles relate to stick breaking [ScV]. The cycle structure of random permutations has numerous applications to real-world problems such as population genetics (for this and more, see the reference list in [F1]).

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Some years back, Persi Diaconis observed to the author that a probabilistic understanding of conjugacy classes of finite groups of Lie type was missing and urged him to find one. The paper [F2] provided a useful and beautiful picture for the finite general linear and unitary groups, with connections to symmetric function theory, but gave only partial results for the symplectic and orthogonal groups. The purpose of this paper is to complete the program for the finite classical groups. To this two caveats should be added. First, only odd characteristic symplectic and orthogonal groups are considered. As is clear from [W], the even characteristic conjugacy classes have a very complicated description; a different view is given in [FNP]. Second, the current paper lumps together unipotent conjugacy classes with the same underlying Jordan form. As indicated at the end of the paper, this can be remedied, but the resulting formulas seem too complicated to be useful.

Before describing the contents of this paper, it is worth remarking that the probabilistic study of Jordan forms of unipotent upper triangular matrices over a finite field has a fascinating theory behind it. From the theory of wild quivers there is a provable sense in which conjugacy classes of upper triangular matrices over a finite field have no simple description; hence the reduction to Jordan form is necessary. A lovely probabilistic description of Jordan form is given in [K] and is exploited in [B]. The survey [F3] links their work with symmetric function theory and potential theory.

The main motivation for the current paper is [F4], which gave a Markov chain description of the conjugacy classes of the finite general linear and unitary groups. It is worth recalling the general nature of that description, as a variation of it occurs here. The conjugacy classes of $GL(n,q)$ are parametrized by rational canonical form; for each irreducible polynomial $\phi \neq z$ over F_q, one chooses a partition λ_ϕ of an integer $|\lambda_\phi|$, subject to the constraint that $\sum_\phi \deg(\phi)|\lambda_\phi| = n$. One can then define a probability measure M on the set of all partitions of all natural numbers by taking the limit as $n \to \infty$ of λ_{z-1} for a uniformly chosen element of $GL(n,q)$. (The polynomial $z-1$ is taken without loss of generality. For other polynomials, one simply replaces q by q raised to the degree of the polynomial in all formulas. Furthermore, asymptotically the distributions on partitions for different polynomials are independent.) Recall that partitions can be viewed geometrically. For example, the diagram of the partition (5441) is

$$\begin{array}{lllll} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & & & & \end{array}$$

The Markov chain method of sampling from M operates by choosing the size of the first column according to a certain probability distribution; then, given that column i has size a, column $i+1$ will have size b with probability $K(a, b)$. The remarkable fact is that the probabilities $K(a, b)$ are independent of i, yielding a Markov chain. An immediate consequence of this viewpoint was an elementary probabilistic proof of the Rogers–Ramanujan identities, which suggested generalizations to quivers.

The main result of this paper is that a similar description occurs for the symplectic and orthogonal cases, except that now the description will require two Markov chains K_1 and K_2 defined on the natural numbers. These Markov chains have the property that they can never move up, and K_1 has the additional property that it can only move down by an even amount. For the symplectic case, steps with column number i odd use K_1 and steps with column number i even use K_2. For the orthogonal case, steps with column number i odd use K_2 and steps with column number i even use K_1. The Markov chains K_1, K_2 are the same for both cases. The only disappointing aspect of our result is that the product matrices K_1K_2 and K_2K_1 do not seem to have a simple diagonalization; this blocked us from proving Rogers–Ramanujan type identities for the symplectic and orthogonal groups.

The structure of this paper is as follows. Section 2 recalls the conjugacy classes of the symplectic and orthogonal groups and defines measures on partitions from them, giving combinatorially useful rewritings. Section 3 begins with generalizations of formulas of Rudavlis and Shinoda [RShi], [Shi], and proves the aforementioned description in terms of Markov chains. In fact, it is shown that the setup extends to a more general family of measures on partitions with a parameter u.

2. CONJUGACY CLASSES AND MEASURES ON PARTITIONS

Let λ be a partition of some nonnegative integer $|\lambda|$ into parts $\lambda_1 \geq \lambda_2 \geq \cdots$. Let $m_i(\lambda)$ be the number of parts of λ of size i, and let λ' be the partition dual to λ in the sense that $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Let $n(\lambda)$ be the quantity $\sum_{i\geq 1}(i-1)\lambda_i$. It is also useful to define the diagram associated to λ as the set of points $(i, j) \in Z^2$ such that $1 \leq j \leq \lambda_i$. We use the convention that the row index i increases as one goes downward and the column index j increases as one goes to the right. So the diagram of the partition (5441) is as in the introduction.

The following combinatorial lemma about partitions will be helpful in what follows. For a proof, one simply uses the fact that $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$.

LEMMA 1.

$$\sum_{h<i} 2hm_h(\lambda)m_i(\lambda) + \sum_i (i-1)m_i(\lambda)^2 = \sum_i (\lambda_i')^2 - \sum_i m_i(\lambda)^2.$$

We also recall the following formulas for the sizes of finite symplectic and orthogonal groups in odd characteristic,

$$|Sp(2n,q)| = q^{n^2}\prod_{i=1}^{n}(q^{2i}-1),$$

$$|O^{\pm}(2n+1,q)| = 2q^{n^2}\prod_{i=1}^{n}(q^{2i}-1),$$

$$|O^{\pm}(2n,q)| = 2q^{n^2-n}(q^n \mp 1)\prod_{i=1}^{n-1}(q^{2i}-1).$$

2.1. *Symplectic Groups*

Wall [W] parametrized the conjugacy classes of the finite symplectic groups and found formulas for their sizes. Let us recall his parametrization for the case of odd characteristic. Given a polynomial $\phi(z)$ with coefficients in F_q and nonvanishing constant term, define a polynomial $\bar{\phi}$ by

$$\bar{\phi} = \frac{z^{\deg(\phi)}\phi(1/z)}{\phi(0)}.$$

Wall showed that a conjugacy class of $Sp(2n,q)$ corresponds to the following data. To each monic, nonconstant, irreducible polynomial $\phi \neq z \pm 1$, associate a partition λ_ϕ of some nonnegative integer $|\lambda_\phi|$. To ϕ equal to $z-1$ or $z+1$, associate a symplectic signed partition $\lambda(\pm)_\phi$, by which is meant a partition of some natural number $|\lambda(\pm)_\phi|$ such that the odd parts have even multiplicity, together with a choice of sign for the set of parts of size i for each even $i > 0$.

Example of a Symplectic Signed Partition.

$$\begin{array}{cccccc}
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & & \\
 & \cdot & \cdot & \cdot & & \\
- & \cdot & \cdot & & & \\
 & \cdot & \cdot & & &
\end{array}$$

Here the + corresponds to the parts of size 4 and the − corresponds to the parts of size 2. This data represents a conjugacy class of $Sp(2n,q)$ if and only if

(1) $|\lambda_z| = 0$,

(2) $\lambda_\phi = \lambda_{\bar{\phi}}$,

(3) $\sum_{\phi=z\pm1} \mid \lambda(\pm)_\phi \mid + \sum_{\phi\neq z\pm1} \mid \lambda_\phi \mid \deg(\phi) = 2n.$

Let

$$A_{Sp}(\phi^i) = \begin{cases} |Sp(m_i(\lambda(\pm)_\phi), q)| & \text{if } i \text{ odd}, \ \ \phi = z\pm 1, \\ q^{m_i(\lambda(\pm)_\phi)/2}|O(m_i(\lambda(\pm)_\phi), q)| & \text{if } i \text{ even}, \ \ \phi = z\pm 1, \\ |U(m_i(\lambda_\phi), q^{\deg(\phi)/2})| & \text{if } \ \phi = \bar{\phi} \neq z\pm 1, \\ |GL(m_i(\lambda_\phi), q^{\deg(\phi)})|^{1/2} & \text{if } \ \phi \neq \bar{\phi}, \end{cases}$$

where $O(m_i(\lambda_\phi), q)$ is the orthogonal group with the same sign as the sign associated to the parts of size i.

Theorem 1 is implicit in the discussion in [F1]. The three ingredients in its proof are Wall's formulas for conjugacy class sizes [W], the deduction that the cycle index of the symplectic groups factors, and the fact that the formulas in the statement of Theorem 1 define probability measures (i.e., the asserted probabilities sum to 1). This third fact will be deduced in the proof of Theorem 4, using only an identity of Cauchy. It is worth emphasizing that neither Steinberg's count of unipotent elements nor the formulas of Rudvalis and Shinoda in Section 3 are needed to prove the third fact.

THEOREM 1. *Fix some value of u with $0 < u < 1$. Then pick a non-negative, even integer with the probability of getting $2n$ equal to $(1-u^2)u^{2n}$ and pick uniformly in $Sp(2n, q)$. Let $\Lambda(\pm)_{z-1}, \Lambda(\pm)_{z+1}, \Lambda_\phi$ be the random variables corresponding to the conjugacy class data of the chosen element of $Sp(2n, q)$. Then, aside from the fact that $\Lambda_\phi = \Lambda_{\bar{\phi}}$, these random variables are independent, with probability laws*

$$\mathrm{Prob}(\Lambda(\pm)_{z-1} = \lambda(\pm)_{z-1}) = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{|\lambda(\pm)_{z-1}|}}{q^{[\sum_{h<i} hm_h(\lambda(\pm)_{z-1})m_i(\lambda(\pm)_{z-1})+1/2\sum_i(i-1)m_i(\lambda(\pm)_{z-1})^2]}\prod_i A_{Sp}((z-1)^i)},$$

$$\mathrm{Prob}(\Lambda(\pm)_{z+1} = \lambda(\pm)_{z+1}) = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{|\lambda(\pm)_{z+1}|}}{q^{[\sum_{h<i} hm_h(\lambda(\pm)_{z+1})m_i(\lambda(\pm)_{z+1})+1/2\sum_i(i-1)m_i(\lambda(\pm)_{z+1})^2]}\prod_i A_{Sp}((z+1)^i)},$$

$$\mathrm{Prob}(\Lambda_\phi = \lambda_\phi) = \frac{\prod_{r=1}^{\infty}(1+(-1)^r(u^{\deg(\phi)}/q^{\deg(\phi)r/2}))u^{\deg(\phi)\cdot|\lambda_\phi|}}{q^{\deg(\phi)[\sum_{h<i} hm_h(\lambda_\phi)m_i(\lambda_\phi)+1/2\sum_i(i-1)m_i(\lambda_\phi)^2]}\prod_i A_{Sp}(\phi^i)} \ \ \textit{if} \ \ \phi = \bar{\phi} \neq z\pm 1,$$

$$\mathrm{Prob}(\Lambda_\phi = \lambda_\phi) = \frac{\prod_{r=1}^{\infty}(1-(u^{2\deg(\phi)}/q^{\deg(\phi)r}))u^{2\deg(\phi)\cdot|\lambda_\phi|}}{q^{2\deg(\phi)[\sum_{h<i} hm_h(\lambda_\phi)m_i(\lambda_\phi)+1/2\sum_i(i-1)m_i(\lambda_\phi)^2]}\prod_i A_{Sp}(\phi^i)} \ \ \textit{if} \ \ \phi \neq \bar{\phi}.$$

Furthermore, setting $u = 1$ in these formulas yields the laws arising from the $n \to \infty$ limit of conjugacy classes of a uniformly chosen element of $Sp(2n, q)$, and the random variables corresponding to different polynomials are independent, up to the fact that $\Lambda_{\phi} = \Lambda_{\bar{\phi}}$.

From Theorem 1, one sees that if $\phi = \bar{\phi}$, then the corresponding measures on partitions are specializations of those for the unitary groups treated in [F2]. Similarly, if $\phi \neq \bar{\phi}$, then the corresponding measures on partitions are specializations of those for the general linear groups treated in [F2]. As the formulas for $z \pm 1$ are the same, for the rest of this paper only the partition corresponding to $z - 1$ will be studied.

Combining Theorem 1 with Lemma 1 leads one to the following measure on symplectic signed partitions:

$$M^{\pm}_{Sp,\,u}(\lambda(\pm)) = \prod_{r=1}^{\infty}(1 - u^2/q^{2r-1}) \frac{u^{|\lambda(\pm)|}}{q^{1/2[\sum_i (\lambda(\pm)'_i)^2 - \sum_i m_i(\lambda(\pm))^2]} \prod_i A_{Sp}((z-1)^i)}.$$

Forgetting about signs (i.e., lumping together some conjugacy classes) yields a measure on underlying shapes which will be denoted by $M_{Sp.u}$. Using the formulas for the sizes of the finite symplectic and orthogonal groups given at the beginning of this section, one arrives at the expression

$$\begin{aligned} M_{Sp,\,u}(\lambda) \;=\; & \frac{\prod_{r=1}^{\infty}(1 - u^2/q^{2r-1}) u^{|\lambda|}}{q^{1/2[\sum_i (\lambda'_i)^2 - \sum_i m_i^2]} \prod_{i=1 \bmod 2}\left(q^{m_i^2/4} \prod_{l=1}^{m_i/2}(q^{2l}-1)\right)} \\ & \cdot 1 \Big/ \Bigg[\prod_{\substack{i=0 \bmod 2 \\ m_i=0 \bmod 2}} \left(q^{m_i^2/4 - m_i/2} \prod_{l=1}^{m_i/2} (q^{2l} - 1) \right) \\ & \cdot \prod_{\substack{i=0 \bmod 2 \\ m_i=1 \bmod 2}} \left(q^{(m_i^2+1)/4} \prod_{l=1}^{(m_i-1)/2} (q^{2l} - 1) \right) \Bigg]. \end{aligned}$$

2.2. *Orthogonal Groups*

Wall [W] parametrized the conjugacy classes of the finite orthogonal groups and found formulas for their sizes. Let us recall his parametrization for the case of odd characteristic. To each monic, nonconstant, irreducible polynomial $\phi \neq z \pm 1$, associate a partition λ_{ϕ} of some nonnegative integer $|\lambda_{\phi}|$. To ϕ equal to $z - 1$ or $z + 1$, associate an orthogonal signed partition $\lambda(\pm)_{\phi}$, by which is meant a partition of some natural number $|\lambda(\pm)_{\phi}|$ such that all even parts have even multiplicity, and all odd $i > 0$ have a choice of sign. For $\phi = z - 1$ or $\phi = z + 1$ and odd $i > 0$, we denote by $\Theta_i(\lambda(\pm)_{\phi})$ the Witt type of the orthogonal group on a vector space of dimension $m_i(\lambda(\pm)_{\phi})$ and sign the choice of sign for i.

Example of an Orthogonal Signed Partition.

$$\begin{array}{ccccc}
 & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot \\
- & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & & \\
 & \cdot & \cdot & & \\
+ & \cdot & & & \\
 & \cdot & & &
\end{array}$$

Here the $-$ corresponds to the part of size 3 and the $+$ corresponds to the parts of size 1. The data $\lambda(\pm)_{z-1}, \lambda(\pm)_{z+1}, \lambda_\phi$ represents a conjugacy class of some orthogonal group if

(1) $|\lambda_z| = 0$,

(2) $\lambda_\phi = \lambda_{\bar{\phi}}$,

(3) $\sum_{\phi=z\pm 1} \mid \lambda(\pm)_\phi \mid + \sum_{\phi \neq z\pm 1} \mid \lambda_\phi \mid \deg(\phi) = n$.

In this case, the data represents the conjugacy class of exactly one orthogonal group $O(n, q)$, with sign determined by the condition that the group arises as the stabilizer of a form of Witt type,

$$\sum_{\phi=z\pm 1} \sum_{i \text{ odd}} \Theta_i(\lambda(\pm)_\phi) + \sum_{\phi\neq z\pm 1} \sum_{i\geq 1} i m_i(\lambda_\phi)\omega,$$

where ω is the Witt type of the quadratic form $x^2 - \delta y^2$ with δ a fixed nonsquare in F_q.

Let

$$A_O(\phi^i) = \begin{cases} q^{-m_i(\lambda(\pm)_\phi)/2}|Sp(m_i(\lambda(\pm)_\phi), q)| & \text{if } i \text{ even}, \quad \phi = z \pm 1, \\ |O(m_i(\lambda(\pm)_\phi), q)| & \text{if } i \text{ odd}, \quad \phi = z \pm 1, \\ |U(m_i(\lambda_\phi), q^{\deg(\phi)/2})| & \text{if } \quad \phi = \bar{\phi} \neq z \pm 1, \\ |GL(m_i(\lambda_\phi), q^{deg(\phi)})|^{1/2} & \text{if } \quad \phi \neq \bar{\phi}, \end{cases}$$

where $O(m_i(\lambda_\phi), q)$ is the orthogonal group with the same sign as the sign associated to the parts of size i.

Theorem 2 is implicit in [F1]. The three ingredients in its proof are Wall's formulas for conjugacy class sizes [W], the deduction that the cycle index for the sum of $+$ and $-$ types of the orthogonal groups factors, and the fact that the formulas in the statement of Theorem 2 define probability measures. As in the symplectic case, this third fact will be deduced in the proof of Theorem 5, using only an identity of Cauchy.

Theorem 2. *Fix some value of u with* $0 < u < 1$. *Then pick a nonnegative integer with the probability of getting* 0 *equal to* $(1-u)/(1+u)$ *and probability of getting* $n > 0$ *equal to* $[2u^n(1-u)]/(1+u)$. *Choose either* $O^+(n, q)$ *or* $O^-(n, q)$ *with probability* $\frac{1}{2}$. *Finally, select an element uniformly within the* *chosen orthogonal group and let* $\Lambda(\pm)_{z-1}, \Lambda(\pm)_{z+1}, \Lambda_{\phi}$ *be the random variables corresponding to its conjugacy class data. Then aside from the fact that* $\Lambda_{\phi} = \Lambda_{\bar{\phi}}$, *these random variables are independent, with probability laws*

$$\begin{aligned}
&\mathrm{Prob}(\Lambda(\pm)_{z-1} = \lambda(\pm)_{z-1})\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{|\lambda(\pm)_{z-1}|}}{(1+u)q^{[\sum_{h<i} hm_h(\lambda(\pm)_{z-1})m_i(\lambda(\pm)_{z-1})+1/2\sum_i(i-1)m_i(\lambda(\pm)_{z-1})^2]}\prod_i A_O((z-1)^i)},\\
&\mathrm{Prob}(\Lambda(\pm)_{z+1} = \lambda(\pm)_{z+1})\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{|\lambda(\pm)_{z+1}|}}{(1+u)q^{[\sum_{h<i} hm_h(\lambda(\pm)_{z+1})m_i(\lambda(\pm)_{z+1})+1/2\sum_i(i-1)m_i(\lambda(\pm)_{z+1})^2]}\prod_i A_O((z+1)^i)},\\
&\mathrm{Prob}(\Lambda_{\phi} = \lambda_{\phi})\\
&\quad = \frac{\prod_{r=1}^{\infty}(1+(-1)^r(u^{\deg(\phi)}/q^{\deg(\phi)r/2}))u^{\deg(\phi)\cdot|\lambda_{\phi}|}}{q^{\deg(\phi)[\sum_{h<i} hm_h(\lambda_{\phi})m_i(\lambda_{\phi})+1/2\sum_i(i-1)m_i(\lambda_{\phi})^2]}\prod_i A_O(\phi^i)} \ \ \text{if } \phi = \bar{\phi} \neq z \pm 1,\\
&\mathrm{Prob}(\Lambda_{\phi} = \lambda_{\phi})\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-(u^{2\deg(\phi)}/q^{\deg(\phi)r}))u^{2\deg(\phi)\cdot|\lambda_{\phi}|}}{q^{2\deg(\phi)[\sum_{h<i} hm_h(\lambda_{\phi})m_i(\lambda_{\phi})+1/2\sum_i(i-1)m_i(\lambda_{\phi})^2]}\prod_i A_O(\phi^i)} \ \ \text{if } \phi \neq \bar{\phi}.
\end{aligned}$$

Furthermore, setting $u = 1$ *in these formulas yields the laws arising from the* $n \to \infty$ *limit of conjugacy classes of a uniformly chosen element of* $O(n, q)$, *where the* $+, -$ *sign is chosen with probability* $\frac{1}{2}$. *The random variables corresponding to different polynomials are independent, up to the fact that* $\Lambda_{\phi} = \Lambda_{\bar{\phi}}$.

For the same reasons as with the symplectic groups, the only case remaining to be understood is the measure of the partition corresponding to the polynomial $z - 1$. Combining Theorem 1 with Lemma 1 leads one to the following measure on orthogonal signed partitions:

$$\begin{aligned}
M^{\pm}_{O,u}(\lambda(\pm)) &= \frac{1}{1+u}\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})\\
&\quad \times \frac{u^{|\lambda(\pm)|}}{q^{1/2[\sum_i(\lambda(\pm)'_i)^2-\sum_i m_i(\lambda(\pm))^2]}\prod_i A_O((z-1)^i)}.
\end{aligned}$$

Forgetting about signs (i.e., lumping together some conjugacy classes) yields a measure on underlying shapes which will be denoted by $M_{O,u}$. Using the formulas for the sizes of the finite symplectic and orthogonal groups given at the beginning of this section, one arrives at the expression

$$\begin{aligned} M_{O,u}(\lambda) \;=\; & \frac{1}{1+u}\prod_{r=1}^{\infty}(1-u^2/q^{2r-1}) \\ & \times \frac{u^{|\lambda|}}{q^{1/2[\sum_i (\lambda_i')^2 - \sum_i m_i^2]}\prod_{i=0 \bmod 2}\left(q^{m_i^2/4-m_i/2}\prod_{l=1}^{m_i/2}(q^{2l}-1)\right)} \\ & \cdot 1\Bigg/\Bigg[\prod_{\substack{i=1 \bmod 2\\ m_i=0 \bmod 2}}\left(q^{m_i^2/4-m_i}\prod_{l=1}^{m_i/2}(q^{2l}-1)\right) \\ & \qquad \times \prod_{\substack{i=0 \bmod 2\\ m_i=1 \bmod 2}}\left(q^{(m_i^2-1)/4}\prod_{l=1}^{(m_i-1)/2}(q^{2l}-1)\right)\Bigg]. \end{aligned}$$

3. MARKOV CHAIN DESCRIPTION

This section proves the Markov chain descriptions of conjugacy classes as advertised in the introduction. The first goal is Theorem 3, which generalizes work of Rudvalis and Shinoda [RShi] (proved by different methods and not in the language of partitions). First, a lemma of Cauchy is needed.

LEMMA 2. (*See* [An], *p.* 20). *If* $|q| > 1$,

$$\begin{aligned} \prod_{m=0}^{\infty}(1-z/q^m)^{-1} = 1 + \sum_{n=1}^{\infty} z^n \Bigg/ \Bigg[& q^{n^2-n}(1-1/q)(1-1/q^2)\cdots(1-1/q^n) \\ & \times(1-z)(1-z/q)\cdots(1-z/q^{n-1})\Bigg]. \end{aligned}$$

Let G denote either Sp or O, and let $P_{G,u}(i)$ be the probability that a partition chosen from the measure $M_{G,u}$ has i parts. Let

$$P'_{Sp,u}(i) = \frac{P_{Sp,u}(i)}{\prod_{i=1}^{\infty}(1-u^2/q^{2i-1})},$$

$$P'_{O,u}(i) = \frac{(1+u)P_{O,u}(i)}{\prod_{i=1}^{\infty}(1-u^2/q^{2i-1})}.$$

THEOREM 3.

$$\begin{aligned} P_{Sp,u}(2k) = & \prod_{i=1}^{\infty}(1-u^2/q^{2i-1}) \\ & \cdot \frac{u^{2k}}{q^{2k^2+k}(1-u^2/q)(1-1/q^2)\cdots(1-u^2/q^{2k-1})(1-1/q^{2k})}, \end{aligned}$$

$$P_{Sp,u}(2k+1)=\prod_{i=1}^{\infty}(1-u^2/q^{2i-1})$$
$$\cdot\frac{u^{2k+2}}{\left[q^{2k^2+3k+1}(1-u^2/q)(1-1/q^2)\cdots(1-1/q^{2k})(1-u^2/q^{2k+1})\right]},$$

$$P_{O,u}(2k)=\frac{\prod_{i=1}^{\infty}(1-u^2/q^{2i-1})}{1+u}$$
$$\cdot\frac{u^{2k}}{q^{2k^2-k}(1-u^2/q)(1-1/q^2)\cdots(1-u^2/q^{2k-1})(1-1/q^{2k})},$$

$$P_{O,u}(2k+1)=\frac{\prod_{i=1}^{\infty}(1-u^2/q^{2i-1})}{1+u}$$
$$\cdot\frac{u^{2k+1}}{q^{2k^2+k}(1-u^2/q)(1-1/q^2)\cdots(1-1/q^{2k})(1-u^2/q^{2k+1})}.$$

Proof. Using only the facts that $M_{Sp,u}$ and $M_{O,u}$ define a measure (i.e., not necessarily a probability measure), the proofs of Theorems 4 and 5 will establish the equations

$$P'_{Sp,u}(a)=\sum_{\substack{b\leq a\\ a-b\ \text{even}}}\frac{u^a P'_{O,u}(b)}{q^{(a^2-b^2+2(a+1)b)/4}(q^{a-b}-1)\cdots(q^4-1)(q^2-1)},$$

$$P'_{O,u}(a)=\sum_{\substack{b\leq a\\ a-b\ \text{even}}}\frac{u^a P'_{Sp,u}(b)q^{(a-b)^2/4}}{q^{(a^2+b-2a)/2}(q^{a-b}-1)\cdots(q^4-1)(q^2-1)}$$
$$+\sum_{\substack{b\leq a\\ a-b\ \text{odd}}}\frac{u^a P'_{Sp,u}(b)q^{((a-b)^2-1)/4}}{q^{(a^2-a)/2}(q^{a-b-1}-1)\cdots(q^4-1)(q^2-1)}.$$

To get a recurrence relation for the $P'_{Sp,u}(a)$'s, one simply plugs the second equation into the first. Similarly, one obtains a recurrence relation for the $P'_{O,u}(a)$'s. These recurrences allow one to solve for $P'_{G,u}(a)$ in terms of $P'_{G,u}(0)$, implying that the formulas for $P_{G,u}(a)$ are proportional to the asserted values. Thus it is enough to prove that the asserted formulas for $P_{G,u}(a)$ satisfy the equation $\sum_{a\geq 0}P_{G,u}(a)=1$. This follows readily from Lemma 2. ∎

Before continuing, we pause to indicate how the formulas of Theorem 3 can be used to deduce group theoretic results which are normally proved by techniques such as character theory and Möbius inversion. The first set of results, Corollary 1, considers only the symplectic groups. The same technique would give results for the sum of $+, -$ type orthogonal groups. Since the cycle index for the difference of orthogonal groups also factors, one could rework all of the paper until now to give measures corresponding to the difference of $+, -$ type orthogonal groups, apply the same technique, and then average the results to get theorems about groups over a given $+$ or $-$ type. This does not deserve to be done publicly.

COROLLARY 1. 1. (*see Steinberg, p.* 156 *of* [H]). *The number of unipotent elements in* $Sp(2n, q)$ *is* q^{2n^2}.

2. (*See* [RShi]). *The probability that a randomly chosen element of* $Sp(2n, q)$ *has a* $2k$*-dimensional fixed space is*

$$\frac{1}{|Sp(2k, q)|} \sum_{i=0}^{n-k} \frac{(-1)^i (q^2)^{\binom{i}{2}}}{|Sp(2i, q)| q^{2ik}}.$$

The probability that a randomly chosen element of $Sp(2n, q)$ *has a* $(2k+1)$*-dimensional fixed space is*

$$\frac{1}{|Sp(2k, q)| q^{2k+1}} \sum_{i=0}^{n-k-1} \frac{(-1)^i (q^2)^{\binom{i}{2}}}{|Sp(2i, q)| q^{2i(k+1)}}.$$

Proof. The arguments are completely analogous to those for $GL(n, q)$ in [F2, Corollary 1 and Theorem 6], using the cycle index of the finite symplectic groups [F1]. ∎

Rudvalis and Shinoda [RShi] also considered the probability that the fixed space of a random element of a finite classical group has a given isometry type. For the finite unitary groups, the isometry classes are parametrized by pairs (s, t) of natural numbers such that $s + 2t \leq n$. Here a subspace W of V has type (s, t) if $\dim(W/\mathrm{rad}(W)) = s$ and $\dim(\mathrm{rad}(W)) = t$. Theorem 2 uses cycle index techniques to give new proofs of their results for the finite unitary groups. Exactly the same methods work for the finite symplectic and orthogonal groups, but we spare the reader the details.

COROLLARY 2. *The probability that an element of* $U(n, q)$ *has isometry type corresponding to the pair* (s, t) *is*

$$\sum_{i=0}^{n-2s-t} \frac{(-1/q)^{(t+1)i} (-1/q)^{\binom{i}{2}}}{(1+1/q)(1-1/q^2)\cdots(1-(-1)^i/q^i)}$$
$$\Big/ \Big[q^{s^2+2st} (1+1/q)(1-1/q^2)\cdots(1-(-1)^s/q^s)$$
$$\times (1+1/q)(1-1/q^2)\cdots(1-(-1)^t/q^t) \Big].$$

In the $n \to \infty$ limit, this converges to

$$\prod_{r=0}^{\infty}(1/(1+1/q^{2r+1}))\Big/\Big[q^{s^2+2st}(1+1/q)(1-1/q^2)\cdots(1-(-1)^s/q^s) \times (1-1/q^2)(1-1/q^4)\cdots(1-1/q^{2t})\Big].$$

Proof. The most important observation (see [FNP] for a readable proof) is that the fixed space of an element α of $U(n,q)$ has isometry type (s,t) precisely when the partition corresponding to the polynomial $z-1$ in the rational canonical form of α satisfies $\lambda'_1 = s+t$ and $\lambda'_2 = t$. In other words, the partition has $s+t$ parts and s 1's. Let $[u^n]f(u)$ denote the coefficient of u^n in some polynomial $f(u)$. Then one uses the cycle index for the unitary groups as in [F2] to see that the sought probability for $U(n,q)$ is

$$[u^n]\frac{1}{1-u}\sum_{\lambda:\lambda'_1=s+t,\ \lambda'_2=t} M_{U,u}(\lambda).$$

Using the fact that

$$M_{U,u}(\lambda) = \prod_{r=1}^{\infty}(1+u/(-q)^r) \times \frac{u^{|\lambda|}}{q^{\sum_i(\lambda'_i)^2}\prod_i(1+1/q)(1-1/q^2)\cdots(1+(-1)^{m_i+1}/q^{m_i})},$$

this becomes

$$[u^n]\frac{u^{s+t}}{(1-u)q^{(s+t)^2}(1+1/q)(1-1/q^2)\cdots(1-(-1)^s/q^s)}\sum_{\lambda:\lambda'_1=t} M_{U,u}(\lambda)$$

$$= \frac{1}{q^{(s+t)^2}(1+1/q)(1-1/q^2)\cdots(1-(-1)^s/q^s)} \times [u^{n-s-t}]\frac{1}{1-u}\sum_{\lambda:\lambda'_1=t} M_{U,u}(\lambda)$$

$$= \sum_{i=0}^{n-2s-t}\frac{(-1/q)^{(t+1)i}(-1/q)^{\binom{i}{2}}}{(1+1/q)(1-1/q^2)\cdots(1-(-1)^i/q^i)} \Big/\Big[q^{s^2+2st}(1+1/q)(1-1/q^2)\cdots(1-(-1)^s/q^s) \times (1+1/q)(1-1/q^2)\cdots(1-(-1)^t/q^t)\Big],$$

where the last equality is in the proof of Theorem 6 in [F2].

The formula for the $n \to \infty$ limit follows from the well-known identity

$$\prod_{r=1}^{\infty}(1 - v/w^r) = \sum_{n=0}^{\infty} \frac{(-v)^n}{(w^n - 1)\cdots(w-1)}.$$

Alternatively, it follows from the principle that the limit as $n \to \infty$ of $f(u)/(1-u)$ is $f(1)$ if f has a Taylor expansion around zero converging in a circle of radius 1, together with the formula for

$$\sum_{\lambda:\lambda_1'=t} M_{U,1}(\lambda)$$

given in Theorem 5 of [F2]. ∎

Lemma 3 is crucial and motivated the combinatorial moves made in rewriting the formulas for $M_{Sp,u}$ and $M_{O,u}$ in Section 2. In all that follows, $M_{G,x}$ will denote the probability of an event X under the measure $M_{G,u}$ with G equal to Sp or O.

Lemma 3. 1. *If i is odd, then*

$$\begin{aligned}&M_{Sp,u}(\lambda_1' = s_1, \ldots, \lambda_{i-1}' = s_{i-1}, \lambda_i' = k)\\ &\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}P'_{Sp,u}(k)}{q^{(s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{Sp}((z-1)^{m_j})}.\end{aligned}$$

2. *If i is even, then*

$$\begin{aligned}&M_{Sp,u}(\lambda_1' = s_1, \ldots, \lambda_{i-1}' = s_{i-1}, \lambda_i' = k)\\ &\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}P'_{O,u}(k)}{q^{(k+s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{Sp}((z-1)^{m_j})}.\end{aligned}$$

3. *If i is odd, then*

$$\begin{aligned}&M_{O,u}(\lambda_1' = s_1, \ldots, \lambda_{i-1}' = s_{i-1}, \lambda_i' = k)\\ &\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}P'_{O,u}(k)}{(1+u)q^{(s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{O}((z-1)^{m_j})}.\end{aligned}$$

4. *If i is even, then*

$$\begin{aligned}&M_{O,u}(\lambda_1' = s_1, \ldots, \lambda_{i-1}' = s_{i-1}, \lambda_i' = k)\\ &\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}q^{k/2}P'_{Sp,u}(k)}{(1+u)q^{(s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{O}((z-1)^{m_j})}.\end{aligned}$$

Proof. The idea for all of the proofs is the same; hence we prove part two as follows,

$$\begin{aligned}
&M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_{i-1} = s_{i-1},\ \lambda'_i = k)\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}}{q^{(s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{Sp}((z-1)^{m_j})}\\
&\qquad \cdot \sum_{\lambda'_i=k\geq\lambda'_{i+1}\geq\cdots\geq 0}\frac{u^{\sum_{j\geq i}\lambda'_j}}{q^{\sum_{j\geq i}(\lambda'^2_j-m_j^2)/2}\prod_{j\geq i}A_{Sp}((z-1)^{m_j})}\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}}{q^{(k+s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{Sp}((z-1)^{m_j})}\\
&\qquad \cdot \sum_{\lambda'_i=k\geq\lambda'_{i+1}\geq\cdots\geq 0}\frac{u^{\sum_{j\geq i}\lambda'_j}}{q^{\sum_{j\geq i}(\lambda'^2_j-m_j^2)/2}\prod_{j\geq i}A_{O}((z-1)^{m_j})}\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}}{q^{(k+s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{Sp}((z-1)^{m_j})}\\
&\qquad \cdot \sum_{\lambda'_1=k\geq\lambda'_{2}\geq\cdots\geq 0}\frac{u^{\sum_{j\geq 1}\lambda'_j}}{q^{\sum_{j\geq 1}(\lambda'^2_j-m_j^2)/2}\prod_{j\geq 1}A_{O}((z-1)^{m_j})}\\
&\quad = \frac{\prod_{r=1}^{\infty}(1-u^2/q^{2r-1})u^{s_1+\cdots+s_{i-1}}}{q^{(k+s_1^2+\cdots+s_{i-1}^2-m_1^2-\cdots-m_{i-1}^2)/2}\prod_{j=1}^{i-1}A_{Sp}((z-1)^{m_j})}P'_{O,u}(k),
\end{aligned}$$

as desired. Note that the meat of the lemma is the second equality, which follows from the formulas for A_{Sp} and A_O. The third equality is simply a relabelling of subscripts. ∎

For the statements of Theorems 4 and 5, we define two Markov chains on the integers. Recall that

$$\begin{aligned}
P'_{Sp,u}(2k) &= \frac{u^{2k}}{q^{2k^2+k}(1-u^2/q)(1-1/q^2)\cdots(1-u^2/q^{2k-1})(1-1/q^{2k})},\\
P'_{Sp,u}(2k+1) &= \frac{u^{2k+2}}{q^{2k^2+3k+1}(1-u^2/q)(1-1/q^2)\cdots(1-1/q^{2k})(1-u^2/q^{2k+1})},\\
P'_{O,u}(2k) &= \frac{u^{2k}}{q^{2k^2-k}(1-u^2/q)(1-1/q^2)\cdots(1-u^2/q^{2k-1})(1-1/q^{2k})},\\
P'_{O,u}(2k+1) &= \frac{u^{2k+1}}{q^{2k^2+k}(1-u^2/q)(1-1/q^2)\cdots(1-1/q^{2k})(1-u^2/q^{2k+1})}.
\end{aligned}$$

The chains K_1, K_2 are defined on the natural numbers with transition probabilities

$$K_1(a,b) = \begin{cases} \dfrac{u^a P'_{O,u}(b)}{P'_{Sp,u}(a) q^{(a^2-b^2+2(a+1)b)/4}(q^{a-b}-1)\cdots(q^4-1)(q^2-1)} & \text{if } a-b \text{ even,} \\ 0 & \text{if } a-b \text{ odd,} \end{cases}$$

$$K_2(a,b) = \begin{cases} \dfrac{u^a P'_{Sp,u}(b) q^{(a-b)^2/4}}{P'_{O,u}(a) q^{(a^2+b)/2-a}(q^{a-b}-1)\cdots(q^4-1)(q^2-1)} & \text{if } a-b \text{ even,} \\ \dfrac{u^a P'_{Sp,u}(b) q^{((a-b)^2-1)/4}}{P'_{O,u}(a) q^{(a^2-a)/2}(q^{a-b-1}-1)\cdots(q^4-1)(q^2-1)} & \text{if } a-b \text{ odd.} \end{cases}$$

The fact that these transition probabilities add up to 1 will follow from the proof of Theorem 4.

THEOREM 4. *Starting with λ'_1 distributed as $P_{Sp,u}$, define in succession $\lambda'_2, \lambda'_3, \ldots$ according to the rules that if $\lambda'_i = a$, then $\lambda'_{i+1} = b$ with probability $K_1(a,b)$ if i is odd and probability $K_2(a,b)$ if i is even. The resulting partition is distributed according to $M_{Sp,u}$.*

Proof. The $M_{Sp,u}$ probability of choosing a partition with $\lambda'_i = s_i$ for all i is

$$M_{Sp,u}(\lambda'_1 = s_1) \prod_{i=1}^{\infty} \frac{M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_{i+1} = s_{i+1})}{M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_i = s_i)}.$$

Since $M_{Sp,u}(\lambda'_1 = s_1)$ is equal to $P_{Sp,u}(s_1)$ by definition, it is enough to prove two claims: first, that for every choice of $i, a, b, s_1, \ldots, s_{i-1}$,

$$\frac{M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_{i-1} = s_{i-1},\ \lambda'_i = a,\ \lambda'_{i+1} = b)}{M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_{i-1} = s_{i-1},\ \lambda'_i = a)}$$

is equal to the asserted transition rule probability for moving from $\lambda'_i = a$ to $\lambda'_{i+1} = b$, and second, that the transition rule probabilities sum to 1.

The first claim follows from Lemma 3. For the second claim, observe that

$$\sum_{b \le a} \frac{M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_{i-1} = s_{i-1},\ \lambda'_i = a,\ \lambda'_{i+1} = b)}{M_{Sp,u}(\lambda'_1 = s_1, \ldots, \lambda'_{i-1} = s_{i-1},\ \lambda'_i = a)} = 1,$$

because $M_{Sp,u}$ is a measure and the columns of a partition are nonincreasing in size as one moves to the right. Since $\sum_{i \ge 0} P_{Sp,u}(i) = 1$, it follows that $M_{Sp,u}$ is a probability measure, as promised earlier. ∎

Theorem 5 gives the analogous result for the orthogonal groups. As the proof method is the same as for the symplectic groups, we merely record the result.

THEOREM 5. *Starting with λ_1' distributed as $P_{O,u}$, define in succession $\lambda_2', \lambda_3', \ldots$ according to the rules that if $\lambda_i' = a$, then $\lambda_{i+1}' = b$ with probability $K_2(a,b)$ if i is odd and $K_1(a,b)$ if i is even. The resulting partition is distributed according to $M_{O,u}$.*

We close the paper with the following remarks.

REMARKS. 1. Theorems 4 and 5 allow one to draw exact samples from the measures $M_{Sp,u}$ or $M_{O,u}$. First recall that sampling from discrete distributions P with known formulas is straightforward; simply pick U uniformly in $[0,1]$ and find the value of j such that $\sum_{i=0}^{j} P(i) < U < \sum_{i=0}^{j+1} P(i)$. This allows one to sample from $P_{Sp,u}$ or $P_{O,u}$. Then move according to the appropriate Markov chains.

2. For the case of $M_{Sp,u}$, one can view the algorithm of Theorem 4 slightly differently. One starts with an imaginary zeroth column of size approaching infinity, and then gets λ_1' by transitioning according to the chain K_2. It is straightforward to verify that the resulting distribution of the first column size agrees with $P_{Sp,u}$. This viewpoint was useful in the general linear and unitary cases [F4].

3. As noted in the introduction, it is possible that the measures $M_{Sp,u}$ and $M_{O,u}$ are related to generalizations of the Rogers–Ramanujan identities, in analogy with the corresponding measures for $GL(n,q)$. In this regard observe that

$$\begin{aligned}\sum_{\lambda:\lambda_2'=0} M_{Sp,u}(\lambda) &= \prod_{r=1}^{\infty}(1-u^2/q^{2r-1})\sum_{n=0}^{\infty}\frac{u^{2n}}{q^{n^2}(q^{2n}-1)\cdots(q^2-1)}\\ &= \prod_{r=1}^{\infty}(1-u^2/q^{2r-1})\sum_{n=0}^{\infty}\frac{u^{2n}}{q^{2n^2+n}(1-1/q^2)\cdots(1-1/q^{2n})}.\end{aligned}$$

For the value $u = q^{-1/2}$, the sum has a product expansion by a Rogers–Ramanujan identity.

4. Although a probabilistic understanding of $M_{Sp,u}$ and $M_{O,u}$ has been given, a viewpoint explaining the products in Theorem 3 in terms of certain random variables being independent would be desirable. This was possible for the finite general linear and unitary groups [F2].

5. As mentioned in the introduction, the clumping of conjugacy classes given by looking at the underlying shape was not necessary. To modify things to take signs into account, K_1 starts at a number a but outputs an ordered pair $(b, \pm)$, that is a choice of sign associated with b. In the + (resp. −) case, the expression $P_{O,u}'(b)$ in the numerator of the definition of K_1 is replaced by the probability that the partition under the measure $M_{O,u}$ has b parts and a + (resp. −) choice for the parts of size 1.

Unfortunately, we do not know of simple formulas for these probabilities analogous to Theorem 3. Formulas can be inferred from the paper [RShi], but the result involves unpleasant sums and does not seem useful. The transition probabilities for K_2 are also affected: the $P'_{O,u}(a)$ are replaced the same way as for K_1, and letting ϵ be the sign associated to a, the transition probabilities are multiplied by an additional factor of

$$\frac{1}{|O^{\epsilon}(a-b,q)|}\frac{1}{1/(|O^{+}(a-b,q)|)+1/(|O^{-}(a-b,q)|)}.$$

It is not necessarily surprising that the theory is nicer when conjugacy classes are lumped; the cycle index only factors for sums or differences of orthogonal groups.

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