

# Generalized Orbital Integrals of Some Euler–Poincaré Functions of a Reductive $p$ -Adic Group<sup>1</sup>

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Schneider and Stuhler have defined Euler–Poincaré functions of irreducible representations of reductive  $p$ -adic groups and calculated their orbital integrals. Orbital integrals belong to a larger family of invariant distributions appearing in the geometric side of the Arthur–Selberg trace formula. We calculate the value of these distributions on the Euler–Poincaré functions of tempered representations using the Arthur local trace formula. © 2001 Academic Press

*Key Words:* orbital integral; Euler–Poincaré function; reductive  $p$ -adic group.

## 1. INTRODUCTION

The Bruhat–Tits building of a reductive  $p$ -adic group is an analogue of the global symmetric space of a Lie group [5]. Given an irreducible representation of a  $p$ -adic reductive group, Schneider and Stuhler have defined a coefficient system on the Bruhat–Tits building and the Euler–Poincaré function [8]. Among others, they showed that the elliptic orbital integral of the Euler–Poincaré function is the character value of the contragredient representation.

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Orbital integrals belong to a larger family of invariant distributions appearing in the geometric side of the Arthur–Selberg trace formula, which is a formula on the trace of a convolution operator acting on the discrete spectrum of some  $L^2$ -space. We call these distributions generalized orbital integrals in the title. This is somewhat inappropriate because the support of this distribution is not confined to the orbit. We have an expansion of the generalized orbital integral in terms of tempered characters [3]. Using this formula, we calculate generalized orbital integral of the Euler–Poincaré function of an irreducible tempered representation (Theorem 4.1). The result is a linear combination of characters of the contragredient and some other induced representations, whose coefficients are explicitly given in terms of multiplicities of representations of  $R$ -groups.

The generalized orbital integrals are obtained from the weighted orbital integrals by making the latter invariant. It would be interesting to calculate the weighted orbital integrals of Euler–Poincaré functions. It is a problem Peter Schneider suggested some time ago. The first author thanks him for suggesting the problem and for many helpful discussions. The authors also thank the referee for some helpful comments.

## 2. EULER-POINCARÉ FUNCTIONS

We review parts of [8]. Let  $K$  be a non-Archimedean local field of characteristic zero and let  $G$  be the set of  $K$ -rational points of a connected reductive algebraic group  $\mathbf{G}$  defined over  $K$ . We will assume that the center of  $G$  is compact for simplicity. We will keep these notations and the assumption throughout the paper.

For a facet  $F$  of the Bruhat–Tits building  $X_G$  of  $G$ , let  $P_F^\dagger = \{g \in G \mid gF = F\}$  be the stabilizer of  $F$  in  $G$ . In [8], a decreasing filtration  $U_F^{(1)} \supset U_F^{(2)} \supset \cdots$  of normal subgroups of  $P_F^\dagger$  is defined.

Let  $(\pi, V)$  be a smooth (or algebraic) representation of  $G$  of finite length. Fix a positive integer  $e$  such that  $V$  is generated as a  $G$ -module by the subspace  $V^{U_F^{(e)}}$  of  $U_F^{(e)}$ -fixed vectors. For each facet  $F$ , put  $V_F = V^{U_F^{(e)}}$ . Since  $U_F^{(e)}$  is a normal subgroup of  $P_F^\dagger$ ,  $P_F^\dagger$  acts on the finite dimensional vector space  $V_F$ . Denote its character by  $\tau_{F,e}^V$ . Let  $\varepsilon$  be the sign character of  $P_F^\dagger$  acting on  $F$ , i.e.,  $\varepsilon(k) = \pm 1$  according to whether the action of  $k \in P_F^\dagger$  on  $F$  is orientation preserving.

The Euler–Poincaré function  $f_{\text{EP}}^V$  of  $(\pi, V)$  is defined as

$$f_{\text{EP}}^V = \sum_{q=0}^{\dim X_G} \sum_{F \in \mathcal{F}_q} (-1)^q \text{vol}(P_F^\dagger)^{-1} \overline{\tau_{F,e}^V} \varepsilon_F,$$

where  $\mathcal{F}_q$  is a set of representatives of  $G$ -orbits in the set of  $q$ -dimensional facets of  $X_G$ .

Let  $C_c^\infty(G)$  be the space of a locally constant compactly supported functions on  $G$ . For  $f \in C_c^\infty(G)$ , let  $\pi(f)$  denote the finite rank operator  $v \mapsto \pi(f)v = \int_G f(g)\pi(g)v dg$ . The distribution  $f \mapsto \text{trace } \pi(f)$  is given by a locally  $L^1$ -function  $\Theta_\pi$  on  $G$  [6]; i.e.,  $\text{trace } \pi(f) = \int_G f(g)\Theta_\pi(g) dg$  for all  $f$ . The function  $\Theta_\pi$  (also denoted by  $\Theta(\pi)$ ) is called the character of  $(\pi, V)$ .

**THEOREM 2.1** [8, III.4.1, 4.16]. (i) *If  $(\pi', V')$  is an admissible representation of  $G$ , then*

$$\text{trace } \pi'(f_{\text{EP}}^V) = \sum_{q \geq 0} (-1)^q \dim \text{Ext}_{\text{Alg}(G)}^q(V, V').$$

(ii) *If  $x \in G$  is regular elliptic, then we have*

$$\int_G f_{\text{EP}}^V(gxg^{-1}) dg = \Theta_\pi(x^{-1}).$$

(iii) *The orbital integral of  $f_{\text{EP}}^V$  over any regular non-elliptic conjugacy class of  $G$  vanishes.*

The definition of the Euler–Poincaré function of  $\pi$  depends on the choice of  $e$  and  $\mathcal{F}_q$ 's. But the above theorem shows that  $\text{trace } \pi'(f_{\text{EP}}^V)$  as well as the orbital integrals of  $f_{\text{EP}}^V$  are independent of the choice.

### 3. EXPANSION OF $I_M(\gamma)$

In this section we review some results of [2]. Fix a minimal parabolic subgroup  $P_0$  of  $G$  and a Levi component  $M_0$  of it. By a (standard) parabolic subgroup we always mean a parabolic subgroup containing  $P_0$  and by a (standard) Levi subgroup, the Levi component of such a parabolic subgroup containing  $M_0$ . Let  $P$  be a parabolic subgroup of  $G$  and let  $M$  be the Levi subgroup of it. Let  $W_M = W_M^G = \text{Norm}_G(M)/M$  be the Weyl group of  $(G, M)$  and let  $\sigma$  be an irreducible square integrable representation of  $M$ . Put  $W_\sigma = \{w \in W_M \mid w\sigma \cong \sigma\}$ . To each  $w \in W_\sigma$ , a normalized intertwining operator  $R(w, \sigma)$  of the induced representation  $I_M^G(\sigma)$  is defined. The  $R$ -group  $R_\sigma$  of  $\sigma$  is the quotient of  $W_\sigma$  by  $W_\sigma^0 = \{w \in W_\sigma \mid R(w, \sigma) \text{ is a scalar}\}$ . The mapping  $r \mapsto R(r, \sigma)$  defines a projective representation of  $R_\sigma$  on the space of  $I_M^G(\sigma)$ . There exists a finite central extension of  $R_\sigma$

$$1 \rightarrow Z_\sigma \rightarrow \tilde{R}_\sigma \rightarrow R_\sigma \rightarrow 1$$

and a function  $\xi_\sigma: \tilde{R}_\sigma \rightarrow \mathbf{C}^*$  such that  $r \mapsto \tilde{R}(r, \sigma) := \xi_\sigma(r)^{-1} R(r, \sigma)$  defines a unitary representation of  $\tilde{R}_\sigma$  on the space of  $I_M^G(\sigma)$  with a central character  $\chi_\sigma^{-1}$  on  $Z_\sigma$ . Then there is a bijection  $\rho \leftrightarrow \pi_\rho$  between the set  $\Pi(\tilde{R}_\sigma, \chi_\sigma)$  of irreducible representations of  $\tilde{R}_\sigma$  with the  $Z_\sigma$ -central character  $\chi_\sigma$  and the set  $\Pi_\sigma(G)$  of irreducible constituents of  $I_M^G(\sigma)$ . More precisely, the representation  $\mathcal{R}$  of  $\tilde{R}_\sigma \times G$  on  $I_M^G(\sigma)$  can be decomposed as

$$\mathcal{R} = \bigoplus_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \rho^\vee \otimes \pi_\rho.$$

In terms of characters, it can be written as

$$\text{trace}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)) = \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \text{tr}(\rho^\vee(r)) \text{tr}(\pi_\rho(f)).$$

We consider triples  $\tau = (M, \sigma, r)$  consisting of a Levi subgroup  $M$ , an irreducible square integrable representation  $\sigma$  of it, and  $r \in \tilde{R}_\sigma$ . Let  $T(G)$  be the collection of *essential* triples  $\tau = (M, \sigma, r)$  modulo conjugation by  $W^G = W_{M_0}^G$ .  $T(G)$  has a structure of a (singular) manifold having infinitely many connected components, all of which are compact. The  $R$ -group  $R_\sigma$  can be identified with a subgroup of  $W_\sigma$ . Actually  $W_\sigma$  is isomorphic to the semidirect product of  $W_\sigma^0$  and  $R_\sigma$ . Hence  $R_\sigma$  acts on  $a_M = \text{Hom}(X^*(M), \mathbf{R})$  and so does  $\tilde{R}_\sigma$  via the projection onto  $R_\sigma$ . Let  $\tilde{R}_{\sigma, \text{reg}}$  be the set of elements  $r \in \tilde{R}_\sigma$  which fixes only  $a_G \subset a_M$  and let  $T_{\text{ell}}(G) = \{\tau = (M, \sigma, r) \in T(G) \mid r \in \tilde{R}_{\sigma, \text{reg}}\}$ .  $T_{\text{ell}}(G)$  is a union of connected components of the minimal dimension (in our case it is zero since we are assuming the center of  $G$  is compact). For  $\tau = (M, \sigma, r) \in T(G)$  let  $\Theta(\tau)$  be the distribution given as follows:

$$\Theta(\tau, f) = \text{tr}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)).$$

We have seen that  $\Theta(\tau)$  is a linear combination of tempered characters. The characters of irreducible representations and orbital integrals are the most important invariant distributions on  $G$ . They belong to two larger families of invariant distributions  $I_L(\pi)$ ,  $\pi \in \Pi_{\text{temp}}(L)$  and  $I_M(\gamma)$ ,  $\gamma \in M \cap G_{\text{reg}}$  which are defined by Arthur and appear in the (local and global) Arthur-Selberg trace formula.  $I_G(\pi)$  and  $I_G(\gamma)$  are the ordinary tempered character and the orbital integral. For precise definitions of them for  $L, M \neq G$ , see [3, Sect. 3].

We say that  $f \in C_c^\infty(G)$  is cuspidal if the orbital integral of  $f$  over any regular non-elliptic conjugacy class vanishes. For example, the Euler-Poincaré functions are cuspidal. For  $\tau = (M, \sigma, r)$ , we put  $d(\tau) = d(r) =$

$\det(1 - r)_{a_M}$  and  $\tau^\vee = (M, \sigma^\vee, r)$ . ( $\sigma^\vee$  denotes the contragredient of  $\sigma$ . We can take  $\tilde{R}_{\sigma^\vee} = \tilde{R}_\sigma$  and  $\chi_{\sigma^\vee} = \chi_\sigma^{-1}$ .) We have an expansion of  $I_M(\gamma)$  in terms of tempered characters [3, Theorem 4.1]. When  $f \in C_c^\infty(G)$  is cuspidal, it can be written as follows.

**THEOREM 3.1** [2, Theorem 5.1]. *Let  $f \in C_c^\infty(G)$  be a cuspidal function. When  $\gamma$  is a  $G$ -regular elliptic element of  $M$ ,*

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} |D(\gamma)|^{1/2} \\ \times \int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau^\vee, \gamma) \Theta(\tau, f) d\tau$$

and  $I_M(\gamma, f)$  vanishes if  $\gamma$  is not elliptic in  $M$ .

In the above equation,  $A_M$  is the split component of the center of  $M$  and  $D(\gamma) = \det(1 - \text{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$  is the Weyl discriminant. Note in our case that the above integral is actually a summation.

#### 4. EVALUATION OF $I_M(\gamma, F_{EP}^V)$

Let  $\theta$  be a  $W^G$ -invariant function on  $T(G)$  which is supported on only finite many components of  $T_{\text{ell}}(G)$  and such that  $\theta(z\tau) = \chi_\tau(z)\theta(\tau)$  for all  $\tau \in T_{\text{ell}}(G)$  and  $z \in Z_\tau$ , where we have written  $\chi_\tau$  (resp.  $Z_\tau$ ) for  $\chi_\sigma$  (resp.  $Z_\sigma$ ) if  $\tau = (M, \sigma, r)$ . Such a function is called a cuspidal test function. Then

$$\Theta(f) = \int_{T_{\text{ell}}(G)} \theta(\tau) \Theta(\tau, f) d\tau$$

defines an invariant distribution, which is a linear combination of elliptic tempered characters. In particular,  $\Theta$  is a locally integrable function on  $G$ .

Till the end of this paper, we fix a Levi subgroup  $M$ , an irreducible square integrable representation  $\sigma$  of  $M$ , and an irreducible representation  $\rho$  of  $\tilde{R}_\sigma$  with the  $Z_\sigma$ -central character  $\chi_\sigma$ . Let  $\pi = \pi_\rho$  be the corresponding irreducible constituent of  $I_M^G(\sigma)$ . Let  $\phi_\rho$  be the cuspidal test function given as

$$\phi_\rho(\tau) = |\tilde{R}_{\sigma, r}|^{-1} \text{tr } \rho(r)$$

if  $\tau$  is  $W^G$ -conjugate to  $(M, \sigma, r)$  with  $r \in \tilde{R}_{\sigma, \text{reg}}$  and zero otherwise. Here  $\tilde{R}_{\sigma, r}$  denotes the centralizer of  $r$  in  $\tilde{R}_\sigma$ . Let  $\Phi_\pi$  be the tempered distribution corresponding to  $\phi_\rho$  as before. Since  $\text{tr } \rho(r) = 0$  if  $(M, \sigma, r)$  is

an inessential triple, we have for any  $f \in C_c^\infty(G)$ ,

$$\Phi_\pi(f) = |\tilde{R}_\sigma|^{-1} \sum_{r \in \tilde{R}_{\sigma, \text{reg}}} \text{tr}(\rho(r)) \text{tr}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)). \quad (1)$$

LEMMA 4.1. *On the set  $G_{\text{ell}}$  of regular elliptic elements of  $G$ , we have  $\Phi_\pi = \Theta_\pi$ , the character of  $\pi$ .*

*Proof.* For any  $f \in C_c^\infty(G)$  we have

$$\begin{aligned} \Theta_\pi(f) &= \sum_{\lambda \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \sum_{r \in \tilde{R}_\sigma} |\tilde{R}_\sigma|^{-1} \text{tr}(\rho(r)) \text{tr}(\lambda^\vee(r)) \text{tr}(\pi_\lambda(f)) \\ &= \sum_{r \in \tilde{R}_\sigma} |\tilde{R}_\sigma|^{-1} \text{tr}(\rho(r)) \text{tr}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)). \end{aligned}$$

If  $f$  is supported on  $G_{\text{ell}}$ , then  $f$  is cuspidal and the lemma follows from the fact that

$$\text{tr}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)) = 0 \quad \text{if } r \notin \tilde{R}_{\sigma, \text{reg}},$$

which was shown in [2, p. 97]. ■

PROPOSITION 4.1. *Let  $f_\pi$  be the Euler–Poincaré function of  $\pi^\vee$ .*

(i) *We have for any  $\tau \in T(G)$ ,*

$$\Theta(\tau, f_\pi) = |d(\tau)| \phi_\rho(\tau^\vee).$$

(ii) *Let  $N$  be a standard Levi subgroup of  $G$ . If  $\gamma \in N$  is  $G$ -regular elliptic, then*

$$\Phi_\pi(\gamma) = \int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau^\vee, \gamma) \Theta(\tau, f_\pi) d\tau$$

$$I_N(\gamma, f_\pi) = (-1)^{\dim A_N} |D(\gamma)|^{1/2} \Phi_\pi(\gamma).$$

*Proof.* (i) Let  $\theta$  be a cuspidal test function and let  $\Theta$  be the distribution associated to it. On  $G_{\text{ell}}$ ,  $\Theta$  is given as orbital integrals of some function as follows. There exists an  $f \in C_c^\infty(G)$  such that  $\Theta(\tau, f) = |d(\tau)| \theta(\tau^\vee)$  for all  $\tau \in T(G)$  by the trace Paley–Wiener theorem [4]. Such an  $f$  is unique modulo  $[C_c^\infty(G), C_c^\infty(G)]$  [7, Theorem 0] and it is easy to show that  $f$  is cuspidal. It follows from Theorem 3.1 with  $M = G$  (see the proof of [2, Theorem 6.1]) that for all  $\gamma \in G_{\text{ell}}$

$$\int_G f(g\gamma g^{-1}) dg = \Theta(\gamma).$$

The converse is also true by [7, Theorem 0] again: If a cuspidal function  $f \in C_c^\infty(G)$  satisfies the above equality for all  $\gamma \in G_{\text{ell}}$ , then  $\Theta(\tau, f) = |d(\tau)|\theta(\tau^\vee)$  for all  $\tau \in T(G)$ .

Now the orbital integral of  $f_\pi$  over an elliptic conjugacy class  $\gamma^G$  of  $G$  is  $\Theta_\pi(\gamma) = \Phi_\pi(\gamma)$  by Theorem 2.1 and the above lemma. Since  $f_\pi$  is cuspidal, this implies that for any  $\tau \in T(G)$ ,  $\Theta(\tau, f_\pi) = |d(\tau)|\phi_\rho(\tau^\vee)$ .

(iii) By Theorem 3.1, if  $\gamma \in N$  is  $G$ -regular elliptic,

$$\begin{aligned} I_N(\gamma, f_\pi) &= (-1)^{\dim A_N} |D(\gamma)|^{1/2} \int_{T_{\text{ell}}(G)} |d(\tau)|^{-1} \Theta(\tau^\vee, \gamma) \Theta(\tau, f_\pi) d\tau \\ &= (-1)^{\dim A_N} |D(\gamma)|^{1/2} \int_{T_{\text{ell}}(G)} \phi_\rho(\tau^\vee) \Theta(\tau^\vee, \gamma) d\tau \\ &= (-1)^{\dim A_N} |D(\gamma)|^{1/2} \int_{T_{\text{ell}}(G)} \phi_\rho(\tau) \Theta(\tau, \gamma) d\tau \\ &= (-1)^{\dim A_N} |D(\gamma)|^{1/2} \Phi_\pi(\gamma). \end{aligned}$$

The proposition is proven.  $\blacksquare$

Let  $r \in \tilde{R}_\sigma$ . Then  $a_M^r = a_L$  for some Levi subgroup  $L$  containing  $M$  and  $W_M^L \cap R_\sigma$  can be identified with the  $R$ -group  $R_\sigma^L$  of  $\sigma$  relative to  $L$  [2, Sect. 2]. Let  $\tilde{R}_\sigma^L$  be the inverse image of  $R_\sigma^L$  under the map  $\tilde{R}_\sigma \rightarrow R_\sigma$ . Then  $r$  is an element of  $\tilde{R}_\sigma^L \subset \tilde{R}_\sigma$ . As before, the induced representation  $I_M^L(\sigma)$  as an  $\tilde{R}_\sigma^L \times L$ -module is isomorphic to  $\bigoplus_{\lambda \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)} (\lambda^\vee \otimes \pi_\lambda)$ . Since  $I_M^G(\sigma) = I_M^G(I_M^L(\sigma))$  and the action of the  $R$ -group is compatible with induction, we have the following decomposition of an  $\tilde{R}_\sigma^L \times G$ -module:

$$I_M^G(\sigma) = \bigoplus_{\lambda \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)} \lambda^\vee \otimes I_L^G(\pi_\lambda).$$

Suppose  $R_{\sigma, \text{reg}} = \emptyset$ . Then  $I_N(\gamma, f_\pi) = 0$  for any Levi subgroup  $N$  and  $\gamma \in N \cap G_{\text{reg}}$  since  $\phi_\rho \equiv 0$  and  $\Phi_\pi \equiv 0$ . From now on we suppose that  $R_{\sigma, \text{reg}} \neq \emptyset$ . In this case  $W_\sigma^0$  is trivial and  $R_\sigma = W_\sigma$  [2, p. 97]. This in turn implies that for any Levi subgroup  $L$  containing  $M$ , the  $R$ -group  $R_\sigma^L$  of  $\sigma$  relative to  $L$  can be identified with the subgroup  $W_M^L \cap R_\sigma$  of  $R_\sigma$  as above. ( $a_\sigma^+$  in the notation of [2, p. 88] is the whole  $a_M$ .) The following lemma is a reminiscence of [1, Proposition 1.1].

**LEMMA 4.2.** *Let  $M$  be a Levi subgroup of  $G$  and let  $\sigma$  be an irreducible square integrable representation of  $M$  such that  $R_{\sigma, \text{reg}} \neq \emptyset$ . Suppose  $F$  is a*

function on  $\tilde{R}_\sigma$ . Then

$$\sum_{L \in \mathcal{L}(M)} (-1)^{\dim(A_L/A_G)} \sum_{r \in \tilde{R}_\sigma^L} F(r) = \sum_{r \in \tilde{R}_{\sigma, \text{reg}}} F(r),$$

where  $\mathcal{L}(M)$  denotes the set of Levi subgroups of  $G$  containing  $M$ .

*Proof.* Let  $r \in \tilde{R}_\sigma$  and let  $L_r \in \mathcal{L}(M)$  be such that  $a_M^r = a_{L_r}$ . It is the smallest Levi subgroup containing  $M$  such that  $r \in \tilde{R}_{\sigma}^{L_r}$ . There is a one-to-one order-preserving correspondence between the set of Levi subgroups containing  $L_r$  and the set of subsets of simple roots  $\Delta_r$  of  $A_{L_r}$ . And if  $L \leftrightarrow \Delta$  under this correspondence then  $\dim(A_{L_r}/A_L) = |\Delta|$ . Hence if  $r \notin \tilde{R}_{\sigma, \text{reg}}$ , i.e.,  $L_r \neq G$ , then  $\sum_{L \in \mathcal{L}(L_r)} (-1)^{\dim(A_L/A_G)} = \pm \sum_{\Delta \subset \Delta_r} (-1)^{|\Delta|} = 0$ . But this is exactly the coefficient of  $F(r)$  in the left hand side of the equation of the lemma. ■

By the above lemma, for any  $f \in C_c^\infty(G)$ ,  $\Phi_\pi(f)$  is equal to

$$|\tilde{R}_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} (-1)^{\dim A_L} \sum_{r \in \tilde{R}_\sigma^L} \text{tr}(\rho(r)) \text{tr}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)).$$

We have seen that for  $r \in \tilde{R}_\sigma^L$  we have

$$\text{tr}(\tilde{R}(r, \sigma) I_M^G(\sigma, f)) = \sum_{\lambda \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)} \text{tr}(\lambda^\vee(r)) \text{tr}(I_L^G(\pi_\lambda, f)).$$

On the other hand, for  $\lambda \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)$

$$|\tilde{R}_\sigma^L|^{-1} \sum_{r \in \tilde{R}_\sigma^L} \text{tr}(\rho(r)) \text{tr}(\lambda^\vee(r)) = \langle \rho|_{\tilde{R}_\sigma^L}, \lambda \rangle_{\tilde{R}_\sigma^L}$$

is just the multiplicity of  $\lambda$  in the restriction of  $\rho$  to  $\tilde{R}_\sigma^L$ , where  $\langle \cdot, \cdot \rangle_{\tilde{R}_\sigma^L}$  denotes the scalar product on the space of class functions on  $\tilde{R}_\sigma^L$ . We have proven the following

LEMMA 4.3. *As a distribution given by a locally  $L^1$ -function,*

$$\Phi_\pi = \sum_{L \in \mathcal{L}(M)} (-1)^{\dim A_L} \frac{|R_\sigma^L|}{|R_\sigma|} \sum_{\lambda \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)} \langle \rho|_{\tilde{R}_\sigma^L}, \lambda \rangle_{\tilde{R}_\sigma^L} \Theta(I_L^G(\pi_\lambda)).$$

The induced character  $\Theta(I_L^G(\sigma))$  of any irreducible representation  $\sigma$  of  $L$  is supported on the union of conjugates of  $L$ . We claim that if  $N$  is a standard Levi subgroup of  $G$ , then  $\Theta(I_L^G(\sigma))$  vanishes on  $N_{\text{ell}}$  unless  $L$  contains a  $W^G$ -conjugate of  $N$ . This follows from two lemmas below.



LEMMA 4.4. *Let  $L$  and  $N$  be (not necessarily standard) Levi subgroups of  $G$ . If  $L$  contains a  $G$ -regular elliptic element of  $N$ , then  $N \subset L$ .*

*Proof.* The connected centralizer  $Z_G(\gamma)^0$  of  $\gamma$  in  $G$  is an elliptic torus  $T$  of  $N$ . Since  $A_N$  is the split component of  $T$  and  $A_L \subset Z_G(\gamma)^0 = T$ ,  $A_L \subset A_N$ . So  $N = Z_G(A_N) \subset Z_G(A_L) = L$ . ■

LEMMA 4.5. *Let  $L$  and  $N$  be (standard) Levi subgroups. If  $L$  contains a  $G$ -conjugate of  $N$ , then  $L$  contains a  $W^G$ -conjugate of  $N$ .*

*Proof.* From the proof of the last lemma,  $L \supset N^g := g^{-1}Ng$  if and only if  $A_L \subset A_N^g$ . Write  $g = uvv^{-1}m$  with  $u \in U_0 \cap wU_0^-w^{-1}$ ,  $v \in U_0$ , and  $m \in M_0$ , where  $U_0$  is the unipotent radical of  $P_0$  and  $U_0^-$  is that of the opposite parabolic subgroup  $P_0^-$ . Then  $A_L^u \subset A_N^{uw}$ . Hence for any  $b \in A_L$ , there exists  $a \in A_N$  such that  $bb^{-1}v^{-1}bv = w^{-1}aww^{-1}a^{-1}u^{-1}auw$ . Since  $b, w^{-1}aw \in M_0$ ,  $b^{-1}v^{-1}bv \in U_0$ , and  $w^{-1}a^{-1}u^{-1}auw \in U_0^-$ ,  $b = w^{-1}aw$  and  $b^{-1}v^{-1}bv = w^{-1}a^{-1}u^{-1}auw = 1$ . In particular,  $A_L \subset A_N^w$ . ■

Replacing  $\pi$  by  $\pi^\vee$ , we have proven the following theorem.

THEOREM 4.1. *Let  $M$  be a standard Levi subgroup of  $G$  and let  $\sigma$  be an irreducible square integrable representation of  $M$ . Let  $\pi$  be the irreducible constituent of the induced representation  $I_M^G(\sigma)$  which corresponds to  $\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)$  and let  $f_{\text{EP}}^\pi$  be its Euler–Poincaré function. Suppose  $\gamma$  is a  $G$ -regular element of a standard Levi subgroup  $N$ .*

(i) *If  $\gamma$  is elliptic in  $N$  and  $\tilde{R}_{\sigma, \text{reg}} \neq \emptyset$ , then  $I_N(\gamma, f_{\text{EP}}^\pi)$  is equal to the product of  $(-1)^{\dim A_N} |D(\gamma)|^{1/2}$  and*

$$\sum_L (-1)^{\dim A_L} \frac{|R_\sigma^L|}{|R_\sigma|} \sum_\lambda \langle \rho |_{\tilde{R}_\sigma^L}, \lambda \rangle_{\tilde{R}_\sigma^L} \Theta(I_L^G(\pi_\lambda), \gamma^{-1}),$$

*where the summations are over  $L \in \mathcal{L}(M)$  which contains a  $W^G$ -conjugate of  $N$  and over  $\lambda \in \Pi(\tilde{R}_\sigma^L, \chi_\sigma)$ , respectively.*

(ii) *If  $\gamma$  is not elliptic in  $N$  or  $\tilde{R}_{\sigma, \text{reg}} = \emptyset$ , then  $I_N(\gamma, f_{\text{EP}}^\pi) = 0$ .*

Note the term corresponding to  $L = G$  is  $(-1)^{\dim A_N} |D(\gamma)|^{1/2} \Theta_\pi(\gamma^{-1})$ .

Remark 4.1. Obviously, the above theorem holds when  $f_{\text{EP}}^\pi$  is replaced by any  $f \in C_c^\infty(G)$  having the same values on irreducible characters as  $f_{\text{EP}}^\pi$  (or equivalently, having the same orbital integrals) since  $I_N(\gamma)$  is supported on characters [3, Theorem 4.1].

Remark 4.2. Let  $\pi$  be any irreducible representation of  $G$ . In an appropriate Grothendieck group,  $\pi$  is a linear combination of irreducible tempered representations  $\pi_i$  and properly induced representations [7, Proposition 1.1]. Since the Euler–Poincaré characteristic of a properly

induced representation vanishes ([8, III.4.18], which is due to Kazhdan),  $I_N(\gamma, f_{\text{EP}}^\pi)$  is a linear combination of  $I_N(\gamma, f_{\text{EP}}^{\pi_i})$ . Applying the above theorem, we obtain an expansion of  $I_N(\gamma, f_{\text{EP}}^\pi)$ . The sum of the terms corresponding to  $L = G$  in this expansion is  $\Theta_\pi(\gamma^{-1})$  (multiplied by  $(-1)^{\dim A_N} |D(\gamma)|^{1/2}$ ). But the meaning of the other terms remains to be investigated.

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