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Modular Categories and Hopf Algebras

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The modularity of a ribbon Hopf algebra is characterized by the Drinfeld map. An elementary approach to Etingof and Gelaki's (1998, *Math. Res. Lett.* **5**, 119–197) result on the dimensions of irreducible modules is given by deducing the necessary identities involving the matrix (S_{ij}) from the well-known orthogonal relations of Hopf algebra characters. © 2001 Academic Press

Key Words: modular category; Hopf algebra; ribbon Hopf algebra; the Drinfeld map; weakly factorizable Hopf algebra.

The notion of a modular category was introduced by Turaev [10, 11] in order to construct some topological invariants of 3-manifolds. It was used by Etingof and Gelaki [2] to solve Kaplansky's sixth conjecture on semisimple quasi-triangular Hopf algebras. It was later pointed out by Schneider [9] that the theory of modular categories is unnecessary to prove this result. He uses the class equation theorem due to Kac and Zhu instead of the modular category and formulates the main theorem for factorizable semisimple Hopf algebras. In this paper we give another elementary approach to the theorem of Etingof and Gelaki. We do not use the class equation theorem, but rather follow the original approach of Etingof and Gelaki. Some matrix (S_{ij}) appears in a modular category and some of its properties were used in the original proof. We show that those properties follow directly from the well-known orthogonal relations of Hopf algebra characters. As a result, we can formulate the main theorem for weakly factorizable Hopf algebras.

Further, we analyze the connection of modularity and the Drinfeld map. We show that a semisimple ribbon Hopf algebra H is modular if and only if the Drinfeld map induces an isomorphism $C_q(H) \tilde{\to} Z(H)$. If this is the case, we show that H is cosemisimple, hence $S^2 = id$ by [3].

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We work over a fixed field k of arbitrary characteristic. In Section 1, we review the notion of a ribbon Hopf algebra and associated ribbon categories. In Section 2, we review the Drinfeld map and factorizable Hopf algebras. In Section 3, we introduce the notion of quantum characters for a ribbon Hopf algebra. In Section 4, we characterize semisimple modular Hopf algebras by using the Drinfeld map and quantum characters. In Section 5, we introduce the notion of weakly factorizable Hopf algebras and formulate the theorem of Etingof and Gelaki in this context. We deduce the necessary properties of the matrix (S_{ij}) from Hopf algebra character theory.

1. RIBBON HOPF ALGEBRAS AND RIBBON CATEGORIES

We briefly review ribbon Hopf algebras, ribbon categories, and associated notions following [4, 11].

A ribbon Hopf algebra is a triple (H, R, v), where H is a Hopf algebra over k and $R \in H \otimes H$ and $v \in H$ are units satisfying the following conditions:

$$\Delta^{op}(x) = R\Delta(x)R^{-1}, \qquad x \in H, \tag{1.1}$$

$$(\Delta \otimes id)(R) = R_{13}R_{23}, \tag{1.2}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12}, \tag{1.3}$$

$$v \text{ is in the center of } H, \tag{1.4}$$

$$v = Sv, \qquad \text{where } S \text{ is the antipode of } H, \tag{1.5}$$

$$\Delta(v) = (R_{21}R)(v \otimes v). \tag{1.6}$$

Thus (H, R) is a quasi-triangular Hopf algebra. We put as usual:

$$u = \sum_i (Sb_i)a_i, \qquad \text{where } R = \sum_i a_i \otimes b_i. \tag{1.7}$$

It is known that u is a unit and we have

$$S^2x = uxu^{-1}, \qquad x \in H, \tag{1.8}$$

$$\Delta(u) = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1}. \tag{1.9}$$

It follows that $G = uv$ is a group-like element and that we have

$$S^2x = GxG^{-1}, \qquad x \in H. \tag{1.10}$$

EXAMPLE. Assume that (H, R) is a finite-dimensional semisimple quasi-triangular Hopf algebra such that $S^2 = id$. Then we have $u = Su$ and u^{-1} is a ribbon element. This observation is due to [2], where the characteristic of k is assumed to be zero. Since the dimensions of irreducible modules are not zero in k [5, Theorem 2.8], this fact holds in any characteristic.

Let (H, R, v) be a ribbon Hopf algebra and let ${}_H\mathbf{mod}$ be the category of left H modules which are finite-dimensional over k. This category has the following structures.

(1) It is a monoidal category. For V, W in ${}_H\mathbf{mod}$, we let H act on $V \otimes W$ via the comultiplication $\Delta: H \to H \otimes H$.

(2) Every $V \in {}_H\mathbf{mod}$ has left dual $V^* \in {}_H\mathbf{mod}$, where the left H module structure is defined by

$$\langle xf, p\rangle = \langle f, S(x)p\rangle, \qquad x \in H, f \in V^*, p \in V.$$

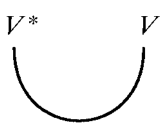
We illustrate the evaluation

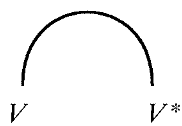
$$e: V^* \otimes V \to k, \qquad f \otimes p \mapsto \langle f, p\rangle$$

and the coevaluation

$$c: k \to V \otimes V^*, \qquad 1 \mapsto \sum_i e_i \otimes e_i^*,$$

where $\{e_i\}$ and $\{e_i^*\}$ are dual bases for V and V^*, by the diagrams

$e:$ 

, $c:$ 

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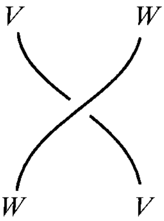
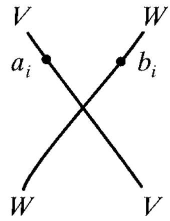
All such diagrams mean linear maps being read downward.

(3) It is a braided category. For $V, W \in {}_H\mathbf{mod}$, the braiding $c_{V,W}$: $V \otimes W \to W \otimes V$ is defined by

$$c_{V,W}(p \otimes q) = \sum_i b_i q \otimes a_i p, \qquad p \in V, q \in W,$$

where $R = \Sigma_i a_i \otimes b_i$. This is illustrated by diagrams as

$c_{V,W}$: 

$=$ Σ_i 

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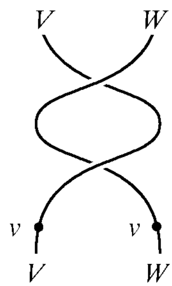
(4) Finally, each $V \in {}_H\mathbf{mod}$ has a twist map

$$\theta_V : V \to V, \qquad p \mapsto \upsilon p,$$

the multiplication by the ribbon element. This is H linear by (1.4), and $(\theta_V)^* = \theta_{V^*}$ by (1.5). Condition (1.6) means we have

$\theta_{V \otimes W}$: 

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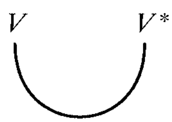
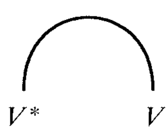
Summarizing the above, ${}_H\mathbf{mod}$ forms a ribbon category.

In a ribbon category, the left dual V^* can also be viewed as a right dual canonically. In other words, right versions of e and c,

e': 

, c': 

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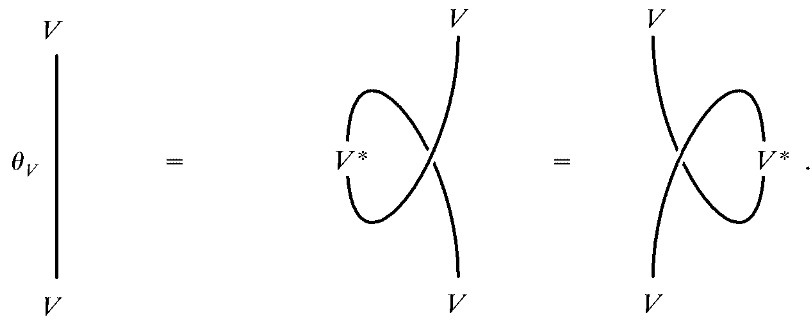
are canonically defined. These maps are characterized by the identities



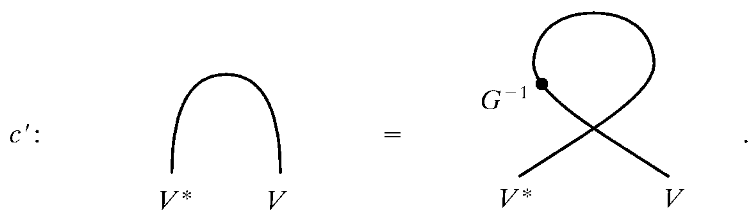
In the present case ${}_H\mathbf{mod}$, it is easy to see that

(1.11) $$e'(p \otimes f) = \langle f, Gp \rangle$$

and

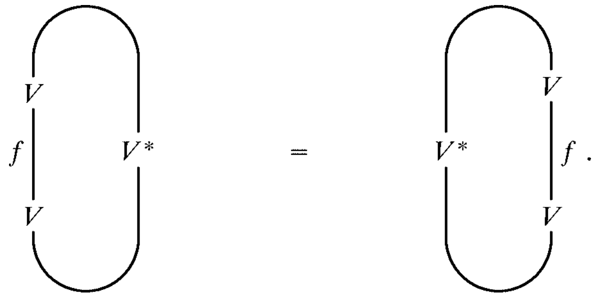
(1.12) $$c'(1) = \sum_i e_i^* \otimes G^{-1} e_i$$

with the previous notation. Thus



Note that these are H linear.

If $f: V \to V$ is an endomorphism in ${}_H\mathbf{mod}$, we have



This quantity in k is called the quantum trace of f and is denoted by $tr_q(f)$. We put

$$\dim_q(V) = tr_q(id_V)$$

and call it the quantum dimension.

2. THE DRINFELD MAP AND FACTORIZABLE HOPF ALGEBRAS

The Drinfeld map was introduced in [1]. We review its main properties, following [9].

Let (H, R) be a finite-dimensional quasi-triangular Hopf algebra. The linear map

$$\Phi: H^* \to H, \qquad \Phi(f) = (id \otimes f)(R_{21}R)$$

is called the Drinfeld map. We put

$$C_q(H) = \{ f \in H^* \mid f(xy) = f(yS^2(x)),\ x, y \in H \},$$

which is a subalgebra of H^*.

EXAMPLE 2.1. If H is unimodular, the space of left integrals in H^*, $I_l(H^*)$, is contained in $C_q(H)$.

PROPOSITION 2.2. *Φ maps $C_q(H)$ into the center $Z(H)$ and induces an algebra map $C_q(H) \to Z(H)$.*

This is the main property of Φ (see [9, Theorem 2.1]). Later, in Section 3, we give a diagrammatic explanation of this fact.

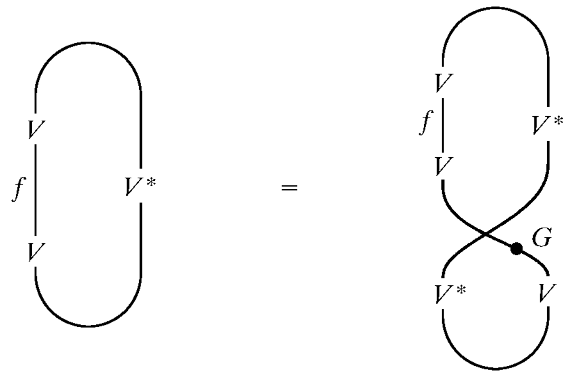
DEFINITION 2.3. The quasi-triangular Hopf algebra (H, R) is called factorizable if Φ is an isomorphism $H^* \tilde{\to} H$ [7].

For example, the Drinfeld double $D(H)$ is factorizable.

PROPOSITION 2.4 [9, Theorem 2.3]. *If (H, R) is factorizable, then H is unimodular and Φ induces an algebra isomorphism $C_q(H) \tilde{\to} Z(H)$.*

3. QUANTUM CHARACTERS

Let (H, R, v) be a finite-dimensional ribbon Hopf algebra with a special group-like element G. For a linear endomorphism f of V in ${}_H\mathbf{mod}$ we have



by (1.11); i.e., we have

$$tr_q(f) = tr(Gf).$$

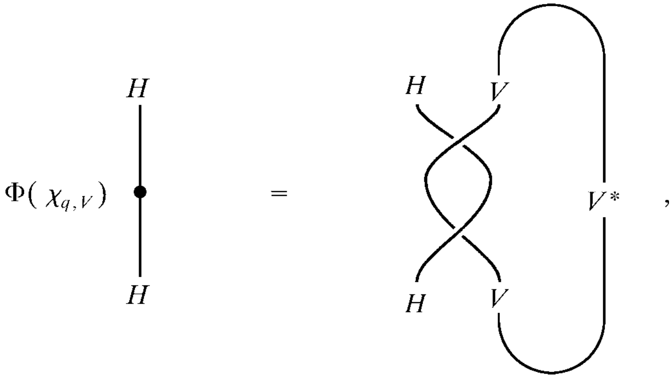
For $a \in H$, we put

$$\chi_{q,V}(a) = tr_q(a_V) = \chi_V(Ga),$$

where χ_V is the usual character of V. The map $\chi_{q,V}\colon H \to k$ is called the quantum character of V. We have $\chi_{q,V} \in C_q(H)$, since

$$\begin{aligned}\chi_{q,V}(ab) &= \chi_V(Gab) = \chi_V(GbGaG^{-1})\\ &= \chi_V\big(GbS^2(a)\big) = \chi_{q,V}\big(bS^2(a)\big), \qquad a, b \in H.\end{aligned}$$

We have



which is a map in ${}_H\mathbf{mod}$. This explains why $\Phi(\chi_{q,V})$ is in the center of H.

4. MODULAR HOPF ALGEBRAS

Assume that k is algebraically closed. Let (H, R, v) be a finite-dimensional semisimple ribbon Hopf algebra. Let $V_0 = k$ and $V_1, \ldots, V_n$ be a complete set of irreducible H modules up to isomorphisms. Let $\chi_0, \chi_1, \ldots, \chi_n$ be the corresponding characters. They form a basis for the character algebra $C(H)$ over k. There are orthogonal central idempotents $e_0, e_1 \ldots, e_n$ such that

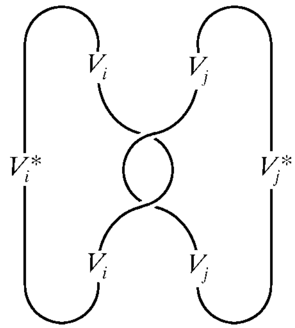
$$H = He_0 \oplus He_1 \oplus \cdots \oplus He_n,$$

$$He_i \cong M_{m_i}(k) \qquad \text{with } m_i = \dim(V_i).$$

We put $\chi_{q,i} = \chi_i \leftharpoonup G$, the quantum character of V_i. We see that $\chi_{q,0}, \chi_{q,1}, \ldots, \chi_{q,n}$ form a basis for $C_q(H)$, since $\dim C_q(H) = \dim Z(H)$ [9, Lemma 2.2]. We put

$$S_{ij} = \langle \chi_{q,i}, \Phi(\chi_{q,j}) \rangle,$$

which is illustrated by



We have $S_{ij} = S_{ji}$, as is easily checked.

DEFINITION 4.1 [11]. The ribbon category ${}_H\mathbf{mod}$ is called modular if the matrix (S_{ij}) is invertible.

If this is the case, we also call (H, R, v) a modular Hopf algebra. Note that this terminology is different from [8; 11, Chap. XI.5]. We should say that it is a semisimple modular Hopf algebra.

The center $Z(H)$ has basis $e_0, e_1, \ldots, e_n$ so that the Drinfeld map Φ: $C_q(H) \to Z(H)$ can be written

$$\Phi(\chi_{q,j}) = \sum_i a_{ij} e_i.$$

PROPOSITION 4.2. *We have*

$$S_{ij} = \dim_q(V_i) a_{ij}.$$

Proof. This follows easily from

$$\chi_{q,j}(e_i) = \begin{cases} \dim_q(V_i) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

which follows from the definition. Q.E.D.

COROLLARY 4.3. (H, R, v) *is modular if and only if* Φ: $C_q(H) \xrightarrow{\sim} Z(H)$ *and* $\dim_q(V_i) \neq 0$ *for* $0 \leq i \leq n$.

We show that the latter condition is unnecessary.

PROPOSITION 4.4. *If* Φ: $C_q(H) \xrightarrow{\sim} Z(H)$, *then* H *is cosemisimple*.

Proof. Take a nonzero left integral λ in H^*. Since H is unimodular, λ is in $C_q(H)$ and we have

$$\lambda^2 = \langle \lambda, 1 \rangle \lambda.$$

The algebra $C_q(H)$ has no nonzero nilpotent element, since it is isomorphic to $Z(H)$, a finite direct product of k. Hence $\langle \lambda, 1 \rangle \neq 0$. Q.E.D.

THEOREM 4.5 [3]. *Semisimple and cosemisimple Hopf algebras are involutory.*

Assume Φ: $C_q(H) \tilde{\to} Z(H)$. It follows that $S^2 = id$, hence u is central and $u = Su$ (Section 1, Example). Therefore

$$G^{-1} = (Su)v = uv = G.$$

Thus G is a central group-like element with $G^2 = 1$. It follows that $Ge_i = \pm e_i$, hence

$$\dim_q(V_i) = \pm m_i.$$

But we know that $m_i \neq 0$ in k by [5, Theorem 2.8]. Hence we have the folloving conclusion.

THEOREM 4.6. *Let (H, R, v) be a finite-dimensional semisimple ribbon Hopf algebra over the algebraically closed field k.*

(1) *It is modular if and only if Φ: $C_q(H) \tilde{\to} Z(H)$.*

(2) *If it is modular, then $S^2 = id$ and (H, R, u^{-1}) is also modular.*

(3) *It is modular if (H, R) is factorizable.*

5. WEAKLY FACTORIZABLE HOPF ALGEBRAS

DEFINITION 5.1. A finite-dimensional quasi-triangular Hopf algebra (H, R) is called weakly factorizable if the Drinfeld map Φ induces an isomorphism $C_q(H) \tilde{\to} Z(H)$.

Factorizable Hopf algebras are weakly factorizable (Proposition 2.4). If (H, R) is weakly factorizable and semisimple, then it is also cosemisimple (Proposition 4.4), hence $S^2 = id$. If, further, k is algebraically closed, (H, R, v) becomes modular for any ribbon element v. In pairticular, (H, R, u^{-1}) is modular for the canonical ribbon element u^{-1}.

Assume that k is algebraically closed and let (H, R) be a semisimple weakly factorizable Hopf algebra. We are interested in the modular ribbon Hopf algebra (H, R, u^{-1}). Then $G = 1$ so that quantum trace, quantum dimension, and quantum characters reduce to the usual trace, dimension and character. We use the notation of Section 4. In particular, we have

$$\Phi(\chi_j) = \sum_i a_{ij} e_i, \tag{5.2}$$

$$S_{ij} = m_i a_{ij}, \tag{5.3}$$

where m_i is the degree of the irreducible character χ_i.

Define i^* by $V_{i^*} \cong V_i^*$. We have the identities [11, 3.2.2 Lemma]

$$\sum_i m_i S_{ij} = \dim(H)\,\delta_{0j}, \tag{5.4}$$

$$\sum_j S_{ij} S_{jk} = \dim(H)\,\delta_{i,k^*}. \tag{5.5}$$

We show later that these identities follow from the well-known orthogonal relations of Hopf characters.

Etingof and Gelaki [2] have shown by using these identities that $m_i^2 \mid \dim(H)$ if $\mathrm{char}(k) = 0$.

Proposition 5.6 [2, Lemma 1.2]. *If* $\mathrm{char}(k) = 0$, a_{ij} *are all algebraic integers.*

This follows easily from the assumption that $\Phi\colon C(H) \xrightarrow{\sim} Z(H)$ and the fact that $\mathbb{Z}\chi_0 + \mathbb{Z}\chi_1 + \cdots + \mathbb{Z}\chi_n$ forms a $\mathbb{Z}$ form of the character algebra $C(H)$.

Since $S_{jk} = S_{kj}$, (5.5) implies that

$$\sum_j m_i a_{ij} m_k a_{kj} = \dim(H)\,\delta_{i,k^*}.$$

Since $m_k = m_{k^*}$, this implies that

$$\frac{\dim(H)}{m_i^2} = \sum_j a_{ij} a_{i^*,j}.$$

Hence we have

Theorem 5.7. *Let* (H, R) *be a finite-dimensional semisimple weakly factotizable quasi-triangular Hopf algebra over an algebraically closed field* k *of characteristic* 0. *Then* $\dim(V)^2$ *divides* $\dim(H)$ *for every irreducible* H *module* V.

This is a slight generalization of [2, Theorem 1.4; 9, Theorem 3.2]. Note that the big Theorem 4.5 is unnecessary in characteristic 0, by [6].

Getting back to arbitrary characteristic, let $t \in H$ be an integral such that $\varepsilon(t) = 1$. The identities

$$\langle \chi_j, t \rangle = \delta_{0j}, \tag{5.8}$$

$$\langle \chi_j \chi_{k^*}, t \rangle = \delta_{jk} \tag{5.9}$$

are well-known [5, Theorem 2.7]. We show that (5.4) and (5.5) follow from (5.8) and (5.9). We have

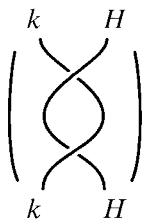
$$\chi_H = m_0 \chi_0 + \cdots + m_n \chi_n. \tag{5.10}$$

Since H is semisimple and $S^2 = id$, it is an integral in H^* [5, Proposition 4.1]. Thus we have

$$\chi_H^2 = \dim(H)\,\chi_H \tag{5.11}$$

since $\langle \chi_H, 1 \rangle = \dim(H)$. Since $C(H)$, which is isomorphic to $Z(H)$, has no nonzero nilpotent element, it follows that $\dim(H) \neq 0$ in k. Now, $\frac{1}{\dim(H)}\chi_H$ is a primitive idempotent in $C(H)$, since it is in H^*. Hence $\Phi(\frac{1}{\dim(H)}\chi_H) = e_i$ for some i. But we have

$$\langle \chi_0, \Phi(\chi_H) \rangle = \mathrm{tr}$$



$$= \dim(H).$$

This means $i = 0$. Thus we have

$$\Phi(\chi_H) = \dim(H) e_0. \tag{5.12}$$

This implies that we have a commutative diagram

$$\begin{tikzcd} C(H) \arrow[rr, "\Phi", "\sim"'] \arrow[dr, "\dim(H)t"'] & & Z(H) \arrow[dl, "\chi_H"] \\ & k & \end{tikzcd}$$

Proof of (5.8) $\Rightarrow$ (5.4).

We have

$$\langle \chi_H, \Phi(\chi_j) \rangle = \dim(H) \langle \chi_j, t \rangle.$$

Since $\langle \chi_H, e_i \rangle = m_i^2$, we have

$$\sum_i m_i^2 a_{ij} = \sum_i m_i S_{ij} = \dim(H)\,\delta_{0j}.$$

Proof of (5.9) $\Rightarrow$ (5.5).

We have

$$\left\langle \chi_H, \Phi(\chi_j)\Phi(\chi_k)\right\rangle = \dim(H)\langle \chi_j \chi_k, t\rangle.$$

Since we have

$$\Phi(\chi_j)\Phi(\chi_k) = \sum_i a_{ij} a_{ik} e_i,$$

we have

$$\sum_i m_i^2 a_{ij} a_{ik} = \sum_i S_{ij} S_{ik} = \dim(H)\,\delta_{j,k^*}.$$

REFERENCES

1. V. G. Drinfeld, On almost cocommutative Hopf algebras, *Leningrad Math. J.* **1** (1990), 321–342.
2. P. Etingof and S. Gelaki, Some properties of finite dimensional semisimple Hopf algebras, *Math. Res. Lett.* **5** (1998), 191–197.
3. P. Etingof and S. Gelaki, On finite dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, *Int. Math. Res. Notes* **16** (1998), 851–864.
4. C. Kassel, "Quantum Groups," Graduate Texts in Mathematics, Vol. 155, Springer, New York, 1995.
5. R. G. Larson, Characters of Hopf algebras, *J. Algebra* **17** (1971), 352–368.
6. R. G. Larson and D. E. Radford, Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, *J. Algebra* **117** (1988), 267–289.
7. N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, *J. Geom. Phys.* **5** (1988), 533–550.
8. N. Reshetikhin and V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–598.
9. H.-J. Schneider, Some properties of factorizable Hopf algebras, preprint.
10. V. Turaev, Modular categories and 3-manifold invariants, *Int. J. Modern Phys. B* **6** (1992), 1807–1824.
11. V. Turaev, "Quantum Invariants of Knots and 3-Manifolds," de Gruyter Studies in Mathematics, Vol. 18, de Gruyter, Berlin, 1994.