



Asymptotics of Plancherel-type random partitions

Alexei Borodin^a, Grigori Olshanski^{b,*}

^a *Mathematics 253-37, Caltech, Pasadena, CA 91125, USA*

^b *Dobrushin Mathematics Laboratory, Institute for Information Transmission Problems, Bolshoy Karetny 19, 127994 Moscow GSP-4, Russia*

Received 15 September 2006

Available online 24 January 2007

Communicated by Victor Kac, Ruth Kellerhals, Friedrich Knop, Peter Littelmann and Dmitri Panyushev

Dedicated to E.B. Vinberg

Abstract

We present a solution to a problem suggested by Philippe Biane: we prove that a certain Plancherel-type probability distribution on partitions converges, as partitions get large, to a new determinantal random point process on the set \mathbb{Z}_+ of nonnegative integers. This can be viewed as an edge limit transition. The limit process is determined by a correlation kernel on \mathbb{Z}_+ which is expressed through the Hermite polynomials, we call it the discrete Hermite kernel. The proof is based on a simple argument which derives convergence of correlation kernels from convergence of unbounded self-adjoint difference operators. Our approach can also be applied to a number of other probabilistic models. As an example, we discuss a bulk limit for one more Plancherel-type model of random partitions.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Plancherel measure; Random partitions; Determinantal processes; Correlation kernel

Introduction

This work appeared as our attempt to solve a problem posed by Philippe Biane. In [Bi2] he considered a model of random partitions arising from decomposition of tensor spaces $(\mathbb{C}^N)^{\otimes n}$ (the Schur–Weyl duality between representations of the symmetric group S_n and the unitary group $U(N)$). The partitions in question have at most N nonzero parts which sum to n , and the

* Corresponding author.

E-mail addresses: borodin@caltech.edu (A. Borodin), olsh@online.ru (G. Olshanski).

weight of a partition λ is proportional to the product of dimensions of the irreducible representations of S_n and $U(N)$ indexed by λ .

Biane discovered that as n and N go to infinity so that $\sqrt{n} \sim cN$ then the boundary of the Young diagram associated to the random partition λ , suitably scaled, tends to a nonrandom limit curve given by an explicit formula. The limit curve depends on the parameter $c > 0$.

If n is fixed while $N \rightarrow \infty$ then the model turns into the well-known Plancherel model of random partitions of n . This agrees with the fact that as c approaches 0, Biane’s limit curve turns into the celebrated Vershik–Kerov–Logan–Shepp limit curve for the Plancherel model found in [VK1, VK2, LS].

Biane’s formulas show that the value $c = 1$ is special: the tangent line to the limit curve at one of its endpoints has (in appropriate coordinates) slope -1 for $c < 1$, 0 for $c = 1$, and $+1$ for $c > 1$. Biane’s question concerned the local structure of the boundary of the random Young diagram at $c = 1$ near this point of the limit shape.

We address this question in a modified form. Namely, we replace the initial probability distribution on partitions by its poissonization with respect to parameter n . This procedure is well known, we explain it in Section 4. One expects that poissonization does not affect the asymptotics but we leave the discussion of this issue out of this paper.

After poissonization, the probability distribution lives on all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ with at most N nonzero parts and without any constraints on $|\lambda| = \lambda_1 + \dots + \lambda_N$. It is convenient to interpret λ as an N -point configuration on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ via

$$\lambda \mapsto \{x_1, \dots, x_N\}, \quad x_i = \lambda_i + N - i.$$

The weight of λ now depends on the poissonization parameter $\nu > 0$ (which replaces n) and has the form

$$\text{const} \cdot \prod_{i=1}^N \frac{(\nu/N)^{x_i}}{x_i!} \cdot \prod_{1 \leq i < j \leq N} (x_i - x_j)^2.$$

This is the so-called Charlier orthogonal polynomial ensemble.

Our main result is the following statement (see Theorem 4.1 and Proposition 3.3).

Theorem. Fix an arbitrary $s \in \mathbb{R}$. Let $N = 1, 2, \dots$ and assume that the parameter $\nu = \nu(N)$ depends on N in such a way that

$$\nu = N^2 + (s + o(1))N^{3/2}, \quad N \rightarrow \infty.$$

As $N \rightarrow \infty$, the probability distribution of $\{x_1, \dots, x_N\}$ converges to a probability measure on $2^{\mathbb{Z}_+}$, the set of all point configurations Y on \mathbb{Z}_+ . The correlation functions of the limit measure have the form ($k = 1, 2, \dots$)

$$\text{Prob}\{Y \supset \{y_1, \dots, y_k\}\} = \det[K_s(y_i, y_j)]_{1 \leq i, j \leq k}, \quad y_1, \dots, y_k \in \mathbb{Z}_+,$$

where, for $x, y \in \mathbb{Z}_+$,

$$\begin{aligned}
 K_s(x, y) &= (\pi x!y!2^{x+y})^{-1/2} \int_{s/\sqrt{2}}^{+\infty} e^{-t^2} H_x(t)H_y(t) dt \\
 &= (\pi x!y!2^{x+y})^{-1/2} e^{-s^2/2} \frac{xH_{x-1}(s/\sqrt{2}) \cdot H_y(s/\sqrt{2}) - H_x(s/\sqrt{2}) \cdot yH_{y-1}(s/\sqrt{2})}{x - y} \\
 &= (4\pi x!y!2^{x+y})^{-1/2} e^{-s^2/2} \frac{H_{x+1}(s/\sqrt{2})H_y(s/\sqrt{2}) - H_x(s/\sqrt{2})H_{y+1}(s/\sqrt{2})}{x - y}.
 \end{aligned}$$

Here H_m is the classical Hermite polynomial, see [KS].

The introduction of the additional parameter s above is also due to Biane.

The determinantal structure of the correlation functions means that the limit measure belongs to the class of determinantal random point processes which arise in a variety of probabilistic models, see, e.g., [So1,So2,BHKPV,Ly]. In particular, the determinantal processes arise in connection with the Plancherel measure, see [Jo1,BOO].

We call $K_s(x, y)$ (the kernel of the determinantal formula) the *discrete Hermite kernel*. Many similar examples of correlation kernels are known, however, to our best knowledge, the discrete Hermite kernel is new.

To prove the theorem we have to check that the Charlier correlation kernel (which is essentially the N th Christoffel–Darboux kernel for the orthogonal Charlier polynomials) converges to the discrete Hermite kernel. Usually such facts are verified using asymptotics of orthogonal polynomials (see, e.g., [Jo1] for a different limit regime for the Charlier kernel). Such an approach is applicable to our problem as well. However, we take another path and extract the needed convergence from an abstract theorem concerning strong resolvent convergence of unbounded self-adjoint operators. These self-adjoint operators appear as difference operators on \mathbb{Z}_+ associated to the Charlier polynomials. This approach seems to be new,¹ and it appears to be much less technical comparing to the traditional one.

To demonstrate the effectiveness of this approach we apply it to another model of representation-theoretic origin. This model appeared in the works of Biane [Bi1] and of Pittel and Romik [PR], it turns out to be related to the so-called Krawtchouk orthogonal polynomial ensemble. In Section 5 we sketch a proof of the convergence of the Krawtchouk kernel to the discrete sine kernel. This result cannot be viewed as new: it can be extracted from [IS,BKMM]. The point is that our argument is short and direct. We show that the result allows one to predict the form of the limit shape obtained in [Bi1,PR].

1. Preliminaries and the Plancherel model

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ be the set of nonnegative integers. Recall that a *partition* is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonincreasing numbers from \mathbb{Z}_+ with finitely many nonzero terms. The sum of the terms is denoted as $|\lambda|$. We say that λ is a partition of n if $n = |\lambda|$.

Following [Ma] we identify partitions and Young diagrams. A *Young diagram* is a finite collection of unit squares in the quarter plane with coordinates (i, j) , where the i -axis is di-

¹ Even though the method of deriving the asymptotics of special functions through differential equations is well known, we have never seen the same idea applied to correlation kernels.

rected downward and the j -axis is directed to the right; a square with lower right corner $(i, j) \in \{1, 2, \dots\} \times \{1, 2, \dots\}$ enters the diagram of λ if and only if $\lambda_i \geq j$. Thus, the diagram λ has λ_i squares in the i th row with the numbering of rows ranging from the top to the bottom, and the total number of squares is equal to $|\lambda|$. The set of all partitions (= Young diagrams) will be denoted as \mathbb{Y} and the subset of partitions of $n \in \mathbb{Z}_+$ will be denoted as \mathbb{Y}_n .

The *conjugate* partition λ' is obtained, in terms of Young diagrams, by transposing the coordinate axes. Clearly, λ'_1 coincides with the number of nonzero terms λ_i (or the number of nonvoid rows in the diagram); this number is also denoted as $\ell(\lambda)$.

The *boundary* of a diagram λ is a broken line going from the point $(i, j) = (\lambda'_1, 0)$ to the point $(i, j) = (0, \lambda_1)$. It is convenient to add to the boundary those parts of the coordinate axes that are below $(\lambda'_1, 0)$ and to the right of $(0, \lambda_1)$. The boundary of λ will be denoted as $\partial\lambda$.

Partitions $\lambda \in \mathbb{Y}$ can also be regarded as *particle configurations* on a 1-dimensional lattice. Let $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ denote the set of (proper) half-integers. We assign to λ an infinite subset of \mathbb{Z}' :

$$X(\lambda) = \left\{ \lambda_i - i + \frac{1}{2} \right\}, \quad i = 1, 2, \dots, \tag{1.1}$$

and we regard $X(\lambda)$ as a configuration of *particles* sitting at nodes of the lattice \mathbb{Z}' . The unoccupied nodes of \mathbb{Z}' will be called *holes*. Note a duality relation between particles and holes: reflecting the configuration of holes about 0 we get $X(\lambda')$.

The boundary $\partial\lambda$ (with parts of the coordinate axes included) can be viewed as a doubly infinite sequence of horizontal and vertical unit segments, and the particle configuration $X(\lambda)$ is a convenient way to encode that sequence. Specifically, a node $k \in \mathbb{Z}'$ is occupied by a particle from $X(\lambda)$ if and only if the line $j - i = k$ meets the boundary $\partial\lambda$ at the midpoint of a vertical segment. Likewise, the holes correspond to the midpoints of horizontal segments. This correspondence makes evident the particle/hole duality mentioned above.

Let S_n denote the symmetric group of degree n . The irreducible S_n -modules are parametrized by the Young diagrams with n squares. The set of such diagrams will be denoted as \mathbb{Y}_n . For an arbitrary diagram $\lambda \in \mathbb{Y}_n$, let $\dim \lambda$ denote the dimension of the corresponding irreducible module. Equivalently, $\dim \lambda$ equals the number of standard Young tableaux of the shape λ .

By Burnside’s theorem,

$$\sum_{\lambda \in \mathbb{Y}_n} (\dim \lambda)^2 = n!$$

The *Plancherel measure* on \mathbb{Y}_n , denoted as $M_n^{\text{Plancherel}}$, is defined as the probability measure with the weights

$$M_n^{\text{Plancherel}}(\lambda) = \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n.$$

Regard diagrams $\lambda \in \mathbb{Y}_n$ as random objects defined on the probability space $(\mathbb{Y}_n, M_n^{\text{Plancherel}})$. As $n \rightarrow \infty$, the boundary of the random diagram, scaled by the factor of $n^{-1/2}$, tends to a (nonrandom) limit curve. This is a well-known result due to Logan–Shepp [LS] and Vershik–Kerov [VK1, VK2] (see also Kerov’s book [Ke2]). Specifically, if i and j denote the initial row

and column coordinates then the scaled coordinates are defined as $x = i \cdot n^{-1/2}$ and $y = j \cdot n^{-1/2}$, and the equation of the limit curve in the (x, y) plane can be written as

$$x + y = \Omega(y - x), \quad -2 \leq y - x \leq 2,$$

where

$$\Omega(u) = \frac{2}{\pi} \left(u \arcsin \frac{u}{2} + \sqrt{4 - u^2} \right).$$

This result leads to the following conclusions:

- (a) Observe that the limit curve meets the coordinate axes $x = 0$ and $y = 0$ at points $(0, 2)$ and $(2, 0)$, respectively. This suggests that the first row λ_1 and the first column λ'_1 of the typical Plancherel diagram $\lambda \in \mathbb{Y}_n$ should grow as $2\sqrt{n}$, which is indeed true: see [VK1, VK2] for a precise statement. Moreover, the same holds for each of the largest row and column lengths λ_k, λ'_k , where the index k is arbitrary but fixed.
- (b) Let, as above, i and j be the row and column coordinates. Fix $u \in (-2, 2)$ and let a_n be a sequence of positive numbers such that $a_n \rightarrow \infty$ and $a_n/\sqrt{n} \rightarrow 0$. Then, as n gets large, the proportion of horizontal (respectively, vertical) steps of the random boundary, contained in the strip $|j - i - u \cdot \sqrt{n}| \leq a_n$, should be close to $(1 + \Omega'(u))/2$ (respectively, to $(1 - \Omega'(u))/2$), where $\Omega'(u)$ is the derivative of $\Omega(u)$. This statement just means that the slope of the boundary of our random Young diagram approximates the slope of the limit curve.

A somewhat different but essentially equivalent formulation is as follows. Let k_n be a sequence of half-integers such that $k_n/\sqrt{n} \rightarrow u \in (-2, 2)$. Given n , look at the intersection of the line $j - i = k_n$ with the boundary of the random diagram $\lambda \in \mathbb{Y}_n$. This is a midpoint of a boundary segment, which can be either horizontal or vertical. Then, for n large, the probability to find a horizontal segment should be close to $(1 + \Omega'(u))/2$. This is indeed true, see [BOO].

- (c) As u ranges over $(-2, 2)$, the quantity $(1 + \Omega'(u))/2$ monotonically increases, so that that the probability of finding horizontal fragments increases, too. At the endpoints -2 and 2 the quantity $(1 + \Omega'(u))/2$ takes values 0 and 1 (that is, the limit curve is tangent to the coordinate axes $y = 0$ and $x = 0$). This suggests that, typically, each of the differences $\lambda_k - \lambda_{k+1}$ or $\lambda'_k - \lambda'_{k+1}$ (with k fixed) should grow as $n \rightarrow \infty$. A much more precise statement can be found in [Ok,BOO,Jo1]. In particular, it turns out that the order of growth of these differences is $n^{1/6}$.

In the next sections we will consider two other models of random Young diagrams which may be viewed as deformations of the Plancherel model.

2. Biane’s model

There is a close relationship between the Plancherel model and the biregular representation of the symmetric group. Indeed, the group S_n acts on itself by left and right shifts. The corresponding representation of the group $S_n \times S_n$ in the space of functions on S_n has simple spectrum indexed by diagrams $\lambda \in \mathbb{Y}_n$, and $(\dim \lambda)^2$ is just the dimension of the irreducible component

indexed by λ . Since $n!$ is the dimension of the whole representation space, the Plancherel weight of λ is equal to the relative dimension of the irreducible component indexed by λ .

Now we apply the same construction but starting with a different representation with simple spectrum. Let n and N be two positive integers. Consider the tensor space $(\mathbb{C}^N)^{\otimes n}$ as a bimodule with respect to the natural commuting actions of the symmetric group S_n and the unitary group $U(N)$. By the Schur–Weyl duality, the representation of the group $S_n \times U(N)$ in $(\mathbb{C}^N)^{\otimes n}$ has simple spectrum which is indexed by Young diagrams λ such that $|\lambda| = n$ and $\ell(\lambda) \leq N$. Let $\mathbb{Y}_n(N)$ stand for the set of such diagrams. For $\lambda \in \mathbb{Y}_n(N)$, the dimension of the corresponding irreducible component equals $\dim \lambda \cdot \text{Dim}_N \lambda$, where by $\text{Dim}_N \lambda$ we denote the dimension of the irreducible polynomial $U(N)$ -module with highest weight $(\lambda_1, \dots, \lambda_N)$. This serves as a prompt for introducing a probability measure on $\mathbb{Y}_n(N)$:

$$M_{n,N}^{\text{Schur-Weyl}}(\lambda) = \frac{\dim \lambda \cdot \text{Dim}_N \lambda}{N^n}, \quad \lambda \in \mathbb{Y}_n(N) \tag{2.1}$$

(the factor N^n in the denominator is the dimension of the whole tensor space).

Let us take $M_{n,N}^{\text{Schur-Weyl}}$ as the distribution law for a random ensemble of diagrams $\lambda \in \mathbb{Y}_n(N)$ and ask about the asymptotic properties of this ensemble as n and N go to infinity.

For the first time, this question was addressed by Sergei Kerov [Ke1] (see also [Ke2, Chapter III, Section 3]). He showed that if n and N have the same order of growth (that is, n/N tends to a positive constant) then, after scaling by the factor of $n^{-1/2}$, the boundary of the random diagram tends to a limit shape, which is exactly the same as in the case of the Plancherel model. This result admits the following heuristic explanation.

If $N \geq n$ then $\mathbb{Y}_n(N)$ coincides with \mathbb{Y}_n , and it is readily checked that

$$\lim_{N \rightarrow \infty} M_{n,N}^{\text{Schur-Weyl}}(\lambda) \rightarrow M_n^{\text{Plancherel}}(\lambda), \quad \lambda \in \mathbb{Y}_n.$$

On the other hand, a typical Plancherel diagram $\lambda \in \mathbb{Y}_n$ has approximately $2\sqrt{n}$ rows, which explains why the constraint of type $\ell(\lambda) \leq N = O(n)$ turns out to be asymptotically negligible.

Finer results were obtained later by Philippe Biane [Bi2].² He examined a family of limit regimes depending on parameter $c \in (0, +\infty)$:

$$n \rightarrow \infty, \quad N \sim c^{-1}\sqrt{n},$$

and discovered that, for c fixed, the scaled random diagrams concentrate near a limit curve $x + y = P_c(y - x)$ depending on c . The curves $v = P_c(u)$ are explicitly described in [Bi2, Section 3.1], they provide an interesting deformation of the Plancherel curve $v = \Omega(u)$, which appears as the limit case $c = 0$.

Look at the intersection of the curve $v = P_c(u)$ with the line $v = -u$, which happens at $u = c - 2$. A close examination of Biane’s formulas (see the end of Section 3.1 in [Bi2]) reveals the following fact:

$$\left. \frac{dP_c(u)}{du} \right|_{u=c-2} = \begin{cases} -1, & c < 1, \\ 0, & c = 1, \\ +1, & c > 1. \end{cases}$$

² We strongly encourage the reader to look at this paper for a better understanding of what follows.

Our interest is what happens at the critical value $c = 1$ corresponding to the limit regime $n \sim N^2$.

Assign to $\lambda \in \mathbb{Y}_n(N)$ an N -particle configuration on \mathbb{Z}_+ :

$$\tilde{X}(\lambda) = \{x_1, \dots, x_N\}, \quad x_i = \lambda_i + N - i \tag{2.2}$$

(note a difference from (1.1); due to the restriction $\ell(\lambda) \leq N$, λ is uniquely determined by $\tilde{X}(\lambda)$). Then the probability space $(\mathbb{Y}_n(N), M_{n,N}^{\text{Schur-Weyl}})$ gives rise to an ensemble of random N -particle configurations on \mathbb{Z}_+ . More generally, we will deal with ensembles of infinite random particle configurations as well. Such ensembles are examples of what is called a *random point process* (or random point field). For a discrete state space \mathfrak{X} (in our concrete case $\mathfrak{X} = \mathbb{Z}_+$), a random point process in \mathfrak{X} is determined by specifying a probability measure \mathcal{P} on the space $\text{Conf}(\mathfrak{X}) = 2^{\mathfrak{X}}$ of all subsets in \mathfrak{X} . Note that $\text{Conf}(\mathfrak{X})$ is a compact topological space in the natural topology.³

Conjecture 2.1 (Biane). *Consider the random Young diagram λ distributed according to the measure $M_{n,N}^{\text{Schur-Weyl}}$ on $\mathbb{Y}_n(N)$. Assume that $n \rightarrow \infty$ and*

$$N = n^{1/2} - \frac{1}{2}s \cdot n^{1/4} + o(n^{1/4}),$$

where s is an arbitrary fixed real number. Equivalently,

$$n = N^2 + (s + o(1))N^{3/2}.$$

Then the random configuration $\tilde{X}(\cdot)$ converges to a nontrivial random point process on \mathbb{Z}_+ depending on s .

Here convergence means weak convergence of probability measures on the compact space $\text{Conf}(\mathbb{Z}_+)$. The limit process is nontrivial in the sense that the limiting measure on $\text{Conf}(\mathbb{Z}_+)$ does not reduce to the delta measure on the empty configuration or on the configuration coinciding with the whole set \mathbb{Z}_+ .

In Section 3 we introduce the random point processes that appear as limit processes for Biane’s model, and in Section 4 we verify Biane’s conjecture in a modified formulation.

3. The discrete Hermite kernel

We start with some necessary generalities. Let \mathfrak{X} be a discrete space and \mathcal{P} be a random point process in \mathfrak{X} (that is, a probability measure on $\text{Conf}(\mathfrak{X})$). The *correlation functions* ρ_n of \mathcal{P} ($n = 1, 2, \dots$) are probabilities for random configurations $X \in \text{Conf}(\mathfrak{X})$ to contain a given finite set $\{x_1, \dots, x_n\}$:

$$\rho_n(x_1, \dots, x_n) = \text{Prob}\{\{x_1, \dots, x_n\} \subseteq X\}.$$

The initial measure \mathcal{P} is uniquely determined by the correlation functions ρ_1, ρ_2, \dots . Indeed, for any finite subset $A \subset \mathfrak{X}$, there is a natural projection $\text{Conf}(\mathfrak{X}) \rightarrow \text{Conf}(A)$ given by taking

³ About random point processes in general, see, e.g., [DVJ,So1].

intersection: $X \mapsto X_A = X \cap A$. Let \mathcal{P}_A be the pushforward of \mathcal{P} under this projection; this is a probability measure on the finite set $\text{Conf}(A)$. Using the inclusion–exclusion principle it is readily seen that \mathcal{P}_A is determined by the values of the correlation functions on A . For instance, for $A = \{a, b\}$ we have

$$\begin{aligned} \text{Prob}\{X_A = \{a, b\}\} &= \rho_2(a, b), \\ \text{Prob}\{X_A = \{a\}\} &= \rho_1(a) - \rho_2(a, b), \\ \text{Prob}\{X_A = \{b\}\} &= \rho_1(b) - \rho_2(a, b), \\ \text{Prob}\{X_A = \emptyset\} &= 1 - \rho_1(a) - \rho_1(b) + \rho_2(a, b). \end{aligned}$$

On the other hand, the initial measure \mathcal{P} is clearly determined by collection of the measures \mathcal{P}_A .

We say that a sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ of random point processes in \mathfrak{X} converges to a random point process \mathcal{P} in the same space, $\mathcal{P}_k \rightarrow \mathcal{P}$, if the corresponding probability measures weakly converge. This happens if and only if the correlation functions of the processes \mathcal{P}_k pointwise converge to the respective correlation functions of the process \mathcal{P} . Indeed, by the definition of the topology in $\text{Conf}(\mathfrak{X})$, we have $\mathcal{P}_k \rightarrow \mathcal{P}$ if and only if $(\mathcal{P}_k)_A \rightarrow \mathcal{P}_A$ for any finite $A \subset \mathfrak{X}$, and the latter is clearly equivalent to convergence of correlation functions.

A random point process in \mathfrak{X} is said to be *determinantal* if there exists a function $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$ such that the correlation functions are given by the determinantal formula

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)], \quad n = 1, 2, \dots,$$

where the determinant in the right-hand side has the order $n \times n$. Then K is called the *correlation kernel* of \mathcal{P} . Thus, a determinantal process is uniquely determined by its correlation kernel. If a kernel K is Hermitean-symmetric, $K(x, y) = \overline{K(y, x)}$, then it serves as a correlation kernel of a random point process if and only if $\|K\| \leq 1$, that is, K corresponds to a contractive operator in the Hilbert space $\ell^2(\mathfrak{X})$. Indeed, this is a very special case of [So1, Theorem 3].

In particular, any projection kernel (that is, the kernel corresponding to a self-adjoint projection operator in $\ell^2(\mathfrak{X})$) determines a random point process in \mathfrak{X} . We will introduce now a concrete family of projection kernels which we will need in the sequel.

Consider the semi-infinite Jacobi matrix

$$D^{\text{Hermite}} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.1}$$

(the origin of this matrix and of its notation will become clear soon). We agree that the rows and columns are indexed by the nonnegative integers $x \in \mathbb{Z}_+$. Then the matrix D^{Hermite} determines a symmetric operator in the Hilbert space $\ell^2(\mathbb{Z}_+)$: by definition, the domain of the operator is the algebraic span of the basis elements $\{\delta_x\}$, $x \in \mathbb{Z}_+$. We will denote this operator by the same symbol D^{Hermite} .

Lemma 3.1. *The operator D^{Hermite} is essentially self-adjoint. Its closure $\overline{D^{\text{Hermite}}}$ has simple purely continuous Lebesgue spectrum. For any Borel set $B \subseteq \mathbb{R}$, the corresponding spectral projection operator P_B is given by the kernel*

$$P_B(x, y) = (P_B \delta_y, \delta_x) = (2\pi x!y!2^{x+y})^{-1/2} \int_B e^{-t^2/2} H_x(t/\sqrt{2}) H_y(t/\sqrt{2}) d\sigma, \quad (3.2)$$

where x, y range over \mathbb{Z}_+ and $H_x(t)$ denotes the Hermite polynomial of degree x .

Proof. We will need a few basic facts concerning the classical moment problem. Let ρ be a measure on \mathbb{R} with infinite support and with finite moments of all orders. Then the space $\mathbb{C}[t]$ of polynomials in one variable can be viewed as a subspace of the Hilbert space $L^2(\mathbb{R}, \rho)$. Let $\overline{\mathbb{C}[t]}$ denote the closure of this subspace. Finally, let $m_n = \int t^n \rho(dt)$ be the moments of ρ , $n = 0, 1, 2, \dots$

(A) The operator of multiplication by t with domain of definition $\mathbb{C}[t]$ is essentially self-adjoint in $\overline{\mathbb{C}[t]}$ if and only if the moment problem associated with the sequence $\{m_n\}$ is determinate (that is, ρ is a unique measure on \mathbb{R} with moments m_n). See, e.g., [Si, Theorem 2, p. 86].

(B) If the moment problem associated with $\{m_n\}$ is determinate then $\overline{\mathbb{C}[t]}$ coincides with the whole space $L^2(\mathbb{R}, \rho)$. See, e.g. [Si, Proposition 4.15, p. 131] or [Ak, Corollary 2.3.3].

(C) The moment problem associated with $\{m_n\}$ is determinate if the moments grow not too fast. For instance, a simple sufficient condition says that the moment problem is determinate if the exponential generating series

$$g(z) = \sum_{n=0}^{\infty} m_n \frac{z^n}{n!}$$

is analytic in a neighborhood of $z = 0$ (which holds if the function $t \mapsto e^{zt}$ is ρ -integrable for sufficiently small z). See, e.g., [Si, Proposition 1.5, p. 88].

Finally, let p_0, p_1, \dots stand for the orthogonal polynomials with respect to ρ , normalized so that $\int p_n^2(t) \rho(dt) = 1$. Then $\{p_n\}$ is an orthonormal basis in $\overline{\mathbb{C}[t]}$. The polynomials p_n satisfy a three-term recurrence relation, which means that in the basis $\{p_n\}$, the matrix of the operator of multiplication by t is a (symmetric) tridiagonal matrix.

Now we return to the proof of the lemma. Take as ρ the normal distribution

$$\rho^{\text{Hermite}}(dt) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

The corresponding polynomials p_n are

$$\tilde{H}_n(t) = (n!2^n)^{-1/2} H_n(t/\sqrt{2}),$$

where the H_n s are the Hermite polynomials in the standard normalization, see [KS, Section 1.13]. The three-term recurrence relation for the H_n s has the form

$$H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t) = 0, \quad (3.3)$$

see [KS, Section 1.13]. Rewriting this in terms of the \tilde{H}_n s we get

$$t\tilde{H}_n(t) = \sqrt{n+1}\tilde{H}_{n+1}(t) + \sqrt{n}\tilde{H}_{n-1}(t).$$

Thus, the matrix of multiplication by t in the basis $\{\tilde{H}_n\}$ is just the Jacobi matrix D^{Hermite} as defined in (3.1).

It is readily checked that ρ^{Hermite} satisfies condition (C) above. By (B), the space of polynomials is dense in $L^2(\mathbb{R}, \rho^{\text{Hermite}})$. Consequently, $\{\tilde{H}_n\}$ is an orthonormal basis in $L^2(\mathbb{R}, \rho^{\text{Hermite}})$, and the correspondence $\delta_x \leftrightarrow \tilde{H}_x$ makes it possible to identify the Hilbert spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}, \rho^{\text{Hermite}})$. By (A), our operator D^{Hermite} is essentially self-adjoint.⁴ Now it is clear that we have obtained the explicit spectral decomposition for the corresponding self-adjoint operator $\overline{D^{\text{Hermite}}}$, the closure of D^{Hermite} . Namely, for any bounded real Borel function χ on \mathbb{R} , the operator $\chi(\overline{D^{\text{Hermite}}})$ is realized as the operator of multiplication by χ in $L^2(\mathbb{R}, \rho^{\text{Hermite}})$. In particular, taking as χ the characteristic function χ_B of a Borel set $B \subset \mathbb{R}$ we see that the corresponding spectral projection P_B is the operator of multiplication by χ_B . The matrix of P_B in the basis $\{\tilde{H}_n\}$ is given by formula (3.3), which concludes the proof. \square

Definition 3.2. The *discrete Hermite kernel* with parameter $s \in \mathbb{R}$ is the above spectral kernel corresponding to the set $B = [s, \infty)$. That is

$$K_s^{\text{Hermite}}(x, y) = (2\pi x!y!2^{x+y})^{-1/2} \int_s^{+\infty} e^{-t^2/2} H_x(t/\sqrt{2})H_y(t/\sqrt{2}) dt, \tag{3.4}$$

where $x, y \in \mathbb{Z}_+$.

The next proposition provides alternative expressions for this kernel.

Proposition 3.3. For $x \neq y$, the discrete Hermite kernel can also be written as

$$(\pi x!y!2^{x+y})^{-1/2} e^{-s^2/2} \cdot \frac{xH_{x-1}(s/\sqrt{2}) \cdot H_y(s/\sqrt{2}) - H_x(s/\sqrt{2}) \cdot yH_{y-1}(s/\sqrt{2})}{x - y} \tag{3.5}$$

or equivalently as

$$(4\pi x!y!2^{x+y})^{-1/2} e^{-s^2/2} \cdot \frac{H_{x+1}(s/\sqrt{2}) \cdot H_y(s/\sqrt{2}) - H_x(s/\sqrt{2}) \cdot H_{y+1}(s/\sqrt{2})}{x - y} \tag{3.6}$$

Proof. The equivalence of (3.5) and (3.6) follows from the three-term relation (3.3).

Next, we will employ two relations for Hermite polynomials (see [KS, (1.13.6) and (1.13.8)]):

$$(H_{n+1}(t))' = 2(n+1)H_n(t), \tag{3.7}$$

$$(e^{-t^2} H_n(t))' = -e^{-t^2} H_{n+1}(t). \tag{3.8}$$

⁴ Of course, all these facts are well known.

Abbreviating

$$C(x, y) = (2\pi x!y!2^{x+y})^{-1/2}$$

we have

$$\begin{aligned} K_s^{\text{Hermite}}(x, y) &= C(x, y) \int_s^{+\infty} e^{-t^2/2} H_x(t/\sqrt{2}) H_y(t/\sqrt{2}) dt \\ &= C(x, y) \sqrt{2} \int_{s/\sqrt{2}}^{+\infty} e^{-t^2} H_x(t) H_y(t) dt. \end{aligned}$$

Multiplying by $x - y = (x + 1) - (y + 1)$ and using (3.7) we get

$$(x - y) K_s^{\text{Hermite}}(x, y) = \frac{C(x, y)}{\sqrt{2}} \int_{s/\sqrt{2}}^{+\infty} e^{-t^2} (H'_{x+1}(t) H_y(t) - H_x(t) H'_{y+1}(t)) dt.$$

Integrating by parts and using (3.8) we finally get

$$\begin{aligned} (x - y) K_s^{\text{Hermite}}(x, y) &= \frac{C(x, y)}{\sqrt{2}} e^{-s^2/2} (H_{x+1}(s/\sqrt{2}) H_y(s/\sqrt{2}) - H_x(s/\sqrt{2}) H_{y+1}(s/\sqrt{2})), \end{aligned}$$

which equals the expression (3.6) multiplied by $x - y$. \square

4. Proof of modified Biane’s conjecture

We will apply a well-known trick called *poissonization*. Its general idea is to make a large parameter n random and obeying the Poisson distribution on \mathbb{Z}_+ with large parameter ν . Due to the asymptotic concentration of the Poisson distribution one believes that the large n limit regime and the large ν limit regime are equivalent (of course, this claim has to be justified in each concrete situation). On the other hand, the latter regime often turns out to be easier to study.

For instance, as shown in [Jo1,BOO], application of the poissonization procedure to the Plancherel measures $M_n^{\text{Plancherel}}$ leads to determinantal point processes. The same happens for the measures $M_{n,N}^{\text{Schur-Weyl}}$ (Lemma 4.2).

By definition, the poissonized version of $M_{n,N}^{\text{Schur-Weyl}}$, denoted as $M_{\nu,N}^{\text{Poisson-Schur-Weyl}}$, lives on the set

$$\mathbb{Y}(N) = \bigcup_{n=0}^{\infty} \mathbb{Y}_n(N) = \{ \lambda \in \mathbb{Y} \mid \ell(\lambda) \leq N \}$$

of all Young diagrams with at most N rows. This new measure depends on a positive parameter ν , and is given by

$$M_{\nu, N}^{\text{Poisson-Schur-Weyl}}(\lambda) = e^{-\nu} \frac{\nu^{|\lambda|}}{|\lambda|!} M_{|\lambda|, N}^{\text{Schur-Weyl}}(\lambda), \quad \lambda \in \mathbb{Y}(N). \tag{4.1}$$

Clearly, $M_{\nu, N}^{\text{Poisson-Schur-Weyl}}$ is a probability measure. In the present paper we do not justify the poissonization procedure and simply replace the measures $M_{n, N}^{\text{Schur-Weyl}}$ by their poissonized versions in Biane’s conjecture. Theorem 4.1 stated below proves the conjecture and identifies the limit process.

Let $X_{\nu, N}$ be the random N -particle configuration on \mathbb{Z}_+ obtained via the correspondence (2.2) from the random Young diagram λ distributed according to the measure $M_{\nu, N}^{\text{Poisson-Schur-Weyl}}$ on $\mathbb{Y}(N)$. That is, if $\{x_1, \dots, x_N\} = \tilde{X}(\lambda)$ then

$$\text{Prob}(\{x_1, \dots, x_N\}) = M_{\nu, N}^{\text{Poisson-Schur-Weyl}}(\lambda). \tag{4.2}$$

Theorem 4.1. Fix an arbitrary $s \in \mathbb{R}$. Let $N = 1, 2, \dots$ and assume that the parameter $\nu = \nu(N)$ depends on N in such a way that

$$\nu = N^2 + (s + o(1))N^{3/2}, \quad N \rightarrow \infty. \tag{4.3}$$

As $N \rightarrow \infty$, $X_{\nu(N), N}$ converges to the determinantal point process on \mathbb{Z}_+ with the correlation kernel $K_s^{\text{Hermite}}(x, y)$ as defined in Section 3.

Proof. It suffices to verify that the correlation functions of $X_{\nu(N), N}$ converge to the respective correlation functions given by correlation kernel $K_s^{\text{Hermite}}(x, y)$ (see the beginning of Section 3). To do this we will prove that $X_{\nu(N), N}$ is a determinantal process (Lemma 4.2) and its correlation kernel pointwise converges to $K_s^{\text{Hermite}}(x, y)$ (Lemma 4.4), which implies the claim of the theorem.

Consider the weight function for the Charlier polynomials with parameter $\theta > 0$:

$$W_\theta^{\text{Charlier}}(x) = \frac{\theta^x}{x!}, \quad x \in \mathbb{Z}_+,$$

see [KS, Section 1.12]. The N -particle Charlier ensemble is formed by random N -particle configurations $\{x_1, \dots, x_N\} \subset \mathbb{Z}_+$ such that

$$\text{Prob}(\{x_1, \dots, x_N\}) = \text{const}(N, \theta) \cdot \prod_{i=1}^N W_\theta^{\text{Charlier}}(x_i) \cdot \prod_{1 \leq i < j \leq N} (x_i - x_j)^2, \tag{4.4}$$

where $\text{const}(N, \theta)$ is the normalization constant (it can be evaluated explicitly but we do not need the precise expression).

Let $C_m(x; \theta)$ denote the Charlier polynomial of degree m , and let $\|C_m(\cdot; \theta)\|$ be its norm in the weighted ℓ^2 space with the weight function $W_\theta^{\text{Charlier}}$:

$$\|C_m(\cdot; \theta)\|^2 = \sum_{x=0}^\infty C_m^2(x; \theta) W_\theta^{\text{Charlier}}(x).$$

The normalized functions

$$\tilde{C}_m(x; \theta) = (W^{\text{Charlier}}(x))^{1/2} \|C_m(\cdot; \theta)\|^{-1} C_m(x; \theta), \quad m = 0, 1, 2, \dots$$

form an orthonormal system in the ordinary space $\ell^2(\mathbb{Z}_+)$.

As is well known, the Charlier ensemble is a determinantal point process with the correlation kernel

$$K_{N,\theta}^{\text{Charlier}}(x, y) = \sum_{m=0}^{N-1} \tilde{C}_m(x; \theta) \tilde{C}_m(y; \theta).$$

See, e.g., [Jo1,Ko]. This is a projection kernel: the corresponding operator is the projection in $\ell^2(\mathbb{Z}_+)$ on the N -dimensional subspace spanned by the functions $\tilde{C}_0, \dots, \tilde{C}_{N-1}$.

Lemma 4.2. *For any $\nu > 0$ and $N = 1, 2, \dots$, the random process $X_{\nu,N}$ coincides with the N -particle Charlier ensemble with parameter $\theta = \nu/N$.*

Proof. Let us compare the right-hand sides of (4.2) and (4.4). The right-hand side of (4.2) is defined by (2.1) and (4.1), this gives

$$e^{-\nu} \frac{\nu^{|\lambda|} \dim \lambda \text{Dim}_N \lambda}{|\lambda|! N^{|\lambda|}}.$$

We have

$$\frac{\dim \lambda}{|\lambda|!} = \frac{\prod_{1 \leq i < j \leq N} (x_i - x_j)}{\prod_{1 \leq i \leq N} x_i!} \tag{4.5}$$

(Frobenius’ formula, see, e.g., [Ma]) and

$$\text{Dim}_N \lambda = \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{j - i}$$

(Weyl’s character formula). Finally,

$$|\lambda| = x_1 + \dots + x_N - \frac{N(N - 1)}{2}.$$

Using these expressions we obtain the right-hand side of (4.4) with an appropriate constant. \square

Given $\theta > 0$, consider the semi-infinite Jacobi matrix

$$D_\theta^{\text{Charlier}} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & -1/\sqrt{\theta} & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & -2/\sqrt{\theta} & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & -3/\sqrt{\theta} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We also regard it as a symmetric operator in $\ell^2(\mathbb{Z}_+)$ whose domain of definition is the set of finite linear combination of the basis elements.

Lemma 4.3. *The operator $D_\theta^{\text{Charlier}}$ is essentially self-adjoint. Its closure $\overline{D_\theta^{\text{Charlier}}}$ has purely point spectrum $\{\frac{\theta-m}{\sqrt{\theta}} \mid m = 0, 1, \dots\}$. The kernel $K_{N,\theta}^{\text{Charlier}}$ coincides with the kernel of the spectral projection operator corresponding to the part of spectrum*

$$\left\{ \frac{\theta - m}{\sqrt{\theta}}, m = 0, 1, \dots, N - 1 \right\}. \tag{4.6}$$

Proof. The Charlier polynomials with parameter θ satisfy the difference equation

$$\theta C_m(x + 1; \theta) - x C_m(x; \theta) + x C_m(x - 1; \theta) = (\theta - m) C_m(x; \theta),$$

see [KS, (1.12.5)]. From the expression for the weight function it follows that for the normalized functions $\tilde{C}_m(x; \theta)$, this equation is transformed into

$$\sqrt{x + 1} \tilde{C}_m(x + 1; \theta) - \frac{x}{\sqrt{\theta}} \tilde{C}_m(x; \theta) + \sqrt{x} \tilde{C}_m(x - 1; \theta) = \frac{\theta - m}{\sqrt{\theta}} \tilde{C}_m(x; \theta).$$

We see that $D_\theta^{\text{Charlier}}$ is precisely the difference operator on \mathbb{Z}_+ standing in the left-hand side.

We claim that $D_\theta^{\text{Charlier}}$ is essentially self-adjoint and $\{\tilde{C}_m\}$ is the complete set of the eigenvectors of the self-adjoint operator $\overline{D_\theta^{\text{Charlier}}}$. Indeed, the three-term recurrence relation

$$-x C_m(x; \theta) = \theta C_{m+1}(x; \theta) - (m + \theta) C_m(x; \theta) + m C_{m-1}(x; \theta)$$

(see [KS, (1.12.3)]) and the explicit expression for the norm

$$\|C_m\|^2 = \theta^{-m} e^\theta m!$$

(see [KS, (1.12.2)]) imply that the same Jacobi matrix corresponds to the operator of multiplication by $(\theta - x)/\sqrt{\theta}$ in the basis $\{C_m/\|C_m\|\}$ of the space of polynomials. It is readily checked that the Charlier weight, viewed as a measure on \mathbb{Z}_+ , satisfies the sufficient condition (C), see the proof of Lemma 3.1. Then the same argument as in that lemma shows that the space of polynomials is dense in the weighted space $\ell^2(\mathbb{Z}_+, W_\theta^{\text{Charlier}})$ and the above multiplication operator is essentially self-adjoint.

This is equivalent to saying that the functions \tilde{C}_m form an orthonormal basis in $\ell^2(\mathbb{Z}_+)$ and $D_\theta^{\text{Charlier}}$ is essentially self-adjoint. Then it follows from the difference equation that $\overline{D_\theta^{\text{Charlier}}}$ has \tilde{C}_m as an eigenvector with eigenvalue $\frac{\theta-m}{\sqrt{\theta}}$. The last claim of the lemma is now obvious. \square

Lemma 4.4. *Let $s \in \mathbb{R}$ be fixed and assume $\theta = \theta(N) = N + (s + o(1))N^{1/2}$. Then*

$$\lim_{N \rightarrow \infty} K_{N,\theta(N)}^{\text{Charlier}}(x, y) = K_s^{\text{Hermite}}(x, y), \quad x, y \in \mathbb{Z}_+.$$

Here the assumption on θ comes from (4.3) and the relation $\theta = \nu/N$ (Lemma 4.2).

Proof. Consider the self-adjoint operators $\overline{D_\theta^{\text{Charlier}}}$ (where $\theta > 0$) and $\overline{D^{\text{Hermite}}}$ in $\ell^2(\mathbb{Z}_+)$. Let $\ell_0^2(\mathbb{Z}_+)$ denote the algebraic linear span of the basis elements δ_x , $x \in \mathbb{Z}_+$. By Lemmas 3.1 and 4.3, all these operators have $\ell_0^2(\mathbb{Z}_+)$ as a common essential domain. Moreover, it is evident that if $\theta \rightarrow \infty$ then $D_\theta^{\text{Charlier}} \rightarrow D^{\text{Hermite}}$ on $\ell_0^2(\mathbb{Z}_+)$. It follows that $\overline{D_\theta^{\text{Charlier}}} \rightarrow \overline{D^{\text{Hermite}}}$ in the strong resolvent sense (see [RS, Theorem VIII.25]).

Let us regard $K_{N,\theta}^{\text{Charlier}}$ and K_s^{Hermite} as operators in $\ell^2(\mathbb{Z}_+)$. By Lemma 3.1, the latter operator is the spectral projection of $\overline{D^{\text{Hermite}}}$ corresponding to the set $[s, +\infty)$. By Lemma 4.3, the former operator is the spectral operator of $\overline{D_\theta^{\text{Charlier}}}$ corresponding to the set (4.6). Next, it follows from the description of the spectrum of $\overline{D_\theta^{\text{Charlier}}}$ in Lemma 4.3 that instead of the finite set (4.6) we can equally well take the continuous interval

$$\left[\frac{\theta - N + 1}{\sqrt{\theta}}, +\infty \right).$$

If $\theta = \theta(N) = N + (s + o(1))N^{1/2}$ then the left end of this interval can be written as $s + \varepsilon_N$ where $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. Since $\overline{D^{\text{Hermite}}}$ has purely continuous spectrum, the strong resolvent convergence implies that the spectral projection of $\overline{D_{\theta(N)}^{\text{Charlier}}}$ corresponding to $[s + \varepsilon_N, +\infty)$ strongly converges to the spectral projection of $\overline{D^{\text{Hermite}}}$ corresponding to $[s, +\infty)$: this is proved exactly as claim (b) in [RS, Theorem VIII.24]. \square

Note that Lemma 4.4 could be obtained from the known asymptotics for the Laguerre polynomials [Te] and the well-known connection between the Laguerre and Charlier polynomials.

Lemma 4.4 completes the proof of Theorem 4.1. \square

5. Another model

Let N and M be two natural numbers, and (M^N) be the rectangular Young diagram with N rows and M columns. Given a Young diagram $\lambda \subseteq (M^N)$, we denote by $(M^N)/\lambda$ the skew diagram which is the difference of (M^N) and λ . Reading this skew diagram from the bottom to the top we get an ordinary Young diagram which will be denoted by $\hat{\lambda}$:

$$(\hat{\lambda}_1, \dots, \hat{\lambda}_N) = (M - \lambda_N, \dots, M - \lambda_1).$$

Let $\pi_{(M^N)}$ denote the irreducible representation of the symmetric group S_{NM} of degree NM , indexed by (M^N) . For any $n = 0, 1, \dots, NM$, the restriction of $\pi_{(M^N)}$ to the Young subgroup $S_n \times S_{NM-n}$ has simple spectrum consisting of the irreducible representations of the form $\pi_\lambda \otimes \pi_{\hat{\lambda}}$ (the outer tensor product of the irreducible representations of S_n and S_{NM-n} , indexed by λ and $\hat{\lambda}$, respectively). Indeed, this follows from the fact that the skew Schur function $s_{(M^N)/\lambda}$ equals the ordinary Schur function $s_{\hat{\lambda}}$, as is readily verified using the Jacobi–Trudi formula (see [Ma, Chapter I, (5.4)]).

Let $\mathbb{Y}_n(N, M)$ stand for the set of Young diagrams with n squares, contained in the rectangle (M^N) , $n = 0, 1, \dots, NM$. The above claim shows that the following expression defines a probability measure on $\mathbb{Y}_n(N, M)$, which will be denoted as $M_{n,N,M}$:

$$M_{n,N,M}(\lambda) = \frac{\dim \lambda \cdot \dim \hat{\lambda}}{\dim(M^N)}, \quad \lambda \in \mathbb{Y}_n(N, M).$$

It turns out that if the triple of parameters n, N, M goes to infinity in an appropriate way then the boundary of the random Young diagram distributed according to the measure $M_{n,N,M}$, after a suitable scaling, tends to a nonrandom curve: this is a particular case of the results in Biane [Bi1, Theorem 3.1.2] and Pittel and Romik [PR, Sections 1.1 and 1.5]. Biane’s approach uses free probability. The derivation of Pittel and Romik of the explicit form of the limit curve is based on the variational principle. Here we sketch a simple alternative argument. It does not rigorously prove the existence of the limit curve but allows one to guess what it is.

In what follows we will assume $M = N$ and abbreviate $M_{n,N} = M_{n,N,N}$. The case of a rectangle (M^N) can be handled in a similar way. We stick to the square case $M = N$ to simplify the notation only.

Assume that N and n go to infinity in such a way that $n \sim pN^2$, where $p \in (0, 1)$ is a fixed parameter. Instead of $M_{n,N}$ we will be dealing with a modified measure, which is obtained by a mixing procedure similar to poissonization: all values $n = 0, 1, \dots, N^2$ are mixed by making use of the binomial distribution on $\{0, 1, \dots, N^2\}$ with parameter p . Like the Poisson distribution, the binomial distribution possesses the concentration property: as N gets large, the main contribution comes from those n ’s which are close to pN^2 . Thus, one may believe that mixing does not affect the asymptotics.

The resulting measure lives on the set $\mathbb{Y}(N, N)$ of all Young diagrams contained in (N^N) (no constraints on $|\lambda|$ are imposed):

$$M_{p,N}^{\text{Mix}}(\lambda) = \binom{N^2}{|\lambda|} p^{|\lambda|} (1-p)^{N^2-|\lambda|} M_{|\lambda|,N}(\lambda), \quad \lambda \in \mathbb{Y}(N, N). \tag{5.1}$$

The next claim, which is similar to Lemma 4.2, shows that the measure $M_{p,N}^{\text{Mix}}$ leads to the N -particle *Krawtchouk ensemble*.

Denote by $W_{p,L}^{\text{Krawtchouk}}$ the weight function of the Krawtchouk orthogonal polynomials on the finite set of integers $\{0, 1, \dots, L\}$ and depending on the parameter $p \in (0, 1)$:

$$W_{p,L}^{\text{Krawtchouk}}(x) = \binom{L}{x} p^x (1-p)^{L-x}, \quad x = 0, 1, \dots, L.$$

Lemma 5.1. *Under the correspondence $\lambda \leftrightarrow \{x_1, \dots, x_N\}$ defined by (2.2), random Young diagrams $\lambda \in \mathbb{Y}(N, N)$ distributed according to the measure $M_{p,N}^{\text{Mix}}$ turn into random N -particle configurations in $\{0, 1, \dots, L\}$ with $L = 2N - 1$ and such that*

$$\text{Prob}(\{x_1, \dots, x_N\}) = \text{const}(p, N) \cdot \prod_{i=1}^N W_{p,L}^{\text{Krawtchouk}}(x_i) \cdot \prod_{1 \leq i < j \leq N} (x_i - x_j)^2. \tag{5.2}$$

Proof. Recall that

$$M_{|\lambda|, N}(\lambda) = \frac{\dim \lambda \cdot \dim(\hat{\lambda})}{\dim(N^N)}.$$

Applying Frobenius’ dimension formula (4.5) and using (5.1) we obtain the desired expression. \square

The following result describes the limit behavior of the Krawtchouk ensemble “in the bulk.”

Fix a real number c such that $|c| < 2\sqrt{p(1-p)}$. Let a_N be an arbitrary sequence of integers such that $a_N \sim cN$. Given a random configuration $\{x_1, \dots, x_N\}$ of the N -particle Krawtchouk ensemble (5.2), we shift it by $N + a_N$ to the left and obtain in this way a new random N -particle configuration $\{x'_1, \dots, x'_N\} \subset \mathbb{Z}$:

$$x'_i = x_i - N - a_N, \quad 1 \leq i \leq N. \tag{5.3}$$

Proposition 5.2. *Under the above assumptions, the random configuration (5.3) converges as $N \rightarrow \infty$ to the translation invariant determinantal random point process on \mathbb{Z} with the correlation kernel*

$$K_\varphi^{\text{discrete sine}}(x, y) = \frac{\sin(\varphi(x - y))}{\pi(x - y)}, \quad x, y \in \mathbb{Z}, \tag{5.4}$$

where

$$\varphi = \arccos\left(\frac{c(1 - 2p)}{2\sqrt{(1 - c^2)p(1 - p)}}\right). \tag{5.5}$$

The assumption $|c| < 2\sqrt{p(1-p)}$ just means that

$$\left| \frac{c(1 - 2p)}{2\sqrt{(1 - c^2)p(1 - p)}} \right| < 1,$$

so that φ is well defined.

The kernel (5.4), called the *discrete sine kernel*, first appeared in connection with the Plancherel model, see [BOO]. This kernel is a lattice counterpart of the celebrated sine kernel

$$K^{\text{sine}}(u, v) = \frac{\sin(\pi(u - v))}{\pi(u - v)}, \quad u, v \in \mathbb{R}.$$

The result of the proposition is a manifestation of a general phenomenon studied in [BKMM]: the discrete sine kernel is the universal correlation kernel arising in the bulk limit of discrete orthogonal polynomial ensembles.

Sketch of proof. The argument is very similar to the proof of Theorem 4.1. We give the formal computation below; the justification is omitted.

Let $K_m(x; p, L)$ denote the Krawtchouk polynomial of degree m (necessary information about these polynomials can be found in [KS, Section 1.10]). The normalized functions

$$\tilde{K}_m(x; p, L) = (W_{p,L}^{\text{Krawtchouk}}(x))^{1/2} \|K_m(\cdot; p, L)\|^{-1} K_m(x; p, L)$$

form an orthonormal basis in the ℓ^2 space on the finite set $\{0, 1, \dots, L\}$. The N -particle Krawtchouk ensemble (5.2) is a determinantal point process with the correlation kernel

$$K_{p,L}^{\text{Krawtchouk}}(x, y) = \sum_{m=0}^{N-1} \tilde{K}_m(x; p, L) \tilde{K}_m(y; p, L). \tag{5.6}$$

The Krawtchouk polynomials $K_m(x; p, L)$ satisfy the difference equation

$$p(L - x)K_m(x + 1; p, L) + x(2p - 1)K_m(x; p, L) + x(1 - p)K_m(x - 1; p, L) = (pL - m)K_m(x; p, L).$$

In terms of the normalized functions the difference equation takes the form

$$\begin{aligned} \frac{\sqrt{(L-x)(x+1)}}{L} \tilde{K}_m(x+1; p, L) + \frac{x(2p-1)}{L\sqrt{p(1-p)}} \tilde{K}_m(x; p, L) \\ + \frac{\sqrt{(L-x+1)x}}{L} \tilde{K}_m(x-1; p, L) = \frac{pL-m}{L\sqrt{p(1-p)}} \tilde{K}_m(x; p, L). \end{aligned}$$

Let D denote the difference operator defined by the left-hand side of this equation. The correlation kernel (5.6) corresponds to the projection on the following part of the spectrum of the operator D :

$$\left\{ \frac{pL - m}{L\sqrt{p(1-p)}}, m = 0, \dots, N - 1 \right\}. \tag{5.7}$$

Recall that $L = 2N - 1$ and $x \approx N + cN + x'$. For large N , the three coefficients of our difference operator are approximately equal to

$$\frac{1}{2}\sqrt{1 - c^2}, \quad \frac{(1 + c)(2p - 1)}{2\sqrt{p(1 - p)}}, \quad \frac{1}{2}\sqrt{1 - c^2},$$

and the set (5.7) approximates the interval

$$\left[\frac{2p - 1}{2\sqrt{p(1 - p)}}, \frac{2p}{2\sqrt{p(1 - p)}} \right]. \tag{5.8}$$

Thus, in the limit $N \rightarrow \infty$ we get the difference operator

$$\frac{1}{2}\sqrt{1 - c^2} f(x' + 1) + \frac{(1 + c)(2p - 1)}{2\sqrt{p(1 - p)}} f(x') + \frac{1}{2}\sqrt{1 - c^2} f(x' - 1), \quad x' \in \mathbb{Z},$$

and the spectral projection corresponding to the interval (5.8).

Subtracting the scalar operator $f \mapsto \frac{(1+c)(2p-1)}{2\sqrt{p(1-p)}} f$ and dividing by $\frac{1}{2}\sqrt{1-c^2}$ we finally arrive to the difference operator

$$f(x' + 1) + f(x' - 1), \quad x' \in \mathbb{Z}, \tag{5.9}$$

and the spectral interval

$$\left[\frac{c(1-2p)}{\sqrt{(1-c^2)p(1-p)}}, \frac{c(1-2p)+1}{\sqrt{(1-c^2)p(1-p)}} \right]. \tag{5.10}$$

The corresponding spectral projection is given by the discrete sine kernel (5.4). Indeed, to study the difference operator (5.9) it is convenient to make the Fourier transform from $\ell^2(\mathbb{Z})$ to the L^2 space on the unit circle $|z| = 1$. Then (5.9) becomes the operator of multiplication by the function $z + \bar{z} = 2\Re z$. Hence, we see that our operator has purely continuous (double) spectrum ranging from -2 to 2 . It is readily seen that the right end of the interval (5.10) is always ≥ 2 , while the left end is somewhere inside this interval (here we use the assumption $|c| < 2\sqrt{p(1-p)}$). Thus, in the L^2 space on the circle, our spectral projection becomes the operator of multiplication by the characteristic function of the arc going in the counterclockwise direction from $e^{-i\varphi}$ to $e^{i\varphi}$, where φ is given by (5.5). In the $\ell^2(\mathbb{Z})$ -realization, this is the integral operator with the discrete sine kernel (5.4). \square

Corollary 5.3. *Let $\lambda \in \mathbb{Y}(N, N)$ be the random Young diagram distributed according to the probability measure $M_{p,N}^{\text{Mix}}$. Assuming that the boundary of λ in the scaled coordinates $x = i/N$, $y = j/N$ has a nonrandom limit described by a curve $x + y = F(y - x)$ we can explicitly find F from the equation*

$$\frac{1 - F'(c)}{2} = \frac{\varphi}{\pi} = \frac{1}{\pi} \arccos\left(\frac{c(1-2p)}{2\sqrt{(1-c^2)p(1-p)}}\right),$$

where c ranges over the interval $(-2\sqrt{p(1-p)}, 2\sqrt{p(1-p)})$.

An additional condition is that the area bounded by the limit curve and the coordinate axes in the (x, y) plane has to be equal to p .

Idea of proof. We observe that the density function (that is, the first correlation function) of the point process with discrete sine correlation kernel is the constant φ/π . Then we use the same argument as in item (b) of Section 1 (see also [BOO, Remark 1.7]). \square

One can verify that this result agrees with the formulas in [PR]. The endpoints of the interval $(-2\sqrt{p(1-p)}, 2\sqrt{p(1-p)})$ correspond to the endpoints of the limit curve that lie on the coordinate axes.

Note that the result of Proposition 5.2 can be obtained using asymptotics of Krawtchouk polynomials obtained in [IS]. The case $p = 1/2$ is also handled in [Jo2, Lemma 2.8]. These papers contain much finer results on the asymptotics but obtaining them requires substantially more work.

Acknowledgments

We are very grateful to Philippe Biane for posing the problem and sharing his insights with us. We are also grateful to Barry Simon for helpful advice. The research of A. Borodin was partially supported by the NSF grant DMS-0402047. Both authors were also supported by the CRDF grant RUM1-2622-ST-04.

References

- [Ak] N.I. Akhiezer, *The Classical Moment Problem*, Fizmatgiz, Moscow, 1961 (in Russian); English translation: Hafner, New York, 1965.
- [BKMM] J. Baik, T. Kriecherbauer, K.T.-R. McLaughlin, P.D. Miller, Uniform asymptotics for polynomials orthogonal with respect to a general class of discrete weights and universality results for associated ensembles, arXiv: math.CA/0310278.
- [BHKPV] J. Ben Hough, M. Krishnapur, Y. Peres, B. Virag, Determinantal processes and independence, *Probab. Surveys* 3 (2006) 206–229.
- [Bi1] P. Biane, Representations of symmetric groups and free probability, *Adv. Math.* 138 (1998) 126–181.
- [Bi2] P. Biane, Approximate factorization and concentration for characters of symmetric groups, *Int. Math. Res. Not.* 2001 (4) (2001) 179–192.
- [BOO] A. Borodin, A. Okounkov, G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, *J. Amer. Math. Soc.* 13 (2000) 481–515, arXiv: math.CO/9905032.
- [DVJ] D.J. Daley, D. Vere-Jones, *An Introduction to the Theory of Point Processes*, Springer Ser. Statist., Springer, 1988.
- [IS] M.E.H. Ismail, P. Simeonov, Strong asymptotics for Krawtchouk polynomials, *J. Comput. Appl. Math.* 100 (1998) 121–144.
- [Jo1] K. Johansson, Discrete orthogonal polynomial ensembles and the Plancherel measure, *Ann. of Math.* (2) 153 (1) (2001) 259–296, arXiv: math.CO/9906120.
- [Jo2] K. Johansson, Non-intersecting paths, random tilings and random matrices, *Probab. Theory Related Fields* 123 (2) (2002) 225–280, arXiv: math.PR/0011250.
- [Ke1] S.V. Kerov, Distribution of symmetry types of high rank tensors, *Zap. Nauchn. Sem. LOMI* 155 (1986) 181–186 (in Russian); English translation: *J. Soviet Math. (New York)* 41 (2) (1988) 995–999.
- [Ke2] S.V. Kerov, *Asymptotic Representation Theory of the Symmetric Group and Its Applications in Analysis*, Amer. Math. Soc., Providence, RI, 2003, 201 pp.
- [KS] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998, available via <http://aw.twi.tudelft.nl/~koekoek/askey.html>.
- [Ko] W. König, Orthogonal polynomial ensembles in probability theory, *Probab. Surveys* 2 (2005) 385–447.
- [LS] B.F. Logan, L.A. Shepp, A variational problem for random Young tableaux, *Adv. Math.* 26 (1977) 206–222.
- [Ly] R. Lyons, Determinantal probability measures, *Publ. Math. Inst. Hautes Études Sci.* 98 (2003) 167–212.
- [Ma] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford Univ. Press, 1995.
- [Ok] A. Okounkov, Random matrices and random permutations, *Int. Math. Res. Not.* 2000 (20) (2000) 1043–1095.
- [PR] B. Pittel, D. Romik, Limit shapes for random square Young tableaux and plane partitions, arXiv: math.PR/0405190.
- [RS] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. Vol. I. Functional Analysis*, Academic Press, New York, 1972.
- [Si] B. Simon, The classical moment problem as a self-adjoint finite difference operator, *Adv. Math.* 137 (1998) 82–203.
- [So1] A. Soshnikov, Determinantal random point fields, *Russian Math. Surveys* 55 (2000) 923–975.
- [So2] A. Soshnikov, Determinantal random fields, in: J.-P. Francoise, G. Naber, T.S. Tsun (Eds.), *Encyclopedia Math. Phys.*, vol. 2, Elsevier, Oxford, 2006, pp. 47–53.
- [Te] N.M. Temme, Asymptotic estimates for Laguerre polynomials, *Z. Angew. Math. Phys.* 41 (1990) 114–126.

- [VK1] A.M. Vershik, S.V. Kerov, Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux, *Dokl. Akad. Nauk SSSR* 233 (6) (1977) 1024–1027; English translation: *Soviet Math. Dokl.* 18 (1977) 527–531.
- [VK2] A.M. Vershik, S.V. Kerov, Asymptotic behavior of the largest and the typical dimensions of irreducible representations of the symmetric group, *Funct. Anal. Appl.* 19 (1) (1985) 21–31.