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Prüfer $*$-multiplication domains, Nagata rings, and Kronecker function rings

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Abstract

Let D be an integrally closed domain, $*$ a star-operation on D, X an indeterminate over D, and $N_* = \{f \in D[X] \mid (A_f)^* = D\}$. For an $e.a.b.$ star-operation $*_1$ on D, let $Kr(D, *_1)$ be the Kronecker function ring of D with respect to $*_1$. In this paper, we use $*$ to define a new $e.a.b.$ star-operation $*_c$ on D. Then we prove that D is a Prüfer $*$-multiplication domain if and only if $D[X]_{N_*} = Kr(D, *_c)$, if and only if $Kr(D, *_c)$ is a quotient ring of $D[X]$, if and only if $Kr(D, *_c)$ is a flat $D[X]$-module, if and only if each $*$-linked overring of D is a Prüfer v-multiplication domain. This is a generalization of the following well-known fact that if D is a v-domain, then D is a Prüfer v-multiplication domain if and only if $Kr(D, v) = D[X]_{N_v}$, if and only if $Kr(D, v)$ is a quotient ring of $D[X]$, if and only if $Kr(D, v)$ is a flat $D[X]$-module.

Keywords: ($e.a.b.$) $*$-operation; Prüfer $*$-multiplication domain; Nagata ring; Kronecker function ring

1. Introduction

Let D be an integral domain with quotient field K, and let X be an indeterminate over D. For each polynomial $f \in K[X]$, we denote by A_f the fractional ideal of D generated by the coefficients of f. An overring of D means a ring between D and K. Let $\mathcal{F}(D)$ (respectively, $f(D)$) be the set of nonzero (respectively, nonzero finitely generated) fractional ideals of D; so $f(D) \subseteq \mathcal{F}(D)$.

A map $*: \mathcal{F}(D) \to \mathcal{F}(D)$, $I \mapsto I^*$, is called a *star-operation on* D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in \mathcal{F}(D)$: (i) $(aD)^* = aD$ and $(aI)^* = aI^*$,

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(ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$, and (iii) $(I^*)^* = I^*$. Given a star-operation $*$ on D, we can construct two new star-operations $*_f$ and $*_w$ on D as follows; for each $I \in \mathcal{F}(D)$, $I^{*_f} = \bigcup\{J^* \mid J \subseteq I \text{ and } J \in f(D)\}$ and $I^{*_w} = \{x \in K \mid xJ \subseteq I \text{ for some } J \in f(D) \text{ with } J^* = D\}$ [1, Theorem 2.1]. The simplest example of star-operations is the d-operation, which is the identity map on $\mathcal{F}(D)$, i.e., $I^d = I$ for all $I \in \mathcal{F}(D)$. Other well-known star-operations are the v-, t- and w-operations. The v-operation is defined by $I^v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, for all $I \in \mathcal{F}(D)$, and the t-operation is given by $t = v_f$ and the w-operation is given by $w = v_w$.

A star-operation $*$ on D is said to be *endlich arithmetisch brauchbar* (*e.a.b.*) if, for all $A, B, C \in f(D)$, $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*$. It is well known that if D admits an *e.a.b.* star-operation, then D is integrally closed [16, Corollary 32.8]. Conversely, if D is integrally closed, define $I^b = \bigcap\{IV \mid V \text{ is a valuation overring of } D\}$ for each $I \in \mathcal{F}(D)$; then the map $b : \mathcal{F}(D) \to \mathcal{F}(D)$, given by $I \mapsto I^b$, is an *e.a.b.* star-operation D [16, Theorem 32.5]. Let $*$ be an *e.a.b.* star-operation on D, and define

$$Kr(D, *) = \{0\} \cup \left\{ \frac{f}{g} \;\middle|\; 0 \neq f, g \in D[X] \text{ such that } (A_f)^* \subseteq (A_g)^* \right\}.$$

Then $Kr(D, *)$ is a Bezout domain with quotient field $K(X) = \{\frac{f}{g} \mid f, g \in D[X] \text{ and } g \neq 0\}$ and $Kr(D, *) \cap K = D$ [16, Theorem 32.7]. We will call $Kr(D, *)$ *the Kronecker function ring of D with respect to the star-operation* $*$.

Let $*$ be a star-operation on D, and let $N_* = \{f \in D[X] \mid (A_f)^* = D\}$. An $I \in \mathcal{F}(D)$ is said to be $*$-*invertible* if $(II^{-1})^* = D$, while D is a *Prüfer $*$-multiplication domain* (P$*$MD) if each $I \in f(D)$ is $*_f$-invertible. Arnold proved that if D is integrally closed, then D is a Prüfer domain if and only if $D[X]_{N_d} = Kr(D, b)$, if and only if $Kr(D, b)$ is a quotient ring of $D[X]$ [4, Theorem 4]. This was generalized to PvMDs as follows: if D is a v-domain (D is a v-domain if the v-operation on D is an *e.a.b.* star-operation), then D is a PvMD if and only if $Kr(D, v)$ is a quotient ring of $D[X]$, if and only if $Kr(D, v) = D[X]_{N_v}$, if and only if $Kr(D, v)$ is a flat $D[X]$-module ([15, Theorem 2.5], [5, Theorem 3]). The purpose of this paper is to generalize these results to arbitrary integrally closed domains (note that a v-domain is integrally closed (cf. [16, Theorem 34.6 and Proposition 34.7])).

More precisely, let D be an integrally closed domain, and let $\{V_\alpha\}$ be the set of $*$-linked valuation overrings of D (definition is reviewed in Section 3). In Section 3, we show that the map $*_c : \mathcal{F}(D) \to \mathcal{F}(D)$, given by $I \mapsto I^{*_c} = \bigcap_\alpha IV_\alpha$, is an *e.a.b.* star-operation on D such that $(*_c)_f = *_c$, $*_c$-Max$(D) = *_f$-Max(D), and $I^{*_c} = (ID[X]_{N_*})^{b'} \cap K$ for each $I \in \mathcal{F}(D)$, where b' is the b-operation on $D[X]_{N_*}$. Then we use these results to prove that D is a P$*$MD if and only if $Kr(D, *_c) = D[X]_{N_*}$, if and only if $Kr(D, *_c)$ is a quotient ring of $D[X]$, if and only if $Kr(D, *_c)$ is a flat $D[X]$-module, if and only if each $*$-linked overring of D is a PvMD. We also prove that, for $0 \neq f \in D[X]$, A_f is $*_f$-invertible if and only if $(A_{fg})^{*_w} = (A_f A_g)^{*_w}$ for all $0 \neq g \in D[X]$. As a corollary, we have that D is a P$*$MD if and only if $(A_f A_g)^{*_w} = (A_{fg})^{*_w}$ for all $0 \neq f, g \in D[X]$. This is the star-operation analog of the fact that D is a Prüfer domain if and only if $A_f A_g = A_{fg}$ for all $0 \neq f, g \in D[X]$ [16, Corollary 28.6].

2. Star-operations and P$*$MD

In this section, we review definitions related to star-operations, and then we examine some well-known characterizations of P$*$MDs.

Let D be an integral domain with quotient field K, and let $*$ be a star-operation on D. We say that $*$ is of *finite character* if $*_f = *$. It is clear that $d_f = d_w = d$, $(*_f)_f = *_f$ and $(*_w)_f = *_w = (*_f)_w$; so d, $*_f$ and $*_w$ are of finite character. An $I \in \mathcal{F}(D)$ is called a $*$-*ideal* if $I^* = I$. Let $*$-$\mathrm{Max}(D)$ denote the set of $*$-ideals maximal among proper integral $*$-ideals of D. We know that if $\star$ is a star-operation of finite character on D, then $\star$-$\mathrm{Max}(D) \neq \emptyset$ if D is not a field; each $\star$-ideal in $\star$-$\mathrm{Max}(D)$, called a *maximal* $\star$-*ideal*, is a prime ideal; each proper integral $\star$-ideal is contained in a maximal $\star$-ideal; and each prime ideal minimal over a $\star$-ideal is a $\star$-ideal. Also, $*_f$-$\mathrm{Max}(D) = *_w$-$\mathrm{Max}(D)$ [1, Theorem 2.16]. A $*$-ideal $I \in \mathcal{F}(D)$ is said to be $*$-*finite* if there is a $J \in f(D)$ such that $I = J^*$. It is known that an $I \in \mathcal{F}(D)$ is $*_f$-invertible if and only if I^{*_f} is $*_f$-finite and I is $*_f$-locally principal, i.e., ID_P is principal for all $P \in *_f$-$\mathrm{Max}(D)$ [20, Proposition 2.6].

If $*_1$ and $*_2$ are star-operations on D, we mean by $*_1 \leqslant *_2$ that $I^{*_1} \subseteq I^{*_2}$ for all $I \in \mathcal{F}(D)$. Obviously, $*_f \leqslant *$, $d \leqslant * \leqslant v$, $d \leqslant *_w \leqslant *_f \leqslant t$, and if $*_1 \leqslant *_2$, then $(*_1)_f \leqslant (*_2)_f$ and $(*_1)_w \leqslant (*_2)_w$ (cf. [1, Section 2]). The following lemma follows directly from the definitions; we recall it for the reader's convenience.

Lemma 2.1. *Let $*_1$ and $*_2$ be star-operations of finite character on D. If $I^{*_1} \subseteq I^{*_2}$ for all $I \in f(D)$, then $*_1 \leqslant *_2$. In particular, $*_1 = *_2$ if and only if $I^{*_1} = I^{*_2}$ for all $I \in f(D)$.*

Let $N_* = \{f \in D[X] \mid (A_f)^* = D\}$; then N_* is a saturated multiplicative subset of the polynomial ring $D[X]$. Note that, for each $I \in f(D)$ with $I \subseteq D$, we have $I^* = I^{*_f}$, and $I^{*_f} = D$ if and only if $I \nsubseteq P$ for all $P \in *_f$-$\mathrm{Max}(D)$; hence $N_* = N_{*_f} = N_{*_w}$ by the fact that $*_f$-$\mathrm{Max}(D) = *_w$-$\mathrm{Max}(D)$. It is known that $ID[X]_{N_*} \cap K = I^{*_w}$ for all $I \in \mathcal{F}(D)$ ([13, Proposition 3.4] or [6, Lemma 2.3]) and each invertible ideal of $D[X]_{N_*}$ is principal [20, Theorem 2.14]. If $* = d$, then $D[X]_{N_*}$, denoted by $D(X)$, is called the *Nagata ring of D* (see [16, Section 33]). The Nagata ring $K(X)$ of K is the quotient field of $K[X]$.

It is obvious that if $\star$ is a star-operation of finite character on D, then an $I \in \mathcal{F}(D)$ is $\star$-invertible, i.e., $(II^{-1})^\star = D$ if and only if $II^{-1} \nsubseteq P$ for all $P \in \star$-$\mathrm{Max}(D)$. Note again that $*_f$-$\mathrm{Max}(D) = *_w$-$\mathrm{Max}(D)$ and $(*_f)_f = *_f$; so D is a P$*$MD $\Leftrightarrow$ D is a P$*_f$MD $\Leftrightarrow$ D is a P$*_w$MD (cf. [10, Proposition 3.12]). The P$*$MDs have been studied by many authors (see, for example, [8–10,17,19,20,23]). We next review some well-known characterizations of P$*$MDs.

Theorem 2.2. *The following statements are equivalent for an integral domain D.*

(1) *D is a P$*$MD.*
(2) *D_P is a valuation domain for each maximal $*_f$-ideal P of D.*
(3) *D is a PvMD and $*_f = t$.*
(4) *D is a PvMD and $*_w = t$.*
(5) *D is a PvMD and $*_f$-$\mathrm{Max}(D) = t$-$\mathrm{Max}(D)$.*
(6) *Each $*$-linked overring of D is integrally closed.*
(7) *$D[X]_{N_*}$ is a Prüfer domain.*
(8) *$D[X]_{N_*}$ is a Bezout domain.*
(9) *D is integrally closed and $Q \cap N_* \neq \emptyset$ for each nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (9). See [19, Theorem 1.1]. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7). These are the star-operation versions of [10, Remark 3.14, Proposition 3.15, and Theorem 3.1]. (1) $\Leftrightarrow$ (6).

This is the star-operation version of the equivalence ((iii) $\Leftrightarrow$ (viii)) of [9, Theorem 5.3] since the notion "R is $*$-linked over D" is equivalent to that of "R is t-linked to $(D, *)$" [6, Remark 3.4(3)].
(7) $\Rightarrow$ (8). This follows because each invertible ideal of $D[X]_{N_*}$ is principal [20, Theorem 2.14].
(8) $\Rightarrow$ (7). Clear. $\square$

Let $*$ be a star-operation on D. An $x \in K$ is said to be *$*$-integral over D* if there exists an $I \in f(D)$ such that $xI^* \subseteq I^*$. Let $D^{[*]} = \{x \in K \mid x \text{ is } *\text{-integral over } D\}$; then $D^{[*]}$, called the *$*$-integral closure of D*, is an integrally closed overring of D [24]. If $D = D^{[*]}$, we say that D is *$*$-integrally closed*. Note that $I^* = I^{*_f}$ for each $I \in f(D)$; so $D^{[*]} = D^{[*_f]}$. It is known that D is v-integrally closed if and only if D is a v-domain, if and only if $(II^{-1})^v = D$ for each $I \in f(D)$ (cf. [16, Theorem 34.6]). A valuation overring V of D is called a *$*$-valuation overring of D* if $I^* \subseteq IV$ for each $I \in f(D)$. Halter-Koch proved that the $*$-integral closure is the intersection of all $*$-valuation overrings [18, Theorem 3]. Hence D is $*$-integrally closed if and only if D is the intersection of $*$-valuation overrings of D. It is clear that if $*_1 \leqslant *_2$ are star-operations on D, then a $*_2$-valuation overring of D is a $*_1$-valuation overring, and hence a $*_2$-integrally closed domain is $*_1$-integrally closed. In particular, a $*$-integrally closed domain is integrally closed. The reader can be referred to [3,6,7,12,18,24] for more about $*$-integral closure.

Suppose that D is $*$-integrally closed, and let $I^{*_a} = \bigcap\{IV_\beta \mid V_\beta \text{ is a } *\text{-valuation overring of } D\}$ for each $I \in \mathcal{F}(D)$. Then $*_a$ is an *e.a.b.* star-operation of finite character on D and $(*_a)_a = *_a$ ([18, Propositions 4 and 5] and [11, Proposition 4.5]). Note that the set of $*$-valuation overrings coincides with the set of $*_f$-valuation overrings; hence $(*_f)_a = *_a$. In particular, if $* = d$, then the valuation overrings of D is equal to the $*$-valuation overrings, and thus $*_a = b$.

Next, for a star-operation $*$ on D, let $Kr(D, *) = \{0\} \cup \{\frac{f}{g} \mid 0 \neq f, g \in D[X]$ and there is an $0 \neq h \in D[X]$ such that $(A_f A_h)^* \subseteq (A_g A_h)^*\}$. Then $Kr(D, *)$ is a Bezout domain with quotient field $K(X)$ and $Kr(D, *) \cap K = D^{[*_a]}$ [11, Theorem 5.1, Proposition 4.5(2), and Corollary 3.5]. This was first introduced and studied by Fontana and Loper [11] (in the more general setting of semistar-operations). Clearly, if $*$ is an *e.a.b.* star-operation, then $Kr(D, *)$ is the usual Kronecker function ring (so we use the same notation $Kr(D, *)$). It is clear that $Kr(D, *) = Kr(D, *_f)$ and if $*_1 \leqslant *_2$ are star-operations on D, then $Kr(D, *_1) \subseteq Kr(D, *_2)$; in particular, $Kr(D, d) \subseteq Kr(D, w) \subseteq Kr(D, t) = Kr(D, v)$. Note that a Bezout domain is integrally closed, so if D is not integrally closed, then $D \subsetneq Kr(D, *) \cap K$.

Using this notion of Kronecker function rings, Fontana, Jara and Santos generalized some of the classical characterizations of PvMDs (cf. [5, Theorem 3]).

Theorem 2.3. *(See [10, Theorem 3.1 and Remark 3.1].) Let $*$ be a star-operation on an integral domain D. Then the following statements are equivalent.*

(1) *D is a P$*$MD.*
(2) $D[X]_{N_*} = Kr(D, *_w)$.
(3) *$*_w$ is an e.a.b. star-operation.*
(4) $*_w = *_a$.
(5) *$*_f$ is an e.a.b. star-operation and $(I \cap J)^{*_f} = I^{*_f} \cap J^{*_f}$ for all $I, J \in \mathcal{F}(D)$.*

Let $*$ be an *e.a.b.* star-operation on D, and let $Kr(D, *)$ be the Kronecker function ring. It is known that $IKr(D, *) \cap K = I^*$ for each $I \in f(D)$ [16, Theorem 32.7]. So if $*_1$ and $*_2$ are *e.a.b.* star-operations on D, then $Kr(D, *_1) = Kr(D, *_2)$ if and only if $I^{*_1} = I^{*_2}$ for all $I \in f(D)$ [16, Remark 32.9]. Clearly, $*$ is an *e.a.b.* star-operation if and only if $*_f$ is an *e.a.b.*

star-operation. Hence if $*$ is an *e.a.b.* star-operation on D, then $Kr(D,*) = Kr(D,*_f)$. For more on $Kr(D,*)$, see Fontana and Loper's recent interesting survey article [14] or [13]. Any other undefined terminology or notation is standard, as in [16] or [21].

3. New characterizations of P$*$MD

Throughout this section, D is an integral domain with quotient field K, $*$ is a star-operation on D, and $N_* = \{f \in D[X] \mid (A_f)^* = D\}$.

As in [6], we say that an overring R of D is *$*$-linked over D* if $R = R[X]_{N_*} \cap K$. It is known that R is $*$-linked over D if and only if $(Q \cap D)^{*_f} \subsetneq D$ for each prime t-ideal Q of R, if and only if $I^* = D$ for an $I \in f(D)$ implies $(IR)^v = R$ [6, Proposition 3.2]. This shows that the concepts of $*$-, $*_f$-, and $*_w$-linkedness are all equivalent and that if $*_1 \leqslant *_2$ are star-operations on D, then each $*_2$-linked overring of D is $*_1$-linked over D [6, Remark 3.4(2)]. In particular, each t-linked overring of D is $*$-linked over D. Note that if V is a $*$-valuation overring of D, then, for each $I \in f(D)$ with $I^* = D$, $(IV)^v = V$. Hence V is $*$-linked over D, but a $*$-linked valuation overring need not be a $*$-valuation overring (see Remark 3.5(2)).

Let $*$ be a star-operation on an integrally closed domain D. We first use $*$ to construct a new *e.a.b.* star-operation $*_c$ on D. The $*_c$-operation plays a central role in the study of P$*$MDs of this paper.

Lemma 3.1. *Let D be an integrally closed domain, and let $\{V_\alpha\}$ be the set of $*$-linked valuation overrings of D.*

(1) *The map $*_c : \mathcal{F}(D) \to \mathcal{F}(D)$, given by $I \mapsto I^{*_c} = \bigcap_\alpha IV_\alpha$, is an e.a.b. star-operation on D.*
(2) *$*_c = (*_w)_a$, and hence $*_c$ is of finite character.*
(3) *$*_w = (*_c)_w \leqslant *_c$ and $*_f$-Max$(D) = *_c$-Max(D).*
(4) *If $*_1 \leqslant *_2$ are star-operations on D, then $(*_1)_c \leqslant (*_2)_c$.*
(5) *If D is $*$-integrally closed, then $*_c \leqslant *_a$.*
(6) *$b = d_a = d_c$.*

Proof. (1) Since D is integrally closed, $D = \bigcap_\alpha V_\alpha$ [6, Corollary 4.2]. Thus the map $*_c : \mathcal{F}(D) \to \mathcal{F}(D)$, given by $I \mapsto I^{*_c} = \bigcap_\alpha IV_\alpha$, is an *e.a.b.* star-operation on D [16, Theorem 32.5].

(2) Recall that the $*_w$-valuation overrings coincide with the $*$-linked valuation overrings and an integrally closed domain is $*_w$-integrally closed [6, Corollaries 3.3 and 4.2(1)]. Hence $(*_w)_a = *_c$, and thus $*_c$ is of finite character [11, Proposition 4.5].

(3) Step 1. $*_w \leqslant *_c$. (Proof. Let $*_\alpha$ be the star-operation on V_α defined by $J^{*_\alpha} = JV_\alpha[X]_{N_*} \cap K$ for all $J \in \mathcal{F}(V_\alpha)$ [6, Lemma 3.1]. If $I \in f(D)$, then IV_α is finitely generated, and so IV_α is principal. Hence $(IV_\alpha)^v = IV_\alpha \subseteq (IV_\alpha)^{*_\alpha} \subseteq (IV_\alpha)^v$ or $(IV_\alpha)^{*_\alpha} = IV_\alpha$. So $I^{*_w} = ID[X]_{N_*} \cap K \subseteq IV_\alpha[X]_{N_*} \cap K = (IV_\alpha)^{*_\alpha} = IV_\alpha$ (cf. [6, Lemma 2.3] for the first equality), and hence $I^{*_w} \subseteq \bigcap_\alpha IV_\alpha = I^{*_c}$. Thus, as $*_c$ and $*_w$ are of finite character, we have $*_w \leqslant *_c$ by Lemma 2.1.)

Step 2. $N_{*_c} = N_*$. (Proof. Note that $N_* = N_{*_w}$ and $*_w \leqslant *_c$ by Step 1; so $N_* \subseteq N_{*_c}$. Conversely, let $f \in D[X] \setminus N_*$ be a nonzero polynomial. Then $(A_f)^* \subsetneq D$, and hence there exists a maximal $*_f$-ideal P of D such that $(A_f)^* \subseteq P$. Let V be a valuation overring of D with maximal ideal M such that $M \cap D = P$ [16, Corollary 19.7]. Clearly, V is $*$-linked over D and $A_f V \subseteq M$; so $(A_f)^{*_c} \subsetneq D$ and $f \notin N_{*_c}$. Hence $N_{*_c} \subseteq N_*$, and thus $N_{*_c} = N_*$.)

Step 3. $*_f$-Max$(D) = *_c$-Max(D), and hence $*_w = (*_c)_w$. (Proof. Let P be a nonzero prime ideal of D. If $P^{*_c} = D$, then since $*_c$ is of finite character by (2), there is a nonzero finitely generated ideal $J \subseteq P$ such that $J^{*_c} = D$. Let $f \in D[X]$ with $J = A_f$; then $f \in N_{*_c} = N_*$ by Step 2, and hence $J^* = (A_f)^* = D$. Thus $P^{*_f} = D$. Conversely, if $P^{*_f} = D$, then $P^{*_w} = D$, and since $*_w \leqslant *_c$ by Step 1, we have $P^{*_c} = D$. Thus $*_f$-Max$(D) = *_c$-Max(D), and so $*_w = (*_c)_w$ by definition.)

(4) This follows because each $*_2$-linked overring of D is $*_1$-linked over D.

(5) First, note that since D is $*$-integrally closed $*_a$ is a well-defined star-operation on D. As we noted at the beginning of this section, since each $*$-valuation overring of D is $*$-linked over D, we have $I^{*_c} = \bigcap_\alpha IV_\alpha \subseteq \bigcap\{IW \mid W$ is a $*$-valuation overring of $D\} = I^{*_a}$ for each $I \in \mathcal{F}(D)$. Thus $*_c \leqslant *_a$.

(6) Obviously, the valuation overrings of D are the d-valuation overrings. Hence D is d-integrally closed, and thus d_a is well defined and $b = d_a = d_c$. □

From now on, we denote by $*_c$ the *e.a.b.* star-operation on an integrally closed domain D induced by $*$ as in Lemma 3.1. As we noted, the concepts of $*$-, $*_f$- and $*_w$-linkedness coincide, and so $*_c = (*_f)_c = (*_w)_c$. Also, since $d \leqslant * \leqslant v$, we have $d_c \leqslant *_c \leqslant v_c$, hence $Kr(D, b) \subseteq Kr(D, *_c) \subseteq Kr(D, v_c) \subseteq K(X)$.

Corollary 3.2. *Let $*$ and $*_1$ be star-operations on an integrally closed domain D.*

(1) *If $Kr(D, *_c) = Kr(D, (*_1)_c)$, then $*_c = (*_1)_c$.*
(2) *The following statements are equivalent.*
   (a) *$Kr(D, b) = Kr(D, *_c)$.*
   (b) *$b = *_c$.*
   (c) *$*_w = d$, i.e., each nonzero ideal of D is a $*_w$-ideal.*

Proof. (1) Note that $I^{*_c} = IKr(D, *_c) \cap K = IKr(D, (*_1)_c) \cap K = I^{(*_1)_c}$ for each $I \in f(D)$ [16, Theorem 32.7]. Hence, by Lemma 2.1, $*_c = (*_1)_c$ because $*_c$ and $(*_1)_c$ are of finite character by Lemma 3.1(2).

(2) (a) $\Rightarrow$ (c). Let $I \in f(D)$; then $I^{*_c} = I^{d_c}$ by (1) and Lemma 3.1(6). Next, if $I^* = D$, then $IV_\alpha = (IV_\alpha)^v = V_\alpha$ for each $*$-linked valuation overring V_α of D. So $I^{*_c} = D$, and hence $I = I^{d_c} = D$ (cf. Lemma 3.1(3)). This means that each nonzero maximal ideal of D is a $*_f$-ideal (cf. [8, Theorem 2.6] for the t-operation). Thus $*_w = (*_f)_w = d_w = d$. (c) $\Rightarrow$ (b). $*_c = (*_w)_a = d_a = b$ by Lemma 3.1(2) and (6). (b) $\Rightarrow$ (a). Clear. □

Lemma 3.3.

(1) *If W is a valuation overring of $D[X]_{N_*}$, then $AW \cap K = A(W \cap K)$ for each $A \in \mathcal{F}(D)$.*
(2) *The set of $*$-linked valuation overrings of D is the set $\{W \cap K \mid W$ is a valuation overring of $D[X]_{N_*}\}$.*

Proof. (1) Let $u = \sum a_i w_i \in AW \cap K$, where $a_i \in A$ and $w_i \in W$. Since W is a valuation domain with $D \subseteq W$, there exists an a_k such that each $\frac{a_i}{a_k} \in W$. Note that $\frac{u}{a_k} = \sum \frac{a_i}{a_k} w_i \in W \cap K$, so $u = a_k \sum \frac{a_i}{a_k} w_i \in A(W \cap K)$. Hence $AW \cap K \subseteq A(W \cap K)$, and since $AK = K$, we have $A(W \cap K) = AW \cap K$.

(2) Assume that W is a valuation overring of $D[X]_{N_*}$, and let $I \in f(D)$ with $I^* = D$. Then $ID[X]_{N_*} = D[X]_{N_*}$, and since $D[X]_{N_*} \subseteq W$, we have $IW = W$. Hence $I(W \cap K) = IW \cap K = W \cap K$ by (1), so $(I(W \cap K))^v = W \cap K$. Thus $W \cap K$ is $*$-linked over D.

Conversely, assume that V is a $*$-linked valuation overring of D. Then, for each $f \in N_*$, since $(A_f)^* = D$, we have $A_f V = (A_f V)^v = V$, hence $N_* \subseteq (V[X] \setminus M[X])$, where M is the maximal ideal of V. Hence $D[X]_{N_*} \subseteq V[X]_{M[X]}$, and $V[X]_{M[X]}$ is a valuation domain [16, Proposition 18.7]. Moreover, since $V[X]_{M[X]} \cap K = V$, the proof is completed. □

Let R be a Bezout domain. Then each (nonzero) finitely generated ideal of R is principal, and hence each star-operation on R is an *e.a.b.* star-operation. Also, note that if J is a nonzero finitely generated ideal of R, then $J = J^t$, and hence each nonzero ideal of R is a t-ideal. This implies that the d-operation on R is a unique star-operation of finite character on R, so $d = b$ on R. Now, suppose that D is $*$-integrally closed; then $*_a$ is an *e.a.b.* star-operation of finite character on D and $Kr(D, *_a)$ is a Bezout domain. Hence the b-operation on $Kr(D, *_a)$ is the unique *e.a.b.* star-operation of finite character on $Kr(D, *_a)$. Also, since $*_a$ is of finite character, we have $IKr(D, *_a) \cap K = I^{*_a}$ for all $I \in \mathcal{F}(D)$ [16, Theorem 32.7(c)] (cf. [13, Corollary 5.2]).

Corollary 3.4. *Let D be an integrally closed domain. If b' is the b-operation on $D[X]_{N_*}$, then $I^{*_c} = (ID[X]_{N_*})^{b'} \cap K$ for each $I \in \mathcal{F}(D)$.*

Proof. First, note that $D[X]_{N_*}$ is integrally closed, and so the b-operation b' on $D[X]_{N_*}$ is a well-defined *e.a.b.* star-operation. Therefore, $(ID[X])^{b'} \cap K = \bigcap\{IW \mid W$ is a valuation overring of $D[X]_{N_*}\} \cap K = \bigcap\{IW \cap K \mid W$ is a valuation overring of $D[X]_{N_*}\} = \bigcap\{I(W \cap K) \mid W$ is a valuation overring of $D[X]_{N_*}\} = \bigcap\{IV \mid V$ is a $*$-linked valuation overring of $D\} = I^{*_c}$ by Lemma 3.3. □

Remark 3.5. (1) Recall that $*$ is an *e.a.b.* star-operation on D if and only if $*_f$ is an *e.a.b.* star-operation on D. Also, if $*_w$ is an *e.a.b.* star-operation, then $*_w = *_f$ by Theorems 2.2 and 2.3. But, $*$ being *e.a.b.* does not imply that $*_w$ is an *e.a.b.* star-operation. For example, if D is an integrally closed domain and not a PvMD, then v_c is an *e.a.b.* star-operation, but $w = (v_c)_w$ is not an *e.a.b.* star-operation on D by Lemma 3.1 and Theorem 2.3.

(2) As we noted, a $*$-valuation overring of D is $*$-linked over D, but a $*$-linked valuation overring need not be a $*$-valuation overring. For example, let F be a field, y, z be indeterminates over F, and $V = F(z)[\![y]\!]$ be the power series ring over the field $F(z)$, and $D = F + yF(z)[\![y]\!]$. Then D is a quasi-local domain with maximal ideal $yF(z)[\![y]\!]$ and D is of (Krull) dimension one. Note that $yF(z)[\![y]\!]$ is a t-ideal, and hence each overring of D is t-linked over D [8, Theorem 2.6].

But, let $V_1 = F[z]_{zF[z]}$; then V_1 is a valuation domain with quotient field $F(z)$, and hence $D_1 = V_1 + yF(z)[\![y]\!]$ is a valuation domain [16, Exercise 13, p. 203] such that $D \subsetneq D_1 \subsetneq V$. Let $J = y(F + zF + yF(z)[\![y]\!])$; then $J \in f(D)$ and $J^v = yF(z)[\![y]\!]$ (see the proof of [3, Proposition 1.8(ii)]). But, since $JD_1 \subseteq y(V_1 + yF(z)[\![y]\!]) \subsetneq yF(z)[\![y]\!]$, we have $J^v \nsubseteq JD_1$. Thus D_1 is not a t-valuation overring of D, and since D_1 is an overring of D, D_1 is t-linked over D.

However, since the domain $D = F + yF(z)[\![y]\!]$ is not t-integrally closed [3, Proposition 1.8(ii)], we may not deduce that $*_c \neq *_a$.

(3) Recall that $*_f$-Max$(D) = *_c$-Max(D) by Lemma 3.1(3), hence the $*$-linked valuation overrings of D coincide with the $*_c$-valuation overrings of D. Thus $(*_c)_c = *_c$. But we do not know whether $*_c = *$ when $*$ is an $e.a.b.$ star-operation.

Let $0 \neq f, g \in K[X]$ and let m be the degree of g. Then $A_f^{m+1}A_g = A_f^m A_{fg}$ by the Dedekind–Mertens Lemma [16, Theorem 28.1]. So if A_f is invertible, then $A_f A_f = A_{fg}$. Conversely, in [22], Loper and Roitman proved that A_f is invertible if $A_f A_g = A_{fg}$ for all $0 \neq g \in K[X]$. This cannot be generalized to the v-operation. For example, let D be an integrally closed domain that is not a v-domain (see [16, Exercise 2, p. 429] for such an integral domain). Then there exists a polynomial $0 \neq f \in D[X]$ such that A_f is not v-invertible, but since D is integrally closed, we have $(A_f A_g)^v = (A_{fg})^v$ for all $0 \neq g \in D[X]$ [25, Lemma 1].

Lemma 3.6. *For $0 \neq f \in D[X]$, the following statements are equivalent.*

(1) *A_f is $*_w$-invertible.*
(2) *A_f is $*_f$-invertible.*
(3) *$(A_{fg})^{*_w} = (A_f A_g)^{*_w}$ for all $0 \neq g \in D[X]$.*
(4) *$A_f D[X]_{N_*} = f D[X]_{N_*}$.*

Proof. (1) $\Leftrightarrow$ (2). This follows because $*_w$-Max$(D) = *_f$-Max(D) [1, Theorem 2.16].

(1) $\Rightarrow$ (3). Assume that A_f is $*_w$-invertible, and let m be a positive integer such that $A_f^{m+1}A_g = A_f^m A_{fg}$. Then $(A_f^{m+1}A_g)^{*_w} = (A_f^m A_{fg})^{*_w}$, and since A_f is $*_w$-invertible, we have $(A_{fg})^{*_w} = (A_f A_g)^{*_w}$.

(3) $\Rightarrow$ (1). Let P be a maximal $*_w$-ideal of D. For a nonzero polynomial $g \in D_P[X]$, let $0 \neq s \in D$ such that $sg \in D[X]$; then $(A_{fsg})^{*_w} = (A_f A_{sg})^{*_w}$ by (3). Note that $s(A_{fg})^{*_w} = (sA_{fg})^{*_w} = (A_{fsg})^{*_w} = (A_f A_{sg})^{*_w} = (A_f s A_g)^{*_w} = s(A_f A_g)^{*_w}$. So $(A_f A_g)^{*_w} = (A_{fg})^{*_w}$, and hence $(A_f D_P)(A_g D_P) = (A_f A_g)D_P = (A_f A_g)^{*_w} D_P = (A_{fg})^{*_w} D_P = A_{fg} D_P$ (cf. [1, Corollary 2.10]). So $A_f D_P$ is principal [22, Theorem 4]. Thus A_f is $*_w$-invertible [20, Proposition 2.6].

(2) $\Leftrightarrow$ (4). See [20, Lemma 2.11]. □

Next, we give new characterizations of P$*$MDs, which generalize some of the classical characterizations (in terms of $e.a.b.$ star-operations) of Prüfer domains and PvMDs ([4, Theorem 4] and [5, Theorem 3]).

Theorem 3.7. *The following statements are equivalent for an integrally closed domain D.*

(1) *D is a P$*$MD.*
(2) *$Kr(D, *_c)$ is a quotient ring of $D[X]$.*
(3) *$D[X]_{N_*} = Kr(D, *_c)$.*
(4) *$*_w = *_c$.*
(5) *$Kr(D, *_c)$ is a flat $D[X]$-module.*
(6) *Each $*$-linked overring of D is a PvMD.*
(7) *Each prime ideal of $D[X]_{N_*}$ is extended from D.*
(8) *Each principal ideal of $D[X]_{N_*}$ is extended from D.*
(9) *Each ideal of $D[X]_{N_*}$ is extended from D.*
(10) *$(A_f A_g)^{*_w} = (A_{fg})^{*_w}$ for all $0 \neq f, g \in D[X]$.*

Proof. (1) $\Rightarrow$ (2). Note that $D[X]_{N_*}$ is a Bezout domain by Theorem 2.2 and $D[X]_{N_*} \subseteq Kr(D, *_c) \subseteq K(X)$ (cf. Lemma 3.1(3)). Thus $Kr(D, *_c)$ is a quotient ring of $D[X]_{N_*}$ [16, Theorem 27.5], and hence of $D[X]$.

(2) $\Rightarrow$ (3). Let $S = \{0 \neq f \in D[X] \mid \frac{1}{f} \in Kr(D, *_c)\}$; then $Kr(D, *_c) = D[X]_S$ by (2). Note that $f \in S \Leftrightarrow \frac{1}{f} \in Kr(D, *_c) \Leftrightarrow D = (1) \subseteq (A_f)^{*_c} \subseteq D \Rightarrow (A_f)^{*_c} = D$; so $(A_f)^* = D$ by Lemma 3.1(3) or $\frac{1}{f} \in D[X]_{N_*}$. Hence $Kr(D, *_c) \subseteq D[X]_{N_*}$, and since $D[X]_{N_*} \subseteq Kr(D, *_c)$, we have $D[X]_{N_*} = Kr(D, *_c)$.

(3) $\Rightarrow$ (4). Note that, for each $I \in f(D)$, we have that $I^{*_w} = ID[X]_{N_*} \cap K$ [6, Lemma 2.3] and $IKr(D, *_c) \cap K = I^{*_c}$ [16, Theorem 32.7(c)]. Thus $*_c = *_a$ by Lemma 2.1 because $*_w$ and $*_c$ are of finite character.

(4) $\Rightarrow$ (1). By (4) and Lemma 3.1(1), $*_w$ is an *e.a.b.* star-operation on D, and thus D is a P$*$MD by Theorem 2.3.

(3) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (3). Let $\mathrm{Max}(B)$ denote the set of maximal ideals of a ring B, and recall that an overring R of an integral domain D_1 is a flat D_1-module if and only if $R_M = (D_1)_{M \cap D_1}$ for all $M \in \mathrm{Max}(R)$ [26, Theorem 2] and $\mathrm{Max}(D[X]_{N_*}) = \{P[X]_{N_*} \mid P \in *_f\text{-}\mathrm{Max}(D)\}$ [20, Proposition 2.1].

Let A be an ideal of $D[X]$ such that $AKr(D, *_c) = Kr(D, *_c)$. Then there exists a polynomial $f \in A$ such that $fKr(D, *_c) = Kr(D, *_c)$ (cf. the proof of [16, Theorem 32.7(b)]); so $\frac{1}{f} \in Kr(D, *_c)$, and hence $f \in A \cap N_* \neq \emptyset$ (see the proof of (2) $\Rightarrow$ (3)). Hence, if P_0 is a maximal $*_f$-ideal of D, then $P_0 Kr(D, *_c) \subsetneq Kr(D, *_c)$, and since $P_0[X]_{N_*}$ is a maximal ideal of $D[X]_{N_*}$, there is a maximal ideal M_0 of $Kr(D, *_c)$ such that $M_0 \cap D[X] = (M_0 \cap D[X]_{N_*}) \cap D[X] = P_0[X]_{N_*} \cap D[X] = P_0[X]$. Thus by (5), $Kr(D, *_c)_{M_0} = D[X]_{P_0[X]} = (D[X]_{N_*})_{P_0[X]_{N_*}}$.

Let M_1 be a maximal ideal of $Kr(D, *_c)$, and let P_1 be a maximal $*_f$-ideal of D such that $M_1 \cap D[X]_{N_*} \subseteq P_1[X]_{N_*}$. By the above paragraph, there is a maximal ideal M_2 of $Kr(D, *_c)$ such that $Kr(D, *_c)_{M_2} = (D[X]_{N_*})_{P_1[X]_{N_*}}$. Note that $Kr(D, *_c)_{M_2} \subseteq Kr(D, *_c)_{M_1}$, M_1 and M_2 are maximal ideals, and $Kr(D, *_c)$ is a Prüfer domain; hence $M_1 = M_2$ (cf. [16, Theorem 17.6(c)]) and $Kr(D, *_c)_{M_1} = (D[X]_{N_*})_{P_1[X]_{N_*}}$. Thus

$$Kr(D, *_c) = \bigcap_{M \in \mathrm{Max}(Kr(D, *_c))} Kr(D, *_c)_M = \bigcap_{P \in *_f\text{-}\mathrm{Max}(D)} \big(D[X]_{N_*}\big)_{P[X]_{N_*}} = D[X]_{N_*}.$$

(1) $\Rightarrow$ (6). Let R be a $*$-linked overring of D, and let Q be a maximal t-ideal of R. Then $(Q \cap D)^{*_f} \subsetneq D$, and hence $D_{Q \cap D}$ is a valuation domain by Theorem 2.2. Since $D_{Q \cap D} \subseteq R_Q$, it follows that R_Q is a valuation domain [16, Theorem 17.6]. Thus, again by Theorem 2.2, R is a PvMD (here $* = v$ and $*_f = t$). (Or see the proof of [9, Corollary 5.5].)

(6) $\Rightarrow$ (1). Let P be a maximal $*_f$-ideal of D. For $0 \neq u \in K$, let $R = D[u^2, u^3]_{D \setminus P}$. Then D_P and R are $*$-linked over D [6, Remark 3.4(7)]; so D_P and R are PvMDs (hence integrally closed). Hence $u \in R$, and since $R = D_P[u^2, u^3]$, there exists a polynomial $h \in D_P[X]$ such that $h(u) = 0$ and $A_h D_P = D_P$. So u or u^{-1} is in D_P [21, Theorem 67]. Hence D_P is a valuation domain. Thus D is a P$*$MD by Theorem 2.2.

(1) $\Rightarrow$ (8). Let $0 \neq f \in D[X]$. Then A_f is $*_f$-invertible by (1), and hence $fD[X]_{N_*} = A_f D[X]_{N_*}$ by Lemma 3.6.

(8) $\Rightarrow$ (9). Let A be an ideal of $D[X]_{N_*}$; then $A = \sum_{f \in A} fD[X]_{N_*}$. For each $f \in A$, there exists an ideal I_f of D such that $fD[X]_{N_*} = I_f D[X]_{N_*}$ by (8). Let $I = \sum_{f \in A} I_f$. Then I is

an ideal of D and $A = \sum_{f \in A}(I_f D[X]_{N_*}) = ID[X]_{N_*}$ (cf. [16, Theorem 4.3(4)] for the second equality).

(9) $\Rightarrow$ (8) and (9) $\Rightarrow$ (7). Clear.

(7) $\Rightarrow$ (1). Let Q be a nonzero prime ideal of $D[X]$ such that $Q \cap D = (0)$. Then $QD[X]_{N_*} = D[X]_{N_*}$ by (7), and hence $Q \cap N_* \neq \emptyset$. Thus D is a P$*$MD by Theorem 2.2.

(1) $\Leftrightarrow$ (10). This follows directly from Lemma 3.6. $\square$

The next result gives new characterizations of PvMDs in which some of the statements extend the result [5, Theorem 3] to arbitrary integrally closed domains. This is the v-operation version of Theorem 3.7.

Corollary 3.8. *The following statements are equivalent for an integrally closed domain D.*

(1) *D is a PvMD.*
(2) *$Kr(D, v_c)$ is a quotient ring of $D[X]$.*
(3) *$D[X]_{N_v} = Kr(D, v_c)$.*
(4) *$w = v_c$.*
(5) *$Kr(D, v_c)$ is a flat $D[X]$-module.*
(6) *Each t-linked overring of D is a PvMD.*
(7) *$(A_f A_g)^w = (A_{fg})^w$ for all $0 \neq f, g \in D[X]$.*

Let $*$ be a star-operation on an integral domain D. By Theorems 2.2, 2.3 and 3.7, we have that if D is a P$*$MD, then $*_c = *_a = *_f = *_w = w = t$. In particular, if D is a PvMD, then $v_c = v_a = t = w$, and hence $Kr(D, v) = Kr(D, t) = Kr(D, w) = D[X]_{N_v}$. However, $*_c = *_a$ does not imply P$*$MD. For example, let L be a field, y, z be indeterminates over L, $M = (y, z)$ be a maximal ideal of the polynomial ring $L[y, z]$, and $D = L[y, z]_M$. Then $d_c = d_a$ on D by Lemma 3.1(6), but it is clear that D is not a PdMD.

On January 26, 2007, Zafrullah sent me a preprint of his recent paper with Anderson and Fontana [2] that contains some interesting results on P$*$MDs. In particular, they also proved that D is a P$*$MD if and only if $(A_{fg})^{*_w} = (A_f A_g)^{*_w}$ for all $0 \neq f, g \in K[X]$ (in the more general setting of semistar-operations) [2, Corollary 1.2].

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