



Primitive orthogonal idempotents for R -trivial monoids

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ABSTRACT

We construct a recursive formula for a complete system of primitive orthogonal idempotents for any R -trivial monoid. This uses the newly proved equivalence between the notions of R -trivial monoid and weakly ordered monoid.

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1. Introduction

Recently, Denton [4] gave a formula for a complete system of primitive orthogonal idempotents for the 0-Hecke algebra of type A , the first since the question was raised by Norton [6] in 1979. A complete system of primitive orthogonal idempotents for *left regular bands* was found by Brown [3] and Saliola [9]. Finding such collections is an important problem in representation theory because they decompose an algebra into projective indecomposable modules: if $\{e_J\}_{J \in \mathcal{J}}$ is such a collection for a finite dimensional algebra A , then $A = \bigoplus_{J \in \mathcal{J}} Ae_J$, where each Ae_J is a projective indecomposable module. They also allow for the explicit computation of the quiver, the Cartan invariants, and the Wedderburn decomposition of the algebra (see [2,1]). For example, in [5], Denton, Hivert, Schilling, and Thiéry use a construction of a system of primitive orthogonal idempotents for any J -trivial monoid S to derive combinatorially the Cartan matrix and quiver of S .

Schocker [10] constructed a class of monoids, called *weakly ordered monoids*, to generalize simultaneously 0-Hecke monoids and left regular bands, with the broader aim of finding a complete

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system of orthogonal idempotents for the corresponding monoid algebras. We achieve this goal here.

A key step is to recognize that the notions of weakly ordered monoid and *R-trivial monoid* are one and the same. This was first pointed out to us by Thiéry [13] after an intense discussion between the authors and Denton, Hivert, Schilling, and Thiéry. In Section 2, we fill out an outline of a proof provided by Steinberg [12], who independently made this same observation. In Section 3, we use this equivalence to build a recursive formula for a complete system of primitive orthogonal idempotents for any *R-trivial monoid*. This covers, in particular but not only, the previously known cases of *J-trivial monoids* [5] and left regular bands.

2. Weakly ordered monoids and *R-trivial monoids*

Given any monoid S , that is, a set with an associative multiplication and an identity element, we define a preorder \leq as follows. Given $u, v \in S$, write $u \leq v$ if there exists $w \in S$ such that $uw = v$. We write $u < v$ if $u \leq v$ but $u \neq v$. Unless stated otherwise, the monoids throughout the paper are endowed with this “weak” preorder. In the monoid theory literature, the *dual* of this preorder is known as *Green’s R-preorder*.

Definition 2.1. A finite monoid S is said to be a **weakly ordered monoid** if there is a finite upper semi-lattice (\mathcal{L}, \preceq) together with two maps $C, D : S \rightarrow \mathcal{L}$ satisfying the following axioms:

1. C is a monoid morphism, i.e. $C(uv) = C(u) \vee C(v)$ for all $u, v \in S$.
2. C is a surjection.
3. If $u, v \in S$ are such that $uv \leq u$, then $C(v) \preceq D(u)$.
4. If $u, v \in S$ are such that $C(v) \preceq D(u)$, then $uv = u$.

Remark 2.2. This notion was introduced by Schocker [10] to generalize 0-Hecke monoids and left regular bands, with the broader aim of finding a complete system of orthogonal idempotents for the corresponding monoid algebras. In his paper, he actually calls these *weakly ordered semigroups*. However our understanding is that monoids include an identity element and semigroups do not. So throughout the paper we call these weakly ordered monoids.

Definition 2.3. A monoid S is ***R-trivial*** if, for all $x, y \in S$, $xS = yS$ implies $x = y$.

We restrict our discussion to *finite R-trivial monoids*.

Example 2.4. A monoid S is called a **left regular band** if $x^2 = x$ and $xyx = xy$ for all $x, y \in S$. Left regular bands are *R-trivial*. Indeed, if $xS = yS$, then there exist $u, v \in S$ such that $xu = y$ and $x = yv$. But then, since $uv = uvu$,

$$x = yv = xuv = xuvu = yvu = xu = y.$$

Finitely generated left regular bands are also weakly ordered monoids, see Shocker [10], e.g. 2.4 and Brown [3, Appendix B].

Example 2.5. Let G be a Coxeter group with simple generators $\{s_i : i \in I\}$ and relations:

- $s_i^2 = 1$,
- $\underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij}}$ for some positive integers m_{ij} .

Then the **0-Hecke monoid** $H^G(0)$ has generators $\{T_i: i \in I\}$ and relations:

- $T_i^2 = T_i$,
- $\underbrace{T_i T_j T_i T_j \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j T_i \cdots}_{m_{ij}}$ for some positive integers m_{ij} .

The weakly ordered monoid $H^G(0)$ has maps C and D onto the lattice of subsets of I . The map C is the *content* of an element: $C(T_{i_1} T_{i_2} \cdots T_{i_k}) = \{i_1, i_2, \dots, i_k\}$. The map D is the set of right descents of an element: $D(x) = \{i \in I: xT_i = x\}$. Note that the preorder for this monoid coincides with the weak order on the elements of the Coxeter group G .

Of particular interest is the case when G is the symmetric group \mathfrak{S}_n . Norton [6] gave a decomposition of the monoid algebra $\mathbb{C}H^{\mathfrak{S}_n}(0)$ into left ideals and classified its irreducible representations. She raised the question of constructing a complete system of orthogonal idempotents for the algebra, which was first answered by Denton [4].

Example 2.6. Let S be the monoid with identity generated by the following matrices:

$$g_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $S = \{1, g_1, g_2, g_1 g_2, g_2 g_1\}$ and S is both an R -trivial monoid and a weakly ordered monoid. For example, we can take \mathcal{L} to be usual lattice of subsets of $\{1, 2\}$, with $C: S \rightarrow \mathcal{L}$ given by

$$C(1) = \emptyset, \quad C(g_1) = \{1\}, \quad C(g_2) = \{2\}, \quad C(g_1 g_2) = C(g_2 g_1) = \{1, 2\},$$

and $D: S \rightarrow \mathcal{L}$ given by

$$D(1) = \emptyset, \quad D(g_1) = \{1\}, \quad D(g_2) = D(g_1 g_2) = \{2\}, \quad D(g_2 g_1) = \{1, 2\}.$$

The monoid S , however, is neither a left regular band, since $g_1 g_2$ is not idempotent, nor isomorphic to the 0-Hecke monoid $H^G(0)$ on two generators, since the latter always has an even number of elements.

The fact that the above examples are all weakly ordered and R -trivial is no coincidence: the purpose of this section is to show that these two notions are equivalent.

Remark 2.7. A monoid S is R -trivial if and only if the preorder \leq defined above is a partial order.

Proof. Suppose S is an R -trivial monoid and $x, y \in S$ are such that $x \leq y$ and $y \leq x$. Then there exist $u, v \in S$ such that $xu = y$ and $yv = x$. So $y \in xS$ and $x \in yS$, implying that $yS \subseteq xS$ and $xS \subseteq yS$. That is, $xS = yS$. Since S is R -trivial, $x = y$.

On the other hand, suppose that the given preorder is a partial order, and that $xS = yS$ for some $x, y \in S$. Since $x = x \cdot 1 \in xS = yS$, we have that $x = yu$ for some $u \in S$. So $y \leq x$. Similarly, $y \in xS$ implies that $x \leq y$. The antisymmetry of \leq implies then that $x = y$. So S is R -trivial. \square

Corollary 2.8. A weakly ordered monoid is an R -trivial monoid.

Proof. Let S be a weakly ordered monoid. Lemma 2.1 in [10] shows that the defining conditions of a weakly ordered monoid imply that the preorder on S is a partial order. The result now follows from Proposition 2.7. \square

We will show that any finite R -trivial monoid S is a weakly ordered monoid using an argument outlined by Steinberg [12]. We must establish the existence of an upper semi-lattice \mathcal{L} and two maps C and D from S to \mathcal{L} that satisfy the conditions of Definition 2.1. We gather here the definitions of \mathcal{L} , C and D :

1. \mathcal{L} is the set of left ideals Se generated by idempotents e in S , ordered by reverse inclusion;
2. $C : S \rightarrow \mathcal{L}$ is defined as $C(x) = Sx^\omega$, where x^ω is the idempotent power of x (see Lemma 2.10);
3. $D : S \rightarrow \mathcal{L}$ is defined as $D(u) = C(e)$, where e is some maximal element in the set $\{s \in S : us = u\}$ (with respect to the preorder \leq).

The following lemma is a simple statement about R trivial monoids which is used frequently throughout the paper.

Lemma 2.9. *Suppose S is an R -trivial monoid. If $x, y, z \in S$ are such that $xyz = x$, then $xy = x$.*

Consequently, if $x, y_1, y_2, \dots, y_m \in S$ are such that $xy_1 \cdots y_m = x$, then $xy_i = x$ for all $1 \leq i \leq m$.

Proof. If $xyz = x$ then $xyS = xS$. Therefore $xy = x$ by the definition of S being R -trivial. The second statement immediately follows from the first. \square

The remainder of this section is dedicated to showing that these objects are well defined and that they satisfy the conditions of Definition 2.1. We begin by recalling some classical results from the monoid literature. The following is [7, Proposition 6.1].

Lemma 2.10. *If S is a finite monoid, then for each $x \in S$, there exists a positive integer $\omega = \omega(x)$ such that x^ω is idempotent, i.e. $(x^\omega)^2 = x^\omega$. Furthermore, if S is R -trivial, then we also have $x^\omega x = x^\omega$.*

Proof. Consider the elements x, x^2, x^3, \dots . Since S is finite, there exist positive integers i and p such that $x^{i+p} = x^i$. Then $x^{k+p} = x^k$ for all $k \geq i$, so if we take $\omega = ip$, then $(x^\omega)^2 = x^{\omega+ip} = x^\omega$.

If S is R -trivial, then $x^\omega \leq x^\omega x \leq x^\omega x^\omega = x^\omega$, and so $x^\omega x = x^\omega$. \square

Remark 2.11. In what follows, if $x \in \mathbb{C}S$ and there exists N such that $x^{N+1} = x^N$, we sometimes abuse notation by writing x^ω in place of x^N .

Lemma 2.12. *Let S be a finite R -trivial monoid. For all x and y in S ,*

1. $(xy)^\omega x = (xy)^\omega$;
2. $(xy)^\omega y = (xy)^\omega$;
3. $(xy)^\omega x^\omega = (xy)^\omega$;
4. $(xy)^\omega y^\omega = (xy)^\omega$;
5. $(x^\omega y^\omega)^\omega x^\omega = (x^\omega y^\omega)^\omega$;
6. $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega (xy)$;
7. $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega (xy)^\omega$.

Proof. (1) Since $(xy)^\omega x \in (xy)^\omega S$, it follows that $(xy)^\omega xS \subseteq (xy)^\omega S$. To show the reverse inclusion, note that $(xy)^\omega = (xy)^\omega (xy) = ((xy)^\omega x)y \in (xy)^\omega xS$, where the first equality follows from Lemma 2.10. So $(xy)^\omega S \subseteq (xy)^\omega xS$. Thus $(xy)^\omega xS = (xy)^\omega S$. Since S is an R -trivial monoid, the desired result follows.

(2) Apply (1) and Lemma 2.10:

$$(xy)^\omega = (xy)^\omega (xy) = ((xy)^\omega x)y = (xy)^\omega y.$$

(3) This follows from applying (1) repeatedly.

(4) This follows from applying (2) repeatedly.

(5) Let $u = x^\omega$ and $v = y^\omega$. Now, by (1), $(uv)^\omega u = (uv)^\omega$.

(6) We compute:

$$\begin{aligned}
 (x^\omega y^\omega)^\omega &= (x^\omega y^\omega)^{\omega-1} x^\omega y^\omega \\
 &= (x^\omega y^\omega)^{\omega-1} x^\omega y^\omega y \quad (\text{by Lemma 2.10}) \\
 &= (x^\omega y^\omega)^\omega y \\
 &= (x^\omega y^\omega)^\omega x^\omega y \quad (\text{by (5)}) \\
 &= (x^\omega y^\omega)^\omega x^\omega xy \quad (\text{by Lemma 2.10}) \\
 &= (x^\omega y^\omega)^\omega xy \quad (\text{by (5)}).
 \end{aligned}$$

(7) This follows by repeatedly applying part (6). \square

We are now ready to construct a lattice corresponding to the R -trivial monoid S . Define

$$\mathcal{L} := \{Se : e \in S \text{ such that } e^2 = e\}.$$

That is, \mathcal{L} is the set of left ideals generated by the idempotents of S . Define a partial order on \mathcal{L} by

$$Se \preceq Sf \iff Se \supseteq Sf.$$

Proposition 2.13. *If e, f are idempotents in S , then $S(ef)^\omega$ is the least upper bound of Se and Sf in \mathcal{L} .*

Proof. First, let us show that $S(ef)^\omega$ is an upper bound for Se and Sf . Since, by Lemma 2.12(1), $(ef)^\omega = (ef)^\omega e$, we have that $(ef)^\omega \in Se$. Hence $S(ef)^\omega \subseteq Se$ and $S(ef)^\omega \succcurlyeq Se$. Moreover, $(ef)^\omega = (ef)^{\omega-1} e$, so $S(ef)^\omega \subseteq Sf$ and $S(ef)^\omega \succcurlyeq Sf$. So $S(ef)^\omega$ is an upper bound for Se and Sf .

Next, let us show that $S(ef)^\omega$ is the least upper bound for Se and Sf . Suppose g is an idempotent in S such that Sg is an upper bound for Se and Sf . That is, $Sg \subseteq Se$ and $Sg \subseteq Sf$. Since $Sg \subseteq Se$, $g = te$ for some $t \in S$. But then $ge = (te)e = te^2 = te = g$. Similarly, $Sg \subseteq Sf$ implies that $gf = g$. So $g(ef) = (ge)f = gf = g$ and it follows that

$$g = g(ef) = (g(ef))(ef) = g(ef)^2 = (g(ef))(ef)^2 = g(ef)^3 = \dots = g(ef)^\omega.$$

Consequently, $g \in S(ef)^\omega$, $Sg \subseteq S(ef)^\omega$, and $Sg \succcurlyeq S(ef)^\omega$. So $S(ef)^\omega$ is the least upper bound of Se and Sf . \square

As a result, we may define the join of two elements Se and Sf in \mathcal{L} by

$$Se \vee Sf = S(ef)^\omega.$$

That is, \mathcal{L} is an upper semi-lattice with respect to this join operation. This observation proves the following.

Proposition 2.14. *The map $C : S \rightarrow \mathcal{L}$ defined by $C(x) = Sx^\omega$ is a surjective monoid morphism.*

Proof. Let $x, y \in S$. By Lemma 2.12(5), we know that $(x^\omega y^\omega)^\omega = (x^\omega y^\omega)^\omega (xy)^\omega$. Hence, $(x^\omega y^\omega)^\omega \in S(xy)^\omega$ and $S(x^\omega y^\omega)^\omega \subseteq S(xy)^\omega$.

To show the reverse inclusion, we begin by noting that, by Lemma 2.12(2), $(xy)^\omega = (xy)^\omega x^\omega$. So $(xy)^\omega \in Sx^\omega$ and $S(xy)^\omega \subseteq Sx^\omega$. That is, $S(xy)^\omega \supseteq Sx^\omega$.

Lemma 2.12(4), implies that $(xy)^\omega \in Sy^\omega$, which implies that $S(xy)^\omega \subseteq Sy^\omega$ and $S(xy)^\omega \supseteq Sy^\omega$. In particular, $S(xy)^\omega$ is an upper bound for both Sx^ω and Sy^ω . So $S(xy)^\omega \supseteq Sx^\omega \vee Sy^\omega = S(x^\omega y^\omega)^\omega$, that is, $S(xy)^\omega \subseteq S(x^\omega y^\omega)^\omega$.

Thus $C(xy) = S(xy)^\omega = S(x^\omega y^\omega)^\omega = Sx^\omega \vee Sy^\omega = C(x) \vee C(y)$, and C is a monoid morphism. Finally, we know that every element of \mathcal{L} is of the form Se for some idempotent e in S . But then $C(e) = Se^\omega = Se$; that is, C is a surjective morphism. \square

Here is an alternate and useful characterization of $C(x)$.

Proposition 2.15. $C(x) = \{a \in S : ax = a\}$ for all $x \in S$.

Proof. Take an arbitrary element in $C(x) = Sx^\omega$, say tx^ω . Since $(tx^\omega)x = t(x^\omega x) = tx^\omega$ by Lemma 2.10, we see that $tx^\omega \in \{a \in S : ax = a\}$. On the other hand, take $b \in \{a \in S : ax = a\}$. Then

$$bx^\omega = (bx)x^{\omega-1} = bx^{\omega-1} = (bx)x^{\omega-2} = bx^{\omega-2} = \dots = bx = b.$$

Therefore, $b \in Sx^\omega$. \square

We now define the map $D : S \rightarrow \mathcal{L}$. Given $u \in S$, let $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S : us = u\}$. To check that D is well defined, let e and f be two distinct maximal elements in $\{s \in S : us = u\}$. Since $e \leq ef$ and $u(e f) = (ue)f = uf = u$, by the maximality of e , $e = ef$. Similarly, since $f \leq fe$ and $u(fe) = u$, the maximality of f implies $f = fe$. Then, by Proposition 2.14,

$$C(e) = C(e f) = C(e) \vee C(f) = C(f) \vee C(e) = C(fe) = C(f).$$

Note that the maximality of e and $ue^2 = u$ also implies that $e = e^2$, that is, e is idempotent.

The next proposition shows that the maps C and D interact in precisely the manner given in conditions (2) and (3) in Definition 2.1. The following lemma will help us prove this proposition.

Lemma 2.16. Let $x, y \in S$. If $x \leq y$, then $C(x) \preceq C(y)$.

Proof. If $s \in C(y)$, then $sy = s$. Since $x \leq y$, there exists $t \in S$ such that $y = xt$. So $sxt = s$, implying $sx \leq s$. That is, $s \in C(x)$. Hence $C(y) \subseteq C(x)$, or $C(x) \preceq C(y)$ since $s \leq sx$ and S is R -trivial. \square

Proposition 2.17. Let $u, v \in S$. (i) If $uv \leq u$, then $C(v) \preceq D(u)$. (ii) If $C(v) \preceq D(u)$, then $uv = u$.

Proof. (i) Since $u \leq uv$, $u = uv$. Hence v lies in the set $\{s \in S : us = u\}$. Let e be a maximal element in this set such that $v \leq e$. Then, by Lemma 2.16, $C(v) \preceq C(e) = D(u)$.

(ii) By definition, $D(u) = C(e)$, where e is a maximal element of $\{s \in S : us = u\}$. So if $C(v) \preceq D(u)$, then $C(v) \preceq C(e)$. Hence $C(e) \subseteq C(v)$. Since $ue = u$, u lies in $C(e)$. So u is also a member of $C(v)$; that is, $uv = u$. \square

Propositions 2.14 and 2.17 tell us that an R -trivial monoid is a weakly ordered monoid. Combining this with Corollary 2.8, we have the following result.

Theorem 2.18. A finite monoid S is a weakly ordered monoid if and only if it is an R -trivial monoid.

3. Constructing idempotents

Definition 3.1. Let A be a finite dimensional algebra with identity 1. We say that a set of nonzero elements $\Lambda = \{e_J : J \in \mathcal{I}\}$ of A is a **complete system of primitive orthogonal idempotents for A** if:

1. each e_J is *idempotent*: that is, $e_J^2 = e_J$ for all $J \in \mathcal{I}$;
2. the e_J are pairwise *orthogonal*: $e_J e_K = 0$ for $J, K \in \mathcal{I}$ with $J \neq K$;
3. each e_J is *primitive* (meaning that it cannot be further decomposed into orthogonal idempotents): if $e_J = x + y$ with x and y orthogonal idempotents in A , then $x = 0$ or $y = 0$;
4. $\{e_J : J \in \mathcal{I}\}$ is *complete* (meaning that the elements sum to the identity): $\sum_{J \in \mathcal{I}} e_J = 1$.

Remark 3.2. If Λ is a *maximal* set of nonzero elements satisfying conditions (1) and (2), then Λ is a complete system of primitive orthogonal idempotents (that is, (3) and (4) also hold). Indeed, e_J is primitive, for if e_J could be written as $x + y$, then we could replace e_J in Λ with x and y , contradicting the maximality of Λ . To see (4), we just note that if $\sum_K e_K \neq 1$, then $1 - \sum_K e_K$ is idempotent and orthogonal to all other e_K . Combining this element with Λ would again contradict the maximality of Λ .

Let S denote a finite weakly ordered monoid with C and D being the associated “content” and “descent” maps from S to an upper semi-lattice \mathcal{L} . We let \mathcal{G} denote a set of generators of S . The main goal of this paper is to build a method for finding a complete system of orthogonal idempotents for the monoid algebra $\mathbb{C}S$. In particular, this solves the problem posed by Norton about the 0-Hecke algebra for the symmetric group.

For each $J \in \mathcal{L}$, we define a **Norton element** $A_J T_J$. Let us begin by defining T_J :

$$T_J = \left(\prod_{\substack{g \in \mathcal{G} \\ C(g) \preceq J}} g^\omega \right)^\omega \in S.$$

Remark 3.3. A different ordering of the set \mathcal{G} of generators may produce different T_J 's; so we fix an (arbitrarily chosen) order uniformly for all J .

We now define the A_J in the Norton element $A_J T_J$. First we let

$$B_J = \prod_{\substack{g \in \mathcal{G} \\ C(g) \not\preceq J}} (1 - g^\omega) \in \mathbb{C}S.$$

In the spirit of Lemma 2.10, we would like to raise B_J to a sufficiently high power so that it is idempotent. However, B_J is not an element of the monoid S , so $(B_J)^\omega$ may not be well defined. The following lemma and corollary shows that it actually is.

Definition 3.4. Given $x = \sum_{w \in S} c_w w \in \mathbb{C}S$, the **coefficient** of w in x is c_w . We say that w is a **term** of x if the coefficient of w in x is nonzero.

Lemma 3.5. Let $b \in S$ and suppose $bx^\omega = b$ for some $x \in \mathcal{G}$ with $C(x) \not\preceq J$. If c is a term of bB_J , then $c > b$.

Proof. Let $\mathcal{D} = \{x^\omega : x \in \mathcal{G}, C(x) \not\preceq J, bx^\omega = b\}$. By assumption \mathcal{D} is not empty. Let g_1, g_2, \dots, g_m be the generators which appear in the definition of B_J . Then

$$B_J = \sum_{i_1 < i_2 < \dots < i_k} (-1)^k g_{i_1}^\omega g_{i_2}^\omega \dots g_{i_k}^\omega.$$

It follows from Lemma 2.9 that the coefficient of b in bB_J is counting the terms in B_J where each of g_{i_1}, \dots, g_{i_k} come from \mathcal{D} , weighted with sign $(-1)^k$. If $|\mathcal{D}| = m \geq 1$ then this is $1 - m + \binom{m}{2} - \binom{m}{3} + \dots + (-1)^m = 0$. Therefore $c \neq b$. The statement now follows from the definition of order, as every term c of bB_J must be of the form $c = bz$ for some term z appearing in B_J , and hence $c \geq b$. \square

Lemma 3.6. *For every $J \in \mathcal{L}$, there exists an integer N such that $y^\omega B_J^N = 0$ for all $y \in \mathcal{G}$ with $C(y) \not\leq J$.*

Proof. Let $N = \ell + 1$, where ℓ is the length of the longest chain of elements in the poset (S, \leq) .

Suppose $y^\omega B_J^N \neq 0$. Let c_N be a term of B_J^N . Then c_N is a term of $c_{N-1}B_J$ for some term c_{N-1} in $y^\omega B_J^{N-1}$. Since $y^\omega y^\omega = y^\omega$, Lemma 3.5 implies that y^ω is not a term of $y^\omega B_J^k$ for any $k \geq 1$, so that $c_{N-1} = y^\omega g_1^\omega \dots g_m^\omega$ for some $m \geq 1$ and $g_i \in \mathcal{G}$ with $C(g_i) \not\leq J$. In particular, $c_{N-1} g_m^\omega = c_{N-1}$, and so, again by Lemma 3.5, $c_N > c_{N-1}$. Repeated application of this argument produces a decreasing chain

$$c_N > c_{N-1} > c_{N-2} > \dots > c_1$$

of elements in S , contradicting the fact that the length of the longest chain of elements in (S, \leq) is ℓ . \square

Corollary 3.7. *For every $J \in \mathcal{L}$ there exists an N such that $B_J^{N+1} = B_J^N$.*

Proof. By Lemma 3.6, $(B_J - 1)B_J^N = 0$ for a sufficiently large N since every element of $B_J - 1$ is of the form αy^ω where $\alpha \in \mathbb{C}$, $y \in \mathcal{G}$ and $C(y) \not\leq J$. \square

Remark 3.8. Corollary 3.7 is a special property of an R -trivial monoid, and is not true for a general monoid. For instance if an element x of a semigroup S generates a finite cyclic group of order 2, then $(1 - x)^k = 2^{k-1} - 2^{k-1}x$, so $(1 - x)^{k+1} \neq (1 - x)^k$ for all k .

This now allows us to define $A_J = B_J^\omega$.

Lemma 3.9. *Let $J \in \mathcal{L}$. Then:*

1. $T_J x = T_J$ for all x such that $C(x) \leq J$;
2. $y^\omega A_J = 0$ for all y such that $C(y) \not\leq J$ and $y \in \mathcal{G}$.

Proof. Since $J = C(T_J)$, $C(x) \leq J$ implies $C(x) \supseteq C(T_J)$. We also know that $T_J \in C(T_J)$ because T_J is idempotent. So $T_J \in C(x)$, that is, $T_J x = T_J$.

The second part follows from Lemma 3.6 since $A_J = B_J^N$. \square

Remark 3.10. Although T_J and A_J are idempotents individually, their product, the Norton element z_J , need not be. For example, take the 0-Hecke algebra $H_6(0)$ corresponding to the symmetric group \mathfrak{S}_6 . Let J be the subset $\{1, 4, 5\}$ of $\{1, 2, 3, 4, 5\}$. Then $T_J = T_1 T_4 T_5 T_4$, $A_J = (1 - T_2)(1 - T_3)(1 - T_2)$ and z_J is their product. No power of z_J is idempotent.

Lemma 3.11. *The coefficient of T_J in $z_J = A_J T_J$ is 1. All other terms y in z_J have $C(y) > J$.*

Proof. The coefficient of the identity element 1 in A_J is 1. Each term of $A_J T_J$ is of the form $a T_J$ for a term a of A_J . If $a \neq 1$, then $C(a) \not\leq J$ so $C(a T_J) = C(a) \vee C(T_J) > C(T_J) = J$. Hence the coefficient of T_J in $A_J T_J$ is 1 and all other terms have content greater than J . \square

Lemma 3.12. *If $J \not\leq K$ then $z_J z_K = 0$.*

Proof. Since $J \not\leq K$, there exists a $g \in \mathcal{G}$ with $C(g) \leq J$ but $C(g) \not\leq K$. Then, using Lemma 3.9(1) and Lemma 3.9(2), $z_J z_K = A_J T_J A_K T_K = A_J (T_J g^\omega) A_K T_K = A_J T_J (g^\omega A_K) T_K = 0$. \square

Lemma 3.13. For all $J \in \mathcal{L}$, there exists an N such that $(1 - z_J)^N z_J^2 = 0$.

Proof. To simplify the notation, let us temporarily set $T = T_J$, $A = A_J$ and $z = z_J = AT$. We first note that for any integer $k \geq 0$,

$$\begin{aligned} (1 - z)^k z^2 &= z(1 - z)^k z \\ &= AT(1 - AT)^k AT \\ &= A(T(1 - A)T)^k AT. \end{aligned}$$

We will show that $(T(1 - A)T)^N A = 0$ for $N > \ell$, where ℓ is the length of the longest chain in the poset (S, \leq) .

Let us write $1 - A = \sum_{a \in S} c_a a$ where each term has $c_a \neq 0$ only if $a = g_1^\omega \cdots g_k^\omega$ with $C(g_i) \not\leq J$ for all i . Therefore

$$T(1 - A)T = \sum_{a \in S} c_a TaT = \sum_{\substack{a \in S \\ TaT = Ta}} c_a Ta + \sum_{\substack{a \in S \\ TaT \neq Ta}} c_a TaT.$$

Note that $c_1 = 0$ since 1 is not a term of $(1 - A)$. If $TaT = Ta$, then we have

$$TaT \cdot (T(1 - A)T) = Ta(1 - A)T = Ta - TaAT = Ta$$

since $aA = 0$ by Lemma 3.9. Thus,

$$\begin{aligned} (T(1 - A)T)^N &= \left(\sum_{\substack{a_1 \in S \\ Ta_1T = Ta_1}} c_{a_1} Ta_1 + \sum_{\substack{a_1 \in S \\ Ta_1T \neq Ta_1}} c_{a_1} Ta_1T \right) (T(1 - A)T)^{N-1} \\ &= \sum_{\substack{a_1 \in S \\ Ta_1T = Ta_1}} c_{a_1} Ta_1 + \left(\sum_{\substack{a_1 \in S \\ Ta_1T \neq Ta_1}} c_{a_1} Ta_1T \right) (T(1 - A)T)^{N-1}. \end{aligned}$$

Next, rewrite the second summand above using the same argument:

$$\begin{aligned} \left(\sum_{\substack{a_1 \in S \\ Ta_1T \neq Ta_1}} c_{a_1} Ta_1T \right) (T(1 - A)T)^{N-1} &= \left(\sum_{\substack{a_1 \in S \\ Ta_1T \neq Ta_1}} c_{a_1} Ta_1T \right) \left(\sum_{a_2 \in S} c_{a_2} Ta_2T \right) (T(1 - A)T)^{N-2} \\ &= \left(\sum_{\substack{a_1, a_2 \in S \\ Ta_1T \neq Ta_1}} c_{a_1} c_{a_2} Ta_1Ta_2T \right) (T(1 - A)T)^{N-2} \\ &= \sum_{\substack{Ta_1T \neq Ta_1 \\ Ta_1Ta_2T = Ta_1Ta_2}} c_{a_1} c_{a_2} Ta_1Ta_2 \\ &\quad + \left(\sum_{\substack{Ta_1T \neq Ta_1 \\ Ta_1Ta_2T \neq Ta_1Ta_2}} c_{a_1} c_{a_2} Ta_1Ta_2T \right) (T(1 - A)T)^{N-2}. \end{aligned}$$

Continuing in this way, we can write $(T(1 - A)T)^N$ in the form

$$(T(1 - A)T)^N = \left(\sum c_{a_1} T a_1 + \cdots + \sum c_{a_1} \cdots c_{a_N} T a_1 \cdots T a_N \right) \\ + \sum_{\substack{Ta_1 \cdots Ta_i T \neq T a_1 \cdots T a_i \\ 1 \leq i \leq N}} c_{a_1} \cdots c_{a_N} T a_1 \cdots T a_N T.$$

By Lemma 3.9, we have $a_i A = 0$ for all terms a_i in $1 - A$, and so

$$(T(1 - A)T)^N \cdot A = \left(\sum_{\substack{Ta_1 \cdots Ta_i T \neq T a_1 \cdots T a_i \\ 1 \leq i \leq N}} c_{a_1} \cdots c_{a_N} T a_1 \cdots T a_N T \right) A.$$

This summation is 0 as it ranges over an empty set: indeed, if it is not empty, we would have an increasing chain of length $N > \ell$, namely

$$T a_1 < T a_1 T a_2 < T a_1 T a_2 T a_3 < \cdots < T a_1 T a_2 \cdots T a_N.$$

Therefore, $(T(1 - A)T)^N A = 0$. \square

Definition 3.14. Let $J \in \mathcal{L}$. Let

$$P_J := \sum_{n, m \geq 0} (1 - z_J)^{n+m} z_J^2 = \sum_{k \geq 0} (k+1)(1 - z_J)^k z_J^2.$$

(In Remark 3.20 we establish a summation-free formula for P_J .)

Remark 3.15. Lemma 3.13 shows there are only finitely many terms in the summation of P_J . Therefore P_J is a well-defined element of $\mathbb{C}S$ for each $J \in \mathcal{L}$.

Remark 3.16. A monoid S is called J -trivial if $SxS = SyS$ implies $x = y$ for all $x, y \in S$. When S is J -trivial it suffices to define

$$P_K = \sum_{n \geq 0} (1 - z_K)^n z_K.$$

Lemma 3.17. The coefficient of T_J in P_J is 1 and all other terms y of P_J have $C(y) \succ J$.

Proof. If $n + m > 0$ then, using that T_J is idempotent,

$$A_J T_J A_J T_J (1 - A_J T_J)^{n+m} = A_J T_J A_J (T_J - T_J A_J T_J)^{n+m}.$$

Each term x in $(T_J - T_J A_J T_J)^{n+m}$ has $C(x) \succ J$, so no T_J appears in $z_J^2(1 - z_J)^{n+m}$. The coefficient of T_J in z_J is 1, by Lemma 3.11. Hence T_J appears in $z_J^2(1 - z_J)^0$ with coefficient 1. By Lemma 3.11, since all of the terms $y \neq T_J$ of z_J have $C(y) \succ J$ and P_J is a polynomial in z_J , all other terms w of P_J must have $C(w) \succ J$. \square

Remark 3.18. As polynomials in x we have for any nonnegative integer N :

$$x \sum_{n=0}^N (1-x)^n = 1 - (1-x)^{N+1}.$$

Proposition 3.19. For each $J \in \mathcal{L}$, the element P_J is idempotent.

Proof. Let $J \in \mathcal{L}$ be fixed and let N be such that $(1-z_J)^N z_J^2 = 0$. Let us temporarily denote z_J by z . We can use Lemma 3.18 to rewrite P_J as

$$\begin{aligned} P_J &= \sum_{n,m \geq 0} z^2 (1-z)^{n+m} = \sum_{n=0}^N \sum_{m=0}^{N-n} z^2 (1-z)^{n+m} \\ &= \sum_{n=0}^N (1-z)^n \left(z^2 \sum_{m=0}^{N-n} (1-z)^m \right) = \sum_{n=0}^N (1-z)^n (z - z(1-z)^{N-n+1}) \\ &= z \left(\sum_{n=0}^N (1-z)^n \right) - (N+1)z(1-z)^{N+1} = 1 - (1-z)^{N+1} - (N+1)z(1-z)^{N+1}. \end{aligned}$$

This implies that $z^2 P_J = z^2$ since $z^2(1-z)^{N+1} = 0$, and so

$$P_J^2 = \left(\sum_{n=0}^N \sum_{m=0}^{N-n} (1-z)^{n+m} z^2 \right) P_J = \sum_{n=0}^N \sum_{m=0}^{N-n} (1-z)^{n+m} z^2 = P_J. \quad \square$$

Remark 3.20. As shown in the calculation above, one could define P_J as

$$P_J = 1 - (1 + (N+1)z_J)(1-z_J)^{N+1},$$

where N is the length of the longest chain in the monoid, or even $N = |S|$. For a J -trivial monoid, it suffices to take $P_J = 1 - (1-z_J)^{N+1}$.

Lemma 3.21. For all $J, K \in \mathcal{L}$, with $J \not\preceq K$, $P_J P_K = 0$.

Proof. Follows from Lemma 3.12 and the fact that P_J is a polynomial in z_J with no constant term. \square

Definition 3.22. For each $J \in \mathcal{L}$, let

$$e_J := P_J \left(1 - \sum_{K \succ J} e_K \right).$$

Lemma 3.23. T_J occurs in e_J with coefficient 1. All other terms y of e_J have $C(y) \succ J$. In particular, $e_J \neq 0$.

Proof. We proceed by induction. If J is maximal, then $e_J = P_J$, so the statement is implied by Lemma 3.17.

Now suppose the statement is true for all $M \succ J$. Then $e_J = P_J(1 - \sum_{M \succ J} e_M)$. By induction, all terms x of e_M have $C(x) \succ M \succ J$. So terms y from $P_J e_M$ have $C(y) \succ M \succ J$. The only other terms are those from P_J , for which the statement was proved in Lemma 3.17. \square

Lemma 3.24. $e_K P_J = 0$ for $K \not\leq J$.

Proof. The proof is by a downward induction on the semi-lattice. If K is maximal, then $e_K = P_K$, so by Lemma 3.21, $e_K P_J = P_K P_J = 0$.

Now suppose that for every $L \succ K$, $e_L P_J = 0$ for $L \not\leq J$, and we will show that $e_K P_J = 0$ for $K \not\leq J$. We expand $e_K P_J$:

$$e_K P_J = P_K \left(1 - \sum_{L \succ K} e_L \right) P_J = P_K P_J - \sum_{L \succ K} P_K e_L P_J.$$

Since $K \not\leq J$, we have $P_K P_J = 0$ by Lemma 3.21, and $e_L P_J = 0$ by induction, since $L \succ K$ and $K \not\leq J$ implies $L \not\leq J$. \square

Corollary 3.25. e_J is idempotent.

Proof. We expand $e_J e_J$:

$$\begin{aligned} e_J e_J &= P_J \left(1 - \sum_{M \succ J} e_M \right) P_J \left(1 - \sum_{M \succ J} e_M \right) = P_J \left(P_J - \sum_{M \succ J} e_M P_J \right) \left(1 - \sum_{M \succ J} e_M \right) \\ &\stackrel{(1)}{=} P_J^2 \left(1 - \sum_{M \succ J} e_M \right) \stackrel{(2)}{=} P_J \left(1 - \sum_{M \succ J} e_M \right) = e_J, \end{aligned}$$

where (1) follows from Lemma 3.24, and (2) follows from Lemma 3.19. \square

Lemma 3.26. $e_J e_K = 0$ for $J \neq K$.

Proof. The proof is by downward induction on the lattice \mathcal{L} . For a maximal element $M \in \mathcal{L}$, $e_M = P_M$, so $e_M e_K = P_M P_K (1 - \sum e_L) = 0$ by Lemma 3.21. Now suppose that for all $M \succ J$, $e_M e_K = 0$ for $M \neq K$ and we will show that $e_J e_K = 0$ for $J \neq K$. We expand $e_J e_K$:

$$e_J e_K = P_J \left(1 - \sum_{L \succ J} e_L \right) e_K = P_J \left(e_K - \sum_{L \succ J} e_L e_K \right). \quad (1)$$

If $K \not\leq J$, then $\sum_{L \succ J} e_L e_K = 0$ by our induction hypothesis, so $P_J (e_K - \sum_{L \succ J} e_L e_K) = P_J e_K = P_J P_K (1 - \sum_{M \succ K} e_M) = 0$ by Lemma 3.21.

If $K \succ J$, then $\sum_{L \succ J} e_L e_K = e_K$ since e_K is idempotent and $e_L e_K = 0$ for $L \neq K$ by the inductive hypothesis. Therefore $e_K - \sum_{L \succ J} e_L e_K = 0$ and hence the right hand side of (1) is zero. \square

Theorem 3.27. The set $\{e_J : J \in \mathcal{L}\}$ is a complete system of primitive orthogonal idempotents for $\mathbb{C}S$.

Proof. From [10], we know that the maximal number of such idempotents is the cardinality of \mathcal{L} . The rest of the claim is just Lemma 3.23, Corollary 3.25 and Lemma 3.26. \square

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Appendix A. Two examples

We illustrate the above constructions on two examples.

A.1. Idempotents for the free left regular band on two generators

Let S be the left regular band freely generated by two elements a, b . Then $S = \{1, a, b, ab, ba\}$. All elements of S are idempotent. Also $aba = ab$ and $bab = ba$. The lattice \mathcal{L} has four elements: $\emptyset := S$, $a := Sa$, $b := Sb$ and $ab := Sab = Sba$, where $\emptyset < a < ab$ and $\emptyset < b < ab$, but a and b have no relation. We begin by computing the elements P_J .

$J = \emptyset$: Neither of the generators satisfies $C(g) \preceq J$, so $T_\emptyset = 1 \in S$. $B_\emptyset = (1 - a)(1 - b)$. Also

$$\begin{aligned} B_\emptyset^2 &= (1 - a)(1 - b)(1 - a)(1 - b) = (1 - a - b + ab)(1 - a)(1 - b) \\ &= (1 - a - b + ab)(1 - b) = (1 - a - b + ab) = B_\emptyset. \end{aligned}$$

Therefore $A_\emptyset = B_\emptyset = 1 - a - b + ab$, so $z_\emptyset = 1 - a - b + ab$ is idempotent and

$$P_\emptyset = 1 - a - b + ab.$$

$J = a$: Then $C(a) \preceq a$ and $C(b) \not\preceq a$, so $T_a = a$ and $B_a = 1 - b = A_a$ since $1 - b$ is idempotent. Therefore $z_a = (1 - b)a = a - ba$. $z_a^2 = a - ab$ and one can check that $z_a^3 = z_a^2$, so

$$P_a = z_a^2(1 + (1 - z_a) + (1 - z_a)^2 + \cdots) = z_a^2 = a - ab.$$

One can check that P_a is idempotent.

$J = b$: Similarly,

$$P_b = b - ba.$$

$J = ab$: $C(a), C(b) \preceq ab$, so $T_{ab} = ab$ and $A_{ab} = 1$. $z_{ab} = ab$ is idempotent, so

$$P_{ab} = ab.$$

We can now compute the idempotents e_J . Since ab is maximal,

$$e_{ab} = ab.$$

Since $P_a e_{ab} = (a - ab)ab = ab - ab = 0$,

$$e_a = P_a(1 - e_{ab}) = P_a = a - ab$$

and similarly,

$$e_b = b - ba.$$

Finally, note that $P_\emptyset e_a = (1 - a - b + ab)(a - ab) = 0$ and similarly $P_\emptyset e_b = 0$, so that

$$e_\emptyset = P_\emptyset(1 - e_a - e_b - e_{ab}) = P_\emptyset - P_\emptyset e_{ab} = 1 - a - b + ab - ab + ba = 1 - a - b + ba.$$

One can check that $\{e_\emptyset, e_a, e_b, e_{ab}\}$ is a collection of mutually orthogonal idempotents.

A.2. Idempotents of $H^{\mathfrak{S}_5}(0)$

As mentioned above, $H^{\mathfrak{S}_5}(0)$ has generators T_1, T_2, T_3, T_4 . In this case, the corresponding lattice \mathcal{L} is the lattice of subsets of $\{1, 2, 3, 4\}$. The monoid $H^{\mathfrak{S}_5}(0)$ is actually a J -trivial monoid, so we can use the simplified formula from Remark 3.16. We use the shorthand notation $T_{i_1 \dots i_k}$ to denote the element $T_{i_1} \cdots T_{i_k}$.

If $J = \{1, 2, 3, 4\}$, then $T_J = T_{1234}^\omega = T_{1234123121}$. Also $A_J = 1$, so $z_J = A_J T_J = T_J$. Also, $P_J = z_J$, and since J is maximal, $e_J = P_J$, so

$$e_{\{1,2,3,4\}} = T_{1234123121}.$$

If $J = \{1, 2, 3\}$, then $T_J = T_{123121}$ and $A_J = 1 - T_4$. Then $z_J = (1 - T_4)T_{123121} = T_{123121} - T_{4123121}$. One can check that $z_J^2 = z_J$, so $P_J = z_J$. Also, one can check that P_J is orthogonal to $e_{\{1,2,3,4\}}$. So $e_J = P_J$. Therefore

$$e_{\{1,2,3\}} = T_{123121} - T_{4123121}.$$

Similarly,

$$e_{\{2,3,4\}} = -T_{1234232} + T_{234232}.$$

Now let $J = \{1, 2, 4\}$. Then $T_J = T_{1214}$ and $A_J = (1 - T_3)$. Letting $z_J = A_J T_J$, one can check that $z_J(1 - z_J)^2 = 0$, so $P_J = z_J(1 + (1 - z_J))$. Again P_J is orthogonal to $e_{\{1,2,3,4\}}$, so $e_J = P_J$. Therefore

$$e_{\{1,2,4\}} = -T_{123423121} + T_{12343121} - T_{34121} + T_{4121}.$$

Similarly,

$$e_{\{1,3,4\}} = -T_{123412321} + T_{12342321} - T_{23431} + T_{3431}.$$

When $J = \{1, 2\}$, $T_J = T_{121}$ and $A_J = (1 - T_3)(1 - T_4)(1 - T_3)$. Then z_J is already idempotent, so $P_J = z_J$. One can check that P_J is already orthogonal to $e_{\{1,2,3,4\}}$, $e_{\{1,2,3\}}$, $e_{\{1,2,4\}}$. Therefore,

$$e_{\{1,2\}} = T_{121} - T_{3121} + T_{34121} - T_{343121} - T_{4121} + T_{43121}.$$

Similarly,

$$e_{\{3,4\}} = T_{12343} - T_{123431} - T_{2343} + T_{23431} + T_{343} - T_{3431}.$$

If $J = \{1, 3\}$, $T_J = T_1 T_3$ and $A_J = (1 - T_2)(1 - T_4)$. One can check that $z_J(1 - z_J)^2 = 0$, and $P_J = z_J(1 + 1 - z_J)$ is idempotent. P_J is orthogonal to $e_{\{1,2,3,4\}}$ and $e_{\{1,2,3\}}$, but not orthogonal to $e_{\{1,2,4\}}$. So we define $e_{\{1,3\}} = P_{\{1,3\}}(1 - e_{\{1,2,4\}})$. Then

$$e_{\{1,3\}} = -T_{123121} + T_{12321} - T_{12341231} + T_{123412321} + T_{1234231} - T_{12342321} - T_{231} + T_{2341231} \\ - T_{23412321} + T_{31} - T_{341231} + T_{3412321} + T_{4123121} - T_{412321} + T_{4231} - T_{431}.$$

Similarly,

$$e_{\{2,4\}} = -T_{12342312} + T_{123423121} + T_{1234232} + T_{1234312} - T_{12343121} - T_{123432} + T_{2342312} \\ - T_{23423121} - T_{234232} - T_{234312} + T_{2343121} + T_{23432} + T_{3412} - T_{342} - T_{412} + T_{42}.$$

We continue in this way, constructing all of the idempotents for the algebra. For the sake of completeness, the other idempotents are:

$$e_{\{2,3\}} = -T_{1232} + T_{123412312} - T_{1234123121} + T_{232} - T_{23412312} + T_{234123121} + T_{41232} - T_{4232};$$

$$e_{\{1,4\}} = -T_{1234123121} + T_{123412321} + T_{123423121} - T_{12342321} - T_{12343121} + T_{1234321} + T_{2341} \\ - T_{23421} - T_{341} + T_{3421} + T_{41} - T_{421};$$

$$e_{\{4\}} = -T_{1234} + T_{12341} - T_{123412} + T_{1234121} + T_{12342} - T_{123421} + T_{234} - T_{2341} + T_{23412} \\ - T_{234121} - T_{2342} + T_{23421} - T_{34} + T_{341} - T_{3412} + T_{34121} + T_{342} - T_{3421} + T_4 \\ - T_{41} + T_{412} - T_{4121} - T_{42} + T_{421};$$

$$e_{\{3\}} = T_{123} - T_{1231} + T_{1234123} - T_{12341232} - T_{123423} + T_{1234232} - T_{23} + T_{231} - T_{234123} \\ + T_{2341232} + T_{23423} - T_{234232} + T_3 - T_{31} + T_{34123} - T_{341232} - T_{3423} + T_{34232} \\ - T_{4123} + T_{41231} + T_{423} - T_{4231} - T_{43} + T_{431};$$

$$e_{\{2\}} = -T_{12} + T_{12312} - T_{123121} + T_2 - T_{2312} + T_{23121} + T_{312} - T_{32} - T_{3412} + T_{3412312} \\ - T_{34123121} + T_{342} - T_{342312} + T_{3423121} + T_{34312} - T_{3432} + T_{412} - T_{412312} \\ + T_{4123121} - T_{42} + T_{42312} - T_{423121} - T_{4312} + T_{432};$$

$$e_{\{1\}} = T_1 - T_{21} + T_{231} - T_{2321} - T_{2341} + T_{23421} - T_{234231} + T_{2342321} + T_{23431} - T_{234321} \\ - T_{31} + T_{321} + T_{341} - T_{3421} + T_{34231} - T_{342321} - T_{3431} + T_{34321} - T_{41} + T_{421} \\ - T_{4231} + T_{42321} + T_{431} - T_{4321}.$$

Finally, e_{\emptyset} is just the signed sum of all elements, with sign determined by Coxeter length:

$$e_{\emptyset} = \sum_w (-1)^{\ell(w)} T_w.$$

One can check (ideally not by hand!) that $\{e_J : J \subseteq \{1, 2, 3, 4\}\}$ is a complete system of orthogonal idempotents.

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