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# On the Lie structure of a prime associative superalgebra

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ABSTRACT

In this paper some results on the Lie structure of prime superalgebras are discussed. We prove that, with the exception of some special cases, for a prime superalgebra $A$ over a ring of scalars $\Phi$ with $1/2 \in \Phi$, if $L$ is a Lie ideal of $A$ and $W$ is a subalgebra of $A$ such that $[W, L] \subseteq W$, then either $L \subseteq Z$ or $W \subseteq Z$. Likewise, if $V$ is a submodule of $A$ and $[V, L] \subseteq V$, then either $V \subseteq Z$ or $L \subseteq Z$ or there exists an ideal of $A$, $M$, such that $0 \neq [M, A] \subseteq V$. This work extends to prime superalgebras some results of I.N. Herstein, C. Lanski and S. Montgomery on prime algebras.



## 1. Introduction

An associative superalgebra is just a superalgebra that is associative as an ordinary algebra. If $A = A_0 + A_1$ is a superalgebra, the elements in $A_0 \cup A_1$ are called homogeneous elements.

It is known that, if we take an associative superalgebra $A$, and we change the product in $A$ by the superbracket product $[a, b] = ab - (-1)^{\bar{a}\bar{b}}ba$, where $\bar{a}, \bar{b}$ denote the degree

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of the homogeneous elements $a$ and $b$ in $A = A_0 + A_1$, we obtain a Lie superalgebra, denoted by $A^-$.

The Lie structure of prime associative superalgebras and simple associative superalgebras was investigated by F. Montaner [19] and S. Montgomery [20]. Concerning superalgebras with superinvolution, several papers have also appeared studying the Lie structure of the skewsymmetric elements in relation to the ideals of the superalgebra (see [6,7,12,14,15]).

The results obtained in the above papers have been an extension of a now classical research, developed in the context of non-graded rings and rings with involution. This research was first initiated by I.N. Herstein [8,9] and W.E. Baxter [1], and afterwards several other authors made contributions to complete it: T.E. Erickson [5], C. Lanski [16], W.S. Martindale III and C.R. Miers [18], etc.

The aim of this paper is to prove, in the setting of prime associative superalgebras over a ring of scalars $\Phi$ with $1/2 \in \Phi$, the following results, which are well known in the non-graded case. These results were proved by I.N. Herstein for semiprime, 2 torsion free rings (see Lemma 4 and Theorems 3 and 5 in [10]), and by C. Lanski and S. Montgomery for prime rings without restriction in the characteristic (see Lemma 11 and Theorems 12 and 13 in [17]).

**Lemma 1.1.** *Let $R$ be a prime, 2-torsion-free ring and $U$ a Lie ideal of $R$. Suppose that $A$ is an additive subgroup such that $[U, A] \subseteq A$ and $[A, A] \subseteq Z$. Then $A \subseteq Z$.*

**Theorem 1.2.** *Let $R$ be a prime, 2-torsion-free ring and $W$ a subring of $R$. Suppose that $U$ is a Lie ideal of $R$ such that $[W, U] \subseteq W$. Then either $U \subseteq Z$, or $W \subseteq Z$, or $W$ contains a nonzero ideal of $R$.*

**Theorem 1.3.** *Let $R$ be a prime, 2-torsion-free ring and let $U$ be a Lie ideal of $R$. Suppose that $V$ is an additive subgroup of $R$ such that $[V, U] \subseteq V$. Then either $U \subseteq Z$, or $V \subseteq Z$, or there exists an ideal $M$ of $R$ such that $0 \neq [M, R] \subseteq V$.*

These results have been very useful in rings (see for example [2,3,11], etc.), and have also been used in superalgebras, for example, in the study of the Lie ideals of the set of skewsymmetric elements of an associative superalgebra with superinvolution (see [7,14, 12,13]). As these results have never been proved in superalgebras, we are interested in proving them here. To do that we take advantage of some of the ideas developed in the proofs made in [10,12,17].

For a complete introduction to the basic definitions and examples of superalgebras, superinvolutions and prime and semiprime superalgebras, we refer the reader to [4,6,19].

Throughout the paper, unless otherwise stated, $A$ will denote a nontrivial prime associative superalgebra over a commutative unital ring $\phi$ of scalars with $\frac{1}{2} \in \phi$. By a nontrivial superalgebra we understand a superalgebra with a nonzero odd part. $Z$ will denote the even part of the center of $A$.

If $Z \neq 0$, one can consider the localization $Z^{-1}A = \{z^{-1}a\colon\ 0 \neq z \in Z,\ a \in A\}$. If $A$ is prime, then $Z^{-1}A$ is a prime associative superalgebra over the field $Z^{-1}Z$, whose center is $Z^{-1}Z$. We call this superalgebra the central closure of $A$. We also say that $A$ is a central order in $Z^{-1}A$. This terminology is not the standard one, for which the definition involves the extended centroid. We say that $A$ is a central order in $C(n)$ if $Z \neq 0$ and $Z^{-1}A$ is isomorphic to the Clifford superalgebra of a non-degenerate quadratic space of dimension $n$ over $Z^{-1}Z$ (see Example 1.5 in [6]).

More precisely, in this paper we prove three main results. Let $A$ be a prime associative superalgebra over a ring of scalars $\Phi$ with $1/2 \in \Phi$, such that $A$ is not a central order in $C(n)$, $n = 1, 2, 3$, and let $L$ be a Lie ideal of $A$ then:

(1) If $V$ is a $\Phi$-submodule of $A$ such that $[V, L] \subseteq V$ and $[V, V] \subseteq Z$, then either $L \subseteq Z$ or $V \subseteq Z$.
(2) If $W$ is a subalgebra of $A$ such that $[W, L] \subseteq W$, then either $L \subseteq Z$, or $W \subseteq Z$, or $W$ contains a nonzero ideal of $A$.
(3) If $V$ is a $\Phi$-submodule of $A$ such that $[V, L] \subseteq V$, then either $L \subseteq Z$, or $V \subseteq Z$ or there exists an ideal $M$ of $A$ such that $0 \neq [M, A] \subseteq V$.

As in the nongraded case, we expect that these results will be useful in future studies concerning Lie structures. For example, along the lines of [17] and [20].

In the superalgebra context, by a subalgebra, a submodule, or an ideal we respectively mean a graded subalgebra, submodule and ideal.

Let $A$ be an associative superalgebra and $M$ be a $\Phi$-submodule of $A$. Denote by $\overline{M}$ the subalgebra of $A$ generated by $M$. We will say that $M$ is dense in $A$ if $\overline{M}$ contains a nonzero ideal of $A$.

The following results are instrumental for the paper:

**Lemma 1.4.** *(See [9, Lemma 1.1.9].) Let $A$ be a semiprime algebra and $L$ a Lie ideal of $A$. If $[a, [a, L]] = 0$, then $[a, L] = 0$.*

**Lemma 1.5.** *(See [19, Lemmata 1.2, 1.3].) If $A = A_0 \oplus A_1$ is a semiprime superalgebra, then $A_0$ and $A$ are semiprime algebras. Moreover, if $A$ is prime, then either $A$ is prime or $A_0$ is prime (as algebras).*

**Lemma 1.6.** *(See [7, Theorem 2.1].) Let $A$ be a prime nontrivial associative superalgebra. If $L$ is a Lie ideal of $A$, then either $L \subseteq Z$ or $L$ is dense in $A$, except if $A$ is a central order in $C(2)$.*

**Lemma 1.7.** *(See [12].) Let $A$ be a prime superalgebra, $L$ a Lie ideal of $A$ such that $L$ is dense in $A$, and $v \in A_i$ such that $vLv = 0$, then $v = 0$.*

**Lemma 1.8.** *(See [12].) Let $A$ be a prime superalgebra, $L$ a Lie ideal of $A$ such that $L$ is dense in $A$, and $V$ a Lie subalgebra of $A$ such that $[V, L] \subseteq V$. If $v^2 = 0$ for every $v \in V_i$, then $V_i = 0$.*

We point out that the bracket product in Lemma 1.1, Theorem 1.2, Theorem 1.3 and Lemma 1.4 is the usual one: $[a, b] = ab - ba$, but the bracket product in Lemma 1.8 is the superbracket $[x_i, y_j]_s = x_i y_j - (-1)^{ij} y_j x_i$ for $x_i \in A_i$, $y_j \in A_j$ homogeneous elements. In fact, the superbracket product coincides with the usual bracket if one of the arguments belongs to the even part of $A$. In the following, to simplify the notation, we will denote both in the usual way $[\ ,\ ]$ but we will understand that it is the superbracket if we are in a superalgebra. In other words, we could say that, when we are not using the prefix 'super', we are assuming that the graduation is the trivial one.

Also, from now on, by an element $a \in M$, with $M$ any $\Phi$-submodule of a superalgebra $A$, we will always understand a homogeneous element $a \in M$, that is, $a \in M_0 \cup M_1$, unless otherwise stated.

## 2. Lie structure of an associative superalgebra

We begin with the following useful result.

**Lemma 2.1.** *Let $A$ be a prime superalgebra such that it is not an order in $C(2)$. Let $L$ be a Lie ideal of $A$. Then either $L \subseteq Z$ or $C(L) \subseteq Z$, where $C(L) = \{x \in A\colon\ [x, L] = 0\}$.*

**Proof.** We notice that $C(L)$ is a Lie ideal and a subalgebra of $A$. Since for, $x, y \in C(L)$, $a \in A$ and $u \in L$, we have

$$\big[[x, a], u\big] = -(-1)^{\bar{x}\bar{a}+\bar{x}\bar{u}}\big[[a, u], x\big] - (-1)^{\bar{u}\bar{x}+\bar{u}\bar{a}}\big[[u, x], a\big] = 0,$$
$$[xy, u] = x[y, u] + (-1)^{\bar{u}\bar{y}}[x, u]y = 0.$$

So, by Theorem 4.1 and its proof in [6] either $C(L) \subseteq Z$ or $C(L)$ is dense in $A$. But if $C(L)$ is dense in $A$, then there exists a nonzero ideal $I$ of $A$ such that $[I, L] = 0$, and from Lemma 2.3 in [14] $L \subseteq Z$. $\square$

**Lemma 2.2.** *Let $A$ be a prime superalgebra such that it is not an order in $C(n)$, $n = 1, 2, 3$. Let $W$ be a subalgebra of $A$ and a Lie ideal of $[A, A]$. Then either $W \subseteq Z$ or $W$ is dense in $A$.*

**Proof.** From Theorem 3.3 in [19], we know that either $W \subseteq Z$ or there exists an ideal $I$ of $A$ such that $0 \neq [I, A] \subseteq W$. So suppose that $W \nsubseteq Z$ and let $I$ be an ideal of $A$ such that $0 \neq [I, A] \subseteq W$. Notice that $[I, A]$ is a nonzero Lie ideal of $A$. Therefore, by Lemma 1.6, either $[I, A] \subseteq Z$ or $[I, A]$ is dense in $A$. In the later case, $W$ is dense in $A$. If $[I, A] \subseteq Z$, we can localize $A$ by $Z$ and consider $Z^{-1}A$. Then $0 \neq [Z^{-1}I, Z^{-1}A] \subseteq Z^{-1}Z$.

Therefore $Z^{-1}I$ has invertible elements and so $Z^{-1}I = Z^{-1}A$. But then, since $[I, A] \subseteq Z$, $[Z^{-1}A, Z^{-1}A] \subseteq Z^{-1}Z \cap Z^{-1}I$, that is, $[[Z^{-1}A, Z^{-1}A], Z^{-1}A] = 0$. From Lemma 2.6 in [19], $A$ is $C(n)$ with $n = 1, 2$ or 3, a contradiction. $\square$

**Lemma 2.3.** *Let $A$ be a prime superalgebra which is not an order in $C(n)$ with $n = 1, 2, 3$, and $L, U$ Lie ideals of $A$ such that $[L, U] \subseteq Z$. Then either $L \subseteq Z$ or $U \subseteq Z$.*

**Proof.** Suppose that $L \nsubseteq Z$. Since $[L, U] \subseteq Z$, it follows that $[L_0, U_1] = [L_1, U_0] = 0$ and $[L_0, U_0] + [L_1, U_1] \subseteq Z$. So for every $u \in U_0$ we have $[u, [u, L]] = 0$, and from Lemmata 1.4 and 1.5 we deduce that $[U_0, L_0] = 0$. But $[U_0, L_1] = 0$ and so $[U_0, L] = 0$. From Lemma 2.1 $U_0 \subseteq Z$, and $[L, U] = [L_1, U_1] \subseteq Z$. If $[L, U] = 0$, then by Lemma 2.1 $U \subseteq Z$. And if $0 \neq [L, U] \subseteq Z$, then $Z \neq 0$ and we can consider the localization $Z^{-1}A$ and the Lie ideals $Z^{-1}ZL, Z^{-1}ZU$ in $Z^{-1}A$.

We suppose now that $L \nsubseteq Z$ and $U \nsubseteq Z$. From Theorem 3.2 in [19] there exist nonzero ideals $I, J$ of $A$ such that

$$0 \neq [I, A] \subseteq L, \qquad 0 \neq [J, A] \subseteq U.$$

Notice that if $[I, A] = 0$ or $[J, A] = 0$, then, by Lemma 2.3 in [14], $A \subseteq Z$, a contradiction. Since $[L, U] \subseteq Z$ we have

$$\left[Z^{-1}ZL, Z^{-1}ZU\right] \subseteq Z^{-1}Z,$$

and so

$$\left[\left[Z^{-1}I, Z^{-1}A\right], \left[Z^{-1}J, Z^{-1}A\right]\right] \subseteq Z^{-1}Z.$$

If $[[Z^{-1}I, Z^{-1}A], [Z^{-1}J, Z^{-1}A]] \neq 0$ then $Z^{-1}I, Z^{-1}J$ have invertible elements and $Z^{-1}I, Z^{-1}J = Z^{-1}A$. Therefore

$$\left[\left[\left[Z^{-1}A, Z^{-1}A\right], \left[Z^{-1}A, Z^{-1}A\right]\right], Z^{-1}A\right] = 0.$$

Now, from Lemma 2.6 in [19] we have a contradiction with our hypothesis about $A$ not being a central order in $C(n)$ with $n = 1, 2, 3$ (notice that in [19] the product $a \circ b$ is our product $[a, b]$ when $a, b \in A_1$).

And if $[[Z^{-1}I, Z^{-1}A], [Z^{-1}J, Z^{-1}A]] = 0$, then, by Lemma 2.1,

$$\text{either} \quad \left[Z^{-1}I, Z^{-1}A\right] \subseteq Z^{-1}Z \quad \text{or} \quad \left[\,Z^{-1}J, Z^{-1}A\right] \subseteq Z^{-1}Z.$$

Therefore, since $[I, A] \neq 0$ and $[J, A] \neq 0$,

$$\text{either} \quad Z^{-1}I = Z^{-1}A \quad \text{or} \quad Z^{-1}J = Z^{-1}A.$$

Since $[Z^{-1}I, Z^{-1}A], [Z^{-1}J, Z^{-1}A] \subseteq Z^{-1}Z$, in both cases we have

$$\big[[Z^{-1}A, Z^{-1}A], Z^{-1}A\big] = 0.$$

Again from Lemma 2.6 in [19] we have a contradiction with our hypothesis. □

**Lemma 2.4.** *Let $A$ be a prime superalgebra such that it is not an order in $C(n)$ with $n = 1, 2, 3$, and $L$ a Lie ideal of $A$ such that $[t, L] \subseteq Z$ for some $t \in A$. Then either $t \in Z$ or $L \subseteq Z$.*

**Proof.** Consider $U = \{x \in A\colon\ [x, L] \subseteq Z\}$. We notice that $U$ is a $\Phi$-submodule of $A$, and it is also a Lie ideal because for every $u \in L$, $x \in U$ and $y \in A$

$$\big[[x, y], u\big] = (-1)^{\bar{u}\bar{y}}\big[[x, u], y\big] + (-1)^{\bar{y}\bar{u}}\big[x, [y, u]\big] \in Z.$$

So, $U$ is a Lie ideal of $A$ and from Lemma 2.3 either $U \subseteq Z$ or $L \subseteq Z$. □

**Lemma 2.5.** *Let $A$ be a prime superalgebra such that it is not an order in $C(n)$ with $n = 1, 2, 3$, $L$ a Lie ideal and $V$ a $\Phi$-submodule of $A$ such that $[V, L] \subseteq V$ and $[V, V] \subseteq Z$. Then either $L \subseteq Z$ or $V \subseteq Z$.*

**Proof.** Suppose that $L \nsubseteq Z$. Then, from Theorem 3.2 in [19], there exists a nonzero ideal $I$ of $A$ such that $[I, A] \subseteq L$, and $[I, A] \neq 0$ by Lemma 2.3 in [14].

If $I \cap Z \neq 0$, we localize $A$ by $Z$ and then $Z^{-1}Z \cap Z^{-1}I \neq 0$, so $Z^{-1}I$ has invertible elements and $Z^{-1}I = Z^{-1}A$. Hence $Z^{-1}ZV$ is a Lie ideal of $[Z^{-1}A, Z^{-1}A]$. From Theorem 3.3 in [19] either $Z^{-1}ZV \subseteq Z^{-1}Z$ or there exists a nonzero ideal $N$ of $A$ such that $0 \neq [Z^{-1}N, Z^{-1}A] \subseteq Z^{-1}ZV$. In the second case, since $[V, V] \subseteq Z$, we have

$$\big[[Z^{-1}N, Z^{-1}A], [Z^{-1}N, Z^{-1}A]\big] \subseteq Z^{-1}Z.$$

From Lemma 2.4 we have $[Z^{-1}N, Z^{-1}A] \subseteq Z^{-1}Z$, and since $[Z^{-1}N, Z^{-1}A] \neq 0$, $Z^{-1}N = Z^{-1}A$. So,

$$[Z^{-1}A, Z^{-1}A] \subseteq Z^{-1}Z,$$

and by Lemma 2.3, $Z^{-1}A \subseteq Z^{-1}Z$, a contradiction with our assumptions. Therefore $Z^{-1}ZV \subseteq Z^{-1}Z$ and so $V \subseteq Z$.

If $I \cap Z = 0$, then for every $v \in V_0$ we have

$$\big[v, \big[v, [I, A]_0\big]\big] \subseteq [V, V] \cap I \subseteq Z \cap I = 0.$$

From Lemmata 1.4 and 1.5 we have

$$\big[V_0, [I, A]_0\big] = 0.$$

Now, we consider $W = [V, [I, A]]$. Notice that

$$[W, W] \subseteq [V, V] \cap I \subseteq Z \cap I = 0.$$

So for every $w \in W_1$ we have $w^2 = 0$. From Lemma 1.8 $W_1 = 0$. Therefore

$$W_1 = \bigl[V_0, [I, A]_1\bigr] + \bigl[V_1, [I, A]_0\bigr] = 0.$$

We have $0 \neq [I, A]$, and also $[I, A] \nsubseteq Z$, because if $[I, A] \subseteq Z$, then $[I, A] \subseteq Z \cap I = 0$, a contradiction. Therefore, since $[V_0, [I, A]] = 0$, we have $V_0 \subseteq Z$ because of Lemma 2.1. But we have deduced that $[V_1, [I, A]_0] = 0$, and we observe that

$$\bigl[V_1, [I, A]_1\bigr] \subseteq V_0 \cap I \subseteq Z \cap I = 0.$$

Therefore, we also obtain that $[V_1, [I, A]] = 0$ and, again by Lemma 2.1, $V_1 \subseteq Z$, that is, $V \subseteq Z$. □

We prove now our first theorem.

**Theorem 2.6.** *Let $A$ be a prime superalgebra such that it is not an order in $C(n)$ for $n = 1, 2, 3$. Let $W$ be a subalgebra of $A$, $L$ a Lie ideal of $A$ and $[W, L] \subseteq W$. Then either $L \subseteq Z$, $W \subseteq Z$, or $W$ is dense in $A$ (that is, $W$ contains a nonzero ideal of $A$).*

**Proof.** We suppose that $L \nsubseteq Z$. Because of Lemma 1.6 there exists a nonzero ideal $N$ of $A$ such that $N \subseteq \bar{L}$.

Let $V = [W, L]$. If $V = 0$, then from Lemma 2.1 we have either $W \subseteq Z$ or $L \subseteq Z$. Since $L \nsubseteq Z$ we deduce that $W \subseteq Z$.

So, suppose now that $V \neq 0$. Let $0 \neq u \in V$, $w \in W$. We notice that if $u$ is even, then $[u, u] = 0$, and if $u$ is odd, then $[u, u] = 0$ means that $u^2 = 0$. We will prove that if $t, s \in W$ and $u \in V$, with $[u, u] = 0$ satisfy $[t, u][u, s] \neq 0$, then $W$ is dense. First we see that $[t, u][u, s]A \subseteq W$. We have

$$[u, s]a = [u, sa] - (-1)^{\bar{s}\bar{u}} s[u, a]$$

for every $a \in A$, therefore

$$[t, u][u, s]a = [t, u][u, sa] - (-1)^{\bar{u}\bar{s}} [t, u]s[u, a].$$

But

$$\begin{aligned}[t, u][u, sa] &= \bigl[t, u[u, sa]\bigr] - (-1)^{\bar{t}\bar{u}} u\bigl[t, [u, sa]\bigr] \\ &= (-1)^{\bar{u}} \bigl[t, [u, usa]\bigr] - (-1)^{\bar{t}\bar{u}} u\bigl[t, [u, sa]\bigr] \in W,\end{aligned}$$

because $W$ is a subring and $L$ is a Lie ideal of $A$. And also

$$[t,u]s[u,a] = [t,u]\big[s,[u,a]\big] + (-1)^{\bar{s}(\bar{u}+\bar{a})}[t,u][u,a]s \in W,$$

because

$$\begin{aligned}[t,u][u,a] &= \big[t,u[u,a]\big] - (-1)^{\bar{t}\bar{u}}u\big[t,[u,a]\big] \\ &= (-1)^{\bar{u}}\big[t,[u,ua]\big] - (-1)^{\bar{t}\bar{u}}u\big[t,[u,a]\big] \in W.\end{aligned}$$

Therefore $[t,u]s[u,a] \in W$, and so

$$[t,u][u,s]a \in W \quad \text{for every } t,s \in W,\ a \in A,\ u \in V,\ \text{with } [u,u]=0.$$

Next we will show that

$$\bar{L}[t,u][u,s]A \subseteq W.$$

Since $[W,L] \subseteq W$ it follows that

$$L[t,u][u,s]A \subseteq \big[L,[t,u][u,s]A\big] + [t,u][u,s]AL \subseteq W.$$

Notice also that

$$L^2[t,u][u,s]A \subseteq [L,W] + [t,u],[u,s]A \subseteq W.$$

Using induction over $i$ it is easy to prove that

$$L^i[t,u][u,s]A \subseteq W \quad \text{for } i > 0,$$

and so that

$$\bar{L}[t,u][u,s]A \subseteq W.$$

Hence, since $N$ is a nonzero ideal such that $N \subseteq \bar{L}$, we have $M = N[t,u][u,x]A$, a nonzero ideal such that $M \subseteq W$.

Therefore, either $W$ is dense in $A$, or, if $W$ is not dense in $A$,

$$[t,u][u,s] = 0 \quad \text{for every } t,s \in W,\ u \in V \text{ such that } [u,u]=0,$$

because of the primeness of $A$.

We suppose now that $W$ is not dense, and so

$$[t,u][u,s] = 0 \quad \text{for every } t,s \in W,\ u \in V \text{ such that } [u,u]=0. \tag{$*$}$$

We will show that $V = [W, L] = 0$, a contradiction with our assumption. We prove this in 4 steps. Let $K = [V, V]$.

1. $K = [V, V] = [V_1, V_1]$. Indeed, let $x, y, u \in V$ such that $u^2 = 0$. From our assumption (∗), $[u, x][u, y] = 0$, and expanding this gives

$$uxuy - (-1)^{\bar{y}\bar{u}} uxyu + (-1)^{\bar{x}\bar{u}+\bar{y}\bar{u}} xuyu = 0.$$

Right multiplication by $u$ gives $uxuyu = 0$. Since $[y, l] \in V$ for every $l \in L$, we obtain that $[y, [l, u]] \in V$. So $uxu[y, [l, u]]u = 0$. Expanding this expression yields $uxuluyu = 0$. From Lemma 1.7 we deduce that $uVu = 0$. If $u, u' \in V$ are homogeneous elements with $u^2 = (u')^2 = 0$, we conclude that

$$\left(uu'\right)^2 = uu'uu' \in uVuu' = 0.$$

If $l \in L$ we have

$$0 = u\left[u', l\right]uu' = uu'luu'.$$

So $uu'Luu' = 0$, and, from Lemma 1.7,

$$uu' = 0 \quad \text{for every homogeneous elements } u, u' \in V, \text{ with } u^2 = \left(u'\right)^2 = 0. \qquad (**)$$

Now consider $x, y \in V_1$, $u, v \in V_0$. We have $[x, u]^2 = 0 = [y, v]^2$, and so $[x, u][y, v] = 0$, because of (∗∗). Since $[V_0, V_1]$ is additively generated by the elements $[x, u]$ with $x \in V_1, u \in V_0$, we have $v^2 = 0$ for every $v \in [V_0, V_1]$. From Lemma 1.8,

$$[V_0, V_1] = 0,$$

and

$$[V, V] = [V_0, V_0] + [V_1, V_1].$$

Now consider $X = [V_0, V_0]$. We notice that $X$ is a Lie subalgebra of $A$ and $[X, L] \subseteq X$. From our assumption (∗), for every $x, y, u, v \in V_0$ we have $[x, u]^2 = [y, v]^2 = 0$, and so, by (∗) we obtain that $[x, u][y, v] = 0$. Again, since $[V_0, V_0]$ is additively generated by the elements $[x, u]$ with $x, u \in V_0$, we deduce that for every $v \in X$, $v^2 = 0$. From Lemma 1.8,

$$X = [V_0, V_0] = 0.$$

Therefore $[V, V] = [V_1, V_1]$.

2. $K = [V_1, V_1] \subseteq Z$. From Lemma 1.5, $A_0$ is semiprime. Also, we notice that $L_0$ is a Lie ideal of $A_0$, and it is satisfied that

$$K = [V, V] = [V_1, V_1] \subseteq L_0, \qquad [K, L_0] \subseteq K \quad \text{and} \quad [K, K] \subseteq [V_0, V_0] = 0.$$

From Lemma 4 in [10], $[K, L_0] = 0$. Moreover, since

$$[K, L_1] \subseteq \big[[V_1, V_1], L_1\big] \subseteq [V_1, V_0] = 0,$$

we deduce that $[K, L] = 0$. From Lemma 2.4, $K \subseteq Z$.

3. $K = [V, V] = 0$. Indeed, if $K \neq 0$, then $Z \neq 0$, and we can localize $A$ by $Z$ and consider $Z^{-1}A, Z^{-1}ZW$ and $Z^{-1}ZL$. From Theorem 3.2 in [19], there exists an ideal of $A, I$, such that $0 \neq [Z^{-1}I, Z^{-1}A] \subseteq Z^{-1}ZL$. Notice that $[Z^{-1}I, Z^{-1}A]$ is a Lie ideal of $Z^{-1}A$. We claim that $Z^{-1}I = Z^{-1}A$.

To prove this, we distinguish two cases: when $[Z^{-1}I, Z^{-1}A] \subseteq Z^{-1}Z$, and when $[Z^{-1}I, Z^{-1}A] \nsubseteq Z^{-1}Z$.

If $[Z^{-1}I, Z^{-1}A] \subseteq Z^{-1}Z$, $Z^{-1}I$ has invertible elements and then $Z^{-1}I = Z^{-1}A$.

If $[Z^{-1}I, Z^{-1}A] \nsubseteq Z^{-1}Z$, then, since $K \subseteq Z$, we have

$$\big[\big[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW\big], \big[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW\big]\big] \subseteq Z^{-1}Z.$$

Notice that if $[[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW], [[Z^{-1}I, Z^{-1}A], Z^{-1}ZW]] = 0$, using Lemma 2.5 for $L = [Z^{-1}I, Z^{-1}A]$ and $V = [[Z^{-1}I, Z^{-1}A], Z^{-1}ZW]$, we obtain that

$$\big[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW\big] \subseteq Z^{-1}Z.$$

If $[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW] \neq 0$, then, since $[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW] \subseteq Z^{-1}I$ we have $Z^{-1}I = Z^{-1}A$. And if $[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW] = 0$, then, by Lemma 2.1, $[Z^{-1}I, Z^{-1}A] \subseteq Z^{-1}Z$ or $Z^{-1}ZW \subseteq Z^{-1}Z$. Since $V \neq 0$, $Z^{-1}ZW \nsubseteq Z^{-1}Z$, and so

$$0 \neq \big[Z^{-1}I, Z^{-1}A\big] \subseteq Z^{-1}Z \cap Z^{-1}I,$$

that is, $Z^{-1}A = Z^{-1}I$. So, finally,

$$0 \neq \big[\big[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW\big], \big[[Z^{-1}I, Z^{-1}A], Z^{-1}ZW\big]\big] \subseteq Z^{-1}Z.$$

Then $Z^{-1}I$ has invertible elements and $Z^{-1}I = Z^{-1}A$.

So $Z^{-1}I = Z^{-1}A$, and then $[Z^{-1}A, Z^{-1}A] \subseteq Z^{-1}ZL$. Therefore $Z^{-1}ZW$ is a subalgebra and a Lie ideal of $[Z^{-1}A, Z^{-1}A]$. From Lemma 2.2, either $Z^{-1}ZW \subseteq Z^{-1}Z$ or $Z^{-1}ZW$ is dense in $Z^{-1}A$. If $Z^{-1}ZW \subseteq Z^{-1}Z$, then $W \subseteq Z$, a contradiction because then $V = 0$. Therefore $Z^{-1}ZW$ is dense in $Z^{-1}A$, and there exists a nonzero ideal $J$ of $A$ such that $Z^{-1}J \subseteq Z^{-1}ZW$. Hence, since $K \subseteq Z$,

$$\big[[Z^{-1}J, Z^{-1}ZL], [Z^{-1}J, Z^{-1}ZL]\big] \subseteq Z^{-1}Z.$$

We observe that if $[[Z^{-1}J, Z^{-1}ZL], [Z^{-1}J, Z^{-1}ZL]] = 0$, then by Lemma 2.1

$$\big[Z^{-1}J, Z^{-1}ZL\big] \subseteq Z^{-1}Z.$$

From Lemma 2.3, either $Z^{-1}J \subseteq Z^{-1}Z$ or $Z^{-1}ZL \subseteq Z^{-1}Z$. In the first case, we obtain that $(Z^{-1}L)(Z^{-1}(A_1 + A_1^2)) = 0$, a contradiction with the primeness. In the second, $L \subseteq Z$ and $V = 0$, again a contradiction.

And if $[[Z^{-1}J, Z^{-1}ZL], [Z^{-1}J, Z^{-1}ZL]] \neq 0$, since $[[Z^{-1}J, Z^{-1}ZL], [Z^{-1}J, Z^{-1}ZL]] \subseteq Z^{-1}Z$, then $Z^{-1}J$ has invertible elements, and $Z^{-1}J = Z^{-1}A$. That is,

$$Z^{-1}ZW = Z^{-1}A.$$

But then

$$\left[Z^{-1}A, Z^{-1}A\right] = \left[Z^{-1}I, Z^{-1}A\right] \subseteq Z^{-1}ZL,$$

and

$$K = [V, V] = \left[[W, L], [W, L]\right] \subseteq Z,$$

implies that

$$\left[\left[Z^{-1}A, \left[Z^{-1}A, Z^{-1}A\right]\right], \left[Z^{-1}A, \left[Z^{-1}A, Z^{-1}A\right]\right]\right] \subseteq Z^{-1}Z.$$

From Lemma 2.3,

$$\left[Z^{-1}A, \left[Z^{-1}A, Z^{-1}A\right]\right] \subseteq Z^{-1}Z,$$

and again by Lemma 2.3, $Z^{-1}A \subseteq Z^{-1}Z$, a contradiction. So $K = [V, V] = 0$.

4. Finally, we reach a contradiction. $V$ is $\Phi$-submodule of $A$ and $[V, L] \subseteq V$ and $[V, V] = 0$ by step 3. From Lemma 2.5 we have $V = [W, L] \subseteq Z$, because $L \nsubseteq Z$. Then by Lemma 2.4 $W \subseteq Z$, a contradiction because $V \neq 0$. $\square$

And, now, to finish, we prove our second theorem.

**Theorem 2.7.** *Let $A$ be a prime superalgebra such that it is not an order in $C(n)$ with $n = 1, 2, 3$. Let $L$ be a Lie ideal of $A$ and $V$ a $\Phi$-submodule of $A$ such that $[V, L] \subseteq V$. Then either $L \subseteq Z$ or $V \subseteq Z$ or there exists an ideal $M$ of $A$ such that $[M, A] \subseteq V$.*

**Proof.** Let $K = [V, L]$, and $T = \{x \in A\colon [x, A] \subseteq V\}$. Then $T$ is a subalgebra of $A$ because for every $t, s \in T$ and $a \in A$

$$[ts, a] = [t, sa] + (-1)^{\bar{t}\bar{s} + \bar{a}\bar{t}}[s, at] \in V.$$

Since

$$\left[[K, K], A\right] \subseteq \left[[K, A], K\right] \subseteq [L, V] \subseteq V,$$

it follows that $[K,K] \subseteq T$. If we consider $T'$, the subring generated by $[K,K]$, we have $[T',L] \subseteq T'$, because

$$\begin{aligned}\big[[[K,K],L],A\big] &\subseteq \big[[K,K],[L,A]\big] + \big[[[K,K],A,],L\big] \\ &\subseteq \big[[K,K],L\big] + [V,L] \subseteq V,\end{aligned}$$

and because for every $t,s \in [K,K]$ and $u \in L$ we have

$$[ts,u] = t[s,u] + (-1)^{\bar{s}\bar{u}}[t,u]s \in T'.$$

Now, $T'$ is a subalgebra of $A$ and $[T',L] \subseteq T'$. From Theorem 2.6 either $L \subseteq Z$, or $T' \subseteq Z$ or $T'$ contains a nonzero ideal $M$ of $A$. If $L \subseteq Z$ we have finished. If $L \nsubseteq Z$ and $T' \subseteq Z$, then $[K,K] \subseteq Z$ and then $K \subseteq Z$ by Lemma 2.5. So $[V,L] \subseteq Z$, but then by Lemma 2.4 $V \subseteq Z$. If $M$ is an ideal of $A$ such that $M \subseteq T'$, then $M \subseteq T$ and $[M,A] \subseteq V$. □

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