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PBW-deformations of quantum groups

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ABSTRACT

In this paper, we investigate certain deformations $\mathfrak{B}_q(\mathfrak{g})$ of the negative part $U_q^-(\mathfrak{g})$ of quantized enveloping algebras $U_q(\mathfrak{g})$. An algorithm is established to determine when a given $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. For $\mathfrak{g}$ of type $\mathbb{A}_2$ and $\mathbb{B}_2$, we classify PBW-deformations of $U_q^-(\mathfrak{g})$. Moreover, we explicitly construct some PBW bases for a class of PBW-deformations $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$. As an application, Iorgov–Klimyk's PBW bases for the non-standard quantum deformation $U_q'(\mathfrak{so}(n,\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{so}(n,\mathbb{C}))$ are recovered.

1. Introduction

A filtered algebra $\mathbb{U}$ is called a Poincaré–Birkhoff–Witt-deformation (abbr. PBW-deformation) of a graded algebra $\mathcal{A}$ if its associated graded algebra $\mathrm{gr}(\mathbb{U})$ is isomorphic to $\mathcal{A}$. PBW-deformation theory of graded algebras is extensively studied. For Koszul or $\mathbb{N}$-Koszul algebras, a Jacobi type condition was given for the determination of PBW-deformation (see e.g. [2,4,10,33]). While for an arbitrary graded algebra $\mathcal{A}$ over a field, Cassidy and Shelton [6] extended the above results to a more general Jacobi condition for deciding when certain deformations of $\mathcal{A}$ obtained by altering its defining relations are PBW ones.

For all complex simple Lie algebras $\mathfrak{g}$, Drinfeld [9] and Jimbo [20] introduced the quantum deformations $U_q(\mathfrak{g})$ of universal enveloping algebras $U(\mathfrak{g})$, which are very important in mathematical

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physics (see also [19,21]). Except the cases $U_q^-(\mathfrak{sl}_2)$ and $U_q^-(\mathfrak{sl}_3)$, the negative nilpotent subalgebras $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ are graded algebras with defining relations in mixed degrees (the word 'mixed' means that there exist at least two defining relations whose degrees are different, cf. [6]). Some PBW-deformations of $U_q^-(\mathfrak{g})$ appeared in the investigation of coideal subalgebras of $U_q(\mathfrak{g})$ (cf. [24–27,31,32]) and non-standard quantum deformations of $U(\mathfrak{g})$ (cf. [12,17]). In the theory of quantum groups, Lusztig [28,29] investigated braid group actions on $U_q(\mathfrak{g})$ which allow the definition of root vectors and PBW bases. In analogy to the quantum group case, braid group actions on coideal subalgebras of $U_q(\mathfrak{g})$ were also investigated by many authors (see e.g. [7,22,30]).

Assume that $\mathfrak{g}$ is a complex simple Lie algebra of rank n. Let $A=(a_{ij})_{n\times n}$ be the Cartan matrix of $\mathfrak{g}$ and $H=(h(B_i,B_j))_{n\times n}$ with $h(B_i,B_j)$ in the free algebra generated by $B_1,B_2,\dots,B_n$ and $\mathrm{Deg}(h(B_i,B_j))<2-a_{ij}$. Denote by $\mathfrak{B}_q(\mathfrak{g})$ the quantum algebra with generators B_i $(1\leqslant i\leqslant n)$ and defining relations

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{pmatrix}1-a_{ij}\\ k\end{pmatrix}_i B_i^{1-a_{ij}-k}B_jB_i^k=h(B_i,B_j),\quad 1\leqslant i\neq j\leqslant n.$$

Motivated by the above mentioned research, in this present paper we mainly focus on the following two problems:

(1) Determining when a given deformation $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$ is a PBW-deformation.
(2) Constructing PBW bases for a class of PBW-deformations $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$.

For the problem (1), our main techniques are the Jacobi condition given in [6] and the Bernstein–Gelfand–Gelfand resolution (abbr. BGG-resolution) established in [14]. The Jacobi condition in [6] actually transforms the problem of determining PBW-deformations of a graded algebra $\mathcal{A}$ into a series of linear algebra problems which we denote $(*)$. Though it is a sufficient and necessary condition for judging which deformations $\mathbb{U}$ of $\mathcal{A}$ are PBW ones, there is a homological constant $c(\mathcal{A})$ in it whose accurate value is generally not easy to obtain. In [6], $c(\mathcal{A})$ is called the complexity of $\mathcal{A}$ which in a sense reflects the scale of the sets of linear equations in $(*)$. By Definition 2.1, the size of $c(\mathcal{A})$ is deeply related with the bigraded Yoneda algebra $E(\mathcal{A})=\bigoplus \mathrm{Ext}_{\mathcal{A}}^{r,s}(\mathbb{C},\mathbb{C})$ of $\mathcal{A}$. For finite dimensional semisimple Lie algebras the BGG-resolution was introduced in [3]. The quantum group version of the BGG-resolution was established in [14] and explicitly written down in [13]. In this paper, we compute the complexity $c(U_q^-(\mathfrak{g}))$ of $U_q^-(\mathfrak{g})$ by using the BGG-resolution of the trivial left $U_q^-(\mathfrak{g})$-module ${}_{U_q^-(\mathfrak{g})}\mathbb{C}$. Based on the above ideas, we propose an algorithm to decide if a given algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. In practical use, our algorithm is very technical because the amount of calculations in it is very large for hand computation. So the computer program realization of our algorithm or more conceptional research on the classification of PBW-deformations of $U_q^-(\mathfrak{g})$ is interesting.

The algebras $\mathfrak{B}_q(\mathfrak{g})$ in problem (2) can be viewed as a uniform description of some coideal subalgebras of $U_q(\mathfrak{g})$ in [22] and Iorgov–Klimyk's non-standard quantum deformation $U_q'(\mathfrak{so}(n,\mathbb{C}))$ in [12]. In fact, they were studied by Letzter in more generality in [24–27], and proved to be coideal subalgebras of $U_q(\mathfrak{g})$ and PBW-deformations of $U_q^-(\mathfrak{g})$ in [24]. Our results indicate that Kolb–Pellegrini's braid group actions on $\mathfrak{B}_q(\mathfrak{g})$ also allow the definition of root vectors and some PBW bases $B(w_0)$ for $\mathfrak{B}_q(\mathfrak{g})$. The root vectors of $\mathfrak{B}_q(\mathfrak{g})$ have the same form as those of $U_q(\mathfrak{g})$ for simply laced $\mathfrak{g}$, while for non-simply laced $\mathfrak{g}$ they have some additional terms of lower degree. In our proof of the PBW theorems for $\mathfrak{B}_q(\mathfrak{g})$, it is crucial that $\mathfrak{B}_q(\mathfrak{g})$ are coideal subalgebras of $U_q(\mathfrak{g})$ and PBW-deformations of $U_q^-(\mathfrak{g})$.

This paper is organized as follows. In Section 2, we fix some notations and collect the background material that will be necessary in the sequel. In Section 3, we give an algorithm for problem (1) after theoretical analysis, then apply it to the case $\mathfrak{g}$ of type $\mathbb{A}_2$ and $\mathbb{B}_2$. In Section 4 we explicitly construct some PBW bases $B(w_0)$ for $\mathfrak{B}_q(\mathfrak{g})$ in problem (2). In Section 4.1, we state some properties of the algebra automorphisms τ_i $(1\leqslant i\leqslant n)$ of $\mathfrak{B}_q(\mathfrak{g})$ given by Kolb and Pellegrini, then calculate some formulas about them. In Section 4.2, the root vectors and the set $B(w_0)$ for $\mathfrak{B}_q(\mathfrak{g})$ are defined. In Section 4.3, the relationship between $B(w_0)$ and Lusztig's PBW basis $F(w_0)$ for $U_q^-(\mathfrak{g})$ is described. In Section 4.4, we establish PBW theorems for $\mathfrak{B}_q(\mathfrak{g})$ with $\mathfrak{g}$ of each type, that is, Theorem 4.4. In Section 4.5, we show that Iorgov–Klimyk's PBW bases for the non-standard quantum deformation $U_q'(\mathfrak{so}(n+1,\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{so}(n+1,\mathbb{C}))$ can be recovered by ours.

Throughout, we denote by $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}$ the complex number field, the set of positive integers and the set of integers, respectively. The parameter $q\in\mathbb{C}\backslash\{0\}$ is not a root of unity.

2. Preliminaries

2.1. PBW-deformation theory of graded algebras

Let $\mathcal{T}$ be the free algebra $\mathbb{C}\langle x_1,x_2,\dots,x_n\rangle$ with a standard grading, that is, $\mathrm{Deg}(x_i)=1$ for $1\leqslant i\leqslant n$. Denote by

$$\mathcal{A}=\mathbb{C}\langle x_1,x_2,\dots,x_n\rangle/\langle r_1,r_2,\dots,r_{m_0}\rangle$$

the quotient algebra of $\mathcal{T}$ with m_0 homogeneous relations $r_1,r_2,\dots,r_{m_0}$. Throughout this paper, we assume that $R=\{r_1,r_2,\dots,r_{m_0}\}$ is a minimal set of relations for $\mathcal{A}$ and that none of the relations is linear. By a deformation of $\mathcal{A}$ we mean a $\mathbb{C}$-algebra

$$\mathbb{U}=\mathbb{C}\langle x_1,x_2,\dots,x_n\rangle/\langle r_1+l_1,r_2+l_2,\dots,r_{m_0}+l_{m_0}\rangle$$

with the set of relations $P=\{r_1+l_1,r_2+l_2,\dots,r_{m_0}+l_{m_0}\}$, where $l_1,l_2,\dots,l_{m_0}$ are (not necessarily homogeneous) elements of $\mathcal{T}$ such that $\mathrm{Deg}(l_i)<\mathrm{Deg}(r_i)$ for all i. The algebra $\mathcal{A}$ is graded and the algebra $\mathbb{U}$ is filtered. We denote by $\mathcal{F}^k(\mathbb{U})$ $(k\in\mathbb{Z})$ the filtration of $\mathbb{U}$ and define $\mathrm{gr}(\mathbb{U})=\bigoplus_{k\in\mathbb{Z}}\mathcal{F}^k(\mathbb{U})/\mathcal{F}^{k-1}(\mathbb{U})$ to be the graded algebra associated with $\mathbb{U}$.

Definition 2.1. (See [6].) The non-graded deformation $\mathbb{U}$ of the graded $\mathbb{C}$-algebra $\mathcal{A}$ is said to be a PBW-deformation if its associated graded algebra $\mathrm{gr}(\mathbb{U})$ is isomorphic to $\mathcal{A}$.

Let $\mathcal{A}$-$\underline{\mathrm{Mod}}$ be the category of $\mathbb{Z}$-graded left $\mathcal{A}$-modules. For each object M in $\mathcal{A}$-$\underline{\mathrm{Mod}}$ and $d\in\mathbb{Z}$, the notation $M\{d\}$ denote the graded left $\mathcal{A}$-module M with grading shifted by d, i.e., $M\{d\}_k=M_{d+k}$ for $k\in\mathbb{Z}$. For the graded left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{A}$ and $\vec{d}=(d_1',d_2',\dots,d_{r_0}')\in\mathbb{Z}^{r_0}$ we denote

$$\mathcal{A}\{\vec{d}\}:=\mathcal{A}\{d_1'\}\oplus\mathcal{A}\{d_2'\}\oplus\cdots\oplus\mathcal{A}\{d_{r_0}'\}.$$

For two objects M,N in $\mathcal{A}$-$\underline{\mathrm{Mod}}$, we define

$$\mathrm{Hom}^j_{\mathcal{A}}(M,N):=\{\phi\in\mathrm{Hom}_{\mathcal{A}}(M,N)\mid\phi(M_i)\subseteq N_{i-j}\},$$
$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N):=\bigoplus_{j\in\mathbb{Z}}\mathrm{Hom}^j_{\mathcal{A}}(M,N).$$

Obviously, $\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N)$ is a $\mathbb{Z}$-graded vector space. Fix a minimal projective resolution

$$\cdots\longrightarrow C_2\xrightarrow{\varphi_2}C_1\xrightarrow{\varphi_1}C_0\xrightarrow{\varepsilon}\mathbb{C}\longrightarrow 0\tag{2.1}$$

of the trivial $\mathcal{A}$-module $\mathbb{C}$ in $\mathcal{A}$-$\underline{\mathrm{Mod}}$. The modules C_i are free and can be expressed as the form $\mathcal{A}\{\vec{d}\}=\mathcal{A}\{d_1'\}\oplus\mathcal{A}\{d_2'\}\cdots\oplus\mathcal{A}\{d_{r_0}'\}$, where r_0 may be infinite. If we apply the functor $\underline{\mathrm{Hom}}_{\mathcal{A}}(\cdot,\mathbb{C})$ to the truncated complex $P_\bullet$ of the above resolution (2.1), then the cohomology of the resulting cochain complex $\underline{\mathrm{Hom}}_{\mathcal{A}}(P_\bullet, \mathbb{C})$ of abelian groups equals $E(\mathcal{A}) = \bigoplus \mathrm{Ext}^{r,s}_{\mathcal{A}}(\mathbb{C}, \mathbb{C})$, which is the associated bi-graded Yoneda algebra of $\mathcal{A}$ with r the cohomology degree and $-s$ the internal degree inherited from the grading on $\mathcal{A}$.

Definition 2.2. (See [6].) The complexity of the graded $\mathbb{C}$-algebra $\mathcal{A}$ is defined by

$$c(\mathcal{A}) = \sup\{s \mid \mathrm{Ext}^{3,s}_{\mathcal{A}}(\mathbb{C}, \mathbb{C}) \neq 0\} - 1$$

if the global dimension of $\mathcal{A}$ is at least 3. For global dimension less than 3 we set $c(\mathcal{A}) = 0$.

Denote by V the $\mathbb{C}$-span of the generators $x_1, x_2, \dots, x_n$ of $\mathcal{T}$ and take $\mathcal{F}^k(\mathcal{T}) = \bigoplus_{i \leqslant k} V^{\otimes i}$. Let

$$P_1 = \mathrm{Span}_{\mathbb{C}}\big(P \cap \mathcal{F}^1(\mathcal{T})\big), \tag{2.2}$$

$$P_k = V P_{k-1} + P_{k-1} V + \mathrm{Span}_{\mathbb{C}}\big(P \cap \mathcal{F}^k(\mathcal{T})\big), \quad \text{for } k > 1. \tag{2.3}$$

In [6], a necessary and sufficient condition was given for determining PBW-deformations of $\mathcal{A}$, which is stated as follows.

Theorem 2.3. *(See [6].) Let $\mathcal{A}$ be a graded $\mathbb{C}$-algebra of finite complexity $c(\mathcal{A})$ and let $\mathbb{U}$ be a deformation of $\mathcal{A}$. Then $\mathbb{U}$ is a PBW-deformation of $\mathcal{A}$ if and only if $P_1 = 0$ and the following Jacobi condition is satisfied:*

$$P_{k+1} \cap \mathcal{F}^k(\mathcal{T}) \subset P_k \quad \textit{for all } 1 \leqslant k \leqslant c(\mathcal{A}). \tag{2.4}$$

2.2. Quantized enveloping algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. We write $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ for the triangular decomposition, and $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ for the Borel subalgebra. Let $\Phi \subseteq \mathfrak{h}^*$ denote the corresponding root system and Φ^+ the set of positive roots. Fix the set $\Pi = \{\alpha_i \mid 1 \leqslant i \leqslant n\}$ of simple roots. For any $\alpha = \sum_{i=1}^n n_i \alpha_i \in \mathbb{Z}\Pi$, define the height of α by $\mathrm{Ht}(\alpha) = \sum_{i=1}^n n_i$. Let W be the Weyl group of $\mathfrak{g}$, which is generated by all simple reflections $s_i = s_{\alpha_i}$ for $\alpha_i \in \Pi$. The notation $\mathrm{Br}(\mathfrak{g})$ denotes the braid group corresponding to W, that is, $\mathrm{Br}(\mathfrak{g})$ is generated by $\{\sigma_i \mid 1 \leqslant i \leqslant n\}$ and satisfies the relation

$$\underbrace{\sigma_i \sigma_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{\sigma_j \sigma_i \cdots}_{m_{ij} \text{ factors}}, \tag{2.5}$$

where m_{ij} is the order of $s_i s_j$ in W. Let $(\cdot,\cdot)$ denote the W-invariant scalar product on $\mathbb{Z}\Phi$ such that $(\alpha, \alpha) = 2$ for all short roots $\alpha \in \Phi$. As usual, $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ are just the entries of the Cartan matrix $A = (a_{ij})_{n \times n}$ of $\mathfrak{g}$.

For each $1 \leqslant i \leqslant n$ one defines $q_i = q^{d_i}$, where $d_i = \frac{(\alpha_i, \alpha_i)}{2}$. For $n \in \mathbb{N}$, the q-number is defined as

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

and let $[n]_i! = [n]_i [n-1]_i \cdots [2]_i [1]_i$. If $(\alpha_i, \alpha_i) = 2$ then we will also write $[n]$ and $[n]!$ instead of $[n]_i$ and $[n]_i!$. Moreover, the q-binomial coefficient is defined for any $a, b \in \mathbb{Z}$ with $b > 0$ by

$$\binom{a}{b}_i = \frac{[a]_i [a-1]_i \cdots [a-b+1]_i}{[b]_i [b-1]_i \cdots [1]_i}.$$

Definition 2.4. (See [19].) The quantized enveloping algebra $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$ is defined as the $\mathbb{C}$-algebra with generators $K_i^{\pm 1}$, E_i, F_i $(1 \leqslant i \leqslant n)$ and relations:

$$K_iK_i^{-1} = 1 = K_i^{-1}K_i, \qquad K_iK_j = K_jK_i,$$
$$K_iE_jK_i^{-1} = q^{(\alpha_i,\alpha_j)}E_j, \qquad K_iF_jK_i^{-1} = q^{-(\alpha_i,\alpha_j)}F_j,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_i E_i^{1-a_{ij}-k}E_jE_i^k = 0 \quad \text{for } i \neq j,$$
$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_i F_i^{1-a_{ij}-k}F_jF_i^k = 0 \quad \text{for } i \neq j.$$

Lusztig observed in [28] that there are a series of automorphisms T_i $(1 \leqslant i \leqslant n)$ on $U_q(\mathfrak{g})$, which we call Lusztig symmetries. These automorphisms are important in investigating PBW bases and canonical bases of $U_q(\mathfrak{g})$. They are defined as follows:

$$\begin{cases} T_i(E_i) = -K_i^{-1}F_i, \quad T_i(F_i) = -E_iK_i, \quad T_i(K_i) = K_i^{-1}, \\ \left.\begin{array}{l} T_i(K_j) = K_jK_i^{-a_{ij}}, \\ T_i(E_j) = \displaystyle\sum_{s=0}^{-a_{ij}}(-1)^s q_i^{-s}E_i^{(s)}E_jE_i^{(-a_{ij}-s)}, \\ T_i(F_j) = \displaystyle\sum_{s=0}^{-a_{ij}}(-1)^s q_i^{s}F_i^{(-a_{ij}-s)}F_jF_i^{(s)}, \end{array}\right\} \quad \text{for } 1 \leqslant i \neq j \leqslant n, \end{cases}$$

where $E_i^{(n)} = \frac{E_i^n}{[n]_i!}$ and $F_i^{(n)} = \frac{F_i^n}{[n]_i!}$ for any $n \in \mathbb{N}$. The following theorem holds.

Theorem 2.5. *(See [19,28].) Lusztig symmetries T_i $(1 \leqslant i \leqslant n)$ of $U_q(\mathfrak{g})$ satisfy the braid relation (2.5).*

Denote by $U_q^-(\mathfrak{g})$ the negative nilpotent subalgebra of $U_q(\mathfrak{g})$ and $w_0 = s_{i_1}s_{i_2}\cdots s_{i_{l_0}}$ a reduced expression of the longest element in W. Then the set $\{\gamma_t = s_{i_1}s_{i_2}\cdots s_{i_{t-1}}(\alpha_{i_t}) \mid 1 \leqslant t \leqslant l_0\}$ is just the set of positive roots Φ^+. For each γ_t, define the root vector F_{γ_t} of $U_q(\mathfrak{g})$ as follows:

$$F_{\gamma_t} = T_{i_1}T_{i_2}\cdots T_{i_{t-1}}(F_{i_t}).$$

In [28] Lusztig proved the following PBW theorem for $U_q^-(\mathfrak{g})$.

Theorem 2.6 *(PBW theorem for $U_q^-(\mathfrak{g})$). (See [19,28].) The set*

$$F(w_0) = \left\{F_{\gamma_{l_0}}^{a_{l_0}}\cdots F_{\gamma_2}^{a_2}F_{\gamma_1}^{a_1} \,\middle|\, a_1, a_2, \ldots, a_{l_0} \in \mathbb{Z}^{\geqslant 0}\right\} \tag{2.6}$$

is a PBW basis of $U_q^-(\mathfrak{g})$.

For more details about unexplained concepts, we refer the readers to [5,6,15,19,23,28].

3. PBW-deformation theory of $U_q^-(\mathfrak{g})$

3.1. PBW-deformation theory of $U_q^-(\mathfrak{g})$

Assume that $\mathfrak{g}$ is a complex simple Lie algebra of rank n. We choose $A=(a_{ij})_{n\times n}$ to be the Cartan matrix of $\mathfrak{g}$ and $H=(h(B_i,B_j))_{n\times n}$ with $h(B_i,B_j)\in\mathcal{T}\langle B\rangle=\mathbb{C}\langle B_1,\dots,B_n\rangle$ and $\operatorname{Deg}(h(B_i,B_j))<2-a_{ij}$.

With the above notations we define the quantum algebra $\mathfrak{B}_q(\mathfrak{g})$ as follows.

Definition 3.1. For any given $H=(h(B_i,B_j))_{n\times n}$, the deformation $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators B_i $(1\leqslant i\leqslant n)$ and relations:

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{pmatrix}1-a_{ij}\\ k\end{pmatrix}_i B_i^{1-a_{ij}-k}B_jB_i^k=h(B_i,B_j) \tag{3.1}$$

for all $i\neq j$.

In this subsection, our main aim is to determine when a given $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. To achieve our aim, we firstly calculate the complexity $c(U_q^-(\mathfrak{g}))$. For convenience, we fix some notations. Let ρ be the half sum of positive roots of $\mathfrak{g}$. For $w\in W$, denote by $l(w)$ the length of w and define $w\cdot 0=w(\rho)-\rho$.

Proposition 3.2. *The complexity $c(U_q^-(\mathfrak{g}))$ of the algebra $U_q^-(\mathfrak{g})$ satisfies*

$$c\bigl(U_q^-(\mathfrak{g})\bigr)=\begin{cases}0, & \text{if } \mathfrak{g}=\mathfrak{sl}_2,\\ \max\{-\operatorname{Ht}(w\cdot 0)\mid w\in W, l(w)=3\}-1, & \text{otherwise}.\end{cases} \tag{3.2}$$

Proof. If $\mathfrak{g}=\mathfrak{sl}_2$, it is obvious that $c(U_q^-(\mathfrak{g}))=0$ since the global dimension of $U_q^-(\mathfrak{sl}_2)$ is 1. Otherwise, to obtain the complexity $c(U_q^-(\mathfrak{g}))$, we will calculate the i-th cohomology $\operatorname{Ext}^i_{U_q^-(\mathfrak{g})}(\mathbb{C},\mathbb{C})$. Indeed, the BGG-resolution of the trivial $U_q^-(\mathfrak{g})$-module $\mathbb{C}$

$$\cdots\longrightarrow C_2\xrightarrow{\varphi_2}C_1\xrightarrow{\varphi_1}C_0\xrightarrow{\varepsilon}\mathbb{C}\longrightarrow 0 \tag{3.3}$$

with $C_i=\bigoplus_{w\in W,\, l(w)=i}U_q^-(\mathfrak{g})\{\operatorname{Ht}(w\cdot 0)\}$, which was established in [14] and can be found in [13], is a graded free one. It follows that the boundary maps of the complex

$$\underline{\operatorname{Hom}}_{U_q^-(\mathfrak{g})}(P_\bullet,\mathbb{C}):=0\longrightarrow\underline{\operatorname{Hom}}_{U_q^-(\mathfrak{g})}(C_0,\mathbb{C})\longrightarrow\underline{\operatorname{Hom}}_{U_q^-(\mathfrak{g})}(C_1,\mathbb{C})\longrightarrow\cdots,$$

where $P_\bullet$ is the truncated complex of (3.3), are all zero. Moreover, we have

$$\begin{aligned}\underline{\operatorname{Hom}}_{U_q^-(\mathfrak{g})}(C_i,\mathbb{C})&=\underline{\operatorname{Hom}}_{U_q^-(\mathfrak{g})}\Bigl(\bigoplus_{w\in W,\, l(w)=i}U_q^-(\mathfrak{g})\bigl\{\operatorname{Ht}(w\cdot 0)\bigr\},\mathbb{C}\Bigr)\\ &\simeq\mathbb{C}\bigotimes_{U_q^-(\mathfrak{g})}\bigoplus_{w\in W,\, l(w)=i}U_q^-(\mathfrak{g})\bigl\{\operatorname{Ht}(w\cdot 0)\bigr\}\\ &\simeq\bigoplus_{w\in W,\, l(w)=i}\mathbb{C}\bigl\{\operatorname{Ht}(w\cdot 0)\bigr\}.\end{aligned}$$

Therefore,

$$\mathrm{Ext}^{i}_{U_q^-(\mathfrak{g})}(\mathbb{C},\mathbb{C}) = \bigoplus_{w\in W,\, l(w)=i} \mathbb{C}\{\mathrm{Ht}(w\cdot 0)\}.$$

Since the definition of the bigraded Yoneda algebra $E(U_q^-(\mathfrak{g})) = \bigoplus \mathrm{Ext}^{r,s}_{U_q^-(\mathfrak{g})}(\mathbb{C},\mathbb{C})$ implies that

$$\mathrm{Ext}^{3,s}_{U_q^-(\mathfrak{g})}(\mathbb{C},\mathbb{C}) = \bigoplus_{\substack{w\in W,\, l(w)=3\\ \mathrm{Ht}(w\cdot 0)=-s}} \mathbb{C}\{\mathrm{Ht}(w\cdot 0)\},$$

then by Definition 2.2 the formula (3.2) holds. $\square$

According to (3.2), the accurate values of the complexity $c(U_q^-(\mathfrak{g}))$ can be obtained via case-by-case calculations.

Corollary 3.3. *For $\mathfrak{g}$ of different types, the complexity $c(U_q^-(\mathfrak{g}))$ of the algebra $U_q^-(\mathfrak{g})$ is as follows*:

Dynkin type of $\mathfrak{g}$	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_n$ $(n \geqslant 3)$, $\mathbb{B}_2$, $\mathbb{D}_n$ $(n \geqslant 4)$, $\mathbb{E}_n$ $(n = 6, 7, 8)$	$\mathbb{B}_n$, $\mathbb{C}_n$ $(n \geqslant 3)$, $\mathbb{F}_4$, $\mathbb{G}_2$
$c(U_q^-(\mathfrak{g}))$	0	3	5	7

Now for the negative part $U_q^-(\mathfrak{g})$ of the quantized enveloping algebra, we transform the Jacobi condition (2.4) in Theorem 2.3 into some explicit linear algebra problems. Before doing it, we fix the following notations:

$$\begin{aligned}
f(B_i, B_j) &:= \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_i B_i^{1-a_{ij}-k} B_j B_i^k,\\
f(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t) &:= B_{i_1}\cdots B_{i_l} f(B_i,B_j) B_{i_{l+1}}\cdots B_{i_t},\\
h(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t) &:= B_{i_1}\cdots B_{i_l} h(B_i,B_j) B_{i_{l+1}}\cdots B_{i_t}.
\end{aligned}$$

Then considered as the elements in $\mathcal{T}\langle B\rangle$, $f(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t)$ and $h(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t)$ have the following unique linear expressions:

$$f(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t) = \sum_{\substack{f_t = t+2-a_{ij},\\ 1\leqslant j_1,\ldots,j_{f_t}\leqslant n}} \xi^{i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t}_{j_1,\ldots,j_{f_t}} B_{j_1}\cdots B_{j_{f_t}}, \tag{3.4}$$

$$h(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t) = \sum_{\substack{h_t < t+2-a_{ij},\\ 1\leqslant j_1,\ldots,j_{h_t}\leqslant n}} \eta^{i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t}_{j_1,\ldots,j_{h_t}} B_{j_1}\cdots B_{j_{h_t}}. \tag{3.5}$$

Theorem 3.4. *The algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$ if and only if for all $1 \leqslant k \leqslant c(U_q^-(\mathfrak{g}))$, the set of linear equations in the variables $y(i_1,\ldots,i_l,\dot{i},\bar{j},i_{l+1},\ldots,i_t)$ given by*

$$\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}=m,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} \xi_{j_1,\dots,j_m}^{i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t} y(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)$$
$$-\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ m<t+2-a_{ij}\leqslant k,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} \eta_{j_1,\dots,j_m}^{i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t} y(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)$$
$$=\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}=k+1,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\eta_{j_1,\dots,j_m}^{i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t}, \tag{3.6}$$

where $m<k+1$ and $1\leqslant j_1,\dots,j_m\leqslant n$, is solvable for basic solutions of the set of linear equations in the variables $x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)$ given by

$$\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}=k+1,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} \xi_{j_1,\dots,j_{k+1}}^{i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t} x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)=0 \quad \textit{for } 1\leqslant j_1,\dots,j_{k+1}\leqslant n. \tag{3.7}$$

Proof. For the case $\mathfrak{B}_q(\mathfrak{g})$, $P_1=0$ in (2.2) and for $k>1$, in (2.3) one has

$$P_k=\operatorname{Span}_{\mathbb{C}}\left\{B_{i_1}\cdots B_{i_l}\big[f(B_i,B_j)-h(B_i,B_j)\big]B_{i_{l+1}}\cdots B_{i_t}\ \middle|\ \substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}\leqslant k,\\ 1\leqslant i_1,\dots,i_t\leqslant n,\\ 0\leqslant l\leqslant t}\right\}. \tag{3.8}$$

Thus the Jacobi condition (2.4) in Theorem 2.3 is equivalent to

$$\operatorname{Span}_{\mathbb{C}}\left\{B_{i_1}\cdots B_{i_l}\big[f(B_i,B_j)-h(B_i,B_j)\big]B_{i_{l+1}}\cdots B_{i_t}\ \middle|\ \substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}=k+1,\\ 1\leqslant i_1,\dots,i_t\leqslant n,\\ 0\leqslant l\leqslant t}\right\}\cap\mathcal{F}^k\big(\mathcal{T}\langle B\rangle\big)\subseteq P_k, \tag{3.9}$$

for all $1\leqslant k\leqslant c(U_q^-(\mathfrak{g}))$. For any element

$$\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}=k+1,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)B_{i_1}\cdots B_{i_l}\big[f(B_i,B_j)-h(B_i,B_j)\big]$$
$$\times B_{i_{l+1}}\cdots B_{i_t}\in\mathcal{F}^k\big(\mathcal{T}\langle B\rangle\big),$$

where $x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\in\mathbb{C}$, we have

$$\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}=k+1,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)f(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)=0. \tag{3.10}$$

Then it can be obtained from (3.4) and (3.10) that the coefficients $x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)$ satisfy (3.7). Let

$$I=\left\{(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\ \middle|\ 1\leqslant i\neq j\leqslant n, t+2-a_{ij}=k+1, 1\leqslant i_1,\dots,i_t\leqslant n, 0\leqslant l\leqslant t\right\},$$

and define

$$\vec{X}:=\left(\dots,x(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t),\dots\right)_{(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\in I},\tag{3.11}$$

$$\vec{H}:=\left(\dots,h(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t),\dots\right)^t_{(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\in I}.\tag{3.12}$$

Then the condition (3.9) holds if and only if

$$\vec{X}\cdot\vec{H}\in P_k\quad\text{for any basic solution } \vec{X} \text{ of } (3.7),\tag{3.13}$$

which means that there exist $y(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\in\mathbb{C}$ such that

$$\begin{aligned}\vec{X}\cdot\vec{H}=&\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}\leqslant k,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} y(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)f(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)\\ &-\sum_{\substack{1\leqslant i\neq j\leqslant n,\\ t+2-a_{ij}\leqslant k,\ 0\leqslant l\leqslant t,\\ 1\leqslant i_1,\dots,i_t\leqslant n}} y(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t)h(i_1,\dots,i_l,\dot{i},\bar{j},i_{l+1},\dots,i_t).\end{aligned}\tag{3.14}$$

Noting that $\{B_{j_1}\cdots B_{j_m}\mid 1\leqslant j_1,\dots,j_m\leqslant n\}$ is a basis of $\mathcal{F}^m(\mathcal{T}\langle B\rangle)$, then by (3.4) and (3.5) we can reexpress (3.14) as the solvability problem of (3.6) in the "only if" part. The proof is completed. $\square$

For practical use, we give the following two-step algorithm for judging whether a given algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$.

Step 1. Solve the set of linear equations (3.7) for k. The linear equations in (3.7) can be obtained in the following process: (i) put (3.4) into (3.10), (ii) expand the formula got in (i) then combine like terms, (iii) set the coefficients of the formula obtained in (ii) to be zero. If (3.7) is solvable, some basic solutions $\vec{X}$ in the form (3.11) can be obtained.

Step 2. Determine the solvability of (3.6) for k if (3.7) for k in Step 1 is solvable, otherwise, skip this step. The linear equations in (3.6) can be explicitly written down by the same process as described in Step 1 modified by "(i) put (3.4), (3.5) and $\vec{X}$ obtained in Step 1 into (3.14)".

Iterate Step 1 and Step 2 for $k=2,3,\dots,c(U_q^-(\mathfrak{g}))$ successively. If the set of linear equations (3.6) is solvable for $k=2,3,\dots,c(U_q^-(\mathfrak{g}))$, then $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. $\square$

3.2. PBW-deformations of $U_q^-(\mathfrak{g})$ with $\mathfrak{g}$ of rank 2

Now we start to apply the above algorithm to the algebras $\mathfrak{B}_q(\mathfrak{g})$ for $\mathfrak{g}$ of type $\mathbb{A}_2$ and $\mathbb{B}_2$. To begin with, we give a condition in the form (3.13) for determining PBW-deformations of $U_q^-(\mathfrak{g})$, which finishes all the work in Step 1 and 2 except that in Step 2 for $k=c(U_q^-(\mathfrak{g}))$. In fact, the condition (3.13) can be considered as a bridge combining Step 1 with Step 2.

Theorem 3.5. *Suppose that the sets P_k for $1\leqslant k\leqslant c(U_q^-(\mathfrak{g}))$ are given by (3.8). Then we have the following results.*

(1) *For $\mathfrak{g}$ of type $\mathbb{A}_2$, the algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$ if and only if*

$$h(\bar{2},\bar{1},1)+h(\bar{1},\bar{2},2)-h(1,\bar{2},\bar{1})-h(2,\bar{1},\bar{2})\in P_3. \tag{3.15}$$

(2) *For $\mathfrak{g}$ of type $\mathbb{B}_2$, the algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$ if and only if $\vec{c_i}\cdot\vec{v_i}\in P_5$ for $1\leqslant i\leqslant 3$, where*

$$\begin{aligned}
\vec{c_1}&=\bigl(-1,q^2+q^{-2},-1,1,-\bigl(q^2+q^{-2}\bigr),1\bigr),\\
\vec{c_2}&=\bigl(q^2+q^{-2},-1,-1,-1,-1,-1,[3],q^2+q^{-2},-[3],1\bigr),\\
\vec{c_3}&=\bigl(1,-1,[3],-[3],-\bigl(q^2+q^{-2}\bigr),1,1,-[3],[3],-1,[3],1,-[3],-1,1\bigr),\\
\vec{v_1}&=\bigl(h(\bar{1},\bar{2},1,1,2),h(\bar{1},\bar{2},1,2,1),h(\bar{1},\bar{2},2,1,1),h(1,1,2,\bar{1},\bar{2}),h(1,2,1,\bar{1},\bar{2}),h(2,1,1,\bar{1},\bar{2})\bigr)^t,\\
\vec{v_2}&=\bigl(h(\bar{2},\bar{1},1,2),h(\bar{2},\bar{1},2,1),h(\bar{1},\bar{2},2,2,2),h(1,\bar{2},\bar{1},2),h(1,2,\bar{2},\bar{1}),h(2,\bar{2},\bar{1},1),\\
&\quad h(2,\bar{1},\bar{2},2,2),h(2,1,\bar{2},\bar{1}),h(2,2,\bar{1},\bar{2},2),h(2,2,2,\bar{1},\bar{2})\bigr)^t,\\
\vec{v_3}&=\bigl(h(\bar{2},\bar{1},1,1),h(\bar{1},\bar{2},1,2,2),h(\bar{1},\bar{2},2,1,2),h(\bar{1},\bar{2},2,2,1),h(1,\bar{2},\bar{1},1),h(1,\bar{1},\bar{2},2,2),\\
&\quad h(1,1,\bar{2},\bar{1}),h(1,2,\bar{1},\bar{2},2),h(1,2,2,\bar{1},\bar{2}),h(2,\bar{1},\bar{2},1,2),h(2,\bar{1},\bar{2},2,1),h(2,1,\bar{1},\bar{2},2),\\
&\quad h(2,1,2,\bar{1},\bar{2}),h(2,2,\bar{1},\bar{2},1),h(2,2,1,\bar{1},\bar{2})\bigr)^t.
\end{aligned}$$

Proof. (1) For $\mathfrak{g}$ of type $\mathbb{A}_2$, the condition (3.13) for $1\leqslant k\leqslant 2$ is trivial, i.e., $0\in P_k$. For $k=3$, the linear problem (3.7) is equivalent to:

$$\begin{cases}
x(\bar{2},\bar{1},1)+x(2,\bar{1},\bar{2})=0,\\
x(\bar{1},\bar{2},2)+x(2,\bar{1},\bar{2})=0,\\
x(1,\bar{2},\bar{1})-x(2,\bar{1},\bar{2})=0,\\
x(\bar{2},\bar{1},2)=x(\bar{1},\bar{2},1)=x(1,\bar{1},\bar{2})=x(2,\bar{2},\bar{1})=0.
\end{cases} \tag{3.16}$$

Hence (3.16) has the unique basic solution $\overrightarrow{X}=(1,0,0,1,-1,0,0,-1)$, where the variables $x(i_1,\ldots,i_l,\bar{i},\bar{j},i_{l+1},\ldots,i_t)$ in $\overrightarrow{X}$ are arranged in the lexicographic ordering of $(i_1,\ldots,i_l,\bar{i},\bar{j},i_{l+1},\ldots,i_t)$ with $\bar{2}<\bar{1}<1<2$. Therefore, $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$ if and only if (3.15) holds.

(2) For $\mathfrak{g}$ of type $\mathbb{B}_2$, the condition (3.13) for $1\leqslant k\leqslant 4$ is trivial, i.e., $0\in P_k$. For $k=5$, the linear problem (3.7) can be simplified to be the following form:

$$\begin{cases}
x(\bar{1},\bar{2},1,1,2)-x(\bar{1},\bar{2},2,1,1)=0,\\
x(\bar{1},\bar{2},1,2,1)+\bigl(q^2+q^{-2}\bigr)x(\bar{1},\bar{2},2,1,1)=0,\\
x(\bar{1},\bar{2},2,1,1)+x(2,1,1,\bar{1},\bar{2})=0,\\
x(1,1,2,\bar{1},\bar{2})-x(2,1,1,\bar{1},\bar{2})=0,\\
x(1,2,1,\bar{1},\bar{2})+\bigl(q^2+q^{-2}\bigr)x(2,1,1,\bar{1},\bar{2})=0,
\end{cases} \tag{3.17}$$

$$\begin{cases}
x(\bar{2},\bar{1},1,2)+\bigl(q^2+q^{-2}\bigr)[3]^{-1}x(2,2,\bar{1},\bar{2},2)=0,\\
x(\bar{2},\bar{1},2,1)-[3]^{-1}x(2,2,\bar{1},\bar{2},2)=0,\\
x(\bar{1},\bar{2},2,2,2)-[3]^{-1}x(2,2,\bar{1},\bar{2},2)=0,\\
x(1,\bar{2},\bar{1},2)-[3]^{-1}x(2,2,\bar{1},\bar{2},2)=0,\\
x(1,2,\bar{2},\bar{1})-[3]^{-1}x(2,2,\bar{1},\bar{2},2)=0,\\
x(2,\bar{2},\bar{1},1)+x(2,2,2,\bar{1},\bar{2})=0,\\
x(2,\bar{1},\bar{2},2,2)+x(2,2,\bar{1},\bar{2},2)=0,\\
x(2,1,\bar{2},\bar{1})+\bigl(q^2+q^{-2}\bigr)[3]^{-1}x(2,2,\bar{1},\bar{2},2)=0,\\
x(2,2,\bar{1},\bar{2},2)+[3]x(2,2,2,\bar{1},\bar{2})=0,
\end{cases} \tag{3.18}$$

$$\begin{cases} x(\bar{2},\bar{1},1,1)+x(2,2,\bar{1},\bar{2},1)=0, \\ x(\bar{1},\bar{2},1,2,2)-x(2,2,\bar{1},\bar{2},1)=0, \\ x(\bar{1},\bar{2},2,1,2)+[3]x(2,2,\bar{1},\bar{2},1)=0, \\ x(\bar{1},\bar{2},2,2,1)-[3]x(2,2,\bar{1},\bar{2},1)=0, \\ x(1,\bar{2},\bar{1},1)-\left(q^2+q^{-2}\right)x(2,2,\bar{1},\bar{2},1)=0, \\ x(1,\bar{1},\bar{2},2,2)+x(2,2,\bar{1},\bar{2},1)=0, \\ x(1,1,\bar{2},\bar{1})+x(2,2,\bar{1},\bar{2},1)=0, \\ x(1,2,\bar{1},\bar{2},2)-[3]x(2,2,\bar{1},\bar{2},1)=0, \\ x(1,2,2,\bar{1},\bar{2})+[3]x(2,2,\bar{1},\bar{2},1)=0, \\ x(2,\bar{1},\bar{2},1,2)+x(2,2,1,\bar{1},\bar{2})=0, \\ x(2,\bar{1},\bar{2},2,1)+[3]x(2,2,\bar{1},\bar{2},1)=0, \\ x(2,1,\bar{1},\bar{2},2)+x(2,2,\bar{1},\bar{2},1)=0, \\ x(2,1,2,\bar{1},\bar{2})-[3]x(2,2,\bar{1},\bar{2},1)=0, \\ x(2,2,\bar{1},\bar{2},1)+x(2,2,1,\bar{1},\bar{2})=0, \end{cases} \tag{3.19}$$

and other $x(i_1,\dots,i_l,\bar{i},\bar{j},i_{l+1},\dots,i_t)$ are zero. Since the basic solutions of (3.17), (3.18) and (3.19) are respectively $\vec{c_1}$, $\vec{c_2}$ and $\vec{c_3}$ (the variables $x(i_1,\dots,i_l,\bar{i},\bar{j},i_{l+1},\dots,i_t)$ in $\vec{c_1}$, $\vec{c_2}$ and $\vec{c_3}$ are also arranged according to the lexicographic ordering of $(i_1,\dots,i_l,\bar{i},\bar{j},i_{l+1},\dots,i_t)$ with $\bar{2}<\bar{1}<1<2$.), then in this case (3.7) has the following three basic solutions:

⟨1⟩ $\vec{X}$ with $x(i_1,\dots,i_l,\bar{i},\bar{j},i_{l+1},\dots,i_t)$ zero except

$$\begin{cases} x(\bar{1},\bar{2},1,1,2)=-1, \quad x(\bar{1},\bar{2},1,2,1)=q^2+q^{-2}, \quad x(\bar{1},\bar{2},2,1,1)=-1, \\ x(1,1,2,\bar{1},\bar{2})=1, \quad x(1,2,1,\bar{1},\bar{2})=-\left(q^2+q^{-2}\right), \quad x(2,1,1,\bar{1},\bar{2})=1; \end{cases} \tag{3.20}$$

⟨2⟩ $\vec{X}$ with $x(i_1,\dots,i_l,\bar{i},\bar{j},i_{l+1},\dots,i_t)$ zero except

$$\begin{cases} x(\bar{2},\bar{1},1,2)=q^2+q^{-2}, \quad x(\bar{2},\bar{1},2,1)=-1, \quad x(\bar{1},\bar{2},2,2,2)=-1, \\ x(1,\bar{2},\bar{1},2)=-1, \quad x(1,2,\bar{2},\bar{1})=-1, \quad x(2,\bar{2},\bar{1},1)=-1, \quad x(2,\bar{1},\bar{2},2,2)=[3], \\ x(2,1,\bar{2},\bar{1})=q^2+q^{-2}, \quad x(2,2,\bar{1},\bar{2},2)=-[3], \quad x(2,2,2,\bar{1},\bar{2})=1; \end{cases} \tag{3.21}$$

⟨3⟩ $\vec{X}$ with $x(i_1,\dots,i_l,\bar{i},\bar{j},i_{l+1},\dots,i_t)$ zero except

$$\begin{cases} x(\bar{2},\bar{1},1,1)=1, \quad x(\bar{1},\bar{2},1,2,2)=-1, \quad x(\bar{1},\bar{2},2,1,2)=[3], \\ x(\bar{1},\bar{2},2,2,1)=-[3], \quad x(1,\bar{2},\bar{1},1)=-\left(q^2+q^{-2}\right), \quad x(1,\bar{1},\bar{2},2,2)=1, \\ x(1,1,\bar{2},\bar{1})=1, \quad x(1,2,\bar{1},\bar{2},2)=-[3], \quad x(1,2,2,\bar{1},\bar{2})=[3], \\ x(2,\bar{1},\bar{2},1,2)=-1, \quad x(2,\bar{1},\bar{2},2,1)=[3], \quad x(2,1,\bar{1},\bar{2},2)=1, \\ x(2,1,2,\bar{1},\bar{2})=-[3], \quad x(2,2,\bar{1},\bar{2},1)=-1, \quad x(2,2,1,\bar{1},\bar{2})=1; \end{cases} \tag{3.22}$$

Now the claim in (2) immediately follows from (3.13) in the proof of Theorem 3.4. □

Remark 3.6. The results in Theorem 3.5 seem more direct if we reformulate them by the coefficients of $h(\bar{i},\bar{j})=h(B_i,B_j)$ in (3.5). For $\mathfrak{g}$ of type $\mathbb{A}_2$, if we write out the set of linear equations (3.6) for $k=3$ according to the process in Step 2 in our algorithm, then we can deduce that (3.6) is solvable if and only if the coefficients of $h(B_i,B_j)$ in (3.5) satisfy the following condition:

$$\begin{cases} \eta_{1,1}^{\bar{1},\bar{2}} - \eta_{1,2}^{\bar{2},\bar{1}} = -\eta_{1,1}^{\bar{1},\bar{2}} + \eta_{2,1}^{\bar{2},\bar{1}} = -\dfrac{1}{q+q^{-1}}\left(\eta_{1,2}^{\bar{2},\bar{1}} - \eta_{2,1}^{\bar{2},\bar{1}}\right), \\ \dfrac{1}{q+q^{-1}}\left(\eta_{1,2}^{\bar{1},\bar{2}} - \eta_{2,1}^{\bar{1},\bar{2}}\right) = \eta_{1,2}^{\bar{1},\bar{2}} - \eta_{2,2}^{\bar{2},\bar{1}} = -\eta_{2,1}^{\bar{1},\bar{2}} + \eta_{2,2}^{\bar{2},\bar{1}}, \\ \left(\eta_{1,1}^{\bar{1},\bar{2}} - \eta_{1,2}^{\bar{2},\bar{1}}\right)\eta_i^{\bar{1},\bar{2}} + \left(\eta_{1,2}^{\bar{1},\bar{2}} - \eta_{2,2}^{\bar{2},\bar{1}}\right)\eta_i^{\bar{2},\bar{1}} = 0, & i = 0, 1, 2, \\ \left(\eta_{1,1}^{\bar{1},\bar{2}} - \eta_{1,2}^{\bar{2},\bar{1}}\right)\eta_{i,i}^{\bar{1},\bar{2}} + \left(\eta_{1,2}^{\bar{1},\bar{2}} - \eta_{2,2}^{\bar{2},\bar{1}}\right)\eta_{i,i}^{\bar{2},\bar{1}} = 0, & i = 1, 2, \\ \left(\eta_{1,1}^{\bar{1},\bar{2}} - \eta_{1,2}^{\bar{2},\bar{1}}\right)\eta_{i,j}^{\bar{1},\bar{2}} + \left(\eta_{1,2}^{\bar{1},\bar{2}} - \eta_{2,2}^{\bar{2},\bar{1}}\right)\eta_{i,j}^{\bar{2},\bar{1}} = (-1)^i\left(\eta_1^{\bar{1},\bar{2}} - \eta_2^{\bar{2},\bar{1}}\right), & (i, j) = (1, 2), (2, 1). \end{cases} \tag{3.23}$$

In other words, for $\mathfrak{g}$ of type $\mathbb{A}_2$, $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$ if and only if (3.23) holds. However, even for $\mathfrak{g}$ of type $\mathbb{B}_2$, the condition like (3.23) is not easy to obtain because the amount of calculations in our algorithm is very large for hand computation.

Next we present some examples of PBW-deformations of $U_q^-(\mathfrak{g})$ for which we can do the work in Step 2 for $k = c(U_q^-(\mathfrak{g}))$. The algebras $\mathfrak{B}_q(\mathfrak{g})$ in the third example will be the main research object in Section 4.

Example 3.7. For any $c \in \mathbb{C}$, let $h(B_1, B_2) = cB_1$ and $h(B_2, B_1) = cB_2$. Then by Theorem 3.5(1) or (3.23) we can check that the algebra $\mathfrak{B}_q(\mathfrak{sl}_3)$ is a PBW-deformation of $U_q^-(\mathfrak{sl}_3)$. Indeed the linear equations (3.6) for $k = 3$ in Step 2 only have the zero solution. In this case $\mathfrak{B}_q(\mathfrak{sl}_3)$ is just the down-up algebra $A(q + q^{-1}, -1, c)$ defined in [1] (see also [34]).

Example 3.8. For any $c_1, c_2 \in \mathbb{C}$, let

$$\begin{cases} h(B_1, B_2) = B_1^2 + B_1B_2 + B_2B_1 + c_1B_2^2, \\ h(B_2, B_1) = c_2B_1^2 + B_1B_2 + B_2B_1 + B_2^2. \end{cases}$$

Then it follows from (3.23) that in this case $\mathfrak{B}_q(\mathfrak{sl}_3)$ is also a PBW-deformation of $U_q^-(\mathfrak{sl}_3)$.

Example 3.9. Let

$$h(B_i, B_j) = \begin{cases} 0, & \text{if } a_{ij} = 0, \\ -q_i^{-1}B_j, & \text{if } a_{ij} = -1, \\ -q^{-1}[2]^2(B_iB_j - B_jB_i), & \text{if } a_{ij} = -2, \\ -q^{-1}([3]^2 + 1)(B_i^2B_j + B_jB_i^2) & \\ \quad + q^{-1}[2]([2][4] + q^2 + q^{-2})B_iB_jB_i - q^{-2}[3]^2B_j, & \text{if } a_{ij} = -3. \end{cases} \tag{3.24}$$

Then one has:

(1) For $\mathfrak{g}$ of type $\mathbb{A}_2$, by Theorem 3.5(1) or (3.23), we can check that the algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. In this case, the linear equations (3.6) only have the zero solution.

(2) For $\mathfrak{g}$ of type $\mathbb{B}_2$, by Theorem 3.5(2), we can check that the algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. Indeed, when $k = 5$, corresponding to the basic solutions (3.20), (3.21) and (3.22) of (3.7), the linear equations (3.6) in Step 2 respectively have the following solutions:

$\langle 1 \rangle$ $y(\bar{1}, \bar{2}, 2) = -q^{-2}$, $y(2, \bar{1}, \bar{2}) = q^{-2}$ and other $y(i_1, \ldots, i_l, \bar{i}, \bar{j}, i_{l+1}, \ldots, i_t)$ are zero.

$\langle 2 \rangle$ $y(\bar{1}, \bar{2}, 2) = -q^{-1}[2]^2$, $y(2, \bar{1}, \bar{2}) = q^{-1}[2]^2$ and other $y(i_1, \ldots, i_l, \bar{i}, \bar{j}, i_{l+1}, \ldots, i_t)$ are zero.

$\langle 3 \rangle$ $y(\bar{2}, \bar{1}) = q^{-2}$, $y(\bar{1}, \bar{2}, 1) = -q^{-1}[2]^2$, $y(1, \bar{1}, \bar{2}) = q^{-1}[2]^2$ and other $y(i_1, \ldots, i_l, \bar{i}, \bar{j}, i_{l+1}, \ldots, i_t)$ are zero.

Remarks 3.10. (1) Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the Chevalley involution, and let $\mathfrak{t} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ be the Lie subalgebra of $\mathfrak{g}$ consisting of elements fixed under θ. If we set $B_i = F_i - K_i^{-1}E_i$ for $1 \leqslant i \leqslant n$. Then the algebra $\mathfrak{B}_q(\mathfrak{g})$ in Example 3.9 is just the subalgebra $U_q'(\mathfrak{t})$ defined in [22], which is called quantum symmetric pair coideal subalgebra of $U_q(\mathfrak{g})$. Letzter's work in [24] showed that $U_q'(\mathfrak{t})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$ (see the proof of Theorem 4.4 in Section 4).

(2) In [12] the authors gave another quantum deformation $U_q'(\mathfrak{so}(n+1,\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{so}(n+1,\mathbb{C}))$. Recall that $U_q'(\mathfrak{so}(n+1,\mathbb{C}))$ is generated by I_i $(1\leqslant i\leqslant n)$ which satisfy the following relations:

$$\begin{aligned}
&I_i^2I_{i+1}-\left(q+q^{-1}\right)I_iI_{i+1}I_i+I_{i+1}I_i^2=-I_{i+1},\\
&I_{i+1}^2I_i-\left(q+q^{-1}\right)I_{i+1}I_iI_{i+1}+I_iI_{i+1}^2=-I_i,\\
&I_iI_j-I_jI_i=0,\quad \text{for } |i-j|>1.
\end{aligned}$$

It is well known that $U_q'(\mathfrak{so}(n+1,\mathbb{C}))\cong\mathfrak{B}_q(\mathfrak{sl}(n+1,\mathbb{C}))$ which maps I_i to $q^{\frac{1}{2}}B_i$ or $q^{\frac{1}{2}}B_{n+1-i}$ for $1\leqslant i\leqslant n$ (see e.g. p. 267 in [25]).

4. Root vectors and PBW theorem for quantum algebras $\mathfrak{B}_q(\mathfrak{g})$

In this section, we always assume that the elements $h(B_i,B_j)$ $(1\leqslant i\neq j\leqslant n)$ in the definition of $\mathfrak{B}_q(\mathfrak{g})$ are given by (3.24). From now on, we are devoted to construct PBW bases for $\mathfrak{B}_q(\mathfrak{g})$.

4.1. The algebra automorphisms of $\mathfrak{B}_q(\mathfrak{g})$

Analogous as Lusztig's construction, Kolb and Pellegrini in [22] established a series of algebra automorphisms τ_i $(1\leqslant i\leqslant n)$ of $\mathfrak{B}_q(\mathfrak{g})$, which satisfy the braid relation (2.5). Precisely, setting

$$\tau_i(B_j)=\begin{cases}
B_j, & \text{if } i=j \text{ or } a_{ij}=0,\\
B_iB_j-q_iB_jB_i, & \text{if } a_{ij}=-1,\\
\sum_{s=0}^{2}(-1)^sq^sB_i^{(2-s)}B_jB_i^{(s)}+B_j, & \text{if } a_{ij}=-2,\\
\sum_{s=0}^{3}(-1)^sq^sB_i^{(3-s)}B_jB_i^{(s)}+\frac{B_iB_j-q^3B_jB_i}{q[3]!}+(B_iB_j-qB_jB_i), & \text{if } a_{ij}=-3.
\end{cases}\tag{4.1}$$

for $1\leqslant i,j\leqslant n$, where $B_i^{(n)}=\frac{B_i^n}{[n]!}$ for any $n\in\mathbb{N}$, we have

Proposition 4.1. *(See [22].)* (1) *For each $1\leqslant i\leqslant n$, there exists a unique algebra automorphism τ_i of $\mathfrak{B}_q(\mathfrak{g})$ such that $\tau_i(B_j)$ is given by (4.1). The inverse of τ_i is given by*

$$\tau_i^{-1}(B_j)=\begin{cases}
B_j, & \text{if } i=j \text{ or } a_{ij}=0,\\
B_jB_i-q_iB_iB_j, & \text{if } a_{ij}=-1,\\
\sum_{s=0}^{2}(-1)^sq^sB_i^{(s)}B_jB_i^{(2-s)}+B_j, & \text{if } a_{ij}=-2,\\
\sum_{s=0}^{3}(-1)^sq^sB_i^{(s)}B_jB_i^{(3-s)}+\frac{(B_jB_i-q^3B_iB_j)}{q[3]!}+(B_jB_i-qB_iB_j), & \text{if } a_{ij}=-3.
\end{cases}$$

(2) *The algebra automorphisms τ_i $(1\leqslant i\leqslant n)$ satisfy the braid relation (2.5).*

Proof. In [22] the authors gave the proof by the computer algebra package QuaGroup [8] within GAP [11] for calculations with quantum enveloping algebras. In addition, the statement follows from results in [30] if $\mathfrak{g}$ is simply laced (see Remarks 3.4 and 3.5 in [22]). $\square$

In the following we list some formulas about τ_i which we will use in the sequel. For the verification of them see Appendix A.

(1) If $a_{ij}=a_{ji}=-1$, we have $\tau_i\tau_j(B_i)=B_j$.

(2) If $a_{ij}=-1$ and $a_{ji}=-2$,

$$\tau_i\tau_j(B_i)=[2]^{-1}\big(B_iB_j^2-q[2]B_jB_iB_j+q^2B_j^2B_i\big)+B_i,\qquad \tau_j\tau_i\tau_j(B_i)=B_i,$$
$$\tau_j\tau_i(B_j)=B_jB_i-q^2B_iB_j,\qquad \tau_i\tau_j\tau_i(B_j)=B_j.$$

(3) If $a_{ij}=-1$ and $a_{ji}=-3$,

$$\tau_i\tau_j(B_i)=\frac{1}{[3]!}\begin{pmatrix}(B_iB_j-q^3B_jB_i)^2B_j+B_j(B_iB_j-q^3B_jB_i)^2\\ -[2](B_iB_j-q^3B_jB_i)B_j(B_iB_j-q^3B_jB_i)\end{pmatrix}$$
$$+\left(\frac{1}{q[3]!}+1\right)\big(B_iB_j-q^3B_jB_i\big)B_i-\left(\frac{q^2}{[3]!}+q\right)B_i\big(B_iB_j-q^3B_jB_i\big),$$
$$\tau_j\tau_i(B_j)=[2]^{-1}B_j^2B_i-q^2B_jB_iB_j+[2]^{-1}q^4B_iB_j^2+q[3][2]^{-1}B_i,$$
$$\tau_j\tau_i\tau_j(B_i)=\frac{1}{[3]}\tau_j\tau_i(B_j)\big(B_jB_i-q^3B_iB_j\big)-\frac{1}{q[3]}\big(B_jB_i-q^3B_iB_j\big)\tau_j\tau_i(B_j)+\frac{[2]}{q[3]}B_j,$$
$$\tau_i\tau_j\tau_i(B_j)=[2]^{-1}q^4B_j^2B_i-q^2B_jB_iB_j+[2]^{-1}B_iB_j^2+q[3][2]^{-1}B_i,$$
$$\tau_i\tau_j\tau_i\tau_j(B_i)=\frac{1}{[3]!}\big(\big(B_iB_j-q^3B_jB_i\big)B_j^2-[2]B_j\big(B_iB_j-q^3B_jB_i\big)B_j+B_j^2\big(B_iB_j-q^3B_jB_i\big)\big)$$
$$-[2]^{-1}(B_jB_i-qB_iB_j)+\frac{[2]}{q[3]}\big(B_iB_j-q^3B_jB_i\big),$$
$$\tau_j\tau_i\tau_j\tau_i\tau_j(B_i)=B_i,\qquad \tau_j\tau_i\tau_j\tau_i(B_j)=B_jB_i-q^3B_iB_j,\qquad \tau_i\tau_j\tau_i\tau_j\tau_i(B_j)=B_j.$$

4.2. Root vectors of $\mathfrak{B}_q(\mathfrak{g})$

In this subsection, we define and investigate root vectors of $\mathfrak{B}_q(\mathfrak{g})$. For any $w\in W$ we set

$$\tau_w=\begin{cases}\mathrm{id}, & \text{if } w=1,\\ \tau_{i_1}\tau_{i_2}\cdots\tau_{i_t}, & \text{if } w=s_{i_1}s_{i_2}\cdots s_{i_t} \text{ is a reduced expression.}\end{cases}\tag{4.2}$$

By Proposition 4.1(2), the right-hand side in (4.2) is independent of the reduced expression of w.

The following lemma makes it possible to define root vectors for $\mathfrak{B}_q(\mathfrak{g})$. For convenience, we set $B_{\alpha_i}\equiv B_i$.

Lemma 4.2. (1) *Let $\alpha_i,\alpha_j\in\Pi$ with $i\neq j$. Let $w\in W$ be in the subgroup W' of W generated by s_i and s_j. Then $\tau_w(B_{\alpha_i})$ is contained in the subalgebra generated by B_{α_i} and B_{α_j}. If $w(\alpha_i)\in\Pi$, then $\tau_w(B_{\alpha_i})=B_{w\alpha_i}$.*

(2) *Let $w\in W$ and $\alpha_i\in\Pi$. If $w\alpha_i\in\Pi$, then $\tau_w(B_{\alpha_i})=B_{w\alpha_i}$.*

Proof. (1) By the definition of τ_w, we easily see that $\tau_w(B_{\alpha_i})$ is contained in the subalgebra generated by B_{α_i} and B_{α_j}. Moreover, if $w=1$, then all the claims are trivial. Therefore, we only need to check the second claim in the following four cases by assuming $w\neq1$. Denote $W_i:=\{w\in W'\mid w(\alpha_i)\in\Pi\}\backslash\{1\}$. Let m be the order of s_is_j. Then $m=2,3,4$ or 6. Firstly, if $m=2$, then $a_{ij}=a_{ji}=0$ and $W_i=\{s_j\}$. In this case the second claim holds because $\tau_j(B_i)=B_i$. Secondly, when $m=3$, one has $a_{ij}=a_{ji}=-1$ and $W_i=\{s_is_j\}$. It follows from the formula for $a_{ij}=-1$ and $a_{ji}=-1$ in Section 4.1 that $\tau_i\tau_j(B_i)=B_j$. Hence we finish the proof of the second claim in case $m=3$. Thirdly, in case $m=4$, we have $W_i=\{s_js_is_j\}$. Therefore, our claim follows from the second and fourth formulas when $a_{ij}=-1$ and $a_{ji}=-2$ in Section 4.1. Finally, if $m=6$, then $W_i=\{s_js_is_js_is_j\}$ and our claim can be obtained from the sixth and eighth formulas when $a_{ij}=-1$ and $a_{ji}=-3$ in Section 4.1.

(2) This claim can be proved by induction on $l(w)$. Since it is in complete analogy to the proof of Proposition 8.20 in [19], we omit it. □

Lemma 4.2 indicates that we can define root vectors as follows. If $w = s_{i_1} s_{i_2} \cdots s_{i_t} \in W$ is a reduced expression, we set $\gamma_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ and call

$$B_{\gamma_j} = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_{j-1}}(B_{i_j})$$

a root vector of $\mathfrak{B}_q(\mathfrak{g})$. In particular, for a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_{l_0}}$ of the longest element in W, we fix the notation

$$B(w_0) = \left\{ B_{\gamma_{l_0}}^{a_{l_0}} \cdots B_{\gamma_2}^{a_2} B_{\gamma_1}^{a_1} \,\middle|\, a_1, a_2, \ldots, a_{l_0} \in \mathbb{Z}^{\geqslant 0} \right\} \subseteq \mathfrak{B}_q(\mathfrak{g}). \tag{4.3}$$

4.3. Relationship between $B(w_0)$ and $F(w_0)$

In this part, we describe the relationship between the PBW basis $F(w_0)$ of $U_q^-(\mathfrak{g})$ in (2.6) and $B(w_0)$ in (4.3).

Proposition 4.3. *There exist $f_{\{a_i\}}(x_1, \ldots, x_n), f'_{\{a_i\}}(x_1, \ldots, x_n) \in \mathcal{T}$ with the former homogeneous and $\mathrm{Deg}(f'_{\{a_i\}}(x_1, \ldots, x_n)) < \mathrm{Deg}(f_{\{a_i\}}(x_1, \ldots, x_n)) = \sum_{i=1}^{l_0} a_i \,\mathrm{Ht}(\gamma_i)$ such that*

$$\begin{aligned} F_{\gamma_{l_0}}^{a_{l_0}} \cdots F_{\gamma_2}^{a_2} F_{\gamma_1}^{a_1} &= f_{\{a_i\}}(F_1, F_2, \ldots, F_n) \in U_q^-(\mathfrak{g}), \\ B_{\gamma_{l_0}}^{a_{l_0}} \cdots B_{\gamma_2}^{a_2} B_{\gamma_1}^{a_1} &= f_{\{a_i\}}(B_1, B_2, \ldots, B_n) + f'_{\{a_i\}}(B_1, B_2, \ldots, B_n) \in \mathfrak{B}_q(\mathfrak{g}), \end{aligned}$$

where $f'_{\{a_i\}}(x_1, \ldots, x_n) = 0$ for simply laced $\mathfrak{g}$.

Proof. Our proof can be divided into two parts according to type of Lie algebra $\mathfrak{g}$.

(1) If $\mathfrak{g}$ is a simple Lie algebra of type $\mathbb{G}_2$, the longest element w_0 has two reduced expressions: $s_1 s_2 s_1 s_2 s_1 s_2$ and $s_2 s_1 s_2 s_1 s_2 s_1$. The first one leads to the following root vectors for $\mathfrak{B}_q(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$:

$$\begin{array}{cccccc} B_1, & \tau_1(B_2), & \tau_1\tau_2(B_1), & \tau_1\tau_2\tau_1(B_2), & \tau_1\tau_2\tau_1\tau_2(B_1), & B_2 = \tau_1\tau_2\tau_1\tau_2\tau_1(B_2); \\ F_1, & T_1(F_2), & T_1T_2(F_1), & T_1T_2T_1(F_2), & T_1T_2T_1T_2(F_1), & F_2 = T_1T_2T_1T_2T_1(F_2). \end{array}$$

By calculations, when $a_{12} = -1$ and $a_{21} = -3$,

$$\begin{aligned} T_1(F_2) &= F_1F_2 - q^3 F_2F_1, \\ T_1T_2(F_1) &= \frac{1}{[3]!} \begin{pmatrix} (F_1F_2 - q^3F_2F_1)^2 F_2 + F_2(F_1F_2 - q^3F_2F_1)^2 \\ -[2](F_1F_2 - q^3F_2F_1)F_2(F_1F_2 - q^3F_2F_1) \end{pmatrix}, \\ T_1T_2T_1(F_2) &= q^4 F_2^{(2)} F_1 - q^2 F_2F_1F_2 + F_1F_2^{(2)}, \\ T_1T_2T_1T_2(F_1) &= \frac{1}{[3]!}\big((F_1F_2 - q^3F_2F_1)F_2^2 - [2]F_2\big(F_1F_2 - q^3F_2F_1\big)F_2 + F_2^2\big(F_1F_2 - q^3F_2F_1\big)\big), \end{aligned}$$

while when $a_{12} = -3$ and $a_{21} = -1$,

$$\begin{aligned}
T_1(F_2) &= \sum_{s=0}^{3}(-1)^s q^s F_1^{(3-s)} F_2 F_1^{(s)}, \\
T_1T_2(F_1) &= F_1^{(2)}F_2 - q^2 F_1F_2F_1 + q^4 F_2 F_1^{(2)}, \\
T_1T_2T_1(F_2) &= \frac{1}{[3]}\big(F_1^{(2)}F_2 - q^2F_1F_2F_1 + q^4F_2F_1^{(2)}\big)\big(F_1F_2 - q^3F_2F_1\big) \\
&\quad - \frac{q^{-1}}{[3]}\big(F_1F_2 - q^3F_2F_1\big)\big(F_1^{(2)}F_2 - q^2F_1F_2F_1 + q^4F_2F_1^{(2)}\big), \\
T_1T_2T_1T_2(F_1) &= F_1F_2 - q^3F_2F_1.
\end{aligned}$$

Now we finish the proof by comparing the formulas for root vectors of $U_q^-(\mathfrak{g})$ with those of $\mathfrak{B}_q(\mathfrak{g})$ in Section 4.1. Similarly, this proposition holds for the second reduced expression of w_0.

(2) If $\mathfrak{g}$ is not of type $\mathbb{G}_2$, it is sufficient to show the following claim.

Claim. *If $w = s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{l(w)}} \in W$ is a reduced expression, then there exist two elements $g(x_1,\dots,x_n)$, $g'(x_1,\dots,x_n) \in \mathcal{T}$ with the former homogeneous and*

$$\mathrm{Deg}\big(g'(x_1,\dots,x_n)\big) < \mathrm{Deg}\big(g(x_1,\dots,x_n)\big) = \mathrm{Ht}\big(s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{l(w)-1}}(\beta_{l(w)})\big)$$

such that

$$\begin{aligned}
&T_{\beta_1}T_{\beta_2}\cdots T_{\beta_{l(w)-1}}(F_{\beta_{l(w)}}) = g(F_1,F_2,\dots,F_n), \\
&\tau_{\beta_1}\tau_{\beta_2}\cdots \tau_{\beta_{l(w)-1}}(B_{\beta_{l(w)}}) = g(B_1,B_2,\dots,B_n) + g'(B_1,B_2,\dots,B_n).
\end{aligned}$$

To prove the claim, we use induction on $l(w)$. For $l(w)=1$, the claim is obvious. Suppose that the claim holds for $l(w)\leqslant l$. We will check the case $l(w)=l+1$. If $(\beta_l,\beta_{l+1})=0$, the claim holds since $T_{\beta_1}T_{\beta_2}\cdots T_{\beta_l}(F_{\beta_{l+1}}) = T_{\beta_1}T_{\beta_2}\cdots T_{\beta_{l-1}}(F_{\beta_{l+1}})$ and $\tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_l}(B_{\beta_{l+1}}) = \tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_{l-1}}(B_{\beta_{l+1}})$. Next we will consider the case $(\beta_l,\beta_{l+1})=-2$, while the case $(\beta_l,\beta_{l+1})=-1$ can be treated analogously. The proof can be done in the following two cases.

(i) If $s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{l-1}}s_{\beta_{l+1}}$ is a reduced expression, then by the formulas in Section 4.1, for $a_{\beta_l,\beta_{l+1}} = -1$ and $a_{\beta_{l+1},\beta_l} = -2$,

$$\begin{aligned}
&T_{\beta_1}T_{\beta_2}\cdots T_{\beta_l}(F_{\beta_{l+1}}) = T_{\beta_1}T_{\beta_2}\cdots T_{\beta_{l-1}}\big(F_{\beta_l}F_{\beta_{l+1}} - q^2F_{\beta_{l+1}}F_{\beta_l}\big), \\
&\tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_l}(B_{\beta_{l+1}}) = \tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_{l-1}}\big(B_{\beta_l}B_{\beta_{l+1}} - q^2B_{\beta_{l+1}}B_{\beta_l}\big),
\end{aligned}$$

and for $a_{\beta_l,\beta_{l+1}} = -2$ and $a_{\beta_{l+1},\beta_l} = -1$,

$$\begin{aligned}
&T_{\beta_1}T_{\beta_2}\cdots T_{\beta_l}(F_{\beta_{l+1}}) = T_{\beta_1}T_{\beta_2}\cdots T_{\beta_{l-1}}\left(\sum_{s=0}^{2}(-1)^s q^s F_{\beta_l}^{(2-s)}F_{\beta_{l+1}}F_{\beta_l}^{(s)}\right), \\
&\tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_l}(B_{\beta_{l+1}}) = \tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_{l-1}}\left(\sum_{s=0}^{2}(-1)^s q^s B_{\beta_l}^{(2-s)}B_{\beta_{l+1}}B_{\beta_l}^{(s)} + B_{\beta_{l+1}}\right).
\end{aligned}$$

Thus by induction the claim holds in this case.

(ii) If $s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{l-1}}s_{\beta_{l+1}}$ is not a reduced expression, by Bourbaki's Exchange Condition (see [16, Section 1.7]), there exists a β_k with $1\leqslant k\leqslant l-1$ such that $s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{l-1}} = s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots s_{\beta_{l-1}}s_{\beta_{l+1}}$, where $\widehat{s_{\beta_k}}$ means that s_{β_k} is omitted. Then we have two reduced expressions $w = s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{l-1}}s_{\beta_l}\times s_{\beta_{l+1}} = s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots s_{\beta_{l-1}}s_{\beta_{l+1}}s_{\beta_l}s_{\beta_{l+1}}$. Now it can be seen from the formulas in Section 4.1, when $a_{\beta_l,\beta_{l+1}} = -1$ and $a_{\beta_{l+1},\beta_l} = -2$,

$$\begin{aligned}
T_{\beta_1}T_{\beta_2}\cdots T_{\beta_l}(F_{\beta_{l+1}}) &= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}T_{\beta_{l+1}}T_{\beta_l}(F_{\beta_{l+1}})\\
&= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}\big(F_{\beta_{l+1}}F_{\beta_l} - q^2F_{\beta_l}F_{\beta_{l+1}}\big)\\
&= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}(F_{\beta_{l+1}})T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}(F_{\beta_l})\\
&\quad - q^2T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}(F_{\beta_l})T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}(F_{\beta_{l+1}}),\\
\tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_l}(B_{\beta_{l+1}}) &= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}\tau_{\beta_{l+1}}\tau_{\beta_l}(B_{\beta_{l+1}})\\
&= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}\big(B_{\beta_{l+1}}B_{\beta_l} - q^2B_{\beta_l}B_{\beta_{l+1}}\big)\\
&= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}(B_{\beta_{l+1}})\tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}(B_{\beta_l})\\
&\quad - q^2\tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}(B_{\beta_l})\tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}(B_{\beta_{l+1}}),
\end{aligned}$$

and when $a_{\beta_l,\beta_{l+1}} = -2$ and $a_{\beta_{l+1},\beta_l} = -1$,

$$\begin{aligned}
T_{\beta_1}T_{\beta_2}\cdots T_{\beta_l}(F_{\beta_{l+1}}) &= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}T_{\beta_{l+1}}T_{\beta_l}(F_{\beta_{l+1}})\\
&= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots T_{\beta_{l-1}}\big(F_{\beta_{l+1}}F_{\beta_l}^{(2)} - qF_{\beta_l}F_{\beta_{l+1}}F_{\beta_l} + q^2F_{\beta_l}^{(2)}F_{\beta_{l+1}}\big),\\
\tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_l}(B_{\beta_{l+1}}) &= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}\tau_{\beta_{l+1}}\tau_{\beta_l}(B_{\beta_{l+1}})\\
&= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\tau_{\beta_{l-1}}\big(B_{\beta_{l+1}}B_{\beta_l}^{(2)} - qB_{\beta_l}B_{\beta_{l+1}}B_{\beta_l} + q^2B_{\beta_l}^{(2)}B_{\beta_{l+1}} + B_{\beta_{l+1}}\big).
\end{aligned}$$

Thus by induction the claim holds if $s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots s_{\beta_{l-1}}s_{\beta_l}$ is a reduced expression. Otherwise, by Bourbaki's Exchange Condition, there exists a $\beta_{k'}$ with $1 \leqslant k \neq k' \leqslant l-1$ such that

$$s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots s_{\beta_{l-1}} = s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots\widehat{s_{\beta_{k'}}}\cdots s_{\beta_{l-1}}s_{\beta_l},$$

where $\widehat{s_{\beta_k}}$ and $\widehat{s_{\beta_{k'}}}$ mean that s_{β_k} and $s_{\beta_{k'}}$ are omitted. Then we have two reduced expressions $w = s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots\widehat{s_{\beta_{k'}}}\cdots s_{\beta_{l-1}}s_{\beta_l}s_{\beta_{l+1}}s_{\beta_l}s_{\beta_{l+1}} = s_{\beta_1}s_{\beta_2}\cdots\widehat{s_{\beta_k}}\cdots\widehat{s_{\beta_{k'}}}\cdots s_{\beta_{l-1}}s_{\beta_{l+1}}s_{\beta_l}s_{\beta_{l+1}}s_{\beta_l}$. Since

$$\begin{aligned}
T_{\beta_1}T_{\beta_2}\cdots T_{\beta_l}(F_{\beta_{l+1}}) &= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots\widehat{T_{\beta_{k'}}}\cdots T_{\beta_{l-1}}T_{\beta_l}T_{\beta_{l+1}}T_{\beta_l}(F_{\beta_{l+1}})\\
&= T_{\beta_1}T_{\beta_2}\cdots\widehat{T_{\beta_k}}\cdots\widehat{T_{\beta_{k'}}}\cdots T_{\beta_{l-1}}(F_{\beta_{l+1}}),\\
\tau_{\beta_1}\tau_{\beta_2}\cdots\tau_{\beta_l}(B_{\beta_{l+1}}) &= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\widehat{\tau_{\beta_{k'}}}\cdots\tau_{\beta_{l-1}}\tau_{\beta_l}\tau_{\beta_{l+1}}\tau_{\beta_l}(B_{\beta_{l+1}})\\
&= \tau_{\beta_1}\tau_{\beta_2}\cdots\widehat{\tau_{\beta_k}}\cdots\widehat{\tau_{\beta_{k'}}}\cdots\tau_{\beta_{l-1}}(B_{\beta_{l+1}}),
\end{aligned}$$

then by induction the claim also holds in this case. □

4.4. PBW theorem for $\mathfrak{B}_q(\mathfrak{g})$

Now we are ready to state our main result which gives a PBW basis for $\mathfrak{B}_q(\mathfrak{g})$ via root vectors defined by the algebra automorphisms τ_i $(1 \leqslant i \leqslant n)$.

Theorem 4.4 *(PBW theorem for $\mathfrak{B}_q(\mathfrak{g})$). Let $w_0 = s_{i_1}s_{i_2}\cdots s_{i_{l_0}}$ be a reduced expression of the longest element in W, and $B_{\gamma_j} = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_{j-1}}(B_{i_j})$ be the root vector corresponding to the positive root $\gamma_j = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j})$ for $1 \leqslant j \leqslant l_0$. Then*

$$B(w_0) = \left\{ B_{\gamma_{l_0}}^{a_{l_0}} \cdots B_{\gamma_2}^{a_2} B_{\gamma_1}^{a_1} \;\middle|\; a_1, a_2, \ldots, a_{l_0} \in \mathbb{Z}^{\geqslant 0} \right\}$$

is a basis of $\mathfrak{B}_q(\mathfrak{g})$ for $\mathfrak{g}$ of each type.

Proof. We will show the claim in two steps. In the following the symbol $\mathcal{F}^*$ is also used to denote the filtrations of $\mathfrak{B}_q(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$.

In the first step, we prove that the dimension of $\mathcal{F}^m(\mathfrak{B}_q(\mathfrak{g}))$ coincides with that of $\mathcal{F}^m(U_q^-(\mathfrak{g}))$, that is,

$$\dim\bigl(\mathcal{F}^m\bigl(\mathfrak{B}_q(\mathfrak{g})\bigr)\bigr) = \dim\bigl(\mathcal{F}^m\bigl(U_q^-(\mathfrak{g})\bigr)\bigr). \tag{4.4}$$

On one hand, it can be seen from the defining relations of $\mathfrak{B}_q(\mathfrak{g})$ that the associated graded algebra $\mathrm{gr}(\mathfrak{B}_q(\mathfrak{g}))$ is a quotient of $U_q^-(\mathfrak{g})$. Thus $\dim(\mathcal{F}^m(\mathfrak{B}_q(\mathfrak{g}))) \leqslant \dim(\mathcal{F}^m(U_q^-(\mathfrak{g})))$. On the other hand, noting that $\mathfrak{B}_q(\mathfrak{g})$ is a coideal subalgebra of $U_q(\mathfrak{g})$ for $\mathfrak{g}$ of each type (see Remarks 3.10(1)), we can deduce from the arguments in [24] (see the explanations after formula (7.17) in [24]) that $\dim(\mathcal{F}^m(\mathfrak{B}_q(\mathfrak{g}))) \geqslant \dim(\mathcal{F}^m(U_q^-(\mathfrak{g})))$. Hence the formula (4.4) holds and $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$.

In the second step, we verify that the elements in $B(w_0)$ are linearly independent and that they span $\mathfrak{B}_q(\mathfrak{g})$. Define $U'(w_0) := \mathrm{Span}_{\mathbb{C}}\, B(w_0)$ and $\mathcal{F}^m(U'(w_0)) := U'(w_0) \bigcap \mathcal{F}^m(\mathfrak{B}_q(\mathfrak{g}))$. By Proposition 4.3 and (4.4) we know that the canonical map

$$\Psi: \bigoplus_{k=0}^{m} \mathcal{F}^k\bigl(U'(w_0)\bigr)/\mathcal{F}^{k-1}\bigl(U'(w_0)\bigr) \to \bigoplus_{k=0}^{m} \mathcal{F}^k\bigl(\mathfrak{B}_q(\mathfrak{g})\bigr)/\mathcal{F}^{k-1}\bigl(\mathfrak{B}_q(\mathfrak{g})\bigr) \cong \mathcal{F}^m\bigl(U_q^-(\mathfrak{g})\bigr)$$

satisfies $\Psi(B_{\gamma_{l_0}}^{a_{l_0}} \cdots B_{\gamma_2}^{a_2} B_{\gamma_1}^{a_1}) = F_{\gamma_{l_0}}^{a_{l_0}} \cdots F_{\gamma_2}^{a_2} F_{\gamma_1}^{a_1}$, where the degree of $B_{\gamma_{l_0}}^{a_{l_0}} \cdots B_{\gamma_2}^{a_2} B_{\gamma_1}^{a_1}$ is no more than m. It follows from the PBW theorem of $U_q^-(\mathfrak{g})$ that Ψ is surjective and that the elements in $B(w_0)$ are linearly independent. Moreover, by (4.4) we obtain that

$$\dim\bigl(\mathcal{F}^m\bigl(\mathfrak{B}_q(\mathfrak{g})\bigr)\bigr) = \dim\bigl(\mathcal{F}^m\bigl(U_q^-(\mathfrak{g})\bigr)\bigr) \leqslant \sharp\bigl(B(w_0) \cap \mathcal{F}^m\bigl(\mathfrak{B}_q(\mathfrak{g})\bigr)\bigr),$$

which implies that $B(w_0) \cap \mathcal{F}^m(\mathfrak{B}_q(\mathfrak{g}))$ spans $\mathcal{F}^m(\mathfrak{B}_q(\mathfrak{g}))$. Therefore, $B(w_0)$ spans $\mathfrak{B}_q(\mathfrak{g})$. $\square$

Remark 4.5. Applying the same procedure in this section to the algebra automorphisms τ_i^{-1} of $\mathfrak{B}_q(\mathfrak{g})$ in Proposition 4.1, we can obtain another PBW basis

$$B^{-1}(w_0) = \left\{ \tau_{i_1}^{-1}\tau_{i_2}^{-1} \cdots \tau_{i_{l_0-1}}^{-1}\bigl(B_{i_{l_0}}^{a_{l_0}}\bigr) \cdots \tau_{i_1}^{-1}\tau_{i_2}^{-1}\bigl(B_{i_3}^{a_3}\bigr)\tau_{i_1}^{-1}\bigl(B_{i_2}^{a_2}\bigr)B_{i_1}^{a_1} \;\middle|\; a_1, a_2, \ldots, a_{l_0} \in \mathbb{Z}^{\geqslant 0} \right\}.$$

4.5. Realization of Iorgov–Klimyk's PBW theorem for $U_q'(\mathfrak{so}(n+1,\mathbb{C}))$

For the algebra $U_q'(\mathfrak{so}(n+1,\mathbb{C}))$ described in Remarks 3.10(2), Iorgov and Klimyk gave two PBW bases in [18]. Recall that they used the notation $I_{l+1,l}^+ \equiv I_{l+1,l}^- \equiv I_l$ and for $k > l+1$ defined recursively

$$I_{k,l}^+ = q^{\frac{1}{2}} I_{l+1,l}^+ I_{k,l+1}^+ - q^{-\frac{1}{2}} I_{k,l+1}^+ I_{l+1,l}^+, \tag{4.5}$$

$$I_{k,l}^- = q^{-\frac{1}{2}} I_{l+1,l}^- I_{k,l+1}^- - q^{\frac{1}{2}} I_{k,l+1}^- I_{l+1,l}^-. \tag{4.6}$$

Then they obtained the following PBW theorem for $U_q'(\mathfrak{so}(n+1,\mathbb{C}))$ by Bergman's Diamond Lemma.

Theorem 4.6 *(PBW theorem for $U'_q(\mathfrak{so}(n+1,\mathbb{C}))$. (See [18].) The following subsets of $U'_q(\mathfrak{so}(n+1,\mathbb{C}))$*

$$I^+(w_0)=\left\{{I^+_{2,1}}^{m_{2,1}}{I^+_{3,1}}^{m_{3,1}}\cdots{I^+_{n+1,1}}^{m_{n+1,1}}\cdots{I^+_{n,n-1}}^{m_{n,n-1}}{I^+_{n+1,n-1}}^{m_{n+1,n-1}}{I^+_{n+1,n}}^{m_{n+1,n}}\;\middle|\;m_{i,j}\in\mathbb{Z}^{\geqslant 0}\right\},$$
$$I^-(w_0)=\left\{{I^-_{2,1}}^{m_{2,1}}{I^-_{3,1}}^{m_{3,1}}\cdots{I^-_{n+1,1}}^{m_{n+1,1}}\cdots{I^-_{n,n-1}}^{m_{n,n-1}}{I^-_{n+1,n-1}}^{m_{n+1,n-1}}{I^-_{n+1,n}}^{m_{n+1,n}}\;\middle|\;m_{i,j}\in\mathbb{Z}^{\geqslant 0}\right\}$$

are PBW bases of $U'_q(\mathfrak{so}(n+1,\mathbb{C}))$.

As a matter of fact, we can prove this theorem by Theorem 4.4. Noting that $U'_q(\mathfrak{so}(n+1,\mathbb{C}))\cong\mathfrak{B}_q(\mathfrak{sl}(n+1,\mathbb{C}))$, we choose the reduced expression

$$w_0=\underbrace{s_n}_{1}\,\underbrace{s_{n-1}s_n}_{2}\,\underbrace{s_{n-2}s_{n-1}s_n}_{3}\cdots\underbrace{s_2\cdots s_n}_{n-1}\,\underbrace{s_1s_2\cdots s_n}_{n},$$

of the longest element w_0 in the symmetry group S_n. Set

$$\begin{aligned}
&\gamma_{n+1,n}=\alpha_n,\\
&\gamma_{n+1,n-1}=s_n(\alpha_{n-1})=\alpha_{n-1}+\alpha_n,\\
&\gamma_{n,n-1}=s_ns_{n-1}(\alpha_n)=\alpha_{n-1},\\
&\cdots,\\
&\gamma_{n+1,1}=s_ns_{n-1}s_ns_{n-2}s_{n-1}s_n\cdots s_2\cdots s_n(\alpha_1)=\alpha_1+\alpha_2+\cdots+\alpha_n,\\
&\cdots,\\
&\gamma_{3,1}=s_ns_{n-1}s_ns_{n-2}s_{n-1}s_n\cdots s_2\cdots s_ns_1\cdots s_{n-2}(\alpha_{n-1})=\alpha_1+\alpha_2,\\
&\gamma_{2,1}=s_ns_{n-1}s_ns_{n-2}s_{n-1}s_n\cdots s_2\cdots s_ns_1\cdots s_{n-1}(\alpha_n)=\alpha_1.
\end{aligned}$$

Then $\{\gamma_{i,j}\mid 1\leqslant j<i\leqslant n+1\}$ is exactly the positive root set of $\mathfrak{sl}(n+1,\mathbb{C})$. The corresponding root vectors $B_{i,j}=B_{\gamma_{i,j}}$ of $\mathfrak{B}_q(\mathfrak{sl}(n+1,\mathbb{C}))$ are as follows:

$$\begin{aligned}
&B_{n+1,n}=B_n,\\
&B_{n+1,n-1}=\tau_n(B_{n-1}),\\
&B_{n,n-1}=\tau_n\tau_{n-1}(B_n)=B_{n-1},\\
&\cdots,\\
&B_{n+1,1}=\tau_n\tau_{n-1}\tau_n\tau_{n-2}\tau_{n-1}\tau_n\cdots\tau_2\cdots\tau_n(B_1),\\
&\cdots,\\
&B_{3,1}=\tau_n\tau_{n-1}\tau_n\tau_{n-2}\tau_{n-1}\tau_n\cdots\tau_2\cdots\tau_n\tau_1\cdots\tau_{n-2}(B_{n-1}),\\
&B_{2,1}=\tau_n\tau_{n-1}\tau_n\tau_{n-2}\tau_{n-1}\tau_n\cdots\tau_2\cdots\tau_n\tau_1\cdots\tau_{n-1}(B_n)=B_1.
\end{aligned}$$

It follows from Theorem 4.4 that

$$B(w_0)=\left\{B_{2,1}^{m_{2,1}}B_{3,1}^{m_{3,1}}\cdots B_{n+1,1}^{m_{n+1,1}}B_{3,2}^{m_{3,2}}B_{4,2}^{m_{4,2}}\cdots B_{n+1,2}^{m_{n+1,2}}\cdots B_{n,n-1}^{m_{n,n-1}}B_{n+1,n-1}^{m_{n+1,n-1}}B_{n+1,n}^{m_{n+1,n}}\;\middle|\;m_{i,j}\in\mathbb{Z}^{\geqslant 0}\right\}$$

forms a basis of $\mathfrak{B}_q(\mathfrak{sl}(n+1,\mathbb{C}))$.

Let us give some accurate relations between root vectors of $\mathfrak{B}_q(\mathfrak{sl}(n+1,\mathbb{C}))$. They are similar to the relation (4.5).

Lemma 4.7. *For $k > l+1$, we have*

$$B_{k,l} = -qB_{l+1,l}B_{k,l+1} + B_{k,l+1}B_{l+1,l}. \tag{4.7}$$

Proof. Since

$$B_{k,l+1} = \tau_n\tau_{n-1}\tau_n\tau_{n-2}\tau_{n-1}\tau_n\cdots\tau_{l+1}\cdots\tau_{n+1-(k-l)}(B_{n+1-(k-l-1)}),$$
$$B_{l+1,l} = \tau_n\tau_{n-1}\tau_n\tau_{n-2}\tau_{n-1}\tau_n\cdots\tau_{l+1}\cdots\tau_{n+1-(k-l)}\tau_{n+1-(k-l-1)}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n),$$
$$B_{k,l} = \tau_n\tau_{n-1}\tau_n\tau_{n-2}\tau_{n-1}\tau_n\cdots\tau_{l+1}\cdots\tau_{n+1-(k-l)}\tau_{n+1-(k-l-1)}\cdots\tau_n\tau_l\cdots\tau_{n+1-(k-l+1)}(B_{n+1-(k-l)}),$$

the relation (4.7) is equivalent to

$$\begin{aligned}
&\tau_{n+1-(k-l-1)}\cdots\tau_n\tau_l\cdots\tau_{n+1-(k-l+1)}(B_{n+1-(k-l)})\\
&\quad = -q\tau_{n+1-(k-l-1)}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n+1-(k-l-1)}\\
&\qquad + B_{n+1-(k-l-1)}\tau_{n+1-(k-l-1)}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n).
\end{aligned} \tag{4.8}$$

The relation (4.8) can be checked by induction on $k-(l+1)$.
Indeed, for $k-(l+1)=1$, (4.8) holds since

$$\begin{aligned}
&-q\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_n + B_n\tau_n\tau_l\cdots\tau_{n-1}(B_n)\\
&\quad = \tau_n\tau_l\cdots\tau_{n-2}\big[-q\tau_{n-1}(B_n)B_n\big] + \tau_n\tau_l\cdots\tau_{n-2}\big[B_n\tau_{n-1}(B_n)\big]\\
&\quad = \tau_n\tau_l\cdots\tau_{n-2}\big[-q\tau_{n-1}(B_n)B_n + B_n\tau_{n-1}(B_n)\big] = \tau_n\tau_l\cdots\tau_{n-2}(B_{n-1}).
\end{aligned}$$

Assume that (4.8) holds for $k-(l+1)=r$, that is,

$$\begin{aligned}
\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-1}(B_{n-r}) &= -q\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r+1}\\
&\quad + B_{n-r+1}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n).
\end{aligned}$$

For the case $k-(l+1)=r+1$, we will check

$$\begin{aligned}
\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}(B_{n-r-1}) &= -q\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r}\\
&\quad + B_{n-r}\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n).
\end{aligned} \tag{4.9}$$

In fact,

$$B_{n-r-1} = B_{n-r}\tau_{n-r-1}(B_{n-r}) - q\tau_{n-r-1}(B_{n-r})B_{n-r}.$$

It follows that

$$\begin{aligned}
&\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}(B_{n-r-1}) \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\big[B_{n-r}\tau_{n-r-1}(B_{n-r}) - q\tau_{n-r-1}(B_{n-r})B_{n-r}\big] \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\big[B_{n-r}\tau_{n-r-1}(B_{n-r})\big] \\
&\qquad+ (-q)\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\big[\tau_{n-r-1}(B_{n-r})B_{n-r}\big] \\
&\quad= B_{n-r+1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\tau_{n-r-1}(B_{n-r}) \\
&\qquad+ (-q)\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\tau_{n-r-1}(B_{n-r})B_{n-r+1} \\
&\quad= B_{n-r+1}\tau_{n-r}\big(-q\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r-1} + B_{n-r+1}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)\big) \\
&\qquad+ (-q)\tau_{n-r}\big(-q\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r+1} + B_{n-r+1}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)\big)B_{n-r+1} \\
&\quad= -qB_{n-r+1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)\tau_{n-r}(B_{n-r+1}) \\
&\qquad+ B_{n-r+1}\tau_{n-r}(B_{n-r+1})\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n) \\
&\qquad+ q^2\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)\tau_{n-r}(B_{n-r+1})B_{n-r+1} \\
&\qquad+ (-q)\tau_{n-r}(B_{n-r+1})\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r+1}.
\end{aligned}$$

Similarly,

$$B_{n-r} = -q\tau_{n-r}(B_{n-r+1})B_{n-r+1} + B_{n-r+1}\tau_{n-r}(B_{n-r+1}),$$

we have

$$\begin{aligned}
&-q\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r} + B_{n-r}\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n) \\
&\quad= -q\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)\big(-q\tau_{n-r}(B_{n-r+1})B_{n-r+1} + B_{n-r+1}\tau_{n-r}(B_{n-r+1})\big) \\
&\qquad+ \big(-q\tau_{n-r}(B_{n-r+1})B_{n-r+1} + B_{n-r+1}\tau_{n-r}(B_{n-r+1})\big)\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n) \\
&\quad= -q\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)B_{n-r+1}\tau_{n-r}(B_{n-r+1}) \\
&\qquad+ B_{n-r+1}\tau_{n-r}(B_{n-r+1})\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n) \\
&\qquad+ q^2\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n)\tau_{n-r}(B_{n-r+1})B_{n-r+1} \\
&\qquad+ (-q)\tau_{n-r}(B_{n-r+1})B_{n-r+1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&B_{n-r+1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n) \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}(B_{n-r})\tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-1}(B_n) \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\big(B_{n-r}\tau_{n-r-1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_{n-1}(B_n)\big) \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\tau_{n-r-1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_{n-1}(B_{n-r-1}B_n) \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_{n-r-2}\tau_{n-r-1}\tau_{n-r}\tau_{n-r+1}\cdots\tau_{n-1}(B_nB_{n-r-1}) \\
&\quad= \tau_{n-r}\tau_{n-r+1}\cdots\tau_n\tau_l\cdots\tau_n(B_n)B_{n-r+1}.
\end{aligned}$$

Therefore, (4.9) holds for the case $k-(l+1)=r+1$. □

If identifying I_i with $q^{\frac{1}{2}}B_i$ under the algebra isomorphism $U'_q(\mathfrak{so}(n+1,\mathbb{C})) \cong \mathfrak{B}_q(\mathfrak{sl}(n+1,\mathbb{C}))$ (see Remarks 3.10(2)), we have

Lemma 4.8. *For $n+1 \geqslant k > l \geqslant 1$, we have*

$$I^+_{k,l} = (-1)^{k-l+1} q^{\frac{1}{2}} B_{k,l}. \tag{4.10}$$

Proof. We show (4.10) by induction on $k-l$.

For $k-l=1$, (4.10) holds since $I^+_{l+1,l} = q^{\frac{1}{2}} B_l = q^{\frac{1}{2}} B_{l+1,l}$.

Assume that (4.10) holds for $k-l=r$. In the case $k-l=r+1$, by induction and Lemma 4.7 we have

$$\begin{aligned}
I^+_{l+r+1,l} &= q^{\frac{1}{2}} I^+_{l+1,l} I^+_{l+r+1,l+1} - q^{-\frac{1}{2}} I^+_{l+r+1,l+1} I^+_{l+1,l} \\
&= q^{\frac{1}{2}} \big(q^{\frac{1}{2}} B_{l+1,l}\big)\big[(-1)^{r+1} q^{\frac{1}{2}} B_{l+r+1,l+1}\big] - q^{-\frac{1}{2}} \big[(-1)^{r+1} q^{\frac{1}{2}} B_{l+r+1,l+1}\big]\big(q^{\frac{1}{2}} B_{l+1,l}\big) \\
&= (-1)^{r+1} q^{\frac{3}{2}} B_{l+1,l} B_{l+r+1,l+1} - (-1)^{r+1} q^{\frac{1}{2}} B_{l+r+1,l+1} B_{l+1,l} \\
&= (-1)^{r+2} q^{\frac{1}{2}} (-q B_{l+1,l} B_{l+r+1,l+1} + B_{l+r+1,l+1} B_{l+1,l}) \\
&= (-1)^{r+2} q^{\frac{1}{2}} B_{l+r+1,l}.
\end{aligned}$$

The proof of the lemma is finished. □

In the end, it can be seen from Lemma 4.8 that up to nonzero scalars $I^+(w_0)$ is just $B(w_0)$ in Theorem 4.4. On the other hand, if we choose

$$w_0 = \underbrace{s_1}_{1}\ \underbrace{s_2 s_1}_{2}\ \underbrace{s_3 s_2 s_1}_{3} \cdots \underbrace{s_{n-1} \cdots s_1}_{n-1}\ \underbrace{s_n s_{n-1} \cdots s_1}_{n},$$

then $I^-(w_0)$ equals $B^{-1}(w_0)$ in Remark 4.5 up to nonzero scalars. Therefore, Iorgov–Klimyk's PBW theorem for $U'_q(\mathfrak{so}(n+1,\mathbb{C}))$, i.e., Theorem 4.6, is recovered by Theorem 4.4. □

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Appendix A

The calculations of the formulas about τ_i in Section 4.1.

(1) If $a_{ij} = a_{ji} = -1$, then

$$\begin{aligned}
\tau_i \tau_j(B_i) &= \tau_i(B_j B_i - q B_i B_j) \\
&= (B_i B_j - q B_j B_i) B_i - q B_i (B_i B_j - q B_j B_i) \\
&= -q\big(B_i^2 B_j - \big(q+q^{-1}\big) B_i B_j B_i + B_j B_i^2\big) \\
&\overset{(3.1)}{\underset{(3.24)}{=}} B_j.
\end{aligned}$$

(2) If $a_{ij}=-1$ and $a_{ji}=-2$, we will check the formulas in (2) one after another. Firstly, since $B_i(B_iB_j-q^2B_jB_i)=q^{-2}(B_iB_j-q^2B_jB_i)B_i-q^{-2}B_j$, then one has

$$\begin{cases} (B_iB_j-q^2B_jB_i)B_i(B_iB_j-q^2B_jB_i)=q^{-2}(B_iB_j-q^2B_jB_i)^2B_i-q^{-2}(B_iB_j-q^2B_jB_i)B_j, \\ B_i(B_iB_j-q^2B_jB_i)^2 \\ \quad =q^{-4}(B_iB_j-q^2B_jB_i)^2B_i-q^{-4}(B_iB_j-q^2B_jB_i)B_j-q^{-2}B_j(B_iB_j-q^2B_jB_i). \end{cases} \tag{A.1}$$

It follows that

$$\begin{aligned} \tau_i\tau_j(B_i) &= \tau_i\big([2]^{-1}\big(B_j^2B_i-q[2]B_jB_iB_j+q^2B_iB_j^2\big)+B_i\big) \\ &= [2]^{-1}\begin{pmatrix} (B_iB_j-q^2B_jB_i)^2B_i+q^2B_i(B_iB_j-q^2B_jB_i)^2 \\ -q[2](B_iB_j-q^2B_jB_i)B_i(B_iB_j-q^2B_jB_i) \end{pmatrix}+B_i \\ &\overset{(A.1)}{=} [2]^{-1}\big(B_iB_j^2-q[2]B_jB_iB_j+q^2B_j^2B_i\big)+B_i. \end{aligned} \tag{A.2}$$

Secondly, one has

$$\begin{aligned} \tau_j\tau_i(B_j) &= \tau_j\big(B_iB_j-q^2B_jB_i\big) \\ &= \tau_j(B_i)B_j-q^2B_j\tau_j(B_i) \\ &= \big([2]^{-1}\big(B_j^2B_i-q[2]B_jB_iB_j+q^2B_iB_j^2\big)+B_i\big)B_j \\ &\quad -q^2B_j\big([2]^{-1}\big(B_j^2B_i-q[2]B_jB_iB_j+q^2B_iB_j^2\big)+B_i\big) \\ &= B_iB_j-q^2B_jB_i-q^2[2]^{-1}\big(B_j^3B_i-[3]B_j^2B_iB_j+[3]B_jB_iB_j^2-B_iB_j^3\big) \\ &\underset{(3.24)}{\overset{(3.1)}{=}} B_iB_j-q^2B_jB_i+q[2](B_jB_i-B_iB_j) \\ &= B_jB_i-q^2B_iB_j. \end{aligned} \tag{A.3}$$

Thirdly, noting that $B_j\tau_j(B_i)=q^{-2}\tau_j(B_i)B_j+B_iB_j-q^{-2}B_jB_i$, we have

$$\begin{cases} B_j\tau_j(B_i)B_j=q^{-2}\tau_j(B_i)B_j^2+B_iB_j^2-q^{-2}B_jB_iB_j, \\ B_j^2\tau_j(B_i)=q^{-4}\tau_j(B_i)B_j^2+q^{-2}B_iB_j^2+(1-q^{-4})B_jB_iB_j-q^{-2}B_j^2B_i, \end{cases} \tag{A.4}$$

Then we obtain

$$\begin{aligned} \tau_j\tau_i\tau_j(B_i) &\overset{(A.2)}{=} \tau_j\big([2]^{-1}\big(B_iB_j^2-q[2]B_jB_iB_j+q^2B_j^2B_i\big)+B_i\big) \\ &= [2]^{-1}\tau_j(B_i)B_j^2-qB_j\tau_j(B_i)B_j+[2]^{-1}q^2B_j^2\tau_j(B_i)+\tau_j(B_i) \\ &\overset{(A.4)}{=} -[2]^{-1}\big(q^2B_iB_j^2-q[2]B_jB_iB_j+B_j^2B_i\big)+\tau_j(B_i) \\ &= B_i. \end{aligned}$$

Finally, we have

$$\begin{aligned}
\tau_i\tau_j\tau_i(B_j) &\overset{(A.3)}{=} \tau_i\big(B_jB_i - q^2B_iB_j\big)\\
&= \big(B_iB_j - q^2B_jB_i\big)B_i - q^2B_i\big(B_iB_j - q^2B_jB_i\big)\\
&= -q^2\big(B_i^2B_j - \big(q^2+q^{-2}\big)B_iB_jB_i + B_jB_i^2\big)\\
&\underset{(3.24)}{\overset{(3.1)}{=}} -q^2\big(-q^{-2}B_j\big)\\
&= B_j.
\end{aligned}$$

(3) When $a_{ij} = -1$ and $a_{ji} = -3$, we also verify the formulas in (3) successively. To begin with, noting that

$$\begin{cases}
B_i(B_iB_j - q^3B_jB_i) = q^{-3}(B_iB_j - q^3B_jB_i)B_i - q^{-3}B_j,\\
(B_iB_j - q^3B_jB_i)B_i(B_iB_j - q^3B_jB_i) = q^{-3}(B_iB_j - q^3B_jB_i)^2B_i - q^{-3}(B_iB_j - q^3B_jB_i)B_j,\\
B_i(B_iB_j - q^3B_jB_i)^2\\
\quad = q^{-6}(B_iB_j - q^3B_jB_i)^2B_i - q^{-6}(B_iB_j - q^3B_jB_i)B_j - q^{-3}B_j(B_iB_j - q^3B_jB_i),\\
B_i(B_iB_j - q^3B_jB_i)^3\\
\quad = q^{-9}(B_iB_j - q^3B_jB_i)^3B_i - q^{-9}(B_iB_j - q^3B_jB_i)^2B_j - q^{-3}B_j(B_iB_j - q^3B_jB_i)^2\\
\qquad - q^{-6}(B_iB_j - q^3B_jB_i)B_j(B_iB_j - q^3B_jB_i),
\end{cases} \tag{A.5}$$

then we obtain that

$$\begin{aligned}
\tau_i\tau_j(B_i) =\ & \tau_i\left(\sum_{s=0}^{3}(-1)^sq^sB_j^{(3-s)}B_iB_j^{(s)}\right) + \frac{1}{q[3]!}\tau_i\big(B_jB_i - q^3B_iB_j\big) + \tau_i(B_jB_i - qB_iB_j)\\
=\ & \frac{1}{[3]!}\big(B_iB_j - q^3B_jB_i\big)^3B_i - \frac{q}{[2]}\big(B_iB_j - q^3B_jB_i\big)^2B_i\big(B_iB_j - q^3B_jB_i\big)\\
& + \frac{q^2}{[2]}\big(B_iB_j - q^3B_jB_i\big)B_i\big(B_iB_j - q^3B_jB_i\big)^2 - \frac{q^3}{[3]!}B_i\big(B_iB_j - q^3B_jB_i\big)^3\\
& + \left(\frac{1}{q[3]!}+1\right)\big(B_iB_j - q^3B_jB_i\big)B_i - \left(\frac{q^2}{[3]!}+q\right)B_i\big(B_iB_j - q^3B_jB_i\big)\\
\overset{(A.5)}{=}\ & \frac{1}{[3]!}\begin{pmatrix}(B_iB_j - q^3B_jB_i)^2B_j + B_j(B_iB_j - q^3B_jB_i)^2\\ -[2](B_iB_j - q^3B_jB_i)B_j(B_iB_j - q^3B_jB_i)\end{pmatrix}\\
& + \left(\frac{1}{q[3]!}+1\right)\big(B_iB_j - q^3B_jB_i\big)B_i - \left(\frac{q^2}{[3]!}+q\right)B_i\big(B_iB_j - q^3B_jB_i\big).
\end{aligned} \tag{A.6}$$

Next, one has

$$\begin{aligned}
\tau_j\tau_i(B_j) &= \tau_j\big(B_iB_j - q^3B_jB_i\big)\\
&= \left(\sum_{s=0}^{3}(-1)^sq^sB_j^{(3-s)}B_iB_j^{(s)} + \frac{1}{q[3]!}\big(B_jB_i - q^3B_iB_j\big) + (B_jB_i - qB_iB_j)\right)B_j\\
&\quad - q^3B_j\left(\sum_{s=0}^{3}(-1)^sq^sB_j^{(3-s)}B_iB_j^{(s)} + \frac{1}{q[3]!}\big(B_jB_i - q^3B_iB_j\big) + (B_jB_i - qB_iB_j)\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{-q^3}{[3]!}\left(B_j^4B_i - [4]B_j^3B_iB_j + \frac{[4][3]}{[2]}B_j^2B_iB_j^2 - [4]B_jB_iB_j^3 + B_iB_j^4\right) \\
&\quad + \frac{1}{q[3]!}\left(B_jB_i - q^3B_iB_j\right)B_j \\
&\quad + (B_jB_i - qB_iB_j)B_j - \frac{q^2}{[3]!}B_j\left(B_jB_i - q^3B_iB_j\right) - q^3B_j(B_jB_i - qB_iB_j) \\
&\overset{(3.1)}{\underset{(3.24)}{=}} \frac{-q^3}{[3]!}\Big(-q^{-1}\left([3]^2+1\right)\left(B_j^2B_i - B_iB_j^2\right) \\
&\quad + q^{-1}[2]\left([2][4] + q^2 + q^{-2}\right)B_jB_iB_j - q^{-2}[3]^2B_i\Big) \\
&\quad - \left(\frac{q^2}{[3]!} + q^3\right)B_j^2B_i + \left(\frac{q^{-1}+q^5}{[3]!} + 1 + q^4\right)B_jB_iB_j - \left(\frac{q^2}{[3]!} + q\right)B_iB_j^2 \\
&= [2]^{-1}B_j^2B_i - q^2B_jB_iB_j + q^4[2]^{-1}B_iB_j^2 + q[3][2]^{-1}B_i. \qquad \text{(A.7)}
\end{aligned}$$

Moreover, if we denote $D := \tau_j\tau_i(B_j) = \tau_j(B_iB_j - q^3B_jB_i)$, and note that

$$B_jD - q^{-1}DB_j = [3]\tau_j(B_i) - q^{-1}[2]\left(B_jB_i - q^3B_iB_j\right), \qquad \text{(A.8)}$$

then we have

$$\begin{aligned}
\tau_j\tau_i\tau_j(B_i) &\overset{(A.6)}{=} \frac{1}{[3]!}\tau_j\begin{pmatrix} (B_iB_j - q^3B_jB_i)^2B_j + B_j(B_iB_j - q^3B_jB_i)^2 \\ -[2](B_iB_j - q^3B_jB_i)B_j(B_iB_j - q^3B_jB_i) \end{pmatrix} \\
&\quad + \left(\frac{1}{q[3]!} + 1\right)\tau_j\left(B_iB_j - q^3B_jB_i\right)\tau_j(B_i) - \left(\frac{q^2}{[3]!} + q\right)\tau_j(B_i)\tau_j\left(B_iB_j - q^3B_jB_i\right) \\
&= \frac{1}{[3]!}D^2B_j - \frac{1}{[3]}DB_jD + \frac{1}{[3]!}B_jD^2 + \left(\frac{1}{q[3]!} + 1\right)D\tau_j(B_i) - \left(\frac{q^2}{[3]!} + q\right)\tau_j(B_i)D \\
&= \frac{1}{[3]!}\begin{pmatrix} -qD(B_jD - q^{-1}DB_j) \\ +(B_jD - q^{-1}DB_j)D \end{pmatrix} + \left(\frac{1}{q[3]!} + 1\right)D\tau_j(B_i) - \left(\frac{q^2}{[3]!} + q\right)\tau_j(B_i)D \\
&\overset{(A.8)}{=} \frac{-q}{[3]!}D\left([3]\tau_j(B_i) - q^{-1}[2]\left(B_jB_i - q^3B_iB_j\right)\right) + \frac{1}{[3]!}\left([3]\tau_j(B_i)\right. \\
&\quad \left. - q^{-1}[2]\left(B_jB_i - q^3B_iB_j\right)\right)D \\
&\quad + \left(\frac{1}{q[3]!} + 1\right)D\tau_j(B_i) - \left(\frac{q^2}{[3]!} + q\right)\tau_j(B_i)D \\
&= \frac{1}{[3]}D\left(B_jB_i - q^3B_iB_j\right) - \frac{1}{q[3]}\left(B_jB_i - q^3B_iB_j\right)D + \frac{[2]}{q[3]}\left(D\tau_j(B_i) - q^3\tau_j(B_i)D\right) \\
&= \frac{1}{[3]}D\left(B_jB_i - q^3B_iB_j\right) - \frac{1}{q[3]}\left(B_jB_i - q^3B_iB_j\right)D \\
&\quad + \frac{[2]}{q[3]}\tau_j\begin{pmatrix} (B_iB_j - q^3B_jB_i)B_i \\ -q^3B_i(B_iB_j - q^3B_jB_i) \end{pmatrix} \\
&\overset{(3.1)}{\underset{(3.24)}{=}} \frac{1}{[3]}D\left(B_jB_i - q^3B_iB_j\right) - \frac{1}{q[3]}\left(B_jB_i - q^3B_iB_j\right)D + \frac{[2]}{q[3]}B_j, \qquad \text{(A.9)}
\end{aligned}$$

$$\begin{aligned}
\tau_i\tau_j\tau_i(B_j) &= \tau_i(D) \overset{(A.7)}{=} \tau_i \begin{pmatrix} [2]^{-1}B_j^2B_i - q^2B_jB_iB_j \\ +q^4[2]^{-1}B_iB_j^2 + q[3][2]^{-1}B_i \end{pmatrix} \\
&= [2]^{-1}\big(B_iB_j - q^3B_jB_i\big)^2B_i - q^2\big(B_iB_j - q^3B_jB_i\big)B_i\big(B_iB_j - q^3B_jB_i\big) \\
&\quad + q^4[2]^{-1}B_i\big(B_iB_j - q^3B_jB_i\big)^2 + q[3][2]^{-1}B_i \\
&\overset{(A.5)}{=} q^{-1}\big(B_iB_j - q^3B_jB_i\big)B_j - q^{-2}[2]^{-1}\big(B_iB_j - q^3B_jB_i\big)B_j \\
&\quad - q[2]^{-1}B_j\big(B_iB_j - q^3B_jB_i\big) + q[3][2]^{-1}B_i \\
&= [2]^{-1}q^4B_j^2B_i - q^2B_jB_iB_j + [2]^{-1}B_iB_j^2 + q[3][2]^{-1}B_i. \qquad \text{(A.10)}
\end{aligned}$$

It follows that

$$\begin{aligned}
\tau_i\tau_j\tau_i\tau_j(B_i) &\overset{(A.9)}{=} \frac{1}{[3]}\tau_i \begin{pmatrix} D(B_jB_i - q^3B_iB_j) \\ -q^{-1}(B_jB_i - q^3B_iB_j)D \end{pmatrix} + \frac{[2]}{q[3]}\tau_i(B_j) \\
&\overset{(A.10)}{\underset{(3.1),\,(3.24)}{=}} \frac{1}{[3]} \begin{pmatrix} [2]^{-1}q^4B_j^2B_i - q^2B_jB_iB_j \\ +[2]^{-1}B_iB_j^2 + q[3][2]^{-1}B_i \end{pmatrix} B_j \\
&\quad - \frac{1}{q[3]}B_j \begin{pmatrix} [2]^{-1}q^4B_j^2B_i - q^2B_jB_iB_j \\ +[2]^{-1}B_iB_j^2 + q[3][2]^{-1}B_i \end{pmatrix} + \frac{[2]}{q[3]}\big(B_iB_j - q^3B_jB_i\big) \\
&= \frac{1}{[3]!} \begin{pmatrix} (B_iB_j - q^3B_jB_i)B_j^2 \\ -[2]B_j(B_iB_j - q^3B_jB_i)B_j \\ +B_j^2(B_iB_j - q^3B_jB_i) \end{pmatrix} - [2]^{-1}(B_jB_i - qB_iB_j) \\
&\quad + \frac{[2]}{q[3]}\big(B_iB_j - q^3B_jB_i\big), \qquad \text{(A.11)}
\end{aligned}$$

and

$$\begin{aligned}
\tau_j\tau_i\tau_j\tau_i\tau_j(B_i) &\overset{(A.11)}{=} \frac{1}{[3]!}\tau_j \begin{pmatrix} (B_iB_j - q^3B_jB_i)B_j^2 \\ -[2]B_j(B_iB_j - q^3B_jB_i)B_j \\ +B_j^2(B_iB_j - q^3B_jB_i) \end{pmatrix} - [2]^{-1}\tau_j(B_jB_i - qB_iB_j) \\
&\quad + \frac{[2]}{q[3]}\tau_j\big(B_iB_j - q^3B_jB_i\big) \\
&= \frac{1}{[3]!}\big(DB_j^2 - [2]B_jDB_j + B_j^2D\big) - [2]^{-1}B_j\tau_j(B_i) + q[2]^{-1}\tau_j(B_i)B_j + \frac{[2]}{q[3]}D \\
&\overset{(A.8)}{=} \frac{1}{[3]}\big(B_jB_i - q^3B_iB_j\big)B_j - \frac{1}{q[3]}B_j\big(B_jB_i - q^3B_iB_j\big) + \frac{[2]}{q[3]}D \\
&\overset{(A.7)}{=} B_i.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\tau_j\tau_i\tau_j\tau_i(B_j) &\overset{(A.10)}{=} \tau_j\big([2]^{-1}q^4B_j^2B_i - q^2B_jB_iB_j + [2]^{-1}B_iB_j^2 + q[3][2]^{-1}B_i\big) \\
&= [2]^{-1}q^4B_j^2\left(\frac{1}{[3]!} \begin{pmatrix} B_j^3B_i - q[3]B_j^2B_iB_j + q^2[3]B_jB_iB_j^2 \\ -q^3B_iB_j^3 + q^{-1}(B_jB_i - q^3B_iB_j) \end{pmatrix} + (B_jB_i - qB_iB_j)\right) \\
&\quad - q^2B_j\left(\frac{1}{[3]!} \begin{pmatrix} B_j^3B_i - q[3]B_j^2B_iB_j + q^2[3]B_jB_iB_j^2 \\ -q^3B_iB_j^3 + q^{-1}(B_jB_i - q^3B_iB_j) \end{pmatrix} + (B_jB_i - qB_iB_j)\right)B_j
\end{aligned}$$

$$
\begin{aligned}
&+[2]^{-1}\left(\frac{1}{[3]!}\begin{pmatrix} B_j^3B_i-q[3]B_j^2B_iB_j+q^2[3]B_jB_iB_j^2 \\ -q^3B_iB_j^3+q^{-1}(B_jB_i-q^3B_iB_j)\end{pmatrix}+(B_jB_i-qB_iB_j)\right)B_j^2 \\
&+q[3][2]^{-1}\left(\frac{1}{[3]!}\begin{pmatrix} B_j^3B_i-q[3]B_j^2B_iB_j+q^2[3]B_jB_iB_j^2 \\ -q^3B_iB_j^3+q^{-1}(B_jB_i-q^3B_iB_j)\end{pmatrix}+(B_jB_i-qB_iB_j)\right) \\
\overset{(3.1)}{\underset{(3.24)}{=}}\ &\frac{-q^3}{[3]![2]}\big(-q^{-1}\big([3]^2+1\big)\big(B_j^2B_i-B_iB_j^2\big) \\
&+q^{-1}[2]\big([2][4]+q^2+q^{-2}\big)B_jB_iB_j-q^{-2}[3]^2B_i\big)B_j \\
&+\frac{q^4}{[3]![2]}B_j\big(-q^{-1}\big([3]^2+1\big)\big(B_j^2B_i-B_iB_j^2\big) \\
&+q^{-1}[2]\big([2][4]+q^2+q^{-2}\big)B_jB_iB_j-q^{-2}[3]^2B_i\big) \\
&+[2]^{-1}q^4B_j^2\left(\frac{q^{-1}}{[3]!}\big(B_jB_i-q^3B_iB_j\big)+(B_jB_i-qB_iB_j)\right) \\
&-q^2B_j\left(\frac{q^{-1}}{[3]!}\big(B_jB_i-q^3B_iB_j\big)+(B_jB_i-qB_iB_j)\right)B_j \\
&+[2]^{-1}\left(\frac{q^{-1}}{[3]!}\big(B_jB_i-q^3B_iB_j\big)+(B_jB_i-qB_iB_j)\right)B_j^2 \\
&+q[3][2]^{-1}\left(\frac{1}{[3]!}\begin{pmatrix} B_j^3B_i-q[3]B_j^2B_iB_j+q^2[3]B_jB_iB_j^2 \\ -q^3B_iB_j^3+q^{-1}(B_jB_i-q^3B_iB_j)\end{pmatrix}+(B_jB_i-qB_iB_j)\right) \\
=\ &B_jB_i-q^3B_iB_j,
\end{aligned}
\tag{A.12}
$$

where the last equality is obtained by combining like terms.

Finally, one has

$$
\begin{aligned}
\tau_i\tau_j\tau_i\tau_j\tau_i(B_j) &\overset{(A.12)}{=} \tau_i\big(B_jB_i-q^3B_iB_j\big) \\
&= \big(B_iB_j-q^3B_jB_i\big)B_i-q^3B_i\big(B_iB_j-q^3B_jB_i\big) \\
&= -q^3\big(B_i^2B_j-\big(q^3+q^{-3}\big)B_iB_jB_i+B_jB_i^2\big) \\
&\overset{(3.1)}{\underset{(3.24)}{=}} -q^3\big(-q^{-3}B_j\big) \\
&= B_j.
\end{aligned}
$$

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