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The degenerate affine walled Brauer algebra

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ABSTRACT

We define a degenerate affine version of the walled Brauer algebra, that has the same role played by the degenerate affine Hecke algebra for the symmetric group algebra. We use it to prove a higher level mixed Schur–Weyl duality for $\mathfrak{gl}_N$. We consider then families of cyclotomic quotients of level two which appear naturally in Lie theory and we prove that they inherit from there a natural grading and a graded cellular structure.

1. Introduction

The classical Schur–Weyl duality [21] is a basic but very important result in representation theory that connects the general Lie algebra $\mathfrak{gl}_N$ of $N \times N$ matrices with the symmetric group $\mathbb{S}_r$. If $V \cong \mathbb{C}^N$ is the vector representation of $\mathfrak{gl}_N$, Schur–Weyl duality states that the natural action of $\mathbb{S}_r$ on $V^{\otimes r}$ permuting the tensor factors is $\mathfrak{gl}_N$-equivariant, and hence induces a map

$$\Phi : \mathbb{C}[\mathbb{S}_r] \longrightarrow \operatorname{End}_{\mathfrak{gl}_N}\left(V^{\otimes r}\right), \tag{1.1}$$

which is always surjective, and is injective if and only if $N \geq r$.

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Many generalizations and variations of Schur–Weyl duality, in which one replaces for example the Lie algebra $\mathfrak{gl}_N$ with some other reductive Lie algebra and/or the representations $V^{\otimes r}$ with other representations, have been studied in the past years. We would like to recall three of them (as a general reference, see [13, §4.2]).

First, there is a version connecting the semisimple Lie algebra $\mathfrak{g}$ of type B, D (or C) and its vector representation W with the Brauer algebra $\mathrm{B}_r(N)$ (or $\mathrm{B}_r(-N)$) [4]:

$$\mathfrak{g} \circlearrowright W^{\otimes r} \circlearrowleft \mathrm{B}_r(\pm N). \tag{1.2}$$

Second, there is a mixed version of (1.1) in which one considers mixed tensor products of the vector representation V and its dual:

$$\mathfrak{gl}_N \circlearrowright \left(V^{\otimes r} \otimes V^{*\otimes t}\right) \circlearrowleft \mathrm{Br}_{r,t}(N). \tag{1.3}$$

Here $\mathrm{Br}_{r,t}(N)$ is the walled Brauer algebra, a subalgebra of the Brauer algebra $\mathrm{B}_{r+t}(N)$ which was introduced independently by Turaev [20] and Koike [15].

Finally, there is a higher version of Schur–Weyl duality (cf. [1]) in which one considers the tensor product of the representation $V^{\otimes r}$ with some (possibly) infinite dimensional module $M \in \mathcal{O}(\mathfrak{gl}_N)$:

$$\mathfrak{gl}_N \circlearrowright \left(M \otimes V^{\otimes r}\right) \circlearrowleft H_r. \tag{1.4}$$

Here H_r is the degenerate affine Hecke algebra of $\mathbb{S}_r$.

The goal of the present paper is to define a degenerate affine version of the walled Brauer algebra $\mathrm{Br}_{r,t}(N)$ in such a way that we get a higher version of mixed Schur–Weyl duality that generalizes both (1.3) and (1.4):

$$\mathfrak{gl}_N \circlearrowright \left(M \otimes V^{\otimes r} \otimes V^{*\otimes t}\right) \circlearrowleft \mathbb{V}\mathsf{Br}_{r,t}(\boldsymbol{\omega}). \tag{1.5}$$

For technical reasons, we will actually need M to be a highest weight module; the parameter $\boldsymbol{\omega}$ is a sequence $(\omega_i)_{i \in \mathbb{N}}$ of complex numbers which depend on M.

The passage $V^{\otimes r} \rightsquigarrow V^{\otimes r} \otimes V^{*\otimes t}$ from (1.4) to (1.5) is quite natural and is motivated for example by the following reason. Brundan and Kleshchev [6] constructed an explicit isomorphism between cyclotomic quotients of the degenerate affine Hecke algebra H_r and the KLR algebra R; but the KLR algebra R is in some sense only one half of the KLR 2-category $\dot{\mathcal{U}}$ [14]. The degenerate affine walled Brauer algebra $\mathbb{V}\mathsf{Br}_{r,t}(\boldsymbol{\omega})$ should correspond to the whole $\dot{\mathcal{U}}$ (see also (1.9) below).

We remark that in (1.3) we may permute the V's and the V^*'s. In particular, we have a version of (1.3) for each (r,t)-sequence $A = (a_1, \ldots, a_{r+t})$, by which we mean a permutation of the sequence

$$(\underbrace{1, \ldots, 1}_{r}, \underbrace{-1, \ldots, -1}_{t}). \tag{1.6}$$

If in (1.3) we replace $V^{\otimes r} \otimes V^{*\otimes t}$ by $V^{\otimes A} = V^{a_1} \otimes \cdots \otimes V^{a_{r+t}}$ with the convention $V^1 = V$ and $V^{-1} = V^*$, then the walled Brauer algebra $\mathrm{Br}_A(N)$ which acts on the right is of course isomorphic to $\mathrm{Br}_{r,t}(N)$, but has a different natural presentation. If moreover we consider all these sequences A together, then it becomes natural to replace the walled Brauer algebra with the *walled Brauer category* $\underline{\mathrm{Br}}_{r,t}(N)$. In our setting, it is easier to define directly the *degenerate affine walled Brauer category* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ and then get the degenerate affine walled Brauer algebras $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ as endomorphism algebras inside our category $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$.

Our first main theorem is:

Theorem. *(See Theorem 5.2.) Let M be a highest weight module for $\mathfrak{gl}_N$ and A be an (r,t)-sequence. Then there is a natural action of the algebra $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ on $M \otimes V^{\otimes A}$, which commutes with the action of $\mathfrak{gl}_N$. Here the parameter $\boldsymbol{\omega} = \boldsymbol{\omega}(M)$ is determined by the highest weight of the module M.*

The action given by the theorem is far from being faithful. To get a faithful action we need to consider cyclotomic walled Brauer algebras as endomorphism algebras inside a cyclotomic quotient of the category $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$. In particular, we will study cyclotomic quotients of level two. Our second main result is:

Theorem. *(See Theorem 6.9.) Let $m, n, N, r, t \in \mathbb{N}$ with $m + n = N$ and $m, n \geq r + t$. Let $\mathfrak{p} \subset \mathfrak{gl}_N$ be the standard parabolic subalgebra of $\mathfrak{gl}_N$ corresponding to the two-blocks Levi $\mathfrak{gl}_m \oplus \mathfrak{gl}_n \subset \mathfrak{gl}_N$. For $\delta \in \mathbb{Z}$ with $\delta \neq m, n$ let $M^{\mathfrak{p}}(\underline{\delta})$ be the parabolic Verma module in $\mathcal{O}^{\mathfrak{p}}(\mathfrak{gl}_N)$ with highest weight $\underline{\delta} = -\delta(\varepsilon_1 + \cdots + \varepsilon_m)$. Then the action from above factors through some cyclotomic quotient $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*)$, and we have an isomorphism of algebras*

$$\mathbb{V}\mathsf{Br}_A\big(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*\big) \cong \mathrm{End}_{\mathfrak{gl}_{n+m}}\big(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}\big). \tag{1.7}$$

The parameters $\boldsymbol{\omega}, \beta_1, \beta_2, \beta_1^*, \beta_2^*$ depend explicitly on δ, m, n.

As a direct corollary of the second theorem, we have that the cyclotomic quotient $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*)$ inherits a natural grading and the structure of a graded cellular algebra. Note that the grading, however, depends on the chosen ordering of the factors, that is on the (r,t)-sequence A.

Our definition of the algebra $\mathbb{V}\mathsf{Br}_{r,t}(\boldsymbol{\omega})$ is inspired by Nazarov's affine Wenzl algebra $\mathbb{V}_r(\boldsymbol{\omega})$ [18], which can be thought as a degenerate affine version of the Brauer algebra $\mathrm{B}_r(N)$. In particular, $\mathbb{V}\mathsf{Br}_{r,t}(\boldsymbol{\omega})$ is generated by the standard generators of the walled Brauer algebra $\mathrm{Br}_{r,t}(\omega_0)$ together with a polynomial ring $\mathbb{C}[y_1, \ldots, y_{r+t}]$. As in [18] we use formal power series to handle the parameters ω_k.

Our result has an analogue for types B, C and D of the form

$$\mathfrak{g} \circlearrowright \big(M \otimes V^{\otimes r}\big) \circlearrowleft \mathbb{V}_r(\boldsymbol{\omega}) \tag{1.8}$$

established in [12], see also [10]; the methods we use and our computations are similar to the ones of these two papers. Note however that although $\mathrm{Br}_{r,t}(N)$ is a subalgebra of $\mathrm{B}_{r+t}(N)$, the degenerate affine version $\mathbb{V}\mathsf{Br}_{r,t}(\boldsymbol{\omega})$ is not (at least in a natural way) a subalgebra of $\mathbb{V}_{r+t}(\boldsymbol{\omega})$. This indicates that there is a close, but non-trivial relationship between type A and type B, C, D Lie algebras.

Using the diagrammatic description of category $\mathcal{O}^{\mathfrak{p}}$ from [7] one can show that the KLR 2-category $\dot{\mathcal{U}}$ [14] acts on $\mathcal{O}^{\mathfrak{p}}$ and moreover there is an isomorphism of algebras

$$\mathrm{End}_{\dot{\mathcal{U}}}(\mathcal{F}\mathbf{1}_\lambda)/I \cong \mathbb{V}\mathsf{Br}_A\big(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*\big), \tag{1.9}$$

where $\mathcal{F}$ is a 1-morphism in $\dot{\mathcal{U}}$ corresponding to the chosen sequence A, the weight λ is determined by the highest weight of the module $M^{\mathfrak{p}}(\underline{\delta})$ of (1.7) and the ideal I indicates that we have to take some cyclotomic quotient, determined also by the highest weight $\underline{\delta}$ or equivalently by the parameters $\beta_1, \beta_2, \beta_1^*, \beta_2^*$. This result will appear as part of joint work with Ehrig and Stroppel.

We point out that degenerate affine walled Brauer algebras have been defined independently by Rui and Su [19]. They prove another version of Schur–Weyl duality involving the Lie superalgebra $\mathfrak{gl}(m|n)$ and their degenerate affine walled Brauer algebra. Our two approaches are connected via the super duality of [9] established in [8] relating $\mathfrak{gl}(m|n)$ with the classical Lie algebra $\mathfrak{gl}_{m+n}$.

Structure of the paper

We define the degenerate affine walled Brauer category and algebras in Section 2. In Section 3 we give a diagrammatic description that allows us to describe a set of generators as a vector space. In Section 4 we compute the center of the degenerate affine walled Brauer category. In Section 5 we define the action on $\mathfrak{gl}_N$-representations and state the first main theorem, whose proof will be presented as a series of lemmas in Section 8. In Section 6 we define cyclotomic quotients and prove our second main result. In Section 7 we will compute explicitly the generalized eigenvalues of the y_i's in the cyclotomic quotients.

2. The degenerate affine walled Brauer category

In this section we will define the degenerate affine walled Brauer category and the degenerate affine walled Brauer algebras. We will indicate by $\mathbb{S}_n$ the symmetric group of permutations of n elements; the simple reflections will be denoted by s_i for $i = 1, \dots, n-1$.

There is an easy correspondence between $\mathbb{C}$-linear categories and $\mathbb{C}$-algebras which we will use in the following and which we now recall.

First, note that a $\mathbb{C}$-algebra A is the same as a $\mathbb{C}$-linear category with only one object. Now, if the algebra A has a (for simplicity finite) set of pairwise orthogonal idempotents

$\mathbf{1}_1, \ldots, \mathbf{1}_n$, then it can be more natural to identify it with a $\mathbb{C}$-linear category $\mathcal{C}$ with n objects $M_1, \ldots, M_n$, one for each idempotent, in such a way that

$$\mathbf{1}_i A \mathbf{1}_j = \mathrm{Hom}_{\mathcal{C}}(M_j, M_i). \tag{2.1}$$

Of course each idempotent truncation $\mathbf{1}_i A \mathbf{1}_i$ is itself an algebra, which corresponds to the full subcategory of $\mathcal{C}$ containing only the object M_i.

Depending from the situation, it can be preferable to use either the algebra or the category language. We will define our main object of study using the language of categories. Anyway, at some point we will need to consider structures and operations which are typical of algebras (like filtrations, gradings, quotients, homomorphisms, generators, central elements). We advise the reader that we will often alternate in considering the degenerate affine walled Brauer category as a category or as an algebra (although we will always call it *category*, because we will leave the term *algebra* for the idempotent truncations).

In the following by an (r,t)-sequence $A = (a_1, \ldots, a_{r+t})$ we will mean a permutation of the sequence

$$(\underbrace{1, \ldots, 1}_{r}, \underbrace{-1, \ldots, -1}_{t}). \tag{2.2}$$

We let $\mathrm{Seq}_{r,t}$ denote the set of (r,t)-sequences.

Definition 2.1. Let $r, t \in \mathbb{N}$ and fix a sequence $\boldsymbol{\omega} = (\omega_k)_{k \in \mathbb{N}}$ of complex parameters. The *degenerate affine walled Brauer category* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ is the category defined as follows. The objects are (r,t)-sequences $A = (a_1, \ldots, a_{r+t}) \in \mathrm{Seq}_{r,t}$. Morphisms are generated as complex vector spaces by the following endomorphisms of A:

$$\begin{aligned} &s_i^{(A)} \quad \text{for all } 1 \leq i \leq r+t-1 \text{ such that } a_i = a_{i+1}, \\ &e_i^{(A)} \quad \text{for all } 1 \leq i \leq r+t-1 \text{ such that } a_i \neq a_{i+1}, \\ &y_i^{(A)} \quad \text{for all } 1 \leq i \leq r+t, \end{aligned} \tag{2.3}$$

and the following two morphisms $A \to A'$

$$\hat{s}_j^{(A)}, \quad \hat{e}_j^{(A)}, \tag{2.4}$$

where $A' = \mathsf{s}_j A$ for some simple reflection $\mathsf{s}_j \in \mathbb{S}_{r+t}$ such that $A' \neq A$ (that is, $a_j \neq a_{j+1}$). We will often omit the superscript A.

We impose the following relations on morphisms (where we use $\dot{s}_i, \dot{e}_i$ to denote both $s_i, \hat{s}_i$ and $e_i, \hat{e}_i$ respectively; the relations are assumed to hold for all possible choices that make sense):

(1) $\dot{s}_i\dot{s}_i = 1$,
(2) (a) $\dot{s}_i\dot{s}_j = \dot{s}_j\dot{s}_i$ for $|i-j| > 1$,
(b) $\dot{s}_i\dot{s}_{i+1}\dot{s}_i = \dot{s}_{i+1}\dot{s}_i\dot{s}_{i+1}$,
(c) $\dot{s}_i y_j = y_j\dot{s}_i$ for $j \neq i, i+1$,
(3) $(e_i^{(A)})^2 = \omega_0 e_i^{(A)}$,
(4) $e_1^{(A)} y_1^k e_1^{(A)} = \omega_k e_1^{(A)}$ for $k \in \mathbb{N}$ if $a_1 = 1, a_2 = -1$,
(5) (a) $\dot{s}_i\dot{e}_j = \dot{e}_j\dot{s}_i$ and $\dot{e}_i\dot{e}_j = \dot{e}_j\dot{e}_i$ for $|i-j| > 1$,
(b) $\dot{e}_i y_j = y_j\dot{e}_i$ for $j \neq i, i+1$,
(c) $y_i y_j = y_j y_i$,
(6) (a) $\hat{s}_i\dot{e}_i = \dot{e}_i = \dot{e}_i\hat{s}_i$,
(b) $\dot{s}_i\dot{e}_{i+1}\dot{e}_i = \dot{s}_{i+1}\dot{e}_i$ and $\dot{e}_i\dot{e}_{i+1}\dot{s}_i = \dot{e}_i\dot{s}_{i+1}$,
(c) $\dot{e}_{i+1}\dot{e}_i\dot{s}_{i+1} = \dot{e}_{i+1}\dot{s}_i$ and $\dot{s}_{i+1}\dot{e}_i\dot{e}_{i+1} = \dot{s}_i\dot{e}_{i+1}$,
(d) $\dot{e}_{i+1}\dot{e}_i\dot{e}_{i+1} = \dot{e}_{i+1}$ and $\dot{e}_i\dot{e}_{i+1}\dot{e}_i = \dot{e}_i$,
(7) (a) $s_i y_i - y_{i+1}s_i = -1$ and $s_i y_{i+1} - y_i s_i = 1$,
(b) $\hat{s}_i y_i - y_{i+1}\hat{s}_i = \hat{e}_i$ and $\hat{s}_i y_{i+1} - y_i\hat{s}_i = -\hat{e}_i$,
(8) (a) $\dot{e}_i(y_i + y_{i+1}) = 0$,
(b) $(y_i + y_{i+1})\dot{e}_i = 0$.

Remark 2.2. Notice that the relation (5b) for $\hat{e}_i$ is implied by the relations (7b), (5c) and (2c). Moreover, the relations (8a)–(8b) for $\hat{e}_i$ are implied by the same relations for e_i together with (6a) and the invertibility of $\hat{s}_i$.

Remark 2.3. Notice that we need to impose the relation (3) only for A, i such that $a_i = 1$, $a_{i+1} = -1$. In fact, consider $A' = \mathsf{s}_i A$; we have $(e_i^{(A')})^2 = e_i^{(A')}e_i^{(A')} = \hat{s}_i e_i^{(A)}\hat{s}_i\hat{s}_i e_i^{(A)}\hat{s}_i = \hat{s}_i(e_i^{(A)})^2\hat{s}_i = \omega_0\hat{s}_i e_i^{(A)}\hat{s}_i = \omega_0 e_i^{(A')}$. In the same way we obtain more generally $\dot{e}_i\dot{e}_i = \omega_0\dot{e}_i$ for all possible choices.

Remark 2.4. One could give a more general definition taking the ω_k's to be formal central parameters. Definition 2.1 would then be a specialized version.

We stress again that we will try to omit the superscript A of the generators of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ as often as possible. The formulas we write then hold for all choices which make sense.

Definition 2.5. Let $A \in \mathrm{Seq}_{r,t}$ and fix a sequence $\boldsymbol{\omega}$ of complex parameters. The *degenerate affine walled Brauer algebra* corresponding to A is

$$\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}) = \mathrm{End}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A). \tag{2.5}$$

Remark 2.6. Notice that $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ is generated by the $s_i^{(A)}, e_i^{(A)}, y_i^{(A)}$ (for all i such that these elements exist). However, these generators satisfy some non-trivial relations that are implied by the relations defining the whole category (that involve also other homomorphisms spaces). Of course, we could try to define the degenerate affine walled Brauer algebra without using Definition 2.1. But then the relations and the proofs that will follow would be much more complicated.

Let for example $A = (1, 1, -1)$, and consider the element $e_2^{(A)} y_2 e_2^{(A)} \in \mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$. Let also $A' = (1, -1, 1)$. Then we have

$$\begin{aligned} e_2^{(A)} y_2 e_2^{(A)} &= e_2^{(A)} y_2 s_1^{(A)} s_1^{(A)} e_2^{(A)} \\ &= e_2^{(A)} s_1^{(A)} y_1 s_1^{(A)} e_2^{(A)} + e_2^{(A)} s_1^{(A)} e_2^{(A)} \\ &= \hat{e}_2^{(A')} e_1^{(A')} \hat{s}_2^{(A)} y_1 \hat{s}_2^{(A')} e_1^{(A')} \hat{e}_2^{(A)} + e_2^{(A)} s_1^{(A)} e_2^{(A)} \\ &= \hat{e}_2^{(A')} e_1^{(A')} y_1 e_1^{(A')} \hat{e}_2^{(A)} + e_2^{(A)} \\ &= \hat{e}_2^{(A')} e_1^{(A')} y_1 e_1^{(A')} \hat{e}_2^{(A)} + e_2^{(A)} \\ &= \omega_1 e_2^{(A)} + e_2^{(A)}. \end{aligned} \tag{2.6}$$

Notice in particular that in order to obtain the relation $e_2^{(A)} y_2 e_2^{(A)} = (1 + \omega_1) e_2^{(A)}$ in $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ we had to use relations involving elements of $\mathrm{Hom}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A, A')$, $\mathrm{Hom}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A', A)$ and $\mathrm{Hom}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A', A')$.

The following result is straightforward:

Lemma 2.7. *All degenerate affine walled Brauer algebras corresponding to (r, t)-sequences are isomorphic.*

Proof. The isomorphisms are given by multiplication with a finite composition of $\dot{s}_i$'s. □

Let us now recall the definition of the walled Brauer category and of the walled Brauer algebras.

Definition 2.8. Let $r, t \in \mathbb{N}$ and fix a complex parameter $\delta \in \mathbb{C}$. The *walled Brauer category* $\underline{\mathrm{Br}}_{r,t}(\delta)$ is the category which has as objects the (r, t)-sequences $A \in \mathrm{Seq}_{r,t}$. Morphisms are generated by the $s_i^{(A)}, e_i^{(A)}, \hat{s}_i^{(A)}, \hat{e}_i^{(A)}$ as in (2.3) and (2.4) subject to the relations (1), (2a)–(2b), (3), (5a) and (6).

For a fixed (r, t)-sequence $A \in \mathrm{Seq}_{r,t}$, the corresponding *walled Brauer algebra* is

$$\mathrm{Br}_A(\delta) = \mathrm{End}_{\underline{\mathrm{Br}}_{r,t}(\delta)}(A). \tag{2.7}$$

In particular, if A is the standard (r, t)-sequence (2.2) then we set $\mathrm{Br}_A(\delta) = \mathrm{Br}_{r,t}(\delta)$. This is the usual walled Brauer algebra. Notice that, as in Lemma 2.7, all walled Brauer algebras (2.7) corresponding to different permutations of A are isomorphic.

As a direct consequence of the definitions we have a homomorphism

$$\iota : \underline{\mathrm{Br}}_{r,t}(\omega_0) \to \underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega}) \tag{2.8}$$

from the walled Brauer category of parameter ω_0 to the degenerate affine walled Brauer category.

Notice that all relations defining the degenerate affine walled Brauer category are symmetric in i except for (4). We will spend the rest of this section to compute $e_i^{(A)} y_i^k e_i^{(A)}$ for all indices i. The following lemma generalizes the relations (7a)–(7b) to higher powers of y_i:

Lemma 2.9. *For all $k \in \mathbb{N}$ we have*

$$s_i y_i^k - y_{i+1}^k s_i = -\sum_{\ell=1}^{k} y_{i+1}^{\ell-1} y_i^{k-\ell}, \tag{2.9}$$

$$s_i y_{i+1}^k - y_i^k s_i = \sum_{\ell=1}^{k} y_{i+1}^{\ell-1} y_i^{k-\ell}, \tag{2.10}$$

$$\hat{s}_i y_i^k - y_{i+1}^k \hat{s}_i = \sum_{\ell=1}^{k} y_{i+1}^{\ell-1} \hat{e}_i y_i^{k-\ell}, \tag{2.11}$$

$$\hat{s}_i y_{i+1}^k - y_i^k \hat{s}_i = -\sum_{\ell=1}^{k} y_i^{\ell-1} \hat{e}_i y_{i+1}^{k-\ell}. \tag{2.12}$$

Proof. Argue by induction, as in [2, Lemma 2.3]. □

Following [18], we introduce formal power series. Let u be a formal variable, and let us work with formal Laurent power series in u^{-1}. Then from (2.9) we have

$$\begin{aligned} s_i \frac{1}{u - y_i} &= u^{-1} s_i \left(\sum_{k=0}^{\infty} y_i^k u^{-k} \right) \\ &= u^{-1} \sum_{k=0}^{\infty} \left(y_{i+1}^k s_i - \sum_{\ell=1}^{k} y_{i+1}^{\ell-1} y_i^{k-\ell} \right) u^{-k} \\ &= \frac{1}{u - y_{i+1}} s_i - \frac{1}{(u - y_{i+1})(u - y_i)}. \end{aligned} \tag{2.13}$$

Similarly, from (2.10), (2.11) and (2.12) we get respectively:

$$s_i \frac{1}{u - y_{i+1}} = \frac{1}{u - y_i} s_i + \frac{1}{(u - y_{i+1})(u - y_i)}, \tag{2.14}$$

$$\hat{s}_i \frac{1}{u - y_i} = \frac{1}{u - y_{i+1}} \hat{s}_i + \frac{1}{u - y_{i+1}} \hat{e}_i \frac{1}{u - y_i}, \tag{2.15}$$

$$\hat{s}_i \frac{1}{u - y_{i+1}} = \frac{1}{u - y_i} \hat{s}_i - \frac{1}{u - y_i} \hat{e}_i \frac{1}{u - y_{i+1}}. \tag{2.16}$$

Set now

$$W_1(u) = \sum_{k=0}^{\infty} \omega_k u^{-k}. \tag{2.17}$$

Choose $A \in \mathrm{Seq}_{r,t}$ with $a_1 = 1$, $a_2 = -1$. Then relation (4) can be rewritten in the compact form

$$\begin{aligned} e_1^{(A)} \frac{1}{u-y_1} e_1^{(A)} &= u^{-1} e_1^{(A)} \left(\sum_{k=0}^{\infty} y_1^k u^{-k} \right) e_1^{(A)} \\ &= u^{-1} \left(\sum_{k=0}^{\infty} \omega_k u^{-k} \right) e_1^{(A)} = \frac{W_1(u)}{u} e_1^{(A)}. \end{aligned} \tag{2.18}$$

We remark that $\frac{W_1(u)}{u} = u^{-1} W_1(u) \in \mathbb{C}[\![u^{-1}]\!]$.

Lemma 2.10. *Let $A' \in \mathrm{Seq}_{r,t}$ be such that $a'_1 = -1$, $a'_2 = 1$. Then*

$$e_1^{(A')} \frac{1}{u-y_1} e_1^{(A')} = \frac{W_1^*(u)}{u} e_1^{(A')} \tag{2.19}$$

where

$$\frac{W_1^*(u)}{u} = \frac{W_1(-u)}{u - W_1(-u)}. \tag{2.20}$$

Proof. Let $A = \mathsf{s}_1 A'$ and compute using (2.15):

$$\begin{aligned} e_1^{(A')} \frac{1}{u-y_1} e_1^{(A')} &= \hat{s}_1 e_1^{(A)} \hat{s}_1 \frac{1}{u-y_1} \hat{s}_1 e_1^{(A)} \hat{s}_1 \\ &= \hat{s}_1 e_1^{(A)} \frac{1}{u-y_2} \hat{s}_1 \hat{s}_1 e_1^{(A)} \hat{s}_1 + \hat{s}_1 e_1^{(A)} \frac{1}{u-y_2} \hat{e}_1^{(A')} \frac{1}{u-y_1} \hat{s}_1 e_1^{(A)} \hat{s}_1 \\ &= \hat{s}_1 e_1^{(A)} \frac{1}{u+y_1} e_1^{(A)} \hat{s}_1 + \hat{s}_1 e_1^{(A)} \frac{1}{u+y_1} e_1^{(A)} \hat{s}_1 \frac{1}{u-y_1} \hat{s}_1 e_1^{(A)} \hat{s}_1 \\ &= \frac{W_1(-u)}{u} \hat{s}_1 e_1^{(A)} \hat{s}_1 + \frac{W_1(-u)}{u} \hat{s}_1 e_1^{(A)} \hat{s}_1 \frac{1}{u-y_1} \hat{s}_1 e_1^{(A)} \hat{s}_1 \\ &= \frac{W_1(-u)}{u} e_1^{(A')} + \frac{W_1(-u)}{u} e_1^{(A')} \frac{1}{u-y_1} e_1^{(A')}. \end{aligned} \tag{2.21}$$

The claim follows. □

Remark 2.11. Notice that the map $* : \mathbb{C}[\![u^{-1}]\!] \to \mathbb{C}[\![u^{-1}]\!]$ defined by

$$f^*(u) = \frac{f(-u)}{1 - u^{-1} f(-u)} \tag{2.22}$$

is an involution, that is $f^{**}(u) = f(u)$ for all $u \in \mathbb{C}[\![u^{-1}]\!]$.

Let $R = \mathbb{C}[y_1, \ldots, y_{r+t}]$. Then the same proof as for Lemma 2.10 gives the following more general result:

Lemma 2.12. *Let $A \in \mathrm{Seq}_{r,t}$ and suppose $a_i \neq a_{i+1}$ for some index i. Suppose that for $A' = \mathsf{s}_i A$ the following holds:*

$$e_i^{(A')} \frac{1}{u - y_i} e_i^{(A')} = \frac{W_i^{(A')}(u)}{u} e_i^{(A')}, \tag{2.23}$$

for some $W_i^{(A')} \in R[\![u^{-1}]\!]$. Then

$$e_i^{(A)} \frac{1}{u - y_i} e_i^{(A)} = \frac{W_i^{(A)}(u)}{u} e_i^{(A)}, \tag{2.24}$$

where

$$\frac{W_i^{(A)}(u)}{u} = \frac{W_i^{(A')}(-u)}{u - W_i^{(A')}(-u)}. \tag{2.25}$$

Lemma 2.13. *Let $A \in \mathrm{Seq}_{r,t}$ with either $(a_i, a_{i+1}, a_{i+2}) = (1, 1, -1)$ or $(a_i, a_{i+1}, a_{i+2}) = (-1, -1, 1)$ for some index i. Suppose that for $A' = \mathsf{s}_{i+1} A$ the following holds:*

$$e_i^{(A')} \frac{1}{u - y_i} e_i^{(A')} = \frac{W_i^{(A')}(u)}{u} e_i^{(A')}, \tag{2.26}$$

for some $W_i^{(A')} \in R[\![u^{-1}]\!]$. Then

$$e_{i+1}^{(A)} \frac{1}{u - y_{i+1}} e_{i+1}^{(A)} = \frac{W_{i+1}^{(A)}(u)}{u} e_{i+1}^{(A)}, \tag{2.27}$$

where $W_{i+1}^{(A)}(u)$ is determined by

$$\frac{W_{i+1}^{(A)}(u) + u}{W_i^{(A')}(u) + u} = \frac{(u - y_i)^2}{(u - y_i)^2 - 1}. \tag{2.28}$$

Proof. The proof is a direct calculation. First, we use $s_i^2 = 1$ and (2.14):

$$\begin{aligned} e_{i+1} \frac{1}{u - y_{i+1}} e_{i+1} &= e_{i+1} s_i s_i \frac{1}{u - y_{i+1}} e_{i+1} \\ &= e_{i+1} s_i \frac{1}{u - y_i} s_i e_{i+1} + e_{i+1} s_i \frac{1}{(u - y_{i+1})(u - y_i)} e_{i+1}. \end{aligned} \tag{2.29}$$

Since $e_{i+1} = s_i \hat{s}_{i+1} e_i \hat{s}_{i+1} s_i$, and since y_i commutes with $\hat{s}_{i+1}$, we can rewrite the first summand as

$$s_i \hat{s}_{i+1} e_i \frac{1}{u-y_i} e_i \hat{s}_{i+1} s_i = \frac{W_i(u)}{u} e_{i+1}. \tag{2.30}$$

For the second summand of (2.29), since $y_{i+1}e_{i+1} = -y_{i+2}e_{i+1}$ and y_{i+2} commutes with s_i, we can write

$$e_{i+1} s_i \frac{1}{(u-y_{i+1})(u-y_i)} e_{i+1} = e_{i+1} \frac{1}{u-y_{i+1}} s_i \frac{1}{u-y_i} e_{i+1} \tag{2.31}$$

and using (2.13) we get

$$e_{i+1} s_i \frac{1}{(u-y_i)^2} e_{i+1} + e_{i+1} \frac{1}{(u-y_{i+1})(u-y_i)^2} e_{i+1}, \tag{2.32}$$

that by the commutativity of y_i and e_{i+1} is

$$\frac{1}{(u-y_i)^2} e_{i+1} + \frac{1}{(u-y_i)^2} e_{i+1} \frac{1}{u-y_{i+1}} e_{i+1}. \tag{2.33}$$

Putting all together, we obtain

$$\left(1 - \frac{1}{(u-y_i)^2}\right) e_{i+1} \frac{1}{u-y_{i+1}} e_{i+1} = \left(\frac{W_i(u)}{u} + \frac{1}{(u-y_i)^2}\right) e_{i+1}, \tag{2.34}$$

that is equivalent to our claim. □

We have then the following generalization of relation (4):

Proposition 2.14. *Let $A \in \mathrm{Seq}_{r,t}$ and let i be an index such that $a_i \neq a_{i+1}$. Then we have*

$$e_i^{(A)} \frac{1}{u-y_i} e_i^{(A)} = \frac{W_i^{(A)}(u)}{u} e_i^{(A)}, \tag{2.35}$$

for some power series $W_i^{(A)}(u) \in \mathbb{C}[y_1, \dots, y_{i-1}][\![u^{-1}]\!]$ which can be determined recursively using (2.28) and (2.25). Moreover, the power series $W_i^{(A)}(u)$ depends only on $(a_1, \dots, a_i)$, that is $W_i^{(A)} = W_i^{(A')}$ if the sequences A and A' coincide up to the index i.

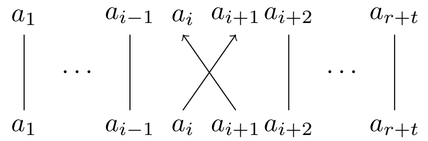
Proof. We prove by induction on i that (2.35) holds for all (r,t)-sequences such that $a_i \neq a_{i+1}$. If $i = 1$ then this follows from (2.18) or (2.19). Now let us suppose that the claim holds for i and consider $i+1$. If $a_i = a_{i+1}$ then we can apply Lemma 2.13. Otherwise we can apply Lemma 2.13 and get the claim for $\mathsf{s}_{i+1}A$, and then deduce the result for A using Lemma 2.12. □

3. A diagrammatic description

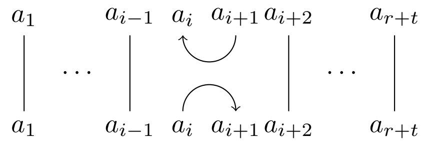
We give now a graphical description of the degenerate affine walled Brauer category, that we will use to describe a set of generators as a vector space.

To $A \in \mathrm{Seq}_{r,t}$ we assign a sequence of $r+t$ oriented points on a horizontal line, numbered from 1 to $r+t$ from left to right. Each point can be oriented upwards or downwards: the point i is oriented upwards if $a_i = 1$ and downwards otherwise.

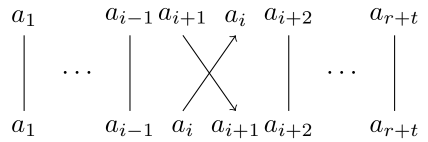
Given $A, A' \in \mathrm{Seq}_{r,t}$, a morphism $\varphi \in \mathrm{Hom}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A, A')$ is a $\mathbb{C}$-linear combination of strand diagrams that connect the point sequence corresponding to A to the point sequence corresponding to A'. In each strand diagram the strands are oriented according to the orientations of the endpoints. The generating morphisms are:

$s_i^{(A)} =$ 

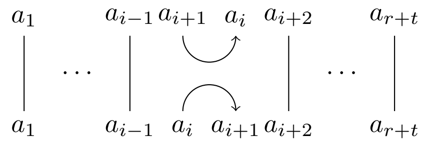
(3.1)

$e_i^{(A)} =$ 

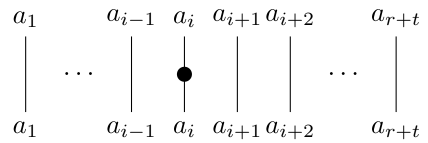
(3.2)

$\hat{s}_i^{(A)} =$ 

(3.3)

$\hat{e}_i^{(A)} =$ 

(3.4)

$y_i^{(A)} =$ 

(3.5)

We did not draw the orientations of all the strands, but as we said they are supposed to be oriented according to the (r,t)-sequences. Moreover, for each of the first four generators we have only depicted the case $a_i = 1$; in the case $a_i = -1$ the orientations are swapped.

Composition of morphisms is obtained by stacking diagram vertically, from the bottom to the top. We will say that a strand is *decorated* if there is at least one dot on it.

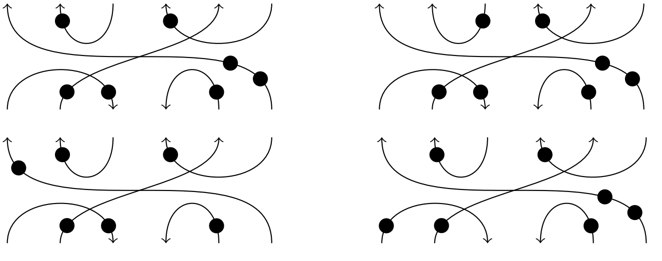
Let us now translate the defining relations of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ into diagrams. Relations (1), (2a)–(2b), (5a) and (6a)–(6d) allow us exactly to stretch undecorated strands. By relations (2c) and (5b)–(5c), dots can be moved vertically as long as they do not step over crossings; by relations (8a)–(8b), if they step over a maximum or minimum, a sign appears. By relation (3) we can remove each closed undecorated circle and multiply by ω_0. Using Proposition 2.14 we can actually remove every decorated circle on the cost of multiplying by the corresponding term of the power series $W_i^{(A)}(u)$. The only relations we have not yet considered are (7a)–(7b), which we can use to make dots step over crossings.

We will call a diagram representing a morphism $A \to B$ an *AB-Brauer diagram*, or an A-Brauer diagram if $A = B$. It is *undecorated* if there are no dots on it, and it is actually an element of the corresponding walled Brauer algebra. An AB-Brauer diagram is called a *monomial* if it is of the type

$$y_1^{\gamma_1} \cdots y_{r+t}^{\gamma_{r+t}} D y_1^{\eta_1} \cdots y_{r+t}^{\eta_{r+t}}, \tag{3.6}$$

where D is an undecorated AB-Brauer diagram. We say that such a monomial is *regular* if $\eta_i = 0$ whenever the point a_i on the bottom of D is the left endpoint of a horizontal arc, and $\gamma_i \neq 0$ implies that the point b_i on the top of D is the left endpoint of a horizontal arc.

Example 3.1. Of the following four AB-Brauer diagrams representing monomials, only the first one is regular.



On $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ we define a filtration (as an algebra)

$$\{0\} = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \tag{3.7}$$

by letting V_i be the vector span of all strand diagrams with at most i dots. As always, given a filtered algebra we can consider the associated graded algebra. Let us denote by $\underline{\mathbb{V}\mathsf{Br}}_{r,t}^{\mathrm{grad}}(\boldsymbol{\omega})$ the associated graded category. Let moreover G be the category with the same generators and the same relations of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ except for (7a)–(7b), which are replaced by

(7′) $\dot{s}_i y_i = y_{i+1} \dot{s}_i$ and $\dot{s}_i y_{i+1} = y_i \dot{s}_i$.

Clearly $\underline{\mathbb{V}\mathsf{Br}}_{r,t}^{\mathrm{grad}}(\boldsymbol{\omega})$ is a quotient of G.

Proposition 3.2. *The regular monomials generate* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ *as a vector space over* $\mathbb{C}$.

Proof. It is straightforward to see that the regular monomials generate G, because in G dots can step over crossings thanks to the relation (7′). It follows that they also generate $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$. □

It follows directly from [5, Theorem 1.2] that regular monomials (which are called *normally ordered dotted oriented diagrams* in [5]) actually give a basis of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$:

Theorem 3.3. *(See [5, Theorem 1.2].) For all choices of parameters* $\boldsymbol{\omega}$*, regular monomials give a basis of* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ *over* $\mathbb{C}$.

This also implies that regular monomials give a basis of G, and that $\mathsf{G} \cong \underline{\mathbb{V}\mathsf{Br}}_{r,t}^{\mathrm{grad}}(\boldsymbol{\omega})$. It follows moreover that the map (2.7) is an embedding of algebras.

We will make use of Theorem 3.3 only in Section 4, in order to compute the center of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$. Our main interest in this paper, however, is the action of degenerate affine walled Brauer algebras on modules in category $\mathcal{O}(\mathfrak{gl}_N)$. Using such action, we will prove in Section 6, independently from the results of [5], that *cyclotomic regular monomials* give a basis of cyclotomic quotients of the degenerate affine walled Brauer category.

4. The center

We are now going to determine the center of the degenerate affine walled Brauer category. Our computations are analogous to the ones in [11]. In this section, we will use several times the fact the fact that regular monomials give a basis of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$, see Theorem 3.3.

Recall that the center of a category is by definition the endomorphism ring of its identity endofunctor. If A is a $\mathbb{C}$-algebra with a finite number of pairwise orthogonal idempotents, then its center coincides with the center of the corresponding $\mathbb{C}$-linear category (see the beginning of Section 2).

Let R be the polynomial ring $R = \mathbb{C}[y_1, \dots, y_{r+t}]$. Observe that for each pair $A, B \in \mathrm{Seq}_{r,t}$, the vector space $\mathrm{Hom}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A, B)$ is an R-bimodule.

Definition 4.1. (See also [11].) We say that a polynomial $p \in R$ satisfies *Q-cancellation* with respect to the variables y_1, y_2 if

$$p(y_1, -y_1, y_3, \dots, y_{r+t}) = p(0, 0, y_3, \dots, y_{r+t}). \tag{4.1}$$

Analogously we say that p satisfies Q-cancellation with respect to the variables y_i, y_j if $w \cdot p$ satisfies (4.1), where $w \in \mathbb{S}_{r+t}$ is the permutation that exchanges 1 with i and 2 with j and $\mathbb{S}_{r+t}$ acts on R permuting the variables.

We have then the following:

Theorem 4.2. *The center of* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ *is isomorphic to the subring of* $(\mathbb{S}_r \times \mathbb{S}_t)$*-invariant polynomials* $p \in R^{\mathbb{S}_r \times \mathbb{S}_t}$ *which satisfy* Q*-cancellation with respect to the variables* y_r, y_{r+1}*. The isomorphism is given by the map*

$$p \mapsto \sum_{A \in \mathrm{Seq}_{r,t}} (w_A \cdot p)\mathrm{id}_A, \tag{4.2}$$

where for each $A \in \mathrm{Seq}_{r,t}$ *the element* w_A *is a permutation such that* $w_A \cdot (1^r, (-1)^t) = A$*.*

Proof. Let Z be the center of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$. By definition an element $f \in Z$ is an element

$$f \in \bigoplus_{A \in \mathrm{Seq}_{r,t}} \mathrm{End}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A) \tag{4.3}$$

such that $f\varphi = \varphi f$ for all morphisms $\varphi \in \mathrm{Hom}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A, A')$ and for all pairs $A, A' \in \mathrm{Seq}_{r,t}$.

Let us pick an $A \in \mathrm{Seq}_{r,t}$. As in the proof of [11, Theorem 4.3], it is easy to show using Theorem 3.3 that for such an f to commute with all endomorphisms of A the component of f in $\mathrm{End}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A)$ has to be a polynomial $p_A \in R$. In particular, we must have $f = \sum_{A \in \mathrm{Seq}_{r,t}} p_A \mathrm{id}_A$.

Let us now fix $A_0 = (1^r, (-1)^t)$. Since p_{A_0} has to be central in $\mathbb{V}\mathsf{Br}_{A_0}(\boldsymbol{\omega}) = \mathrm{End}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A_0)$, it follows from Lemmas 4.4 and 4.5 below that $p_{A_0} \in R^{\mathbb{S}_r \times \mathbb{S}_t}$ and p_{A_0} has to satisfy Q-cancellation with respect to y_r, y_{r+1}. On the other side, it follows by the same lemmas and the fact that the s_i's, e_i's and y_i's generate $\mathbb{V}\mathsf{Br}_{A_0}(\boldsymbol{\omega})$ that such a p_{A_0} is central in $\mathbb{V}\mathsf{Br}_{A_0}(\boldsymbol{\omega})$.

Finally, it follows from Lemma 4.6 below that such an f is central in the whole category if and only if $p_A = w_A \cdot p_{A_0}$ for all $A = w_A \cdot A_0$. $\square$

As a corollary of the theorem (and of its proof), we can describe the center of the degenerate affine walled Brauer algebras:

Corollary 4.3. *Let* $A \in \mathrm{Seq}_{r,t}$ *and* $W_A \subset \mathbb{S}_{r+t}$ *be the subgroup that fixes* A*. Let also* i, j *be two indices such that* $a_i \neq a_j$*. Then the center of* $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}) = \mathrm{End}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})}(A)$ *is the image under* $p \mapsto p \cdot \mathrm{id}_A$ *of the polynomials* $p \in R^{W_A}$ *that satisfy* Q*-cancellation with respect to the variables* y_i, y_j*.*

We conclude the section with the technical lemmas which we used for the proof of Theorem 4.2.

Lemma 4.4. *Let* $A \in \mathrm{Seq}_{r,t}$ *with* $a_i = a_{i+1}$*. The polynomial* $p \in R$ *commutes with* s_i *in* $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ *if and only if* $\mathsf{s}_i \cdot p = p$*.*

Proof. For notational convenience let us suppose $i = 1$. Let $p \in R$ and write

$$p = \sum_{a,b \in \mathbb{N}} y_1^a y_2^b p_{a,b} \quad \text{with } p_{a,b} \in \mathbb{C}[x_3, \ldots, x_{r+t}]. \tag{4.4}$$

Using (2.9) and (2.10) we get

$$s_1 y_1^a y_2^b = y_2^a y_1^b s_1 - \frac{y_1^a y_2^b - y_2^a y_1^b}{y_1 - y_2} \tag{4.5}$$

and hence

$$s_1 p = (\mathsf{s}_1 \cdot p) s_1 - \frac{p - \mathsf{s}_1 \cdot p}{y_1 - y_2}. \tag{4.6}$$

It follows, using Theorem 3.3, that p commutes with s_1 if and only if $\mathsf{s}_1 \cdot p = p$. □

Lemma 4.5. *Let $A \in \mathrm{Seq}_{r,t}$ with $a_i \neq a_{i+1}$. The polynomial $p \in R$ commutes with e_i in $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ if and only if it satisfies Q-cancellation with respect to y_i, y_{i+1}.*

Proof. Let us suppose $i = 1$ and write p as in (4.4). We have

$$e_1 p - p e_1 = e_1 \Bigg(\sum_{\substack{a,b \in \mathbb{N} \\ a+b>0}} y_1^a y_2^b p_{a,b} \Bigg) - \Bigg(\sum_{\substack{a,b \in \mathbb{N} \\ a+b>0}} y_1^a y_2^b p_{a,b} \Bigg) e_1. \tag{4.7}$$

It follows from Theorem 3.3 that the two vector subspaces generated by the elements $\{e_1 y_1^a y_2^b p_{a,b} \mid a, b \in \mathbb{N},\ a+b>0\}$ and $\{y_1^a y_2^b p_{a,b} e_1 \mid a, b \in \mathbb{N},\ a+b>0\}$, respectively, have trivial intersection. Hence (4.7) $= 0$ if and only if both summands of the r.h.s. vanish. Now

$$\begin{aligned} e_1 \Bigg(\sum_{\substack{a,b \in \mathbb{N} \\ a+b>0}} y_1^a y_2^b p_{a,b} \Bigg) &= e_1 \Bigg(\sum_{\substack{a,b \in \mathbb{N} \\ a+b>0}} y_1^a (-y_1)^b p_{a,b} \Bigg) \\ &= e_1 \big(p(y_1, -y_1, y_3, \ldots, y_{r+t}) - p(0, 0, y_3, \ldots, y_{r+t}) \big) \end{aligned} \tag{4.8}$$

vanishes if and only if p satisfies Q-cancellation with respect to y_1, y_2, and similarly for the second summand. □

Lemma 4.6. *Let $A \in \mathrm{Seq}_{r,t}$ with $a_i \neq a_{i+1}$. Then there exists a polynomial $q \in R$ such that $\hat{s}_i p = q \hat{s}_i$ if and only if p satisfies Q-cancellation with respect to y_i, y_{i+1}. In this case we have $q = \mathsf{s}_i \cdot p$.*

Proof. Again, let us suppose $i = 1$, and let us write p as in (4.4). Using (2.11) and (2.12) we get

$$\hat{s}_1 y_1^a y_2^b = y_2^a y_1^b \hat{s}_1 + (-1)^a \sum_{\ell=1}^{a+b} (-1)^\ell y_1^{a+b-\ell} \hat{e}_1 y_1^{\ell-1} \tag{4.9}$$

and hence

$$\hat{s}_1 p = (\mathsf{s}_1 \cdot p)\hat{s}_1 + \sum_{k \in \mathbb{Z}_{>0}} \left(\sum_{\ell=1}^{r} (-1)^\ell y_1^{r-\ell} \hat{e}_1 y_1^{\ell-1} \right) \left(\sum_{b=0}^{k} (-1)^{k-b} p_{k-b,b} \right). \tag{4.10}$$

Using Theorem 3.3, the claim follows since $\sum_{b=0}^{k} (-1)^b p_{k-b,k} = 0$ for every $k > 0$ if and only if p satisfies Q-cancellation with respect to y_i, y_{i+1}. $\square$

5. Action on $\mathfrak{gl}_N$-representations

In this section we will define an action of the degenerate affine walled Brauer category on $\mathfrak{gl}_N$-representations.

Let us fix an integer $N \geq 2$ and let $I = \{1, \dots, N\}$. Let $\mathfrak{gl}_N$ be the Lie algebra of $N \times N$ matrices. We denote by E_{ij} the matrix that has a one at position (i,j) and zeroes elsewhere. We let $\mathfrak{h} \subset \mathfrak{gl}_N$ be the standard Cartan subalgebra of diagonal matrices and $\mathfrak{gl}_N = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ the triangular decomposition of $\mathfrak{gl}_N$. A basis of $\mathfrak{h}$ is given by the matrices $H_i = E_{ii}$. Let ε_i be the basis of $\mathfrak{h}^*$ dual to E_{ii}.

The set of *roots* of $\mathfrak{gl}_N$ is $\Pi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$; a root is positive if $i > j$, negative otherwise. The set of positive (resp. negative) roots will be denoted by Π^+ (resp. Π^-). The *simple roots* are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for all $i = 1, \dots, N-1$. We will denote the root vectors by X_{ij} for all $i \neq j$ or X_α for $\alpha \in \Pi$.

On $\mathfrak{gl}_N$ we consider the non-degenerate symmetric bilinear form defined by

$$(A|B) = \operatorname{tr}(AB). \tag{5.1}$$

Notice that the set $\{H_i, X_\alpha \mid i \in I,\ \alpha \in \Pi\}$ gives an orthonormal basis of $\mathfrak{gl}_N$.

The *vector representation* of $\mathfrak{gl}_N$ is the N-dimensional vector space V with basis $\{v_i \mid i \in I\}$ on which the action of $\mathfrak{gl}_N$ is given by

$$X_{ij} v_k = \delta_{jk} v_i, \qquad H_i v_k = \delta_{ik} v_k, \tag{5.2}$$

where δ_{ij} is the Kronecker delta. The *dual vector representation* V^* has basis $\{v_i^* \mid i \in I\}$, and the action of $\mathfrak{gl}_N$ is given explicitly by

$$X_{ij} v_k^* = -\delta_{ik} v_j^*, \qquad H_i v_k^* = -\delta_{ik} v_k^*. \tag{5.3}$$

We have linear homomorphisms $\sigma_{W,Z} : W \otimes Z \to Z \otimes W$, $\tau_W : W \otimes W^* \to W \otimes W^*$, $\hat{\tau}_W : W \otimes W^* \to W^* \otimes W$, where W, Z are either V or V^* and $V^{**} = V$, defined by

$$\sigma_{W,Z} : w_i \otimes z_j \longmapsto z_j \otimes w_i \tag{5.4}$$

$$\tau_W : w_i \otimes w_j^* \longmapsto \delta_{ij} \sum_{k \in I} (w_k \otimes w_k^*) \tag{5.5}$$

$$\hat{\tau}_W : w_i \otimes w_j^* \longmapsto \delta_{ij} \sum_{k \in I} (w_k^* \otimes w_k) \tag{5.6}$$

where w_i, z_i are either v_i or v_i^*, and $v_i^{**} = v_i$. It is immediate to check that these are indeed homomorphisms of $\mathfrak{gl}_N$-representations.

For an (r,t)-sequence A we let $V^{\otimes A} = V^{a_1} \otimes \cdots \otimes V^{a_{r+t}}$, where $V^1 = V$ and $V^{-1} = V^*$. Let also M be a $\mathfrak{gl}_N$-module. The linear homomorphisms (5.4), (5.5) and (5.6) induce the following endomorphisms of $M \otimes V^{\otimes A}$ for all $1 \leq i, j \leq r+t-1$ such that $a_i = a_{i+1}$, $a_j \neq a_{j+1}$:

$$s_i = \mathrm{id} \otimes \mathrm{id}^{\otimes(i-1)} \otimes \sigma_{V^{a_i}, V^{a_{i+1}}} \otimes \mathrm{id}^{\otimes(r+t-i-1)} \tag{5.7}$$

$$e_j = \mathrm{id} \otimes \mathrm{id}^{\otimes(i-1)} \otimes \tau_{V^{a_j}} \otimes \mathrm{id}^{\otimes(r+t-i-1)} \tag{5.8}$$

and the following homomorphisms $M \otimes V^{\otimes A} \to M \otimes V^{\otimes A'}$ where $A' = \mathsf{s}_j A$ for some simple transposition $\mathsf{s}_j \in \mathbb{S}_{r+t}$:

$$\hat{s}_i = \mathrm{id} \otimes \mathrm{id}^{\otimes(i-1)} \otimes \sigma_{V^{a_i}, V^{a_{i+1}}} \otimes \mathrm{id}^{\otimes(r+t-i-1)} \tag{5.9}$$

$$\hat{e}_j = \mathrm{id} \otimes \mathrm{id}^{\otimes(i-1)} \otimes \hat{\tau}_{V^{a_j}} \otimes \mathrm{id}^{\otimes(r+t-i-1)}. \tag{5.10}$$

Let $U(\mathfrak{gl}_N)$ be the universal enveloping algebra of $\mathfrak{gl}_N$. The *Casimir operator* of $U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N)$ is

$$\Omega = \sum_{i \in I} H_i \otimes H_i + \sum_{\alpha \in \Pi} X_\alpha \otimes X_{-\alpha}. \tag{5.11}$$

The *Casimir element* of $U(\mathfrak{gl}_N)$ is $C = m(\Omega)$, where $m : U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ is the multiplication. Writing $\Omega = \sum x_{(1)} \otimes x_{(2)}$, we define for $0 \leq i < j \leq r+t$

$$\Omega_{ij} = \sum 1 \otimes \cdots \otimes 1 \otimes x_{(1)} \otimes 1 \otimes \cdots \otimes 1 \otimes x_{(2)} \otimes 1 \otimes \cdots \otimes 1, \tag{5.12}$$

where $x_{(1)}$ is at position i and $x_{(2)}$ is at position j, starting with position 0. Multiplication by Ω_{ij} defines an element of $\mathrm{End}_{\mathfrak{gl}_N}(M \otimes V^{\otimes A})$, and we set for $1 \leq i \leq r+t$

$$y_i = \sum_{0 \leq k < i} \Omega_{ki} + \frac{N}{2}. \tag{5.13}$$

Lemma 5.1. *Let M be a highest weight module. Let $A \in \mathrm{Seq}_{r,t}$ with $a_1 = 1, a_2 = -1$. For all $k \in \mathbb{N}$ there exist $\omega_k(M) \in \mathbb{C}$ with $\omega_0(M) = N$ such that $e_1 y_1^k e_1 = \omega_k(M) e_1$ as elements in $\mathrm{End}_{\mathfrak{gl}_N}(M \otimes V^{\otimes A})$.*

Proof. Consider the composition

$$f : M = M \otimes \mathbb{C} \longrightarrow M \otimes V \otimes V^* \xrightarrow{y_1^k} M \otimes V \otimes V^* \longrightarrow M \otimes \mathbb{C} = M, \qquad (5.14)$$

where the first map is the canonical inclusion and the last one is the evaluation. Since M is a highest weight module, f must be a multiple, say $\omega_k(M)$, of the identity. It is then clear that $e_1 y_1^k e_1 = \omega_k(M) e_1$ as elements in $\operatorname{End}_{\mathfrak{gl}_N}(M \otimes V \otimes V^*)$. Adding identities on the following tensor factors, the identity holds also for $M \otimes V^{\otimes A}$ as in the statement.

The fact that $\omega_0(M) = N$ follows by elementary direct computation, and is true for every module M. $\square$

We are ready to state our first main theorem; the computations needed for the proof are collected in Section 8.

Theorem 5.2. *Let M be a highest weight module for $\mathfrak{gl}_N$, and let $\boldsymbol{\omega} = (\omega_k(M))_{k \in \mathbb{N}}$ be the sequence of complex numbers given by Lemma 5.1. Then the assignment $A \mapsto M \otimes V^{\otimes A}$ and formulas (5.7), (5.8), (5.9), (5.10), (5.13) define a functor $\underline{\mathbb{V}\mathsf{Br}}_{r+t}(\boldsymbol{\omega}) \to \mathcal{O}(\mathfrak{gl}_N)$. In particular, we have a well-defined action of $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega})$ on $M \otimes V^{\otimes A}$ for every (r,t)-sequence A.*

Proof. We need to show that the relations of the degenerate affine walled Brauer category are satisfied by definitions (5.7), (5.8), (5.9), (5.10), (5.13) for a highest weight module M. The relations are checked in details in Section 8.

Relation (1) is obvious, as are relations (2a)–(2b). Relation (2c) follows from Lemma 8.1. Relation (3) follows because $\omega_0(M) = N$. Relation (4) is implied by Lemma 5.1 and our choice of $\boldsymbol{\omega}$. Relation (5a) is trivial. Relation (5b) follows from Lemma 8.3 and Remark 2.2. Relation (5c) is Lemma 8.5. Relation (6a) is straightforward, while relations (6b)–(6d) are shortly discussed as Lemma 8.6. Relations (7a) are Lemma 8.8, while relations (7b) are Lemma 8.10. $\square$

As a corollary, we have a non-triviality result for $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$, independent from Theorem 3.3:

Corollary 5.3. *Suppose $N \geq r+t$. If ω is chosen as in Lemma 5.1 then the homomorphism ι from (2.8) is injective. In particular, $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ is non-trivial.*

Proof. Composing ι with the action of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ on $M \otimes V^{\otimes A}$ and forgetting the highest weight module M one obtains the standard action of the walled Brauer category $\underline{\mathrm{Br}}_{r,t}(N)$ on $V^{\otimes A}$. It is known (see [3]) that this action is injective if $N \geq r + t$. In particular, ι has to be injective. $\square$

6. Cyclotomic quotients

We will consider in this section cyclotomic quotients of the degenerate affine walled Brauer category of level two.

Fix two positive integers $m, n \in \mathbb{N}$ and let $N = m + n$ and $I = \{1, \dots, m + n\}$. Let $\mathfrak{gl}_{m+n}$ be the complex general linear Lie algebra with its Levi subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$; let $\mathfrak{p} = (\mathfrak{gl}_m \oplus \mathfrak{gl}_n) + \mathfrak{n}^+$ be the corresponding standard parabolic subalgebra.

Let us set

$$\rho = -\varepsilon_2 - 2\varepsilon_3 - \cdots - (m + n - 1)\varepsilon_{m+n}. \tag{6.1}$$

Let $\mathcal{O}(m, n) = \mathcal{O}^{\mathfrak{p}}_{int}(\mathfrak{gl}_{m+n})$ be the category of finitely generated $\mathfrak{gl}_{n+m}$-modules that are locally finite over $\mathfrak{p}$, semisimple over $\mathfrak{h}$, and have all integral weights (when regarded as $\mathfrak{sl}_{m+n}$-modules). This category is studied extensively in [7]. A full set of representatives for the isomorphism classes of irreducible modules in $\mathcal{O}(m, n)$ is given by the modules $\{L(\lambda) \mid \lambda \in \Lambda(m, n)\}$, where

$$\Lambda(m,n) = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \begin{array}{l} (\lambda + \rho, \varepsilon_i - \varepsilon_j) \in \mathbb{Z} \text{ for all } 1 \le i, j \le m + n, \\ (\lambda + \rho, \varepsilon_1) > \cdots > (\lambda + \rho, \varepsilon_m), \\ (\lambda + \rho, \varepsilon_{m+1}) > \cdots > (\lambda + \rho, \varepsilon_{m+n}) \end{array} \right\} \tag{6.2}$$

and $L(\lambda)$ is the irreducible $\mathfrak{gl}_{n+m}$-module of highest weight λ. The module $L(\lambda)$ is the irreducible head of the parabolic Verma module $M^{\mathfrak{p}}(\lambda)$. This parabolic Verma module is also the largest quotient of the (non-parabolic) Verma module $M(\lambda) \in \mathcal{O}(\mathfrak{gl}_{m+n})$ which lies in the parabolic subcategory $\mathcal{O}(m, n)$.

Notice that the weights of the vector representation V are $\varepsilon_1, \dots, \varepsilon_{n+m}$, while the weights of V^* are $-\varepsilon_1, \dots, -\varepsilon_{n+m}$. By the tensor identity, the module $M(\lambda) \otimes V$ (resp. $M(\lambda) \otimes V^*$) has a filtration with sections isomorphic to $M(\lambda + \varepsilon_j)$ (resp. $M(\lambda - \varepsilon_j)$) for all $j \in I$. It follows, by the characterization of the parabolic Verma modules and because tensoring with V and V^* are endofunctors of $\mathcal{O}(m, n)$ (see [7, Lemma 4.3]), that $M^{\mathfrak{p}}(\lambda) \otimes V$ has a filtration with sections isomorphic to $M^{\mathfrak{p}}(\lambda + \varepsilon_j)$ for all j such that $\lambda + \varepsilon_j \in \Lambda(m, n)$; similarly, $M^{\mathfrak{p}}(\lambda) \otimes V^*$ has a filtration with sections isomorphic to $M^{\mathfrak{p}}(\lambda - \varepsilon_j)$ for all j such that $\lambda - \varepsilon_j \in \Lambda(m, n)$.

For $\delta \in \mathbb{Z}$ we set

$$\underline{\delta} = -\delta(\varepsilon_1 + \cdots + \varepsilon_m). \tag{6.3}$$

Lemma 6.1. *Suppose $m, n \ge 1$ and $\delta \neq m$. Then there is an isomorphism of $\mathfrak{gl}_{n+m}$-modules*

$$M^{\mathfrak{p}}(\underline{\delta}) \otimes V \cong M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_1) \oplus M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_{m+1}). \tag{6.4}$$

This is also an eigenspace decomposition for the action of y_1, with eigenvalues

$$\beta_1 = -\delta + \frac{m+n}{2} \quad \textit{and} \quad \beta_2 = \frac{n-m}{2}. \tag{6.5}$$

Proof. By the discussion above and the definition of $\underline{\delta}$, we have that $M^{\mathfrak{p}}(\underline{\delta}) \otimes V$ has a filtration with parabolic Verma modules $M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_1)$ and $M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_{m+1})$. Since $\underline{\delta} + \varepsilon_1 > \underline{\delta} + \varepsilon_{m+1}$, the term $M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_1)$ is a submodule, hence we have

$$0 \to M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_1) \to M^{\mathfrak{p}}(\underline{\delta}) \otimes V \to M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_{m+1}) \to 0. \tag{6.6}$$

Let C be the Casimir element of $\mathfrak{gl}_{m+n}$ as defined in Section 5. By a straightforward computation, C acts as $\langle \lambda, \lambda + 2\rho \rangle$ on the highest vector of $M^{\mathfrak{p}}(\lambda)$, and hence on the whole module. In particular, as V is the irreducible head of $M^{\mathfrak{p}}(\varepsilon_1)$, C act as $\langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle$ on V. If we denote by Δ the comultiplication of $U(\mathfrak{gl}_{m+n})$, note that $\Delta(C) = C \otimes 1 + 1 \otimes C + 2\Omega$. Hence using the action of the Casimir element we can compute the action of Ω on $M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_1)$, that is given by the scalar

$$\begin{aligned}
&\frac{1}{2}\big(\langle \underline{\delta} + \varepsilon_1, \underline{\delta} + \varepsilon_1 + 2\rho \rangle - \langle \underline{\delta}, \underline{\delta} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle\big) \\
&\quad = \frac{1}{2}\big(\langle \underline{\delta}, \underline{\delta} + 2\rho \rangle + \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle + 2\langle \underline{\delta}, \varepsilon_1 \rangle - \langle \underline{\delta}, \underline{\delta} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle\big) \\
&\quad = \langle \underline{\delta}, \varepsilon_1 \rangle = -\delta,
\end{aligned} \tag{6.7}$$

so that the action of y_1 is given by $-\delta + \frac{m+n}{2}$.

Analogously, the action of Ω on $(M^{\mathfrak{p}}(\underline{\delta}) \otimes V)/M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_1) \cong M^{\mathfrak{p}}(\underline{\delta} + \varepsilon_{m+1})$ is given by the scalar

$$\begin{aligned}
&\frac{1}{2}\big(\langle \underline{\delta} + \varepsilon_{m+1}, \underline{\delta} + \varepsilon_{m+1} + 2\rho \rangle - \langle \underline{\delta}, \underline{\delta} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle\big) \\
&\quad = \frac{1}{2}\big(\langle \underline{\delta}, \underline{\delta} + 2\rho \rangle + \langle \varepsilon_{m+1}, \varepsilon_{m+1} + 2\rho \rangle + 2\langle \underline{\delta}, \varepsilon_{m+1} \rangle - \langle \underline{\delta}, \underline{\delta} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle\big) \\
&\quad = \frac{1}{2}\big(1 + \langle \varepsilon_{m+1}, 2\rho \rangle + 2\langle \underline{\delta}, \varepsilon_{m+1} \rangle - 1 - \langle \varepsilon_1, 2\rho \rangle\big) \\
&\quad = \langle \varepsilon_{m+1} - \varepsilon_1, \rho \rangle + \langle \underline{\delta}, \varepsilon_{m+1} \rangle = -m,
\end{aligned} \tag{6.8}$$

and the action of y_1 is given by $-m + \frac{m+n}{2}$.

Since the two factors (6.7) and (6.8) are different, they are indeed eigenvalues for the action of Ω and the exact sequence (6.6) splits. □

Remark 6.2. If $\delta = m$ then the two eigenvalues β_1 and β_2 coincide. In this case, the short exact sequence (6.6) does not split. The element $(y_1 - \beta_1)^2$ vanishes on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V$.

Lemma 6.3. *Suppose $m, n \geq 1$ and $\delta \neq n$. Then there is an isomorphims of $\mathfrak{gl}_{n+m}$-modules*

$$M^{\mathfrak{p}}(\underline{\delta}) \otimes V^* \cong M^{\mathfrak{p}}(\underline{\delta} - \varepsilon_{m+n}) \oplus M^{\mathfrak{p}}(\underline{\delta} - \varepsilon_m). \tag{6.9}$$

This is also an eigenspace decomposition for the action of y_1, with eigenvalues

$$\beta_1^* = \frac{m+n}{2} \quad \text{and} \quad \beta_2^* = \delta + \frac{m-n}{2}. \tag{6.10}$$

Proof. The proof is analogous to the previous one. We just note that the highest weight of V^* is $-\varepsilon_{m+n}$ and compute the action of Ω on the summand $M^{\mathfrak{p}}(\underline{\delta} - \varepsilon_m)$:

$$\begin{aligned}
&\frac{1}{2}\big(\langle \underline{\delta} - \varepsilon_m, \underline{\delta} - \varepsilon_m + 2\rho\rangle - \langle \underline{\delta}, \underline{\delta} + 2\rho\rangle - \langle -\varepsilon_{m+n}, -\varepsilon_{m+n} + 2\rho\rangle\big) \\
&\quad = \frac{1}{2}\big(\langle \underline{\delta}, \underline{\delta} + 2\rho\rangle - \langle \varepsilon_m, -\varepsilon_m + 2\rho\rangle - 2\langle \underline{\delta}, \varepsilon_m\rangle \\
&\qquad - \langle \underline{\delta}, \underline{\delta} + 2\rho\rangle + \langle \varepsilon_{m+n}, -\varepsilon_{m+n} + 2\rho\rangle\big) \\
&\quad = -\langle \varepsilon_m - \varepsilon_{m+n}, \rho\rangle - \langle \underline{\delta}, \varepsilon_m\rangle = -n + \delta,
\end{aligned} \tag{6.11}$$

so that the action of y_1 is given by $\delta + \frac{m-n}{2}$, and on the summand $M^{\mathfrak{p}}(\underline{\delta} - \varepsilon_{m+n})$:

$$\begin{aligned}
&\frac{1}{2}\big(\langle \underline{\delta} - \varepsilon_{m+n}, \underline{\delta} - \varepsilon_{m+n} + 2\rho\rangle - \langle \underline{\delta}, \underline{\delta} + 2\rho\rangle - \langle -\varepsilon_{m+n}, -\varepsilon_{m+n} + 2\rho\rangle\big) \\
&\quad = \frac{1}{2}\big(\langle \underline{\delta}, \underline{\delta} + 2\rho\rangle - \langle \varepsilon_{m+n}, -\varepsilon_{m+n} + 2\rho\rangle - 2\langle \underline{\delta}, \varepsilon_{m+n}\rangle \\
&\qquad - \langle \underline{\delta}, \underline{\delta} + 2\rho\rangle + \langle \varepsilon_{m+n}, -\varepsilon_{m+n} + 2\rho\rangle\big) \\
&\quad = -\langle \underline{\delta}, \varepsilon_{m+n}\rangle = 0,
\end{aligned} \tag{6.12}$$

and the action of y_1 is given by $\frac{m+n}{2}$. $\square$

Remark 6.4. Also in this case, if $\delta = n$ then $\beta_1^* = \beta_2^*$ and instead of (6.9) we have a short exact sequence that does not split. The element $(y_1 - \beta_1^*)^2$ vanishes on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^*$.

We define now the cyclotomic walled Brauer category of level two:

Definition 6.5. Let $r, t \in \mathbb{N}$ and fix a sequence $\boldsymbol{\omega}$ of complex parameters. Let also β_1, β_2, β_1^*, β_2^* be complex numbers. The *cyclotomic walled Brauer category* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*)$ is the quotient of $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ obtained imposing to the degenerate affine walled Brauer category $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega})$ the following relations:

$$\big(y_1^{(A)} - \beta_1\big)\big(y_1^{(A)} - \beta_2\big) = 0 \quad \text{for every } A \in \mathrm{Seq}_{r,t} \text{ with } a_1 = 1, \tag{6.13}$$

$$\big(y_1^{(A')} - \beta_1^*\big)\big(y_1^{(A')} - \beta_2^*\big) = 0 \quad \text{for every } A' \in \mathrm{Seq}_{r,t} \text{ with } a_1' = -1. \tag{6.14}$$

If A is an (r,t)-sequence, we define the *cyclotomic walled Brauer algebra*

$$\mathbb{V}\mathsf{Br}_A\big(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*\big)=\operatorname{End}_{\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*)}(A). \tag{6.15}$$

Remark 6.6. We remark that we really need to quotient out both (6.13) and (6.14) in order to be sure that we get a finite dimensional quotient. Moreover, we point out that it is important to first take the cyclotomic quotient of the whole category and then define the cyclotomic walled Brauer algebras to be the endomorphism algebras in the cyclotomic category: if we would define the cyclotomic algebras to be the cyclotomic quotients of the degenerate affine algebras by relations (6.13) or (6.14), then they would not be in general finite dimensional.

For general parameters $\boldsymbol{\omega},\beta_1,\beta_2,\beta_1^*,\beta_2^*$ we cannot say anything about the cyclotomic quotient $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*)$, which could even be trivial. However, if the parameters are chosen carefully, we will prove that the cyclotomic walled Brauer algebras are finite dimensional of dimension $2^{r+t}(r+t)!$, as one would expect.

We have the following consequence of the definition and of Theorem 5.2:

Corollary 6.7. *Fix integers r,t with $r\geq 1$, and let $A\in\operatorname{Seq}_{r,t}$. Fix also $\delta\in\mathbb{Z}$ and $m,n\geq 1$. Then the action of Theorem 5.2 factors through the cyclotomic quotient, defining a functor*

$$\begin{aligned}\underline{\mathbb{V}\mathsf{Br}}_{r,t}\big(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*\big)&\longrightarrow\mathcal{O}(m,n)\\ A&\longmapsto M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes A}\end{aligned}\tag{6.16}$$

and in particular an action of $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*)$ on $M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes A}$, where $\boldsymbol{\omega}$ is as in Theorem 5.2, while $\beta_1,\beta_2,\beta_1^*,\beta_2^*$ are given by Lemmas 6.1 and 6.3.

We will need a finite set of generators for the cyclotomic walled Brauer category:

Proposition 6.8. *The regular monomials*

$$y_1^{\gamma_1}\cdots y_{r+t}^{\gamma_{r+t}}By_1^{\eta_1}\cdots y_{r+t}^{\eta_{r+t}}\quad \textit{with } \gamma_i,\eta_j\in\{0,1\}\textit{ for all } i,j \tag{6.17}$$

generate the cyclotomic walled Brauer category $\underline{\mathbb{V}\mathsf{Br}}_A(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*)$.

Proof. As in the proof of Proposition 3.2, it is enough to prove the statement for the associated graded category G' (since the filtration on the degenerate affine walled Brauer category descends to a filtration on the cyclotomic quotient). Of course all regular monomials generate the cyclotomic quotient by Proposition 3.2. Consider a regular monomial: we can move the strands so that every strand at some level happens to be the leftmost strand. Now if some γ_i or η_i is bigger than 1, then there are at least two dots on some strand. Using relations (7′) for G', we can move the two dots along the strand until they reach the level at which there are no other strands on their left. By the graded cyclotomic relation, this monomial is zero in the cyclotomic quotient. □

We will call the elements (6.17) *cyclotomic regular monomials*. We are now ready to prove our second main result:

Theorem 6.9. *Let m, n, r, t, δ be integer numbers with $m, n, r \geq 1$, $t \geq 0$ and $m, n \geq r + t$. Let $\boldsymbol{\omega} = \boldsymbol{\omega}(M^{\mathfrak{p}}(\underline{\delta}))$ be given by Lemma 5.1 and*

$$\beta_1 = -\delta + \frac{m+n}{2}, \qquad \beta_2 = \frac{n-m}{2}, \tag{6.18}$$

$$\beta_1^* = \frac{m+n}{2}, \qquad \beta_2^* = \delta + \frac{m-n}{2} \tag{6.19}$$

as given by Lemmas 6.1 and 6.3.

Then the cyclotomic regular A-monomials of Proposition 6.8 form a basis of $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*)$ and we have an isomorphism of algebras

$$\mathbb{V}\mathsf{Br}_A\big(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*\big) \cong \operatorname{End}_{\mathfrak{gl}_{n+m}}\big(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}\big). \tag{6.20}$$

In particular $\dim_{\mathbb{C}} \mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*) = 2^{r+t}(r+t)!$.

Before proving the theorem, let us state the following important corollary:

Corollary 6.10. *With the hypotheses of Theorem 6.9, the cyclotomic walled Brauer algebra $\mathbb{V}\mathsf{Br}_A(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*)$ inherits a grading and a graded cellular algebra structure, where the graded decomposition numbers are given by parabolic Kazhdan–Lusztig polynomials of type A.*

Proof. By Theorem 6.9 we need to prove the claim for the endomorphism algebra

$$\operatorname{End}_{\mathcal{O}(m,n)}\big(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}\big). \tag{6.21}$$

Notice that the weight $\underline{\delta}$ is either a dominant weight or an anti-dominant weight for the parabolic category $\mathcal{O}(m,n)$ (it is dominant if $\delta \leq n$ and it is anti-dominant weight if $\delta \geq m$, but it could also happen that it is both dominant and anti-dominant). Hence the parabolic Verma module $M^{\mathfrak{p}}(\underline{\delta})$ is either a projective module or a tilting module in $\mathcal{O}(m,n)$.

If $M^{\mathfrak{p}}(\underline{\delta})$ is projective then $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}$ is also a projective module. If $M^{\mathfrak{p}}(\underline{\delta})$ is tilting, then $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}$ is also tilting. Since the blocks of $\mathcal{O}(m,n)$ are Ringel self-dual (for the regular blocks this is [17, Proposition 4.4], and since $\mathfrak{p} \subset \mathfrak{gl}_N$ is a maximal parabolic subalgebra the singular blocks are equivalent to regular blocks for smaller N's), the endomorphism algebra of this tilting module is isomorphic to the endomorphism algebra of a projective module.

In both cases, (6.21) is then isomorphic to an idempotent truncation of the endomorphism algebra of a projective generator of a sum of blocks of $\mathcal{O}(m,n)$. Since blocks of $\mathcal{O}(m,n)$ are graded quasi-hereditary, this idempotent truncation inherits the structure of a graded cellular algebra (see [16, Proposition 4.3]). $\square$

Proof of Theorem 6.9. First, let us compute the action of y_1 on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V$. We indicate with z the highest vector of $M^{\mathfrak{p}}(\underline{\delta})$, and note that $\mathfrak{gl}_m \oplus \mathfrak{gl}_n \subset \mathfrak{p}$ acts as 0 on z, since $M^{\mathfrak{p}}(\underline{\delta}) = U(\mathfrak{gl}_{n+m}) \otimes_{\mathfrak{p}} E(\underline{\delta})$ where $E(\underline{\delta})$ is the simple $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$-module with highest weight $\underline{\delta}$, that by our choice of $\underline{\delta}$ is one-dimensional. Hence $X_{ij} z = 0$ whenever both $i, j \leq m$ or $i, j > m$, and we obtain:

$$y_1(z \otimes v_i) = \begin{cases} (-\delta + \frac{m+n}{2}) z \otimes v_i & \text{if } i \leq m \\ (\frac{m+n}{2}) z \otimes v_i + \sum_{k \in I, k \leq m} X_{ik} z \otimes v_k & \text{if } i > m. \end{cases} \tag{6.22}$$

Analogously, let us compute the action of y_1 on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^*$:

$$y_1\bigl(z \otimes v_j^*\bigr) = \begin{cases} (\delta + \frac{m+n}{2}) z \otimes v_j^* - \sum_{k \in I, k > m} X_{kj} z \otimes v_k^* & \text{if } j \leq m \\ (\frac{m+n}{2}) z \otimes v_i & \text{if } i > m. \end{cases} \tag{6.23}$$

Fix now an index $1 \leq h \leq r+t$ and consider the action of y_h on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}$. By Lemmas 8.7 and 8.9, y_h acts as Ω_{0h} plus some linear combination of A-walled Brauer diagrams. By the PBW Theorem, $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^A$ has basis

$$\bigl\{ pz \otimes v_{b_1}^{a_1} \otimes \cdots \otimes v_{b_{r+t}}^{a_{r+t}} \bigr\} \tag{6.24}$$

for $1 \leq b_i \leq m+n$ (where as usual $v_i^1 = v_i$ and $v_i^{-1} = v_i^*$) and for p that runs over all monomials in the X_{ij}'s for $m < i \leq n$, $1 \leq j \leq m$. (Notice that such X_{ij}'s commute with each other.) By looking at the degree of the monomial p we get a grading on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^A$. Then we have

$$y_h\bigl(z \otimes \cdots v_i^{\chi} \cdots\bigr) = \begin{cases} \sum_{k \in I, k \leq m} X_{ik} z \otimes \cdots v_k \cdots & \text{if } i > m,\ \chi = 1, \\ -\sum_{k \in I, k > m} X_{ki} z \otimes \cdots v_k^* \cdots & \text{if } i \leq m,\ \chi = -1, \\ 0 & \text{otherwise} \end{cases} \tag{6.25}$$

up to terms of degree zero.

Now consider a cyclotomic regular monomial, and draw it as a decorated A-walled Brauer diagram $\aleph$. Remember that we read diagrams from the bottom to the top. We are going to explain a way to label the endpoints of $\aleph$. We consider the oriented arcs of $\aleph$ as arrows, having a source and a target. Order in some way the sources of the arrows of $\aleph$, labeling them sequentially with numbers $m+1, \ldots, m+r+t$. Now for every undecorated arrow label the target with the same label as the source. For every decorated arrow such that the source is labeled with p, label the target with $p-m$. Let τ_i be the label of the target of the arrow with source $i+m$. For an example, see Fig. 1.

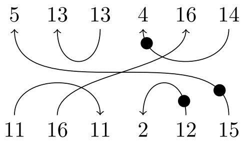


Fig. 1. This figure illustrates the labeling of the proof of Theorem 6.9. Here we suppose $m = 10$.

In this way, we obtain sequences $1 \leq b_h, c_k \leq m+n$ for $h, k = 1, \dots, r+t$, respectively, on the bottom and on the top of our diagram. (This is true because of the assumption $m, n \geq r + t$.)

Let now $v_b^{\otimes A} = v_{b_1}^{a_1} \otimes \cdots \otimes v_{b_{r+t}}^{a_{r+t}}$. Take a zero linear combination $\sum_{\beth} f_{\beth} \beth = 0$ of cyclotomic regular monomials. Pick a diagram $\aleph$ with all arcs decorated and let $b = b(\aleph)$, $c = c(\aleph)$, $\tau = \tau(\aleph)$ be the sequences constructed as before. By (6.25) and by our construction, we have

$$\left\langle \sum_{\beth} f_{\beth} \beth \left(z \otimes v_{b(\aleph)}^{\otimes A} \right), X_{m+1,\tau_1} \cdots X_{m+r+t,\tau_{r+t}} z \otimes v_{c(\aleph)}^{\otimes A} \right\rangle = \pm f_{\aleph}, \tag{6.26}$$

where we have fixed on $M \otimes V^{\otimes A}$ the scalar product with respect to which the basis (6.24) is orthonormal. Hence $f_{\aleph} = 0$.

Now pick a diagram $\aleph$ with all but one arcs decorated, let b, c, τ be the sequences as above. Then Eq. (6.26) again holds, if we do not write the X_{i,τ_i} corresponding to the undecorated arrow. Proceeding in this way, we have that all coefficients $f_{\beth}$ are zero, hence the representation is faithful, or in other words the map

$$\mathbb{V}\mathsf{Br}_A\left(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*\right) \longrightarrow \mathrm{End}_{\mathfrak{gl}_{n+m}}\left(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}\right) \tag{6.27}$$

is injective.

To prove surjectivity, we use a dimension argument. On one side, note that there are $2^{r+t}(r+t)!$ cyclotomic regular monomial, hence

$$\dim \mathbb{V}\mathsf{Br}_A\left(\boldsymbol{\omega}; \beta_1, \beta_2; \beta_1^*, \beta_2^*\right) \leq 2^{r+t}(r+t)! \tag{6.28}$$

By the injectivity of (6.27), this is actually an equality. On the other side, by adjunction we have

$$\mathrm{End}_{\mathfrak{gl}_{m+n}}\left(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}\right) \cong \mathrm{End}_{\mathfrak{gl}_{m+n}}\left(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes (r+t)}\right) \tag{6.29}$$

as vector spaces; the dimension of the r.h.s. of (6.29) can be computed counting standard tableaux, and is well-known to be $2^{r+t}(r+t)!$. $\square$

Putting together the isomorphisms (6.20) for all $A \in \mathrm{Seq}_{r,t}$ one gets the following:

Corollary 6.11. *With the hypotheses of Theorem 6.9, the cyclotomic regular monomials (6.17) give a basis of* $\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*)$ *and we have an isomorphism of algebras*

$$\underline{\mathbb{V}\mathsf{Br}}_{r,t}(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*) \cong \mathrm{End}_{\mathfrak{gl}_{n+m}}\Bigg(\bigoplus_{A\in\mathrm{Seq}_{r,t}} M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes A}\Bigg). \tag{6.30}$$

We conclude this section giving an explicit formula to compute the parameters ω_k in term of β_1,β_2 (and hence in term of m, n and δ).

Lemma 6.12. *The elements* ω_k *in* $\mathbb{V}\mathsf{Br}_{r,t}(\boldsymbol{\omega};\beta_1,\beta_2;\beta_1^*,\beta_2^*)$ *satisfy the following recursion formula*

$$\omega_k = (\beta_1+\beta_2)\omega_{k-1} - \beta_1\beta_2\omega_{k-2} \tag{6.31}$$

with initial data $\omega_0 = m+n$, $\omega_1 = -\delta m + \frac{(m+n)^2}{2}$.

Proof. By Lemma 5.1, $\omega_0 = m+n$. Let $A=(1,-1)$; we have

$$\begin{aligned} e_1^{(A)}y_1e_1^{(A)}\big(z\otimes v_i\otimes v_i^*\big) &= \sum_{j=1}^{m+n} e_1^{(A)}y_1\big(z\otimes v_j\otimes v_j^*\big) \\ &= \sum_{j=1}^{m} e_1^{(A)}\big(-\delta z\otimes v_j\otimes v_j^*\big) + \frac{m+n}{2}\sum_{j=1}^{m+n} e_1^{(A)}\big(z\otimes v_j\otimes v_j^*\big) \\ &= \left(-\delta m + \frac{(m+n)^2}{2}\right)e_1^{(A)}\big(z\otimes v_j\otimes v_j^*\big), \end{aligned} \tag{6.32}$$

hence $\omega_1 = -\delta m + \frac{(m+n)^2}{2}$.

The recursion relation follows from

$$e_1^{(A)}y_1^n e_1^{(A)} = (\beta_1+\beta_2)e_1^{(A)}y_1^{n-1}e_1^{(A)} - \beta_1\beta_2 e_1^{(A)}y_1^{n-2}e_1^{(A)}. \qquad \square \tag{6.33}$$

Using the standard elementary theory of power series defined by recurrence relations, we can explicitly compute $W_1(u)$:

$$W_1(u) = \frac{\omega_0 + (\omega_1 - (\beta_1+\beta_2)\omega_0)u^{-1}}{1-(\beta_1+\beta_2)u^{-1}+\beta_1\beta_2u^{-2}}. \tag{6.34}$$

7. Diagram and partition calculus

We explain now how one can determine which composition factors appear in $M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes A}$ using a partition calculus.

Recall that a *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. The *content* of the box in the

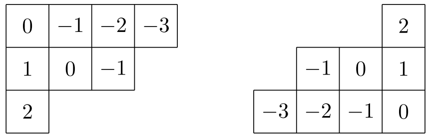


Fig. 2. A Young diagram and a rotated Young diagram, with the contents written in the boxes.

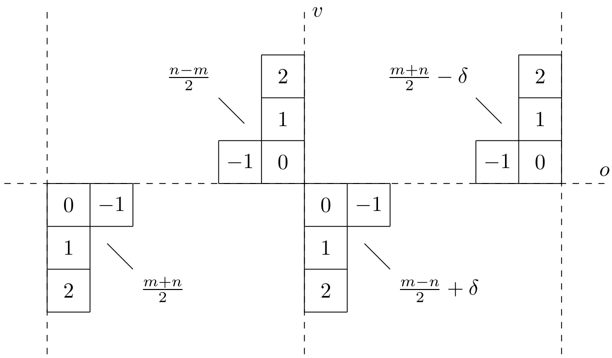


Fig. 3. A 4-Young diagram with the contents in the boxes. The corresponding weight is $3\varepsilon_1 + \varepsilon_2 - \varepsilon_{m-1} - 3\varepsilon_m + 3\varepsilon_{m+1} + \varepsilon_{m+2} - \varepsilon_{m+n-1} - 3\varepsilon_{m+n}$.

r-th row and c-th column (counting from the left to the right and from the top to the bottom, and starting with 0) is $r - c$.

A *rotated Young diagram* is a Young diagram rotated of 180 degrees. The content of a box in a rotated Young diagram is the same as the content of the original box in the original Young diagram (Fig. 2).

Let us now fix positive integers m and n and an integer δ. In a plane let us consider the infinite vertical strip of width equal to $m+n$ boxes. Fix a horizontal line o. Fix also a vertical line v in such a way that there is a space for exactly n boxes on the left of v and for m boxes on the right of v. The lines o and v divide our strip into four regions. Let us number the columns of our vertical strip by the integers $1, \ldots, m+n$ from the right to the left. We define a *4-Young diagram* to be a collection of boxes in this strip, such that in the two regions under the horizontal line o we have two Young diagrams and in the two regions above o we have two rotated Young diagrams, and such that in no column there are boxes both above and under o (see Fig. 3).

By definition a 4-Young diagram is made of four Young diagrams, and every box belongs to exactly one of these. We define the content of a box to be the content in the corresponding Young diagram, translated by the following values:

- the lower left Young diagram by $\frac{m+n}{2}$,
- the upper left Young diagram by $\frac{n-m}{2}$,
- the lower right Young diagram by $\frac{m-n}{2} + \delta$,
- the upper right Young diagram by $\frac{m+n}{2} - \delta$.

See Fig. 3.

Given a 4-Young diagram Y, let $b_i(Y)$ be equal to the number of boxes in the column i of Y, multiplied by -1 if the boxes are under the horizontal line o.

To $\lambda \in \Lambda(m,n)$ we associate the 4-Young diagram $Y(\lambda)$ determined by $b_i(Y(\lambda)) = \langle \lambda, \varepsilon_i \rangle$. More generally, in the same way, to any weight for $\mathfrak{gl}_{m+n}$ we could associate a diagram consisting of boxes in our infinite vertical strip: one can check that this diagram is a 4-Young diagram if and only if the weight is in $\Lambda(m,n)$.

Given a 4-Young diagram Y, we may obtain another 4-Young diagram Y' by adding a box to it (we also say that Y is obtained by removing a box to Y'). Notice that we use the expressions *adding* and *removing boxes* only if the result is again a 4-Young diagram. For an (r,t)-sequence A let us define $\mathcal{Y}_A$ to be the set of sequences

$$\{Y_\bullet = (Y_0, Y_1, \ldots, Y_{r+t})\} \tag{7.1}$$

of 4-Young diagrams such that $Y_0 = Y(\underline{\delta})$ and Y_{i+1} is obtained from Y_i by

- adding a box above o or removing a box below o if $a_i = 1$,
- removing a box above o or adding a box below o if $a_i = -1$.

From the construction and the properties of the functors of tensoring with V and V^* on $\mathcal{O}(m,n)$ we have the following result:

Lemma 7.1. *There is a bijection between $\mathcal{Y}_A$ and the composition factors of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes A}$ (counted with multiplicity). The element $Y_\bullet = (Y_0, \ldots, Y_{r+t}) \in \mathcal{Y}_A$ corresponds to a composition factor isomorphic to $M(\lambda)$, where $Y_{r+t} = Y(\lambda)$.*

Notice that as a consequence we have the following non-trivial combinatorial statement:

Corollary 7.2. *The cardinality of $\mathcal{Y}_A$ is $2^{r+t}(r+t)!$.*

We can now compute the generalized eigenvalues of the y_i's:

Proposition 7.3. *Let $Y_\bullet \in \mathcal{Y}_A$. For $j = 1, \ldots, r+t$ let $\eta_j = 1$ if Y_j has been obtained from Y_{j-1} by adding a box of content i_j, otherwise let $\eta_j = -1$ if Y_j has been obtained from Y_{j-1} by removing a box of content i_j. Then the composition factor $M(\lambda)$ corresponding to $Y_\bullet$ is contained in the generalized eigenspace for the y_i's with generalized eigenvalues $(\eta_1 i_1, \ldots, \eta_{r+t} i_{r+t})$.*

Proof. Remember that we denote by C the Casimir element of $\mathfrak{gl}_{m+n}$. We have

$$\Delta^\ell(C) = \Delta^{\ell-1}(C) \otimes 1 + \Delta^{\ell-1}(1) \otimes C + 2 \sum_{p=0}^{\ell-1} \Omega_{p\ell}. \tag{7.2}$$

For $h = 1, \dots, r+t$ let λ_h be the weight such that $Y_h = Y(\lambda_h)$. Let also $\psi_h = \lambda_h - \underline{\delta}$, and notice that $\psi_h - \psi_{h-1} = (-1)^{a_h} \varepsilon_{\kappa_h}$ for a unique $\kappa_h \in \{1, \dots, n+m\}$. Also, in other words we have

$$\psi_h = (-1)^{a_1} \varepsilon_1 + \dots + (-1)^{a_h} \varepsilon_{\kappa_h} \tag{7.3}$$

for all $h = 1, \dots, r+t$.

Suppose first that $a_\ell = 1$. Then the action of $2 \sum_{p=0}^{\ell-1} \Omega_{p\ell}$ on $M(\lambda) = M(\underline{\delta} + \psi_{r+t})$ is given by

$$\langle \underline{\delta} + \psi_\ell, \underline{\delta} + \psi_\ell + 2\rho \rangle - \langle \underline{\delta} + \psi_{\ell-1}, \underline{\delta} + \psi_{\ell-1} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle. \tag{7.4}$$

Hence y_ℓ acts as

$$\langle \varepsilon_{\kappa_\ell}, \underline{\delta} \rangle + \langle \varepsilon_{\kappa_\ell}, \psi_{\ell-1} \rangle + \langle \varepsilon_{\kappa_\ell} - \varepsilon_1, \rho \rangle + \frac{m+n}{2}. \tag{7.5}$$

Now suppose instead that $a_\ell = -1$. Then the action of $2 \sum_{p=0}^{\ell-1} \Omega_{p\ell}$ on $M(\underline{\delta} + \psi_{r+t})$ is given by

$$\langle \underline{\delta} + \psi_\ell, \underline{\delta} + \psi_\ell + 2\rho \rangle - \langle \underline{\delta} + \psi_{\ell-1}, \underline{\delta} + \psi_{\ell-1} + 2\rho \rangle + \langle \varepsilon_{m+n}, -\varepsilon_{m+n} + 2\rho \rangle. \tag{7.6}$$

Hence y_ℓ acts as

$$\begin{aligned} &-\langle \varepsilon_{\kappa_\ell}, \underline{\delta} \rangle - \langle \varepsilon_{\kappa_\ell}, \psi_{\ell-1} \rangle + \langle \varepsilon_{m+n} - \varepsilon_{k_\ell}, \rho \rangle + \frac{m+n}{2} \\ &\quad = -\langle \varepsilon_{\kappa_\ell}, \underline{\delta} \rangle - \langle \varepsilon_{\kappa_\ell}, \psi_{\ell-1} \rangle - \langle \varepsilon_{\kappa_\ell} - \varepsilon_1, \rho \rangle - \frac{m+n}{2} + 1. \end{aligned} \tag{7.7}$$

Let us now examine (7.5) and (7.7): in both of them, the second term is the index and the third term is (up to a shift) the column index of the box being added/removed at the ℓ-th step. The claim then follows by the definition of the shifts of the contents of the boxes in the 4-Young diagram. $\square$

8. Proofs

We collect in this section the steps of the proof of Theorem 5.2.

First, note that

$$\dot{s}_i \Omega_{jk} = \Omega_{\mathsf{s}_i(j)\mathsf{s}_i(k)} \dot{s}_i \tag{8.1}$$

for every $0 \leq j < k \leq r+t$, where we define $\Omega_{kj} = \Omega_{jk}$ for $k > j$.

Lemma 8.1. *For all $\mathfrak{gl}_N$-modules M and all $A \in \mathrm{Seq}_{r,t}$, on $M \otimes V^{\otimes A}$ we have $\dot{s}_i y_j = y_j \dot{s}_i$ for $j \neq i, i+1$.*

Proof. The statement is obvious for $j < i$. Suppose $j > i+1$: we have

$$\dot{s}_i y_j = \dot{s}_i \left[\sum_{0 \le k < j} \Omega_{kj} + \frac{N}{2} \right] = \left[\sum_{0 \le k < j} \Omega_{\mathsf{s}_i(k)j} + \frac{N}{2} \right] \dot{s}_i = y_j \dot{s}_i. \qquad \square$$

Lemma 8.2. *On $V \otimes V^* \otimes V$ and on $V \otimes V^* \otimes V^*$ the elements $(\Omega_{13} + \Omega_{23})(\tau_V \otimes \mathrm{id})$ and $(\tau_V \otimes \mathrm{id})(\Omega_{13} + \Omega_{23})$ act as zero. Analogously, on $V^* \otimes V \otimes V$ and on $V^* \otimes V \otimes V^*$ the elements $(\Omega_{13} + \Omega_{23})(\tau_{V^*} \otimes \mathrm{id})$ and $(\tau_{V^*} \otimes \mathrm{id})(\Omega_{13} + \Omega_{23})$ act as zero.*

Proof. We prove only that the two actions on $V \otimes V^* \otimes V$ are zero. The other assertions follow analogously. We compute:

$$\begin{aligned}
&(\Omega_{ij} + \Omega_{(i+1)j})(\tau_V \otimes \mathrm{id})\big(v_a \otimes v_b^* \otimes v_c\big) \\
&\quad = (\Omega_{ij} + \Omega_{(i+1)j})\left(\delta_{ab} \sum_{d \in I} v_d \otimes v_d^* \otimes v_c\right) \\
&\quad = \delta_{ab} \sum_{d,e \in I} \big(H_e v_d \otimes v_d^* \otimes H_e v_c + v_d \otimes H_e v_d^* \otimes H_e v_c\big) \\
&\qquad + \delta_{ab} \sum_{d \in I, \alpha \in \Pi} \big(X_\alpha v_d \otimes v_d^* \otimes X_{-\alpha} v_c + v_d \otimes X_\alpha v_d^* \otimes X_{-\alpha} v_c\big) \\
&\quad = \delta_{ab}\big(v_c \otimes v_c^* \otimes v_c - v_c \otimes v_c^* \otimes v_c\big) \\
&\qquad + \delta_{ab} \sum_{d \in I} \left((1 - \delta_{cd})\big(v_c \otimes v_d^* \otimes v_d\big) - \delta_{cd} \sum_{e \in I, e \neq c} \big(v_d \otimes v_e^* \otimes v_e\big) \right) \\
&\quad = \delta_{ab} \sum_{d \neq c} \big(v_c \otimes v_d^* \otimes v_d\big) - \delta_{ab} \sum_{e \neq c} \big(v_c \otimes v_e^* \otimes v_e\big) = 0.
\end{aligned}$$

Next compute:

$$\begin{aligned}
&(\tau_V \otimes \mathrm{id})(\Omega_{ij} + \Omega_{(i+1)j})\big(v_a \otimes v_b^* \otimes v_c\big) \\
&\quad = (\tau_V \otimes \mathrm{id})\Bigg[\sum_{d \in I} \big(H_d v_a \otimes v_b^* \otimes H_d v_c + v_a \otimes H_d v_b^* \otimes H_d v_c\big) \\
&\qquad + \sum_{\alpha \in \Pi} \big(X_\alpha v_a \otimes v_b^* \otimes X_{-\alpha} v_c + v_a \otimes X_\alpha v_b^* \otimes X_{-\alpha} v_c\big) \Bigg] \\
&\quad = (\tau_V \otimes \mathrm{id})\Big[\big(\delta_{ac} v_a \otimes v_b^* \otimes v_a - \delta_{bc} v_a \otimes v_b^* \otimes v_b\big) \\
&\qquad + (1 - \delta_{ac})\big(v_c \otimes v_b^* \otimes v_a\big) - \delta_{bc} \sum_{d \in I, d \neq b} \big(v_a \otimes v_d^* \otimes v_d\big) \Big] \\
&\quad = \sum_{e \in I} \delta_{ac} \delta_{ab} v_e \otimes v_e^* \otimes v_a - \sum_{e \in I} \delta_{bc} \delta_{ab} v_e \otimes v_e^* \otimes v_b
\end{aligned}$$

$$+\sum_{e\in I}\delta_{cb}(1-\delta_{ac})\big(v_e\otimes v_e^*\otimes v_a\big)-\delta_{bc}\sum_{d\in I,d\neq b}\delta_{ad}\sum_{e\in I}\big(v_e\otimes v_e^*\otimes v_d\big)$$
$$=\delta_{bc}(1-\delta_{ac})\sum_{e\in I}\big(v_e\otimes v_e^*\otimes v_a\big)-\delta_{bc}(1-\delta_{ab})\sum_{e\in I}\big(v_e\otimes v_e^*\otimes v_a\big)=0. \qquad \square$$

Lemma 8.3. *For all modules M and (r,t)-sequences A, on $M\otimes V^{\otimes A}$ we have $e_iy_j=y_je_i$ for $j\neq i,i+1$.*

Proof. The case $j<i$ is obvious, and we are left with the case $j>i+1$. It suffices to prove that

$$(\Omega_{ij}+\Omega_{(i+1)j})e_i=e_i(\Omega_{ij}+\Omega_{(i+1)j}),$$

since e_i commutes with all other summands of y_j. This follows immediately from Lemma 8.2. $\square$

Lemma 8.4. *We have $[\Omega_{12},\Omega_{34}]=0$ and $[\Omega_{12}+\Omega_{23},\Omega_{13}]=0$.*

Proof. The first equation is obvious. For the second, we explicitly compute the expression $\Omega_{12}\Omega_{13}+\Omega_{23}\Omega_{13}-\Omega_{13}\Omega_{12}-\Omega_{13}\Omega_{23}$ and we get:

$$\begin{aligned}
&\sum_{\substack{a,b,c\in I\\ b\neq c}}(H_aX_{bc}\otimes H_a\otimes X_{cb}+X_{bc}\otimes H_a\otimes H_aX_{cb}\\
&\quad-H_aX_{bc}\otimes X_{cb}\otimes H_a-H_a\otimes X_{bc}\otimes H_aX_{cb})\\
&\quad+\sum_{a,b,c\in I,b\neq c}(X_{bc}H_a\otimes X_{cb}\otimes H_a+H_a\otimes X_{bc}\otimes X_{cb}H_a\\
&\quad-X_{bc}H_a\otimes H_a\otimes X_{cb}-X_{bc}\otimes H_a\otimes X_{cb}H_a)\\
&\quad+\sum_{\substack{a,b,c,d\in I\\ a\neq b,c\neq d}}(X_{ab}X_{cd}\otimes X_{ba}\otimes X_{dc}+X_{cd}\otimes X_{ab}\otimes X_{ba}X_{dc}\\
&\quad-X_{ab}X_{cd}\otimes X_{dc}\otimes X_{ba}-X_{ab}\otimes X_{cd}\otimes X_{ba}X_{dc})\\
&=\sum_{\substack{a,b,c\in I\\ b\neq c}}\big([H_a,X_{bc}]\otimes H_a\otimes X_{cb}+X_{bc}\otimes H_a\otimes[H_a,X_{cb}]\\
&\quad-[H_a,X_{bc}]\otimes X_{cb}\otimes H_a-H_a\otimes X_{bc}\otimes[H_a,X_{cb}]\big)\\
&\quad+\sum_{\substack{a,b,c,d\in I\\ a\neq b,c\neq d}}\big([X_{ab},X_{cd}]\otimes X_{ba}\otimes X_{dc}-X_{ab}\otimes X_{cd}\otimes[X_{ba},X_{dc}]\big)\\
&=\sum_{\substack{b,c\in I\\ b\neq c}}\big((X_{bc}-X_{bc})\otimes H_a\otimes X_{cb}+X_{bc}\otimes H_a\otimes(X_{cb}-X_{cb})
\end{aligned}$$

$$
\begin{aligned}
&- (X_{bc} - X_{bc}) \otimes X_{cb} \otimes H_a - H_a \otimes X_{bc} \otimes (X_{cb} - X_{cb})\big) \\
&+ \sum_{\substack{a,b,c,d \in I \\ a \neq b, c \neq d}} \big(\big(\delta_{bc}(1-\delta_{ad})X_{ad} - \delta_{ad}(1-\delta_{bc})X_{cb}\big) \otimes X_{ba} \otimes X_{dc} \\
&- X_{ab} \otimes X_{cd} \otimes \big(\delta_{ad}(1-\delta_{bc})X_{bc} - \delta_{bc}(1-\delta_{ad})X_{da}\big)\big) \\
= &\sum_{\substack{a,b,d \in I \\ a\neq b, b\neq d, a\neq d}} (X_{ad} \otimes X_{ba} \otimes X_{db}) - \sum_{\substack{a,b,c \in I \\ a\neq b, a\neq c, b\neq c}} (X_{cb} \otimes X_{ba} \otimes X_{ac}) \\
&- \sum_{\substack{a,b,c \in I \\ a\neq b, a\neq c, b\neq c}} (X_{ab} \otimes X_{ca} \otimes X_{bc}) + \sum_{\substack{a,b,d \in I \\ a\neq b, b\neq d, a\neq d}} (X_{ab} \otimes X_{bd} \otimes X_{da}) = 0,
\end{aligned}
$$

that proves our claim. □

Lemma 8.5. *For $1 \leq i, j \leq r+t$ we have $y_i y_j = y_j y_i$.*

Proof. Let us assume $i > j$, and compute

$$
\begin{aligned}
[y_i, y_j] &= \Big[\sum_{0 \leq h < i} \Omega_{hi}, \sum_{0 \leq k < j} \Omega_{kj}\Big] = \sum_{0 \leq k < j} \Big[\sum_{0 \leq h < i} \Omega_{hi}, \Omega_{kj}\Big] \\
&= \sum_{0 \leq k < j} [\Omega_{ki} + \Omega_{ji}, \Omega_{kj}] = 0
\end{aligned}
$$

using Lemma 8.4. □

Lemma 8.6. *Relations $\dot{s}_i \dot{e}_{i+1} \dot{e}_i = \dot{s}_{i+1} \dot{e}_i$, $\dot{e}_i \dot{e}_{i+1} \dot{s}_i = \dot{e}_i \dot{s}_{i+1}$, $\dot{e}_{i+1} \dot{e}_i \dot{s}_{i+1} = \dot{e}_{i+1} \dot{s}_i$, $\dot{s}_{i+1} \dot{e}_i \dot{e}_{i+1} = \dot{s}_i \dot{e}_{i+1}$, $\dot{e}_{i+1} \dot{e}_i \dot{e}_{i+1} = \dot{e}_{i+1}$ and $\dot{e}_i \dot{e}_{i+1} \dot{e}_i = \dot{e}_i$ hold for $1 \leq i \leq r+t-2$.*

Proof. These relations are very easy to check by hand. Alternatively, they are implied by the standard (trivial) ribbon Hopf algebra structure of $U(\mathfrak{gl}_N)$. □

Lemma 8.7. *We have $\sigma_{V,V} = \Omega$ on $V \otimes V$ and $\sigma_{V^*,V^*} = \Omega$ on $V^* \otimes V^*$.*

Proof. We compute

$$
\begin{aligned}
\Omega(v_i \otimes v_j) &= \sum_{a \in I} H_a v_i \otimes H_a v_j + \sum_{\alpha \in \Pi} X_\alpha v_i \otimes X_{-\alpha} v_j \\
&= \delta_{ij} v_i \otimes v_i + (1 - \delta_{ij}) v_j \otimes v_i = v_j \otimes v_i.
\end{aligned}
$$

Similarly we obtain the second equality. □

Lemma 8.8. *We have $s_i y_i - y_{i+1} s_i = -1$ and $s_i y_{i+1} - y_i s_i = 1$.*

Proof. This is a standard fact (see [1, Lemma 2.1]), but we repeat the proof for completeness. We compute, using (8.1) and Lemma 8.7:

$$\begin{aligned} s_i \sum_{0\le k<i} \Omega_{ki} - \sum_{0\le k<i+1} \Omega_{k(i+1)} s_i &= \sum_{0\le k<i} \Omega_{k(i+1)} s_i - \sum_{0\le k<i+1} \Omega_{k(i+1)} s_i \\ &= -\Omega_{i(i+1)} s_i = -s_i^2 = -1 \end{aligned}$$

and

$$\begin{aligned} \sum_{0\le k<i} \Omega_{ki} s_i - s_i \sum_{0\le k<i+1} \Omega_{k(i+1)} &= \sum_{0\le k<i} \Omega_{ki} s_i - \sum_{0\le k<i} \Omega_{ki} s_i - s_i \Omega_{i(i+1)} \\ &= -s_i^2 = -1. \qquad \square \end{aligned}$$

Lemma 8.9. *We have $\tau_V = -\Omega$ on $V \otimes V^*$ and $\tau_{V^*} = -\Omega$ on $V^* \otimes V$.*

Proof. We compute

$$\begin{aligned} \Omega\big(v_i \otimes v_j^*\big) &= \sum_{a\in I} H_a v_i \otimes H_a v_j^* + \sum_{\alpha\in\Pi} X_\alpha v_i \otimes X_{-\alpha} v_j^* \\ &= -\delta_{ij} v_i \otimes v_i^* - \delta_{ij} \sum_{k\neq i} v_k \otimes v_k^* = -\delta_{ij} \sum_{k\in I} v_k \otimes v_k^*. \end{aligned}$$

Similarly we obtain the second equality. $\square$

Lemma 8.10. *We have $\hat{s}_i y_i - y_{i+1}\hat{s}_i = \hat{e}_i$ and $\hat{s}_i y_{i+1} - y_i \hat{s}_i = -\hat{e}_i$.*

Proof. We compute, using (8.1) and Lemma 8.9:

$$\begin{aligned} \hat{s}_i \sum_{0\le k<i} \Omega_{ki} - \sum_{0\le k<i+1} \Omega_{k(i+1)} \hat{s}_i &= \sum_{0\le k<i} \Omega_{k(i+1)} \hat{s}_i - \sum_{0\le k<i+1} \Omega_{k(i+1)} \hat{s}_i \\ &= -\Omega_{i(i+1)} \hat{s}_i = e_i \hat{s}_i = \hat{e}_i \end{aligned}$$

and

$$\begin{aligned} \sum_{0\le k<i} \Omega_{ki} \hat{s}_i - \hat{s}_i \sum_{0\le k<i+1} \Omega_{k(i+1)} &= \sum_{0\le k<i} \Omega_{ki} \hat{s}_i - \sum_{0\le k<i} \Omega_{ki} \hat{s}_i - \hat{s}_i \Omega_{i(i+1)} \\ &= \hat{s}_i e_i = \hat{e}_i. \qquad \square \end{aligned}$$

Lemma 8.11. *Let M be a highest weight module with one-dimensional highest weight space. Then for all $1 \le i < r+t$ we have $e_i(y_i + y_{i+1}) = 0$ and $(y_i + y_{i+1})e_i = 0$.*

Proof. We start expanding the first relation:

$$\begin{aligned} e_i(y_i + y_{i+1}) &= e_i\left(\sum_{0\le k<i} (\Omega_{ki} + \Omega_{k(i+1)}) + \Omega_{i(i+1)} + N\right) \\ &= e_i\left(\sum_{0\le k<i} (\Omega_{ki} + \Omega_{k(i+1)})\right) - e_i^2 + Ne_i. \end{aligned}$$

We know that $e_i^2 = Ne_i$. Moreover, by Lemma 8.2 we know that $e_i(\Omega_{ki} + \Omega_{k(i+1)})$ acts as 0 if $k > 0$. Hence we are left to show that $e_i(\Omega_{0i} + \Omega_{0(i+1)})$ acts as 0. Let m be a non-zero vector in the highest weight space of M and suppose that at the place i we have V and at the place $i+1$ we have V^* (the other case being analogous). We write only the factors $0, i$ and $i+1$ of the tensor product, and compute $e_i(\Omega_{0i} + \Omega_{0(i+1)})(m \otimes v_h \otimes v_k^*)$:

$$\begin{aligned} & e_i \sum_{a\in I}\big(H_a m \otimes H_a v_h \otimes v_k^* + H_a m \otimes v_h \otimes H_a v_k^*\big) \\ &\qquad + e_i \sum_{\alpha\in\Pi^-} \big(X_\alpha m \otimes X_{-\alpha} v_h \otimes v_k^* + X_\alpha m \otimes v_h \otimes X_{-\alpha} v_k^*\big) \\ &= \delta_{hk} \sum_{b\in I}\big(H_h m \otimes v_b \otimes v_b^* - H_h m \otimes v_b \otimes v_b^*\big) \\ &\qquad + e_i(1-\delta_{hk})\big(X_{hk} m \otimes v_k \otimes v_k^* - X_{hk} m \otimes v_h \otimes v_h\big) = 0, \end{aligned}$$

where in the fourth line we have written only the terms from the second line that survive after applying e_i.

Analogously, for the second relation, we consider

$$\begin{aligned} (y_i + y_{i+1})e_i &= \left(\sum_{0\le k<i} (\Omega_{ki} + \Omega_{k(i+1)}) + \Omega_{i(i+1)} + N\right)e_i \\ &= \left(\sum_{0\le k<i} (\Omega_{ki} + \Omega_{k(i+1)})\right)e_i - e_i^2 + Ne_i \end{aligned}$$

and as before we just need to compute $(\Omega_{0i} + \Omega_{0(i+1)})e_i(m \otimes v_h \otimes v_k^*)$: this can be non-zero only if $h = k$ and in this case we get

$$\begin{aligned} (\Omega_{0i} + \Omega_{0(i+1)})\left(\sum_{l\in I} m \otimes v_l \otimes v_l^*\right) &= \sum_{a,l\in I} \big(H_a m \otimes H_a v_l \otimes v_l^* + H_a m \otimes v_l \otimes H_a v_l^*\big) \\ &\quad + \sum_{\alpha\in\Pi^-} \big(X_\alpha m \otimes X_{-\alpha} v_l \otimes v_l^* + X_\alpha m \otimes v_l \otimes X_{-\alpha} v_l^*\big) \\ &= \sum_{\substack{b,l\in I \\ b\ne l}} \big(X_{bl} m \otimes v_b \otimes v_l^* - X_{bl} m \otimes v_l \otimes v_b^*\big) = 0. \qquad \square \end{aligned}$$

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