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Subgroup-closed lattice formations

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ABSTRACT

A formation $\mathfrak{F}$ of finite groups is called a lattice formation if the set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the lattice of all subgroups in every finite group. The main result of this paper describes the family of all subgroup-closed lattice formations, and it can be regarded as an important step towards the solution of Shemetkov's classification problem.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group.

One of the most striking results in the theory of subnormal subgroups is the celebrated "join" theorem, proved by H. Wielandt in 1939 [12]: the subgroup generated by two

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subnormal subgroups of a finite group is itself subnormal. As a result, the set $sn(G)$ of all subnormal subgroups of a group G is a sublattice of the subgroup lattice.

Wielandt's theorem was developed in formation theory using concepts of $\mathfrak{F}$-subnormality and K-$\mathfrak{F}$-subnormality. We refer to [2, Chapter 6] for a convenient account on the topic.

The first concept was proposed by R. Carter and T. Hawkes [3]. Let $\mathfrak{F}$ be a non-empty formation. A subgroup H of a group G is said to be $\mathfrak{F}$-*subnormal* in G if either $H = G$ or there exists a maximal chain of subgroups

$$H = H_0 \subset H_1 \subset \cdots \subset H_n = G$$

such that $H_i^{\mathfrak{F}} \subseteq H_{i-1}$ for all $i = 1, \ldots, n$. Following [2], the set of all $\mathfrak{F}$-subnormal subgroups of a group G is denoted by $sn_{\mathfrak{F}}(G)$.

It is rather clear that the $\mathfrak{N}$-subnormal subgroups of a group G for the formation $\mathfrak{N}$ of all nilpotent groups are subnormal, and they coincide in the soluble universe. However the equality $sn_{\mathfrak{N}}(G) = sn(G)$ does not hold in general.

To avoid the above situation, O.H. Kegel [7] introduced a somewhat different notion of $\mathfrak{F}$-subnormality. It unites the notions of subnormal and $\mathfrak{F}$-subnormal subgroup.

A subgroup H of a group G is called $\mathfrak{F}$-*subnormal in the sense of Kegel* (or simply K-$\mathfrak{F}$-*subnormal*) in G if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that H_{i-1} is either normal in H_i or $H_i^{\mathfrak{F}} \subseteq H_{i-1}$ for all $i = 1, \ldots, n$. We shall write $H \in sn_{K-\mathfrak{F}}(G)$ and denote by $sn_{K-\mathfrak{F}}(G)$ the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G.

Obviously, $sn_{K-\mathfrak{N}}(G) = sn(G)$ for every group G.

Let $\mathfrak{F}$ be a formation. One might wonder whether the set of $\mathfrak{F}$-subnormal subgroups of a group forms a sublattice of the subgroup lattice. As the Example 6.3.1 in [2] shows, the answer is in general negative.

Therefore the following question naturally arises:

Which are the formations $\mathfrak{F}$ for which the set $sn_{\mathfrak{F}}(G)$ is a sublattice of the subgroup lattice of G for every group G?

This question was first proposed by L.A. Shemetkov in his monograph [9, Problem 12] in 1978 and it appeared in the Kourovka Notebook [8, Problem 9.75] in 1984.

In 1992, A. Ballester-Bolinches, K. Doerk, and M.D. Perez-Ramos [1] gave the answer to that question in the soluble universe for subgroup-closed saturated formations. In 1993, A.F. Vasil'ev, S.F. Kamornikov, and V.N. Semenchuk [11] published the solution of Shemetkov's problem in the general finite universe for subgroup-closed saturated formations. As a significant progress, in 2002, A.F. Vasil'ev and the second author in [10] characterized the subgroup-closed lattice formations which are soluble.

In 1978, O.H. Kegel [7] showed that if $\mathfrak{F}$ is a subgroup-closed formation such that $\mathfrak{F}\mathfrak{F} = \mathfrak{F}$, then the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G is a sublattice of the subgroup lattice of G for every group G. He also asks in [7] for other formations enjoying the lattice property for K-$\mathfrak{F}$-subnormal subgroups:

Which are the formations $\mathfrak{F}$ for which the set $sn_{K-\mathfrak{F}}(G)$ is a sublattice of the subgroup lattice of G for every group G?

In 1993, A.F. Vasil'ev, the second author, and V.N. Semenchuk [11] proved that the problems of O.H. Kegel and L.A. Shemetkov are equivalent for subgroup-closed saturated formations. An analogous result for subgroup-closed formations of soluble groups was obtained by A.F. Vasil'ev and the second author in [10].

This paper can be considered as a further big step of the programme aiming to the classification of all lattice and K-lattice formations. We say that $\mathfrak{F}$ is a *lattice* (respectively, K-*lattice*) formation if the set of all $\mathfrak{F}$-subnormal (respectively, K-$\mathfrak{F}$-subnormal) subgroups is a sublattice of the lattice of all subgroups in every group.

Here we solve the problems of Shemetkov and Kegel for all subgroup-closed formations. It may be worthwhile to note that the hypothesis on the formation being subgroup-closed seems to be quite natural within this framework, since otherwise $\mathfrak{F}$-subnormality (and K-$\mathfrak{F}$-subnormality) does not need to be persistent in intermediate subgroups, which looks as a relevant property for generalized subnormal subgroups. In this sense, the result obtained in the paper provides finally a quite satisfactory approach to the problems of Shemetkov and Kegel.

The main purpose of our paper is to prove the following theorem.

Theorem. *Let $\mathfrak{F}$ be a subgroup-closed formation. The following statements are pairwise equivalent:*

1. *The set of all K-$\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice of every group.*
2. *The set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice of every group.*
3. *$\mathfrak{F} = \mathfrak{M} \times \mathfrak{K} \times \mathfrak{L}$ for some subgroup-closed formations $\mathfrak{M}$, $\mathfrak{K}$ and $\mathfrak{L}$ satisfying the following conditions:*
   (a) *$\pi(\mathfrak{M}) \cap \pi(\mathfrak{K}) = \emptyset$, $\pi(\mathfrak{K}) \cap \pi(\mathfrak{L}) = \emptyset$ and $\pi(\mathfrak{M}) \cap \pi(\mathfrak{L}) = \emptyset$;*
   (b) *$\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ is a saturated formation, and it is an $\mathfrak{M}^2$-normal Fitting class;*
   (c) *every non-cyclic $\mathfrak{M}$-critical group G with $\Phi(G) = 1$ is a primitive group of type 2 such that $G/Soc(G)$ is a cyclic group of prime power order.*
   (d) *there exists a partition $\{\pi_j | j \in J\}$ of $\pi(\mathfrak{K})$ such that $\mathfrak{K} = \times_{j \in J}\mathfrak{S}_{\pi_j}$ and $|\pi_j| > 1$ for all $j \in J$;*
   (e) *$\mathfrak{L} \subseteq \mathfrak{N}_{\pi(\mathfrak{L})}$.*

As immediate deductions we have the following results.

Corollary 1. *(See [2, Theorem 6.3.15], [6, Theorem 3.1.22], [11].) Let $\mathfrak{F}$ be a subgroup-closed saturated formation. Then $\mathfrak{F}$ is a lattice formation if and only if $\mathfrak{F} = \mathfrak{M} \times \mathfrak{H}$*

for some subgroup-closed saturated formations $\mathfrak{M}$ and $\mathfrak{H}$ satisfying the following conditions:

(a) $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$;
(b) $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$, *and* $\mathfrak{M}$ *is an* $\mathfrak{M}^2$*-normal Fitting class;*
(c) *every non-cyclic* $\mathfrak{M}$*-critical group* G *with* $\Phi(G) = 1$ *is a primitive group of type* 2 *such that* $G/Soc(G)$ *is a cyclic group of prime power order;*
(d) *there exists a partition* $\{\pi_i | i \in I\}$ *of* $\pi(\mathfrak{H})$ *such that* $\mathfrak{H} = \times_{i\in I}\mathfrak{S}_{\pi_i}$.

Corollary 2. *(See [2, Theorem 6.3.9], [6, Theorem 3.1.22], [11].) Let* $\mathfrak{F}$ *be a subgroup-closed saturated formation. Then* $\mathfrak{F}$ *is a lattice formation if and only if* $\mathfrak{F}$ *is a K-lattice formation.*

Corollary 3. *(See [2, Theorem 6.3.25], [10].) Let* $\mathfrak{F}$ *be a subgroup-closed formation of soluble groups. The following statements are pairwise equivalent.*

1. *The set of all K-*$\mathfrak{F}$*-subnormal subgroups is a sublattice of the subgroup lattice of every group.*
2. *The set of all* $\mathfrak{F}$*-subnormal subgroups is a sublattice of the subgroup lattice of every group.*
3. *There exists a partition* $\{\pi_i | i \in I\}$ *of the set* $\pi(\mathfrak{F})$ *such that* $\mathfrak{F} = \times_{i\in I}\mathfrak{F}_{\pi_i}$*, where* $\mathfrak{F}_{\pi_i} = \mathfrak{F} \cap \mathfrak{S}_{\pi_i}$*. Moreover,* $\mathfrak{F}_{\pi_i} = \mathfrak{S}_{\pi_i}$ *for all* $i \in I$ *such that* $|\pi_i| > 1$.

Corollary 4. *(See [1], [2, Corollary 6.3.16], [11].) Let* $\mathfrak{F}$ *be a subgroup-closed saturated formation of soluble groups. Then* $\mathfrak{F}$ *is a lattice formation if and only if there exists a partition* $\{\pi_i | i \in I\}$ *of* $\pi(\mathfrak{F})$ *such that* $\mathfrak{F} = \times_{i\in I}\mathfrak{S}_{\pi_i}$.

2. Definitions and preliminary results

The notation and terminology agree with the books [2,4]. We refer the reader to these books for the results on formations.

Our aim in this section is to collect some definitions and results that are needed in the sequel.

Recall that a *formation* is a class of groups which is closed under taking homomorphic images and finite subdirect products. If $\mathfrak{F}$ is a non-empty formation, then each group G has the $\mathfrak{F}$*-residual* $G^{\mathfrak{F}}$, the smallest normal subgroup whose quotient belongs to $\mathfrak{F}$.

We say that $\mathfrak{F}$ is *subgroup-closed* if $\mathfrak{F}$ is closed under taking subgroups.

We use $\mathfrak{E}$, $\mathfrak{S}$ and $\mathfrak{N}$ to denote the class of all groups, the class of all soluble groups and the class of all nilpotent groups, respectively. If π is a set of primes, $\mathfrak{F}_\pi$ denotes the class of all π-groups in $\mathfrak{F}$. In the sequel, $\pi(\mathfrak{F})$ is the set of primes p such that p divides the order of some group in $\mathfrak{F}$.

Let $\mathfrak{F}$ and $\mathfrak{G}$ be formations such that $\pi(\mathfrak{F}) \cap \pi(\mathfrak{G}) = \emptyset$. Denote $\pi_1 = \pi(\mathfrak{F})$ and $\pi_2 = \pi(\mathfrak{G})$. Then

$$\mathfrak{F} \times \mathfrak{G} = (G | G = O_{\pi_1}(G) \times O_{\pi_2}(G), O_{\pi_1}(G) \in \mathfrak{F}, O_{\pi_2}(G) \in \mathfrak{G})$$

is a formation. Moreover, if $\mathfrak{F}$ and $\mathfrak{G}$ are saturated, then $\mathfrak{F} \times \mathfrak{G}$ is saturated and, if $\mathfrak{F}$ and $\mathfrak{G}$ are subgroup-closed, then $\mathfrak{F} \times \mathfrak{G}$ is also subgroup-closed.

The construction $\mathfrak{F} \times \mathfrak{G}$ could be generalized along the following lines: Let I be a non-empty set. For each $i \in I$, let $\mathfrak{F}_i$ be a formation. Assume that $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \emptyset$ for all $i, j \in I$, $i \neq j$. Denote $\pi_i = \pi(\mathfrak{F}_i)$, $i \in I$. Then

$$\times_{i \in I} \mathfrak{F}_i = (G = O_{\pi_{i_1}}(G) \times \ldots \times O_{\pi_{i_n}}(G) | O_{\pi_{i_j}}(G) \in \mathfrak{F}_{i_j}, 1 \leq j \leq n, \{i_1, \ldots, i_n\} \subseteq I)$$

is a formation.

The main properties of $\mathfrak{F}$-subnormal and K-$\mathfrak{F}$-subnormal subgroups are listed in the following lemma.

Lemma 2.1. *(See [2, 6.1.6 and 6.1.7].) Let $\mathfrak{F}$ be a non-empty formation. Let $e \in \{sn_{\mathfrak{F}}, sn_{K-\mathfrak{F}}\}$. Let H and N be subgroups of a group G. Suppose that N is normal in G. Then:*

(1) *If $H \in e(G)$, then $HN/N \in e(G/N)$.*
(2) *If $N \subseteq H$, then $H \in e(G)$ if and only if $H/N \in e(G/N)$.*
(3) *If $\mathfrak{F}$ is subgroup-closed and $H \in e(G)$, then $H \cap K \in e(K)$ for every subgroup K of G.*
(4) *If $\mathfrak{F}$ is subgroup-closed and H contains the $\mathfrak{F}$-residual of G, then $H \in e(G)$.*
(5) *If $H \in e(K)$ and $K \in e(G)$, then $H \in e(G)$.*

Let us denote by $\mathfrak{F}(sub)$ the class of all groups G such that every subgroup of G is $\mathfrak{F}$-subnormal in G. We will need the following information about the properties of the class $\mathfrak{F}(sub)$.

Lemma 2.2. *(See [2, Proposition 6.3.23], [10, Lemmas 2.8, 2.9, 2.10 and 2.14].) Let $\mathfrak{F}$ be a subgroup-closed lattice formation. Then:*

(1) *$\mathfrak{F}(sub)$ is a subgroup-closed formation.*
(2) *$\mathfrak{F} \subseteq \mathfrak{F}(sub)$ and $\pi(\mathfrak{F}) = \pi(\mathfrak{F}(sub))$.*
(3) *$G \in \mathfrak{F}(sub)$ if and only if $1 \in sn_{\mathfrak{F}}(G)$ and $P \in sn_{\mathfrak{F}}(G)$ for every Sylow subgroup P of G.*
(4) *If $\mathfrak{F}$ is soluble, then $\mathfrak{F}(sub)$ is a saturated formation.*

A formation $\mathfrak{F}$ is called *saturated* if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

A formation $\mathfrak{F}$ is said to be *solubly saturated* if $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$ (the symbol $G_{\mathfrak{S}}$ denotes the largest soluble normal subgroup of G). Obviously, every saturated formation is solubly saturated. The converse is not true.

Lemma 2.3. *(See [5, Theorem A].) Any solubly saturated subgroup-closed lattice formation is saturated.*

Lemma 2.4. *Let H be a subgroup of a group G and $\mathfrak{H} \in \{\mathfrak{S}, \mathfrak{E}_{\pi}\}$, where π is a set of primes. Then H is $\mathfrak{H}$-subnormal in G if and only if $G^{\mathfrak{H}} \subseteq H$.*

Proof. The result follows easily from the definition of $\mathfrak{H}$-subnormal subgroup and the fact that $\mathfrak{H}^2 = \mathfrak{H}$. □

Lemma 2.5. *Let $\mathfrak{F}$ be a subgroup-closed formation and $\mathfrak{H} \in \{\mathfrak{S}, \mathfrak{E}_{\pi}\}$, where π is a set of primes. Then*

$$(\mathfrak{F} \cap \mathfrak{H})(sub) = \mathfrak{F}(sub) \cap \mathfrak{H}.$$

Moreover, if $\mathfrak{F}$ is lattice, then $\mathfrak{F} \cap \mathfrak{H}$ is also a subgroup-closed lattice formation.

Proof. For any group G, it follows from Lemma 2.4 that

$$sn_{\mathfrak{F}\cap\mathfrak{H}}(G) = sn_{\mathfrak{F}}(G) \cap \{H \subseteq G | G^{\mathfrak{H}} \subseteq H\}.$$

It is also known that $\mathfrak{F} \cap \mathfrak{H}$ is a subgroup-closed formation. If $\mathfrak{F}$ is lattice, it is clear now that $\mathfrak{F} \cap \mathfrak{H}$ is also lattice.

To complete the proof, we note that, in any case,

$$\mathfrak{F}(sub) \cap \mathfrak{H} \subseteq (\mathfrak{F} \cap \mathfrak{H})(sub).$$

For the converse, if $G \in (\mathfrak{F}\cap\mathfrak{H})(sub)$, then $1 \in sn_{\mathfrak{F}\cap\mathfrak{H}}(G)$, which implies $G \in \mathfrak{H}$. Therefore $G \in \mathfrak{F}(sub) \cap \mathfrak{H}$. Hence $(\mathfrak{F} \cap \mathfrak{H})(sub) = \mathfrak{F}(sub) \cap \mathfrak{H}$. □

Lemma 2.6. *Let $\mathfrak{F}$ be a subgroup-closed lattice formation. Then $\mathfrak{F}(sub)$ is a subgroup-closed lattice saturated formation.*

Proof. By Lemma 2.2, the class $\mathfrak{F}(sub)$ is a subgroup-closed formation. It is clear that a subgroup H of a group G is $\mathfrak{F}$-subnormal in G if and only if H is $\mathfrak{F}(sub)$-subnormal in G. It follows that $\mathfrak{F}(sub)$ is a lattice formation.

We claim that $\mathfrak{F}(sub)$ is solubly saturated. It will follow then that it is saturated by Lemma 2.3.

By Lemma 2.5, we have that $\mathfrak{F} \cap \mathfrak{S}$ is a subgroup-closed lattice formation. Hence $(\mathfrak{F} \cap \mathfrak{S})(sub) = \mathfrak{F}(sub) \cap \mathfrak{S}$ is saturated by Lemma 2.2(4).

To prove the claim we consider a group G such that $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}(sub)$. In order to deduce that $G \in \mathfrak{F}(sub)$ it is enough to prove that $1 \in sn_{\mathfrak{F}}(G)$ and $P \in sn_{\mathfrak{F}}(G)$ for every Sylow subgroup P of G by Lemma 2.2(3).

Let P be a Sylow subgroup of G. We note that $\Phi(G_{\mathfrak{S}}) \subseteq \Phi(G_{\mathfrak{S}}P)$, and $G_{\mathfrak{S}}P/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}(sub) \cap \mathfrak{S}$ because $\mathfrak{F}(sub)$ is subgroup-closed and $G_{\mathfrak{S}}P$ is soluble. Consequently, $G_{\mathfrak{S}}P \in \mathfrak{F}(sub)$. Therefore, $1, P \in sn_{\mathfrak{F}}(G_{\mathfrak{S}}P)$ and $G_{\mathfrak{S}}P \in sn_{\mathfrak{F}}(G)$, which implies that $1, P \in sn_{\mathfrak{F}}(G)$ and the claim follows. $\square$

Lemma 2.7. *Let $\mathfrak{F}$ be a subgroup-closed formation, $\pi = \pi(\mathfrak{F})$ and $|\pi| > 1$. If $\mathfrak{F}(sub) = \mathfrak{S}_\pi\mathfrak{F}(sub)$, then $\mathfrak{F}(sub) = \mathfrak{F}$.*

Proof. Since $\mathfrak{F}$ is subgroup-closed we have, by Statement (2) of Lemma 2.2, that $\mathfrak{F} \subseteq \mathfrak{F}(sub)$. Assume that $\mathfrak{F} \neq \mathfrak{F}(sub)$ and let $G \in \mathfrak{F}(sub) \setminus \mathfrak{F}$ of minimal order. Since $\mathfrak{F}(sub)$ is closed under taking factor groups and $\mathfrak{F}$ is a formation, we can deduce that G has a unique minimal normal subgroup, say N. Set $q \in \pi$. If N is a p-group for a prime p, take $q \neq p$, which is possible since $|\pi| > 1$. Let V be an irreducible and faithful G-module over the finite field of q elements (such module exists by [4, Theorem B.10.9], and consider $X = [V]G$ the corresponding semidirect product. Then X is a primitive group, and G is a maximal subgroup of X with $Core_X(G) = 1$. Since $X \in \mathfrak{S}_\pi\mathfrak{F}(sub) = \mathfrak{F}(sub)$, we deduce that G is $\mathfrak{F}$-subnormal in X, which implies that $X \simeq X/Core_X(G) \in \mathfrak{F}$. Consequently, $G \simeq X/V \in \mathfrak{F}$, a contradiction which concludes the proof. $\square$

3. Proof of the main theorem

Proof. $1 \Rightarrow 2$. Applying [2, Lemma 6.3.7], we have that 1 implies 2.

$2 \Rightarrow 3$. Let $\mathfrak{F}$ be a subgroup-closed lattice formation. Consider the formation $\mathfrak{F}(sub)$. By Lemma 2.6, $\mathfrak{F}(sub)$ is a subgroup-closed lattice saturated formation. Hence, by [6, Theorem 3.1.22] (see also [2, Theorem 6.3.15]), $\mathfrak{F}(sub) = \mathfrak{M} \times \mathfrak{H}$ for some subgroup-closed saturated formations $\mathfrak{M}$ and $\mathfrak{H}$ satisfying the following conditions: $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$; there exists a partition $\{\pi_i | i \in I\}$ of $\pi(\mathfrak{H})$ such that $\mathfrak{H} = \times_{i\in I}\mathfrak{S}_{\pi_i}$; $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$, and $\mathfrak{M}$ is an $\mathfrak{M}^2$-normal Fitting class; every non-cyclic $\mathfrak{M}$-critical group G with $\Phi(G) = 1$ is a primitive group of type 2 such that $G/Soc(G)$ is a cyclic group of prime power order.

Since $\mathfrak{F} \subseteq \mathfrak{F}(sub)$ by Statement (2) of Lemma 2.2, the formation $\mathfrak{F}$ is represented in the form $\mathfrak{F} = \mathfrak{F}_\omega \times \mathfrak{F}_\sigma$, where $\omega = \pi(\mathfrak{M})$, $\sigma = \pi(\mathfrak{H})$. If $J = \{i \in I | \mid \pi_i \mid > 1\}$, then $\mathfrak{F}_\sigma = (\times_{j\in J}\mathfrak{F}_{\pi_j}) \times (\times_{i\in I\setminus J}\mathfrak{F}_{\pi_i})$. Write $\mathfrak{K} = (\times_{j\in J}\mathfrak{F}_{\pi_j})$ and $\mathfrak{L} = (\times_{i\in I\setminus J}\mathfrak{F}_{\pi_i})$.

It is clear that $\mathfrak{L} \subseteq \mathfrak{N}_{\pi(\mathfrak{L})}$.

By Lemma 2.5,

$$\mathfrak{F}_\omega(sub) = (\mathfrak{F} \cap \mathfrak{G}_\omega)(sub) = \mathfrak{F}(sub) \cap \mathfrak{G}_\omega = \mathfrak{M}.$$

Applying Lemma 2.7, $\mathfrak{F}_\omega = \mathfrak{M}$.

By Lemma 2.5,

$$\mathfrak{F}_{\pi_j}(sub) = (\mathfrak{F} \cap \mathfrak{G}_{\pi_j})(sub) = \mathfrak{F}(sub) \cap \mathfrak{G}_{\pi_j} = \mathfrak{S}_{\pi_j}.$$

Applying Lemma 2.7, $\mathfrak{F}_{\pi_j} = \mathfrak{S}_{\pi_j}$. Thus $\mathfrak{K} = \times_{j \in J} \mathfrak{F}_{\pi_j} = \times_{j \in J} \mathfrak{S}_{\pi_j}$.

So $\mathfrak{F}$ is represented in the form $\mathfrak{F} = \mathfrak{M} \times \mathfrak{K} \times \mathfrak{L}$, and the following conditions hold:

(1) $\mathfrak{M}$, $\mathfrak{K}$ and $\mathfrak{L}$ are subgroup-closed formations.
(2) $\pi(\mathfrak{M}) \cap \pi(\mathfrak{K}) = \emptyset$, $\pi(\mathfrak{K}) \cap \pi(\mathfrak{L}) = \emptyset$ and $\pi(\mathfrak{M}) \cap \pi(\mathfrak{L}) = \emptyset$.
(3) $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ is a saturated formation, and it is an $\mathfrak{M}^2$-normal Fitting class.
(4) Every non-cyclic $\mathfrak{M}$-critical group G with $\Phi(G) = 1$ is a primitive group of type 2 such that $G/Soc(G)$ is a cyclic group of prime power order.
(5) There exists a partition $\{\pi_j \mid j \in J\}$ of $\pi(\mathfrak{K})$ such that $\mathfrak{K} = \times_{j \in J} \mathfrak{S}_{\pi_j}$. Moreover, $|\pi_j| > 1$ for all $j \in J$.
(6) $\mathfrak{L} \subseteq \mathfrak{N}_{\pi(\mathfrak{L})}$.

3 ⇒ 1. Assume that $\mathfrak{F} = \mathfrak{M} \times \mathfrak{K} \times \mathfrak{L}$. Moreover, $\mathfrak{M}$, $\mathfrak{K}$ and $\mathfrak{L}$ are subgroup-closed formations, and the conditions (a)–(e) of the Theorem hold. Consider the subgroup-closed formation $\mathfrak{H} = \mathfrak{M} \times \mathfrak{K} \times \mathfrak{N}_{\pi(\mathfrak{L})}$. By [2, Theorem 6.3.15], $\mathfrak{H}$ is a saturated lattice formation and $\pi(\mathfrak{H}) = \pi(\mathfrak{F})$. We aim to show that $sn_{K-\mathfrak{H}}(G) = sn_{K-\mathfrak{F}}(G)$ for every group G. Assume, arguing by contradiction, there exists a group G of minimal order such that $sn_{K-\mathfrak{H}}(G) \neq sn_{K-\mathfrak{F}}(G)$. Clearly $sn_{K-\mathfrak{F}}(G) \subseteq sn_{K-\mathfrak{H}}(G)$ because $\mathfrak{F} \subseteq \mathfrak{H}$. Hence there exists a subgroup $H \in sn_{K-\mathfrak{H}}(G) \setminus sn_{K-\mathfrak{F}}(G)$. Then H is a proper subgroup of G and thus there exists a subgroup M of G such that either M is normal in G or M is an $\mathfrak{H}$-normal maximal subgroup of G. Since H is K-$\mathfrak{H}$-subnormal in M, it follows that H is K-$\mathfrak{F}$-subnormal in M by minimality of G. If M were normal in G, we would have that H would be K-$\mathfrak{F}$-subnormal in G. This would contradict our choice of H. Hence M is an $\mathfrak{H}$-normal maximal subgroup of G and so $G^{\mathfrak{H}}$ is contained in M. Then $G/Core_G(M)$ is a primitive $\mathfrak{H}$-group. Thus $G/Core_G(M) \in \{\mathfrak{M}, \mathfrak{K}\}$ because M is not normal in G. This means that M is $\mathfrak{F}$-normal in G and H is K-$\mathfrak{F}$-subnormal in G, contrary to our initial supposition. Therefore $sn_{K-\mathfrak{F}}(X) = sn_{K-\mathfrak{H}}(X)$ for all groups X. By [2, Theorem 6.3.9], the set $sn_{K-\mathfrak{F}}(X)$ is a sublattice of the subgroup lattice of X for all groups X. □

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