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# Decomposing modular tensor products, and periodicity of ‘Jordan partitions’

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## ABSTRACT

Let  $J_r$  denote an  $r \times r$  matrix with minimal and characteristic polynomials  $(t - 1)^r$ . Suppose  $r \leq s$ . It is not hard to show that the Jordan canonical form of  $J_r \otimes J_s$  is similar to  $J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$  where  $\lambda_1 \geq \cdots \geq \lambda_r > 0$  and  $\sum_{i=1}^r \lambda_i = rs$ . The partition  $\lambda(r, s, p) := (\lambda_1, \dots, \lambda_r)$  of  $rs$ , which depends only on  $r, s$  and the characteristic  $p := \text{char}(F)$ , has many applications including the study of algebraic groups. We prove new periodicity and duality results for  $\lambda(r, s, p)$  that depend on the smallest  $p$ -power exceeding  $r$ . This generalizes results of J.A. Green, B. Srinivasan, and others which depend on the smallest  $p$ -power exceeding the (potentially large) integer  $s$ . It also implies that for fixed  $r$  we can construct a finite table allowing the computation of  $\lambda(r, s, p)$  for all  $s$  and  $p$ , with  $s \geq r$  and  $p$  prime.

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## 1. Introduction

Consider a matrix whose minimal and characteristic polynomials equal  $(t-1)^r$ . To be explicit, take the  $r \times r$  matrix  $J_r$  with 1s in positions  $(i, i)$  for  $1 \leq i \leq r$ , and  $(i, i+1)$  for  $1 \leq i < r$ , and zeros elsewhere. Suppose  $1 \leq r \leq s$ . Then the Jordan canonical form of  $J_r \otimes J_s$  is a direct sum  $J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ , with precisely  $r$  nonempty blocks, see Lemma 9(a). This decomposition depends on the characteristic  $p$  of the underlying field<sup>3</sup>  $F$ , and it determines a partition  $\lambda(r, s, p) = (\lambda_1, \dots, \lambda_r)$  of  $rs$  since  $J_r \otimes J_s$  is an  $rs \times rs$  matrix. We will assume that  $\lambda_1 \geq \cdots \geq \lambda_r > 0$ . The determination of this ‘Jordan partition’<sup>4</sup> has applications to many significant problems. The representation theory of algebraic groups is governed by the behaviour of the unipotent elements, and indeed properties of  $\lambda(r, s, p)$  are particularly useful (when  $p > 0$ ) for the study of exceptional algebraic groups, see [14, 12]. More generally, Lindsey [13, Theorem 1] gives a useful (though somewhat technical) lower bound on the degree of the minimal faithful representation in characteristic  $p$  for certain groups with a prescribed Sylow  $p$ -subgroup structure. Lindsey’s result, in turn, may be applied to the study of primitive permutation groups of  $p$ -power degree, see [19].

The most direct application, and the oldest, is to the study of modular representations of finite cyclic  $p$ -groups. Given two indecomposable modules  $V_r$  and  $V_s$  of a cyclic group  $G$  of order  $p^n$ , the module  $V_r \otimes V_s$  is, by the Krull–Schmidt theorem, a sum of indecomposable modules  $V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}$ . Thus when  $p > 0$ , the partition  $\lambda(r, s, p)$  arises naturally in this context too. The connection with matrices is straightforward:  $G = \langle g \rangle$  has precisely  $p^n$  pairwise nonisomorphic indecomposable modules  $V_1, \dots, V_{p^n}$  which correspond to the matrix representations  $G \rightarrow \mathrm{GL}(r, \mathbb{F}_p): g \mapsto J_r$  where  $1 \leq r \leq p^n$ .

**Definition 1.** The following terminology will be used as convenient abbreviations.

- (a) For integers  $r, s$  with  $1 \leq r \leq s$ , the **standard partition**  $\lambda = (\lambda_1, \dots, \lambda_r)$  of  $rs$  is the partition with  $\lambda_i = r + s - 2i + 1$  for  $1 \leq i \leq r$ , i.e.  $(s + r - 1, \dots, s - r + 1)$ .
- (b) Call  $\lambda = (\lambda_1, \dots, \lambda_r)$  the **( $r$ -)uniform partition** of  $rs$  if  $\lambda_i = s$ , for  $1 \leq i \leq r$ .
- (c) The vector  $\varepsilon(r, s, p) = (\varepsilon_1, \dots, \varepsilon_r)$  with  $\varepsilon_i = \lambda_i - s$ , which measures the deviation of  $\lambda(r, s, p) = (\lambda_1, \dots, \lambda_r)$  from the uniform vector, is called the **deviation vector**.
- (d) The **negative reverse** of  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$  is  $(\overline{\varepsilon_1}, \overline{\varepsilon_2}, \dots, \overline{\varepsilon_r}) := (-\varepsilon_r, \dots, -\varepsilon_2, -\varepsilon_1)$ .
- (e) The  **$k$ -multiple** of  $(\lambda_1, \dots, \lambda_r)$  is the vector  $(k\lambda_1, \dots, k\lambda_1, \dots, k\lambda_r, \dots, k\lambda_r)$  of length  $kr$  where the size, and multiplicity, of each part is multiplied by  $k$ .

In characteristic zero, the partition  $\lambda(r, s, 0)$  was shown to be the standard partition independently by Aitken (1934), Roth (1934), and Littlewood (1936); for more background and references see [18, p. 416]. The change-of-basis matrix exhibiting the Jordan canonical form of  $J_r \otimes J_s$  may be chosen to have rational entries, and so in ‘large’

<sup>3</sup> We may assume that  $F = \mathbb{F}_p$  or  $\mathbb{Q}$  as the Jordan form of  $J_r \otimes J_s$  is invariant under field extensions.

<sup>4</sup> This phrase was used by Dmitri Panyushev in the review MR2728146, but it is not used commonly.

prime characteristic ( $p$  not dividing denominators of the matrix entries), it follows that  $\lambda(r, s, p)$  is also the standard partition. Srinivasan proved that  $\lambda(r, s, p)$  is standard for  $p \geq r + s - 1$ . Our first result generalizes the main results of both [23] and [1].

**Theorem 2.** *If  $r \leq s$ , and  $s \not\equiv 0, \pm 1, \pm 2, \dots, \pm(r-2) \pmod{p}$ , then  $\lambda(r, s, p)$  is the standard partition; i.e. its  $i$ th part is  $\lambda_i = r + s + 1 - 2i$  for  $1 \leq i \leq r$ .*

Throughout this paper we have  $p > 0$  and  $r \leq s$ . It is useful (psychologically) to think of  $r$  as ‘fixed and small’, and  $s$  and  $p$  as ‘variable’, and  $s$  as ‘large’. The seminal paper [8] by J.A. Green led to a series of different algorithms [20,23,15,16,21,22,10,11] for decomposing  $V_r \otimes V_s$  as  $V_{\lambda_1} \oplus \dots \oplus V_{\lambda_r}$ , where  $1 \leq r \leq s \leq p^n$ , and  $p^n$  is the smallest  $p$ -power exceeding  $s$ . One class of algorithms [8,23,15,21,22] involves recursive computations in the modular representation ring (or Green ring) of the cyclic group  $C_{p^n}$ . It is known that  $V_{p^i}$  for  $0 \leq i \leq n$  are generators for the Green ring; however, relations in these generators are rather mysterious. Another class of algorithms is related to  $p$ -adic expansions, and has a more number-theoretic flavour. The algorithms may depend on  $p$ -adic expansions of  $r$  and  $s$  as in [15], or on the values of certain determinants modulo  $p$  as in [23,11], or on so-called  $p$ -ranks [17,10]. Ideally the algorithms can construct a basis relative to which  $J_r \otimes J_s$  is in Jordan canonical form, see [1,18]. No complexity analysis presently exists to compare the time or space requirements of these algorithms.

In contrast to the Green ring results which assume  $1 \leq r \leq s \leq p^n$ , we assume only that  $r \leq \min\{s, p^m\}$ . For maximum impact take  $p^m$  to be the *smallest*  $p$ -power exceeding  $r$ . We first noticed a criterion for  $\lambda(r, s, p)$  to be the uniform partition.

**Proposition 3.** *If  $\text{char}(F) = p > 0$  and  $r \leq \min\{p^m, s\}$ , then  $\lambda(r, s, p)$  is the uniform partition, or equivalently  $\varepsilon(r, s, p) = (0, \dots, 0)$ , if and only if  $s \equiv 0 \pmod{p^m}$ .*

Proposition 3, which is more general than [21, Lemma 2.1], for example, suggested to us to study the deviation  $\varepsilon(r, s, p) = (\lambda_1, \dots, \lambda_r) - (s, \dots, s)$  from the uniform partition  $(s, \dots, s)$ . We found several properties of  $\varepsilon(r, s, p)$  that depend only on the congruence of  $s$  modulo  $p^m$ , where  $m = \lceil \log_p(r) \rceil$ , see Theorem 4. These properties generalize periodicity and duality results for Green rings, cf. [21, Eq. G-1].

**Theorem 4.** *Suppose  $r \leq \min\{s, s', p^m\}$  where  $p = \text{char}(F)$ .*

- (a) [Periodicity] *If  $s \equiv s' \pmod{p^m}$ , then  $\varepsilon(r, s, p) = \varepsilon(r, s', p)$ .*
- (b) [Duality] *If  $s' \equiv -s \pmod{p^m}$ , then  $\varepsilon(r, s', p)$  is the negative reverse of  $\varepsilon(r, s, p)$ .*

Barry [2] has already used Theorem 4 to classify all triples  $(r, s, p)$  for which  $\lambda(r, s, p)$  is standard, and Barry’s result already has an application, namely determining whether so-called ‘Jordan permutations’ are involutions or are trivial, see [5].

Section 2 introduces notation and terminology whilst establishing an important result that is computationally advantageous: computing with the nilpotent matrix  $J_r \otimes I_s + I_r \otimes$

**Table 1**

Values of  $\varepsilon(r, s, p)$  with  $r \leq \min\{5, s\}$  and  $m = \lceil \log_p(r) \rceil$ . See the paragraph beginning ‘How is Table 1 used?’; below  $p'$  is a prime  $\geq 2r - 3$ .

$s \bmod p^m$	0	1	2	3	4
$\varepsilon(1, s, p')$	(0)				
$\varepsilon(2, s, p')$	(0,0)	(1, -1)			
$\varepsilon(3, s, 2)$	(0,0,0)	(2, -1, -1)	(2, 0, -2)		
$\varepsilon(3, s, p')$	(0,0,0)	(2, -1, -1)	(2, 0, -2)		
$\varepsilon(4, s, 2)$	(0,0,0,0)	(3, -1, -1, -1)	(2, 2, -2, -2)		
$\varepsilon(4, s, 3)$	(0,0,0,0)	(3, -1, -1, -1)	(3, 1, -2, -2)	(3, 0, 0, -3)	(3, 1, -1, -3)
$\varepsilon(4, s, p')$	(0,0,0,0)	(3, -1, -1, -1)	(3, 1, -2, -2)	(3, 1, -1, -3)	
$\varepsilon(5, s, 2)$	(0,0,0,0,0)	(4, -1, -1, -1, -1)	(4, 2, -2, -2, -2)	(4, 1, 1, -3, -3)	(4, 0, 0, 0, -4)
$\varepsilon(5, s, 3)$	(0,0,0,0,0)	(4, -1, -1, -1, -1)	(4, 2, -2, -2, -2)	(3, 3, 0, -3, -3)	(4, 2, 0, -2, -4)
$\varepsilon(5, s, 5)$	(0,0,0,0,0)	(4, -1, -1, -1, -1)	(3, 3, -2, -2, -2)		
$\varepsilon(5, s, p')$	(0,0,0,0,0)	(4, -1, -1, -1, -1)	(4, 2, -2, -2, -2)	(4, 2, 0, -3, -3)	(4, 2, 0, -2, -4)

$J_s$  rather than the natural unipotent matrix  $J_r \otimes J_s$ , cf. [24,25]. It is convenient to view the tensor product of vector spaces as a polynomial algebra; then submodules correspond to ideals. In Section 3 we prove the sufficient condition Theorem 2 for  $\lambda(r, s, p)$  to be standard, and also prove:

**Theorem 5.** *If  $r \leq s$  and  $k \geq 0$ , then  $\lambda(p^k r, p^k s, p)$  is the  $p^k$ -multiple of  $\lambda(r, s, p)$ .*

Renaud [21, Lemma 2.2] proved this result (using the language of Green rings) under the additional assumption that  $r \leq s \leq p$  which we do not need. Our  $s$  can be large.

In Section 4 we prove Proposition 3 and Theorem 4, and find  $\varepsilon(r, s, p)$  explicitly when  $s \equiv \pm 1$  or  $\pm 2 \pmod{p^m}$ , see Propositions 13 and 14. In Section 5 we prove the following theorem.

**Theorem 6.** *For a fixed integer  $r \geq 1$ , there are at most  $2^{r-1}$  different deviation vectors  $\varepsilon(r, s, p)$  as both  $s$ , where  $s \geq r$ , and the prime  $p$  vary.*

It was shown in [11, Theorem 2.1.5] that a finite computation is required to determine  $\lambda(r, s, p)$  when both  $r$  and  $s$  are fixed, and  $p$  varies. Table 1 lists the possible deviation vectors  $\varepsilon(r, s, p)$  for  $r \leq 5$ . In Section 5 we prove the generalization (Theorem 7) which allows us to create Table 1. In the proof of Theorem 7 we show that, for a given  $r$ , the number of values of  $s$  and of  $p$  we need to consider are each bounded above in terms of  $r$ .

How is Table 1 used? This table lists the values of  $\varepsilon(r, s, p)$  for  $r \leq \min\{5, s\}$ . It explicitly lists the ‘small’ primes  $p < 2r - 3$ ; these may have  $m > 1$ . The infinitely many ‘large’ primes  $p' \geq 2r - 3$  all have  $m = 1$ . For the small primes  $p$  it suffices, by duality, to list the  $s \pmod{p^m}$  for which  $0 \leq s \pmod{p^m} \leq p^m/2$ . For the large primes  $p'$  it suffices, by Theorem 2 and duality, to list the  $s \pmod{p'}$  for which  $0 \leq s \pmod{p'} \leq r - 2$ . The values of  $s \pmod{p'}$  satisfying  $r - 1 \leq s \pmod{p'} \leq p' - (r - 1)$ , have  $\lambda(r, s, p')$  standard. Thus  $\varepsilon(r, s, p') = (r - 1, r - 3, \dots, -(r - 3), -(r - 1))$  for these  $p' - (2r - 3)$  choices of  $s \pmod{p'}$ . We also list this ‘standard vector’  $\varepsilon(r, s, p')$  for  $s \pmod{p'} = r - 1$ . Note that for  $(r, p') = (3, 3), (4, 5), (5, 7)$ , no value of  $s$  gives rise to this standard vector as

$p' - (2r - 3) = 0$ . When computing  $\varepsilon(4, 17, 3)$  we have  $m = 2$  as  $3 < 4 \leq 3^2$ . The first equality below is by periodicity, the second is by duality, the third is by Table 1, and the fourth is by the definition (see 1(e)) of ‘negative reverse’:

$$\varepsilon(4, 17, 3) = \varepsilon(4, 8, 3) = \overline{\varepsilon(4, 10, 3)} = \overline{(3, -1, -1, -1)} = (1, 1, 1, -3).$$

Also  $\varepsilon(5, s, 11) = (4, 2, 0, -2, -4)$  for  $s = 15, 16, 17, 18$ , and  $p' - (2r - 3) = 11 - 7 = 4$ .

**Theorem 7.** *For fixed  $r$ , a finite computation suffices to compute the values of  $\varepsilon(r, s, p)$  for all  $s$  with  $s \geq r$ , and all primes  $p$ .*

## 2. Notation and basic results

This section introduces notation and establishes facts needed for proofs in subsequent sections. Parts (a)–(c) of Lemma 9 have been proved before in [20, Lemma 2.1], [23, p. 678], and [16, Theorem 2], but because we want to build on their proofs, it is desirable to give new proofs using our polynomial notation. Our alternative proofs, which are based on Lemma 8 and the preamble to Lemma 9, are much shorter than the original proofs.

Fix a positive integer  $r$  and a field  $F$ , and consider the quotient polynomial ring  $B := F[X]/(X^r)$ . Set  $x := X + (X^r)$ . Then  $x^r = 0$  and  $1, x, \dots, x^{r-1}$  is a basis for  $B$ . As usual, right multiplication gives rise to a monomorphism  $\mu: B \rightarrow \text{End}_F(B)$  where for  $b \in B$  the  $F$ -linear map  $\mu_b := \mu(b)$  satisfies  $\mu_b(a) = ab$ . Denote the matrices of  $\mu_x$ ,  $\mu_{1+x}$ , and  $\mu_{\alpha+x}$  relative to the basis  $1, x, \dots, x^{r-1}$  by

$$N_r = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad J_r = \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \quad J_r(\alpha) = \begin{pmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \alpha & 1 \\ & & & \alpha \end{pmatrix},$$

respectively. We say  $N_r$  is *nilpotent*,  $J_r$  is *unipotent*, and  $J_r(\alpha)$  is  $\alpha$ -*potent*, i.e.  $J_r(\alpha) - \alpha I$  is nilpotent. Moreover,  $J_r(\alpha)$  is an  $r \times r$  Jordan block, and its minimal polynomial is  $(t - \alpha)^r$ . The matrix  $J_r(\alpha) \otimes J_s(\beta)$  is an upper-triangular  $\alpha\beta$ -potent matrix, and so its Jordan canonical form is  $\text{JCF}(J_r(\alpha) \otimes J_s(\beta)) = \bigoplus_{t \geq 1} c_{r,s,t} J_t(\alpha\beta)$  where  $c_{r,s,t} \in \mathbb{N}$  denotes the multiplicity of the  $t \times t$  Jordan block  $J_t(\alpha\beta)$ . The Jordan partition  $\lambda := \langle t^{c_{r,s,t}} \rangle$  of  $rs$ , with part size  $t$  occurring with multiplicity  $c_{r,s,t}$ , has been studied by many authors. In the nilpotent case (when  $\alpha\beta = 0$ ) the multiplicities  $c_{r,s,t}$  are easily described (see for example [11, 2.1.2]). Furthermore, the change of basis matrix is known, and is field independent. The invertible case (when  $\alpha\beta \neq 0$ ) reduces to the unipotent case because

$$\text{JCF}(J_r(\alpha) \otimes J_s(\beta)) = \bigoplus_{t \geq 1} c_{r,s,t} J_t(\alpha\beta) \quad \text{if and only if} \quad \text{JCF}(J_r \otimes J_s) = \bigoplus_{t \geq 1} c_{r,s,t} J_t.$$

Thus the same Jordan partitions arise, and the change of basis matrices are easily related.



**Fig. 1.** (a)  $r \times s$  basis elements  $x^i y^j$ ; and (b)–(d) dimensions of sections.

**Table 2**

The action of  $\mu_{(1+x)(1+y)}$  on submodules and quotient modules of  $A$ .

$H_k$	$A/H_k$	$V_k$	$A/V_k$	$H_{k-1}/H_k$	$V_{k-1}/V_k$	$D_{k-1}/D_k$
$J_{r-k} \otimes J_s$	$J_k \otimes J_s$	$J_r \otimes J_{s-k}$	$J_r \otimes J_k$	$J_s$	$J_r$	$I$

The quotient polynomial algebra  $A := F[X, Y]/(X^r, Y^s)$  is an  $rs$ -dimensional  $F$ -vector space with basis  $x^i y^j$ ,  $0 \leq i < r$ ,  $0 \leq j < s$ , where  $x := X + (X^r, Y^s)$  and  $y := Y + (X^r, Y^s)$  satisfy  $x^r = y^s = 0$ . Given  $a \in A$  denote by  $F[a]$  the subalgebra  $\{f(a) \mid f(t) \in F[t]\}$  of  $A$ . Then  $A$ , viewed as a module over the ring  $F[a]$ , is a direct sum  $A = a_1 F[a] \oplus \cdots \oplus a_n F[a]$  of cyclic  $F[a]$ -submodules. To avoid ambiguity, we regard  $A$  as an  $F[a]$ -module rather than more conventionally as an  $F[t]$ -module, see [9]. Indeed,  $F[a]$  is a quotient of the principal ideal domain  $F[t]$ . We are interested in the dimensions of the cyclic submodules when  $a = (1+x)(1+y)$ . Since  $F[a] = F[a-1]$  it is convenient to replace the invertible unipotent element  $(1+x)(1+y)$  with the nilpotent element  $(1+x)(1+y) - 1 = x + y + xy$ . We show that  $x + y + xy$  and  $x + y$  induce *similar* linear transformations on  $A$ . The action of  $x + y$  on the basis  $x^i y^j$  is simple:  $x^i y^j (x + y) = x^{i+1} y^j + x^i y^{j+1}$ . We seek another basis  $f_{i,j}$  for  $A$  such that  $f_{i,j}(x + y + xy) = f_{i+1,j} + f_{i,j+1}$ . An easy calculation shows that  $f_{i,j} = x^i (1+y)^i y^j$ ,  $0 \leq i < r$ ,  $0 \leq j < s$ , is the desired basis. This is a major point in [24,25]. View  $A$  as a module over

$$F[(1+x)(1+y)] = F[x + y + xy], \quad \text{or over} \quad F[x + y]. \quad (1)$$

(Incidentally, when decomposing  $J_r \otimes J_s \otimes J_t$  it is similarly useful to consider a module over  $F[x + y + z]$  instead of  $F[(1+x)(1+y)(1+z)]$  by using  $f_{i,j,k} = x^i (1+y)^i (1+z)^i y^j (1+z)^j z^k$ .)

View the basis elements  $x^i y^j$ ,  $0 \leq i < r$ ,  $0 \leq j < s$ , of  $A$  as placed on an  $r \times s$  rectangle, see Fig. 1(a). Consider the following ‘horizontal’, ‘vertical’, and ‘diagonal’ ideals of  $A$ :

$$\begin{aligned} H_k &= \langle x^i y^j \mid i \geq k \rangle, & A &= H_0 > H_1 > \cdots > H_r = 0, \\ V_k &= \langle x^i y^j \mid j \geq k \rangle, & A &= V_0 > V_1 > \cdots > V_s = 0, \\ D_k &= \langle x^i y^j \mid i + j \geq k \rangle, & A &= D_0 > D_1 > \cdots > D_{r+s-1} = 0. \end{aligned}$$

The matrix of  $\mu_{(1+x)(1+y)}$  is  $J_r \otimes J_s$  relative to the basis  $x^i y^j$ ,  $0 \leq i < r$ ,  $0 \leq j < s$ , ordered lexicographically by  $i$ , then  $j$ . The action of  $\mu_{(1+x)(1+y)}$  on the above ideals and their quotients (relative to this monomial basis) is given in Table 2.

Since  $H_{k-1}/H_k$  is a cyclic  $F[x+y]$ -module (generated by  $x^{k-1} + H_k$ ), it follows that  $A$  is a sum of at most  $r$  cyclic  $F[x+y]$ -submodules. Indeed,  $A = \sum_{i=0}^{r-1} x^i F[x+y]$ . Conversely,  $A$  is a sum of at least  $r$  cyclic  $F[x+y]$ -submodules because  $D_{r-1}/D_r$  is an  $r$ -dimensional vector space, and  $D_{r-1}(x+y) \subseteq D_r$ . Thus  $A$  is a sum of precisely  $r$  nonzero cyclic  $F[x+y]$ -submodules and  $\lambda(r, s, p)$  has precisely  $r$  nonzero parts, see Lemma 9(a) below. The dimensions of the sections  $H_{k-1}/H_k$ ,  $V_{k-1}/V_k$  and  $D_{k-1}/D_k$  can be seen from Fig. 1(b)–(d) to be:

$$\begin{aligned} \dim(H_{k-1}/H_k) &= s & \text{for } 1 \leq k \leq r, \\ \dim(V_{k-1}/V_k) &= r & \text{for } 1 \leq k \leq s, \\ \dim(D_{k-1}/D_k) &= \min\{k, r, r+s-k\} & \text{for } 1 \leq k < r+s. \end{aligned}$$

Define the *annihilator* of an element  $a \in A$  to be  $\text{Ann}(a) = \{b \in A \mid ab = 0\}$ . Then  $\text{Ann}(x+y)$  is an  $r$ -dimensional  $F[x+y]$ -submodule of  $A$  because  $\lambda(r, s, p)$  has precisely  $r$  nonzero parts, as proved above; see also Lemmas 8(b) and 9(a).

Henceforth view  $A$  as an  $F[x+y]$ -module. The dimension of the cyclic submodule  $aF[x+y]$  is the smallest natural number  $n = n(a)$  satisfying  $a(x+y)^n = 0$ . Clearly  $(x+y)^i \neq 0$  for  $0 \leq i < s$ . Simplifying  $(x+y)^n = \sum_{i+j=n} \binom{n}{i} x^i y^j$  using  $x^r = y^s = 0$  gives

$$(x+y)^n = \sum_{i=n-s+1}^{r-1} \binom{n}{i} x^i y^{n-i} \quad \text{for } s \leq n < r+s-1. \quad (2)$$

Eq. (2) implies that  $(x+y)^n = 0$  for  $n \geq r+s-1$  because an empty sum is zero. This shows that  $s \leq \lambda_1 \leq r+s-1$ . Homogeneous polynomials will play an important role.

**Lemma 8.** Suppose  $\text{char}(F) = p$  and  $w = \sum_{i=n-s+1}^{r-1} \alpha_i x^i y^{n-i}$  where  $s-1 \leq n \leq r+s-2$ .

- (a)  $w(x+y) = 0$  holds if and only if  $w$  is a scalar multiple of  $\sum_{i=n-s+1}^{r-1} (-1)^i x^i y^{n-i}$ .
- (b) If  $w_i := \sum_{j=0}^i (-1)^j x^{r-1-j} y^{s-1-i+j}$ , then  $w_0, \dots, w_{r-1}$  is an  $F$ -basis for  $\text{Ann}(x+y)$ .
- (c) If  $\dim(F[x+y]) = n$ , then  $(x+y)^n = 0$  and  $(x+y)^{n-1} \neq 0$ . Moreover,

$$(x+y)^{n-1} = (-1)^{r-1} \binom{n-1}{r-1} \sum_{i=n-s}^{r-1} (-x)^i y^{n-1-i} \quad \text{and} \quad \binom{n-1}{r-1} \neq 0 \text{ in } F.$$

**Proof.** (a) Expanding  $w(x+y)$  and using  $x^r = y^s = 0$  gives

$$\begin{aligned} &\alpha_{n-s+1} x^{n-s+2} y^{s-1} + \alpha_{n-s+2} x^{n-s+3} y^{s-2} + \dots + \alpha_{r-3} x^{r-2} y^{n-r+3} + \alpha_{r-2} x^{r-1} y^{n-r+2} + \\ &\alpha_{n-s+2} x^{n-s+2} y^{s-1} + \alpha_{n-s+3} x^{n-s+3} y^{s-2} + \dots + \alpha_{r-2} x^{r-2} y^{n-r+3} + \alpha_{r-1} x^{r-1} y^{n-r+2}. \end{aligned}$$

Thus  $w(x+y) = 0$  holds if and only if  $\alpha_i = -\alpha_{i+1}$  for  $n-s+1 \leq i \leq r-2$ , as desired.

(b) As  $w_i$  is homogeneous of degree  $r + s - 2 - i$ , it follows that  $w_0, \dots, w_{r-1}$  are  $F$ -linearly independent. The proof of part (a) shows that  $w_i(x + y) = 0$  for each  $i$ . This proves that  $\dim(\text{Ann}(x + y)) \geq r$ . To prove  $\dim(\text{Ann}(x + y)) \leq r$ , we relate  $v$  and  $v' := v(x + y)$ . If

$$v = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} v_{i,j} x^i y^j \quad \text{and} \quad v' = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} v'_{i,j} x^i y^j,$$

then  $v'_{i,n+1-i} = v_{i-1,n-(i+1)} + v_{i,n-i}$ . Suppose there exists  $n < r + s - 2$  and  $i < r$  such that

$$v_{i',j'} = 0 \text{ whenever } i' + j' < n, \quad v_{i',n-i'} = 0 \text{ for } i' < i, \text{ and } v_{i,n-i} \neq 0.$$

Then  $v'_{i,n+1-i} = v_{i,n-i}$ , so  $v'_{i,n+1-i} x^i y^{n+1-i} \neq 0$  and  $v(x + y) \neq 0$ . Hence an element  $v \in \text{Ann}(x + y)$  is a sum  $v = \sum_{n=s-1}^{r+s-2} v_n$ , where each summand is a homogeneous polynomial  $v_n = \sum_{i=n-s+1}^{r-1} v_{i,n-i} x^i y^{n-i}$  of degree  $n$ . Since  $v_n(x + y)$  is homogeneous of degree  $n + 1$ , the equation  $\sum_{n=s-1}^{r+s-2} v_n(x + y) = 0$  implies  $v_n(x + y) = 0$  for  $s - 1 \leq n \leq r + s - 2$ . It now follows from part (a) that  $v$  is a linear combination of  $w_0, w_1, \dots, w_{r-1}$ . Thus  $w_0, w_1, \dots, w_{r-1}$  is a basis for  $\text{Ann}(x + y)$  as claimed.

(c) Now  $\dim(F[x + y]) = n$  implies  $(x + y)^n = 0$ . Since  $(x + y)^{n-1}$  is a homogeneous polynomial in  $\text{Ann}(x + y)$ , part (a) shows that  $(x + y)^{n-1} = \beta \sum_{i=n-s}^{r-1} (-x)^i y^{n-1-i}$  for some  $\beta \in F$ . The binomial theorem shows that  $\beta = (-1)^{r-1} \binom{n-1}{r-1}$ . Since  $(x + y)^{n-1} \neq 0$ , we must have  $\beta \neq 0$  in  $F$ .  $\square$

It follows from Lemma 8(c) that  $\dim(x^0 F[x + y]) = \lambda_1$ , and the coefficient of  $x^{\lambda_1-s} y^{s-1}$  in  $(x + y)^{\lambda_1-1}$  is nonzero. Now  $x^i (x^{\lambda_1-s} y^{s-1}) \neq 0$  for  $0 \leq i \leq r + s - 1 - \lambda_1$ . Thus  $x^i (x + y)^{\lambda_1-1} \neq 0$  and hence  $\dim(x^i F[x + y]) = \lambda_1$ . Therefore the  $F[x + y]$ -submodule  $A' = \bigoplus_{i=0}^{r+s-1-\lambda_1} x^i F[x + y]$  has dimension  $\lambda_1(r + s - \lambda_1)$  as each of the  $r + s - \lambda_1$  summands has dimension  $\lambda_1$ . Any  $F[t]$ -submodule (or  $F[x + y]$ -submodule) of maximal dimension  $\lambda_1$  is known to have an  $F[t]$ -submodule complement, see [9, Ex. 9.2, p. 149]. Hence  $A = A' \oplus A''$  for some  $F[x + y]$ -submodule  $A''$ . Lemma 8(c) gives  $\dim(x^i F[x + y]) < \lambda_1$  for  $i \geq r + s - \lambda_1$ . Since  $A = \sum_{i=0}^{r-1} x^i F[x + y]$ , it follows that  $A''(x + y)^{\lambda_1-1} = 0$  and the multiplicity  $k$  of the part  $\lambda_1$  is  $k = r + s - \lambda_1$ . Substituting  $n = \lambda_1 = r + s - k$  into Lemma 8(c) gives an alternate proof of Lemma 9(c) below. In the next section we prove that Theorem 2 implies Lemma 9(b) when  $p > 0$ , and hence when  $p \geq 0$ . The ideas established in this section will be needed for the proof of our main results.

**Lemma 9.** Suppose  $1 \leq r \leq s$  and  $F$  is a field of characteristic  $p \geq 0$ .

- (a) If  $p \geq 0$ , then the partition  $\lambda(r, s, p)$  of  $rs$  has precisely  $r$  nonzero parts.
- (b) If  $p = 0$  or  $p \geq r + s - 1$ , then  $\lambda_i = r + s - 2i + 1$  for  $1 \leq i \leq r$ .
- (c) If  $p > 0$ , then  $\lambda_1 = r + s - k$  where  $k \geq 1$  is minimal such that  $p \nmid \binom{r+s-1-k}{r-1}$ .  
Moreover, the part  $\lambda_1$  has multiplicity  $k$  in the partition  $\lambda(r, s, p)$ .



**Proof.** Parts (a), (b) and (c) have been proved by Ralley, Srinivasan, and McFall, respectively in [20, Lemma 2.1], [23, p. 678], and [16, Theorem 2].  $\square$

Lemma 9(c) suggests an algorithm for computing the largest part  $\lambda_1$  and its multiplicity. It is sometimes difficult to predict the output of this algorithm, but if  $s$  is ‘not too large’ then  $\lambda_1 = p^m$  where  $m = \lceil \log_p(r) \rceil$ . A precise formulation is given in [4]. More importantly, new symmetries and applications of these symmetries are described in [4].

### 3. The standard partition

Recall that  $r \leq s$  and  $p > 0$  is prime. The hypothesis  $s \not\equiv 0, \pm 1, \dots, \pm(r-2) \pmod{p}$  in Theorem 2 implies that  $p \geq 2r-3$ . As we are thinking of  $s$  as ‘large’ compared to  $r$ , this is a weaker hypothesis than  $p \geq r+s-1$  in [1, Corollary 1].

**Proof of Theorem 2.** The strategy of this proof is to show that there exist elements  $v_0, v_1, \dots, v_{r-1} \in A$  satisfying

- (i)  $\dim(v_i F[x+y]) = r+s-2i+1$ , and
- (ii)  $v_i F[x+y] \cap \sum_{j \neq i} v_j F[x+y] = 0$ .

The sum  $\sum_{i=0}^{r-1} v_i F[x+y]$  is direct by (ii). Since  $\sum_{i=0}^{r-1} (r+s-2i+1) = rs$  holds, it follows by (i) that  $A = \bigoplus_{i=0}^{r-1} v_i F[x+y]$  and  $\lambda_i = r+s-2i+1$ , as claimed.

Choose an  $F$ -basis  $w_0, w_1, \dots, w_{r-1}$  for  $\text{Ann}(x+y)$ . We prove that under the stated hypotheses there exist elements  $v_i \in A$  satisfying  $v_i(x+y)^{s+r-2-2i} = w_i$  for each  $i$ . This implies that (i) holds, and (ii) holds because  $w_i F[x+y]$  is the unique minimal  $F[x+y]$ -submodule of  $v_i F[x+y]$  and  $w_i F[x+y] \cap \bigoplus_{j \neq i} w_j F[x+y] = 0$  holds.

Set  $w_i := \sum_{j=0}^i (-1)^j x^{r-1-j} y^{s-1-i+j}$ . Then  $w_0, w_1, \dots, w_{r-1}$  is a basis for  $\text{Ann}(x+y)$  by Lemma 8(b). Suppose that  $v_i$  has the form  $v_i = \sum_{k=0}^i \nu_k x^{i-k} y^k$  where the  $\nu_k \in F$  are unknowns which (we will see) can be chosen so that  $v_i(x+y)^{s+r-2-2i} = w_i$ . The heart of the proof is that, for each  $i$ , we can solve for the  $\nu_k$ . Each of  $v_i$ ,  $(x+y)^{s+r-2-2i}$  and  $w_i$  is homogeneous, and  $\deg(v_i) + \deg((x+y)^{s+r-2-2i}) = \deg(w_i)$ . The binomial theorem gives

$$\begin{aligned} v_i(x+y)^{s+r-2-2i} &= \left( \sum_{k=0}^i \nu_k x^{i-k} y^k \right) \left( \sum_{\ell=0}^{s+r-2-2i} \binom{s+r-2-2i}{\ell} x^\ell y^{s+r-2-2i-\ell} \right) \\ &= \sum_{k=0}^i \sum_{\ell=0}^{s+r-2-2i} \nu_k \binom{s+r-2-2i}{\ell} x^{i-k+\ell} y^{s+r-2-2i-\ell+k}. \end{aligned}$$

However,  $i-k+\ell \leq r-1$  and  $s+r-2-2i-\ell+k \leq s-1$  implies  $r-1-2i+k \leq \ell \leq r-1-i+k$ . Setting  $j := (r-1-i+k) - \ell$  gives the simpler range  $i \geq j \geq 0$ . Thus

$$\begin{aligned}
 v_i(x+y)^{s+r-2-2i} &= \sum_{k=0}^i \sum_{j=0}^i \nu_k \binom{s+r-2-2i}{r-1-i-j+k} x^{r-1-j} y^{s-1-i+j} \\
 &= \sum_{j=0}^i \left[ \sum_{k=0}^i \nu_k \binom{s+r-2-2i}{s-1-i+j-k} \right] x^{r-1-j} y^{s-1-i+j}
 \end{aligned}$$

As we want  $v_i(x+y)^{s+r-2-2i} = w_i = \sum_{j=0}^i (-1)^j x^{r-1-j} y^{s-1-i+j}$ , equating coefficients of  $x^{r-1-j} y^{s-1-i+j}$  for  $0 \leq j \leq i$  gives the linear system

$$(\nu_0, \nu_1, \dots, \nu_i) A_{i+1} = (1, -1, \dots, (-1)^i) \quad \text{where} \quad A_{i+1} = \left( \binom{s+r-2-2i}{s-1-i+j-k} \right)_{0 \leq j, k \leq i}. \quad (3)$$

The matrix  $A_i = \left( \binom{s+r-2i}{s-i+j-k} \right)_{0 \leq j, k \leq i-1}$  equals the matrix  $M(i-1)$  defined on [11, p. 145]. The determinants  $\delta_i := \det(A_i)$ ,  $1 \leq i \leq r-1$ , play an important role. Since  $A_r$  is upper triangular, we see  $\delta_r = 1$ . The following formula for  $\delta_i$  is given on [11, p. 145]:

$$\delta_i = \det(A_i) = \prod_{j=0}^{i-1} \frac{\binom{r+s-2i+j}{s-i}}{\binom{s-i+j}{s-i}} = \prod_{j=0}^{i-1} \prod_{k=0}^{s-i-1} \frac{r+s-2i+j-k}{s-i+j-k}. \quad (4)$$

However,  $(r+s-2i+j) - (r-i+k) = s-i+j-k$  is a factor of the numerator and the denominator for  $k = 0, 1, \dots, s-r-1$ . We cancel these factors, and use the falling factorial notation  $n^{\underline{i}} := n(n-1) \cdots (n-i+1)$  for  $i > 0$  and  $n^{\underline{0}} := 1$ . For  $0 \leq i \leq r-1$  we have

$$\begin{aligned}
 \delta_i &= \prod_{j=0}^{i-1} \frac{(r+s-2i+j)(r+s-2i+j-1) \cdots (s-i+j+1)}{(r-i+j)(r-i+j-1) \cdots (j+1)} \\
 &= \frac{(r+s-i-1)^{\underline{i}} (r+s-i-2)^{\underline{i}} \cdots s^{\underline{i}}}{(r-1)^{\underline{i}} (r-2)^{\underline{i}} \cdots i^{\underline{i}}} \\
 &= \prod_{k=0}^{r-1-i} \frac{(r+s-1-i-k)^{\underline{i}}}{(r-1-k)^{\underline{i}}}. \quad (5)
 \end{aligned}$$

Equation (5) is most helpful when  $i$  is close to  $r-1$ , and Equation (4) when  $i$  is close to 1. The following variant of (4) uses the identity  $\binom{r+s-2i+j}{r-i+j} \binom{r-i+j}{j} = \binom{r+s-2i+j}{r-i} \binom{s-i+j}{j}$  (see Table 3):

$$\delta_i = \det(A_i) = \prod_{j=0}^{i-1} \frac{\binom{r+s-2i+j}{s-i}}{\binom{s-i+j}{s-i}} = \prod_{j=0}^{i-1} \frac{\binom{r+s-2i+j}{r-i+j} \binom{r-i+j}{j}}{\binom{s-i+j}{r-i} \binom{s-i+j}{j}} = \prod_{j=0}^{i-1} \frac{\binom{r+s-2i+j}{r-i}}{\binom{r-i+j}{r-i}}. \quad (6)$$

**Table 3**Values of  $\delta_i$  computed using Eq. (6) for  $i$  small, and Eq. (5) for  $i$  large.

$\delta_0$	$\delta_1$	$\delta_2$	$\cdots$	$\delta_{r-2}$	$\delta_{r-1}$	$\delta_r$
1	$\binom{r+s-2}{r-1}$	$\frac{1}{r-1} \binom{r+s-4}{r-2} \binom{r+s-3}{r-2}$	$\cdots$	$\binom{s+2}{r-1} \binom{s+1}{r-2}$	$\binom{s}{r-1}$	1

A sufficient condition for  $\lambda(r, s, p)$  to be standard is that  $\delta_1 \delta_2 \cdots \delta_{r-1} \not\equiv 0 \pmod{p}$ . Consider a lower bound for a typical factor  $(r+s-1-i-k)^i = \prod_{\ell=0}^{i-1} (r+s-1-i-k-\ell)$  of the numerator of (5). Using  $0 \leq \ell < i$ ,  $0 \leq k \leq r-1-i$ , and  $1 \leq i \leq r-1$  gives

$$r+s-1-i-k-\ell \geq r+s-2i-k \geq s-i+1 \geq s-(r-2). \quad (7)$$

(Note that a lower bound for a factor of the numerator of (4) is too small. The cancelling required to deduce (5) from (4) was necessary.) An upper bound for a factor of the numerator of (5) can be similarly deduced as follows:

$$r+s-1-i-k-\ell \leq r+s-1-i-k \leq r+s-1-i \leq s+(r-2). \quad (8)$$

Equations (7) and (8) prove that the factors of the numerator of  $\delta_i$  are bounded between  $s-(r-2)$  and  $s+(r-2)$ . Hence the assumption  $s \not\equiv 0, \pm 1, \pm 2, \dots, \pm(r-2) \pmod{p}$  implies that  $\delta_1 \delta_2 \cdots \delta_{r-1} \not\equiv 0 \pmod{p}$ . Therefore  $(\nu_0, \nu_1, \dots, \nu_i)$  in Eq. (3) can be found, and  $\dim(v_i F[x+y]) = r+s-1-2i$  holds by Lemma 8(c). Thus  $\varepsilon(r, s, p)$  is standard.  $\square$

We now prove that  $\lambda(pr, ps, p)$  is the  $p$ -multiple of  $\lambda(r, s, p)$ , see Definition 1(e).

**Proof of Theorem 5.** It suffices to prove the result when  $k = 1$ . Set  $F := \mathbb{F}_p$ , and consider the  $p^2 rs$ -dimensional  $F$ -algebra  $\hat{A}$  with commuting generators  $\hat{x}$  and  $\hat{y}$ , and relations  $\hat{x}^{pr} = \hat{y}^{ps} = 0$ . The  $F$ -subalgebra  $A$  generated by  $x := \hat{x}^p$  and  $y := \hat{y}^p$  has dimension  $rs$  and satisfies  $x^r = y^s = xy - yx = 0$ . By the binomial theorem,  $(\hat{x} + \hat{y})^p = x + y$ . Suppose that the decomposition of  $A$  into cyclic  $F[x+y]$ -submodules gives rise to the partition  $\lambda(r, s, p) = (\lambda_1, \dots, \lambda_r)$ . To determine the partition  $\lambda(pr, ps, p)$  we consider (using (1)) the decomposition of  $\hat{A}$  into cyclic  $F[\hat{x} + \hat{y}]$ -submodules.

Suppose that  $A = \bigoplus_{i=1}^r a_i F[x+y]$  where  $\dim(a_i F[x+y]) = \lambda_i$ . We will prove that  $\hat{A} = \bigoplus_{i=1}^r \bigoplus_{j=0}^{p-1} a_i \hat{x}^j F[\hat{x} + \hat{y}]$  where  $\dim(a_i \hat{x}^j F[\hat{x} + \hat{y}]) = p\lambda_i$  for  $0 \leq j < p$ . Since  $a_i(x+y)^{\lambda_i} = 0$  and  $a_i(x+y)^{\lambda_i-1} \neq 0$ , it follows that  $a_i(\hat{x} + \hat{y})^{p\lambda_i} = 0$  and  $a_i(\hat{x} + \hat{y})^{p(\lambda_i-1)} \neq 0$ . However, to show  $\dim(a_i \hat{x}^j F[\hat{x} + \hat{y}]) = p\lambda_i$ , we must prove that  $a_i \hat{x}^j (\hat{x} + \hat{y})^{p\lambda_i-1} \neq 0$ .

Since  $0 \neq a_i(x+y)^{\lambda_i-1} \in A$  and  $(\hat{x} + \hat{y})^p = x + y$ , there exist scalars  $\alpha_{i,i',j'} \in F$ , not all zero, such that  $a_i(\hat{x} + \hat{y})^{p\lambda_i-p} = \sum_{i'=0}^{r-1} \sum_{j'=0}^{s-1} \alpha_{i,i',j'} \hat{x}^{pi'} \hat{y}^{pj'}$ . Since  $\binom{p-1}{i''}$  equals  $(-1)^{i''}$  in  $\mathbb{F}_p$ , we have  $(\hat{x} + \hat{y})^{p-1} = \sum_{i''=0}^{p-1} (-1)^{i''} \hat{x}^{i''} \hat{y}^{p-1-i''}$ . Multiplying by  $(\hat{x} + \hat{y})^{p-1}$  gives

$$a_i(\hat{x} + \hat{y})^{p\lambda_i-1} = \sum_{i'=0}^{r-1} \sum_{j'=0}^{s-1} \alpha_{i,i',j'} \left( \sum_{i''=0}^{p-1} (-1)^{i''} \hat{x}^{i''} \hat{y}^{p-1-i''} \right) \hat{x}^{pi'} \hat{y}^{pj'}. \quad (9)$$

However, the  $prs$  basis elements  $\hat{x}^k \hat{y}^\ell$  in (9) are nonzero and distinct. Thus, since not all scalars  $\alpha_{i,i',j'}$  are zero, it follows that  $a_i(\hat{x} + \hat{y})^{p\lambda_i - 1} \neq 0$ . Hence  $\dim(W_i) = p\lambda_i$  where  $W_i := a_i F[\hat{x} + \hat{y}]$  for  $1 \leq i \leq r$ .

We now prove that  $\hat{A} = \bigoplus_{i=1}^r \bigoplus_{j=0}^{p-1} (1 + \hat{x})^j W_i$ . The following decompositions

$$F[\hat{x}] = \bigoplus_{i=0}^{p-1} \hat{x}^i F[\hat{x}^p] = \bigoplus_{i=0}^{p-1} (1 + \hat{x})^i F[\hat{x}^p] \quad \text{and} \quad F[\hat{y}] = \bigoplus_{j=0}^{p-1} \hat{y}^j F[\hat{y}^p] = \bigoplus_{j=0}^{p-1} (1 + \hat{y})^j F[\hat{y}^p]$$

imply that  $\hat{A} = F[\hat{x}, \hat{y}] \cong F[\hat{x}] \otimes_F F[\hat{y}]$  may be decomposed as

$$\hat{A} = \bigoplus_{i=0}^{p-1} \bigoplus_{j=0}^{p-1} (1 + \hat{x})^i (1 + \hat{y})^j F[\hat{x}^p] \otimes_F F[\hat{y}^p] = \bigoplus_{i=0}^{p-1} \bigoplus_{j=0}^{p-1} (1 + \hat{x})^i (1 + \hat{y})^j A.$$

Now  $(1 + \hat{x})^{kp} (1 + \hat{y})^{\ell p} A = (1 + x)^k (1 + y)^\ell A = A$  and so  $(1 + \hat{x})^i (1 + \hat{y})^j A = (1 + \hat{x})^{i'} (1 + \hat{y})^{j'} A$  holds if  $i \equiv i' \pmod{p}$  and  $j \equiv j' \pmod{p}$ . Setting  $k = i - j$  gives

$$\hat{A} = \bigoplus_{k=0}^{p-1} (1 + \hat{x})^k \bigoplus_{j=0}^{p-1} (1 + \hat{x})^j (1 + \hat{y})^j A = \bigoplus_{k=0}^{p-1} (1 + \hat{x})^k A F[(1 + \hat{x})(1 + \hat{y})],$$

because  $(1 + \hat{x})^p (1 + \hat{y})^p \in A$ . However, we may replace  $F[(1 + \hat{x})(1 + \hat{y})]$  with  $F[\hat{x} + \hat{y}]$  by (1). Using  $A = \bigoplus_{i=1}^r a_i F[x + y]$  and  $F[x + y] \subseteq F[\hat{x} + \hat{y}]$  now gives

$$\hat{A} = \bigoplus_{k=0}^{p-1} (1 + \hat{x})^k \bigoplus_{i=1}^r a_i F[x + y] F[\hat{x} + \hat{y}] = \bigoplus_{i=1}^r \bigoplus_{k=0}^{p-1} a_i (1 + \hat{x})^k F[\hat{x} + \hat{y}].$$

Finally,  $(1 + \hat{x})^k$  is invertible, and so  $\dim(a_i (1 + \hat{x})^k F[\hat{x} + \hat{y}]) = p\lambda_i$ , as desired.  $\square$

#### 4. Periodicity and duality

Let  $p^m$  be the smallest power of  $p = \text{char}(F)$  satisfying  $r \leq p^m$ . In this section we prove periodicity and a duality results which depend on  $p^m$ . The deviation vector  $\varepsilon(r, s, p) := (\varepsilon_1, \dots, \varepsilon_r)$  in Definition 1(c) satisfies  $\sum_{i=1}^r \varepsilon_i = 0$  since  $\sum_{i=1}^r \lambda_i = rs$ . It turns out that periodicity and a duality are satisfied by the deviation vector  $\varepsilon(r, s, p)$ , but not the partition  $\lambda(r, s, p)$ . The following lemma characterizes when  $\varepsilon(r, s, p) = (0, \dots, 0)$ ; Proposition 3 follows from it.

**Lemma 10.** *Suppose that  $\text{char}(F) = p$  and  $r \leq \min\{p^m, s\}$ .*

- (a) *If  $s \equiv 0 \pmod{p^m}$ , then  $(x + y)^s = 0$ ,  $A = \bigoplus_{i=0}^{r-1} x^i F[x + y]$ , and  $x^i (x + y)^{s-1} \neq 0$  for  $0 \leq i < r$ . Consequently,  $\varepsilon(r, s, p) = (0, 0, \dots, 0)$ .*
- (b) *If  $\varepsilon(r, s, p) = (0, 0, \dots, 0)$ , then  $s \equiv 0 \pmod{p^m}$ .*

**Proof.** (a) Suppose that  $s = kp^m$  where  $k$  is an integer. Then  $(x + y)^{p^m} = x^{p^m} + y^{p^m} = y^{p^m}$  as  $0 = x^r = x^{p^m}$ . Thus  $(x + y)^{kp^m} = y^{kp^m} = y^s = 0$ . It follows from  $x^i(x + y)^s = 0$  that  $\dim(x^i F[x + y]) \leq s$ . However,  $A = \sum_{i=0}^{r-1} x^i F[x + y]$  has dimension  $rs$ , and so the sum must be direct. Thus  $\dim(x^i F[x + y]) = s$ , and  $x^i(x + y)^{s-1} \neq 0$  holds for  $0 \leq i < r$ .

(b) It follows from  $\varepsilon_1 = 0$  that  $\lambda_1 = s$ , and hence that  $(x + y)^s = 0$ . Equation (2) gives  $\sum_{i=1}^{r-1} \binom{s}{i} x^i y^{s-i} = 0$  and thus  $\binom{s}{i} = 0$  in  $\mathbb{F}_p$  for  $1 \leq i \leq r-1$ . A theorem of Lucas [7, p. 2] says that  $\binom{s}{i} \equiv \prod_{k \geq 0} \binom{s_k}{i_k} \pmod{p}$  where  $s = \sum_{k \geq 0} s_k p^k$  and  $i = \sum_{k \geq 0} i_k p^k$  are the base- $p$  expansions of  $s$  and  $i$ , respectively. As  $p^{m-1} < r \leq p^m$ , we have  $p^{m-1} \leq r-1$ . Putting  $i = 1, p, \dots, p^{m-1}$  into Lucas' theorem shows that  $s_0 = s_1 = \dots = s_{m-1} = 0$ . In other words,  $s \equiv 0 \pmod{p^m}$ .  $\square$

The following lemma can be proved naturally using the theory of modules over principal ideal rings, see [3]. However, in the absence of a good reference, our proof makes use of the more familiar theory of modules over principal ideal domains.

**Lemma 11.** Suppose that  $\overline{D} := D/(\alpha^n)$  where  $D$  is a principal ideal domain and  $\alpha \in D$  is prime. Suppose  $1 \leq j \leq r$  and  $M$  is a free  $\overline{D}$ -module with basis  $e_1, \dots, e_r$ , and  $N = \bigoplus_{i=1}^j x_i \overline{D}$  is a submodule of  $M$  with  $\text{Ann}(x_i) = (\alpha^{n_i})/(\alpha^n) \neq \overline{D}$  for  $i = 1, \dots, j$ . Then  $M/N \cong \bigoplus_{i=1}^r D/(\alpha^{n-n_i})$  where  $n_i = 0$  for  $j < i \leq r$ .

**Proof.** We can (and will) initially view  $M$  as a module over the principal ideal domain  $D$ . Since  $x_i \alpha^{n_i} = 0$  for  $i = 1, \dots, j$ , and  $\alpha \in D$  is prime, it follows from the theory of  $D$ -modules [9, Lemma 9.1] that  $x_i = y_i \alpha^{n-n_i}$  for some  $y_i \in M$ . Now view  $M$  as a  $\overline{D}$ -module. Let  $(y_1, \dots, y_j) = (e_1, \dots, e_r)Y$  where  $Y$  is an  $r \times j$  matrix over  $\overline{D}$ . Then

$$\bigoplus_{i=1}^j y_i \overline{\alpha}^{n-1} \overline{D} = \bigoplus_{i=1}^j (y_i \overline{\alpha}^{n-n_i}) \overline{\alpha}^{n_i-1} \overline{D} = \bigoplus_{i=1}^j x_i \overline{\alpha}^{n_i-1} \overline{D}$$

is a  $j$ -dimensional linear space over the field  $\overline{D}/(\overline{\alpha}) = D/(\alpha)$ . We conclude that there exists a  $j \times j$  minor  $Y_1$  of  $Y$  such that  $\det(Y_1)$  is a unit in the local ring  $\overline{D}$ . (The set of units of  $\overline{D}$  equals  $\overline{D} \setminus (\overline{\alpha})$  as  $\alpha$  is prime in  $D$ .) Without loss of generality, suppose that  $Y_1$  comprises the first  $j$  rows (and all  $j$  columns) of  $Y$ . Let  $Y_2$  comprise the bottom  $r-j$  rows of  $Y$ . Then

$$(y_1, \dots, y_j) = (e_1, \dots, e_j)Y_1 + (e_{j+1}, \dots, e_r)Y_2. \quad (10)$$

Postmultiplying (10) by  $Y_1^{-1}$  and rearranging gives

$$(e_1, \dots, e_j) = (y_1, \dots, y_j)Y_1^{-1} - (e_{j+1}, \dots, e_r)Y_2Y_1^{-1}, \text{ and hence}$$

$$(e_1, \dots, e_j, e_{j+1}, \dots, e_r) = (y_1, \dots, y_j, e_{j+1}, \dots, e_r) \begin{pmatrix} Y_1^{-1} & 0 \\ -Y_2Y_1^{-1} & I \end{pmatrix}.$$

The above  $r \times r$  matrix is invertible over  $\overline{D}$ . Hence  $y_1, \dots, y_j, e_{j+1}, \dots, e_r$  is also a  $\overline{D}$ -basis of the free  $\overline{D}$ -module  $M$ . With bases for  $M$  and  $N$  aligned, it follows that

$$M/N \cong \bigoplus_{i=1}^r D/(\alpha^{n-n_i})$$

where  $n_i = 0$  for  $j < i \leq r$ .  $\square$

We now prove  $\varepsilon(r, s, p)$  satisfies the periodicity and duality properties in [Theorem 4](#).

**Proof of Theorem 4.** Our strategy is to prove duality first, as duality implies periodicity.

(b) Suppose that  $s = a + bp^m$  and  $s' = -a + b'p^m$  where  $a, b, b'$  are integers. Now  $s'' := s + s' = (b + b')p^m$  is a multiple of  $p^m$ . Let  $A$  be the homocyclic  $F[x+y]$ -module with relations  $x^r = y^{s''} = xy - yx = 0$ . By [Lemma 10](#), the nilpotent transformation  $\mu_{x+y}$  has minimal polynomial  $t^{s''}$ , and it corresponds to the uniform Jordan partition  $\lambda(r, s'', p) = (s'', \dots, s'')$ . The action of  $\mu_{x+y}$  on  $A$  gives submodules of the same dimension as the action of  $\mu_{(1+x)(1+y)}$  on  $A$  as discussed in [Section 2](#). The restriction of  $\mu_{(1+x)(1+y)}$  to the submodule  $V_s$  of  $A$  is  $J_r \otimes J_{s''-s} = J_r \otimes J_{s'}$  by [Table 2](#), and this corresponds to the Jordan partition  $\lambda(r, s', p) = (s' + \varepsilon'_1, \dots, s' + \varepsilon'_r)$ . Similarly, the restriction of  $\mu_{(1+x)(1+y)}$  to  $A/V_s$  is  $J_r \otimes J_s$  by [Table 2](#), and this corresponds to the Jordan partition  $\lambda(r, s, p) = (s + \varepsilon_1, \dots, s + \varepsilon_r)$ , say. Applying [Lemma 11](#) with  $M = A$ ,  $N = V_s$ ,  $n = s''$ ,  $D = F[t]$ , and  $\alpha = t$  shows that  $\lambda(r, s, p)$  has parts  $s'' - (s' + \varepsilon'_i) = s - \varepsilon'_i$ . Our ordering conventions  $\varepsilon_1 \geq \dots \geq \varepsilon_r$  and  $\varepsilon'_1 \geq \dots \geq \varepsilon'_r$  imply that  $\varepsilon_{r-i+1} = -\varepsilon'_i$ . Thus  $\varepsilon(r, s, p)$  is the negative reverse of  $\varepsilon(r, s', p)$ . This proves part (b).

(a) Suppose that  $s \equiv s' \pmod{p^m}$ . Choose an integer  $s''$  such that  $s'' \geq r$  and  $s'' \equiv -s \equiv -s' \pmod{p^m}$ . By part (b),  $\varepsilon(r, s'', p)$  is the negative reverse of both  $\varepsilon(r, s)$  and  $\varepsilon(r, s')$ . Hence  $\overline{\varepsilon(r, s, p)} = \overline{\varepsilon(r, s', p)}$ , and therefore  $\varepsilon(r, s, p) = \varepsilon(r, s', p)$ .  $\square$

Henceforth, the phrase *by duality* will mean ‘by [Theorem 4\(b\)](#)’, and the phrase *by periodicity* will mean ‘by [Theorem 4\(a\)](#)’.

**Lemma 12.** If  $r \leq s$  and  $\varepsilon(r, s, p) = (\varepsilon_1, \dots, \varepsilon_r)$ , then  $|\varepsilon_i| \leq r - 1$  for  $1 \leq i \leq r$ .

**Proof.** We noted in [Section 2](#) that  $(x+y)^{r+s-1} = 0$ . Hence  $\lambda_i \leq r + s - 1$ , and  $\varepsilon_i \leq r - 1$  for  $1 \leq i \leq r$ . By duality,  $-(r - 1) \leq \varepsilon_i$  and hence  $|\varepsilon_i| \leq r - 1$  for  $1 \leq i \leq r$ .  $\square$

Note that  $|\varepsilon_i| \leq \max\{|\varepsilon_1|, |\varepsilon_r|\}$  as  $\varepsilon_1 \geq \dots \geq \varepsilon_r$ . [Proposition 13](#) shows that the upper bound of  $r - 1$  in [Lemma 12](#) can be attained.

**Proposition 13.** If  $r \leq \min\{s, p^m\}$  and  $s \equiv 1 \pmod{p^m}$ , then

$$\varepsilon(r, s, p) = (r - 1, -1, \dots, -1).$$

By duality, if  $r \leq \min\{s, p^m\}$  and  $s \equiv -1 \pmod{p^m}$ , then  $\varepsilon(r, s, p) = (1, \dots, 1, -(r-1))$ .

**Table 4**The number,  $n_r$ , of different  $\varepsilon(r, s, p)$  vectors as  $s \geq r$  and  $p$  vary.

$r$	1	2	3	4	5	6	7	8	9	10	11	12
$n_r$	1	2	4	8	14	24	28	45	61	78	94	118

**Proof.** Suppose that  $r \leq \min\{s, p^m\}$  and  $s \equiv 1 \pmod{p^m}$ . Let  $q = p^{m+1}$ . Since  $\lambda(1, r, p) = (r)$ , we have by [8, (2.5a)] that  $\lambda(r, q-1, p) = (q, \dots, q, q-r)$ . Subtracting the uniform vector  $(q-1, q-1, \dots, q-1)$  gives  $\varepsilon(r, p^{m+1}-1, p) = (1, \dots, 1, 1-r)$ . Then by duality,  $\varepsilon(r, s, p) = (r-1, -1, \dots, -1)$ , as desired.  $\square$

**Proposition 14.** If  $2 \leq r \leq \min\{s, p^m\}$  and  $s \equiv 2 \pmod{p^m}$ , then

$$\varepsilon(r, s, p) = \begin{cases} (r-2, r-2, -2, \dots, -2) & \text{if } r \equiv 0 \pmod{p}, \\ (r-1, r-3, -2, \dots, -2) & \text{if } r \not\equiv 0 \pmod{p}. \end{cases}$$

By duality,  $2 \leq r \leq \min\{s, p^m\}$  and  $s \equiv -2 \pmod{p^m}$  implies

$$\varepsilon(r, s, p) = \begin{cases} (2, \dots, 2, 2-r, 2-r) & \text{if } r \equiv 0 \pmod{p}, \\ (2, \dots, 2, 3-r, 1-r) & \text{if } r \not\equiv 0 \pmod{p}. \end{cases}$$

**Proof.** Suppose that  $2 \leq r \leq \min\{s, p^m\}$  and  $s \equiv 2 \pmod{p^m}$ . Write  $\varepsilon(2, r, p) = (a, b)$ . Then

$$(a, b) = \begin{cases} (0, 0) & \text{if } r \equiv 0 \pmod{p}, \\ (1, -1) & \text{if } r \not\equiv 0 \pmod{p}, \end{cases}$$

by Table 1, and  $\lambda(2, r, p) = (r+a, r+b)$ . Set  $q = p^{m+1}$ . By [8, (2.5a)], we have  $\lambda(r, q-2, p) = (q, \dots, q, q-r-b, q-r-a)$ . Subtracting the uniform vector with all entries  $q-2$  gives  $\varepsilon(r, q-2, p) = (2, \dots, 2, 2-r-b, 2-r-a)$ . Finally, duality shows that  $\varepsilon(r, s, p) = (r+a-2, r+b-2, -2, \dots, -2)$ , and the proposition follows.  $\square$

## 5. Which partitions $\lambda(r, s, p)$ arise?

This section addresses the question: Which partitions  $\lambda(r, s, p)$  arise? Table 4 lists the number,  $n_r$ , of deviation vectors  $\varepsilon(r, s, p)$  as both  $s \geq r$  and  $p$  vary. Table 4, and parts of Table 1, were generated using Magma computer code available at [6]. Theorem 6 shows that  $n_r \leq 2^{r-1}$ . Although this bound is optimal for  $r \leq 4$ , Table 4 suggests that it may be a gross overestimate for large  $r$ .

The  $i \times i$  matrix  $A_i = \left( \binom{s+r-2i}{s-i+j-k} \right)_{0 \leq j, k \leq i-1}$  in (3) has integer entries. Thus  $\delta_i := \det(A_i)$  is an integer, even though the formulas (5) and (6) appear to give rational values. As we are concerned with the case  $\text{char}(F) = p > 0$ , we henceforth assume that  $\delta_i$ , and

the entries of  $A_i$ , lie in the field  $\mathbb{F}_p$ . The following recurrence for computing  $\lambda(r, s, p)$  is established in Section 2 of [11].

**Theorem 15.** (See [11], Theorem 2.2.9.) Suppose  $r \leq s$  and  $p = \text{char}(F) \geq 0$ . The parts of  $\lambda(r, s, p)$  can be computed recursively (in reverse order  $\lambda_r, \lambda_{r-1}, \dots, \lambda_1$ ) via

$$\lambda_i = \begin{cases} r + s - 2i + d(i) & \text{if } \delta_i \neq 0, \delta_{i-1} = \dots = \delta_{i-(d(i)-1)} = 0 \text{ and } \delta_{i-d(i)} \neq 0, \\ \lambda_{i+1} & \text{if } \delta_i = 0. \end{cases} \quad (11)$$

When  $\text{char}(F) = 0$  the recurrence (11) gives the familiar formula  $\lambda_i = r + s + 1 - 2i$ , because  $d(i) = 1$  and the sequence  $\delta_{i-1}, \dots, \delta_{i-(d(i)-1)}$  is empty. Recall that  $\delta_r = 1$ .

We now prove Theorem 6, which says for fixed  $r \geq 1$ , that there are at most  $2^{r-1}$  different deviation vectors  $\varepsilon(r, s, p)$  as  $s$ , with  $s \geq r$ , and the prime  $p$  vary.

**Proof of Theorem 6.** The recurrence relation in Theorem 15 for the  $i$ th part  $\lambda_i$  of  $\lambda(r, s, p)$  depends on whether or not the determinants  $\delta_1, \dots, \delta_{r-1}$  are zero in  $\mathbb{F}_p$ . Subtracting  $s$  gives a recurrence relation for  $\varepsilon_i$  that depends on whether  $\delta_1, \dots, \delta_{r-1}$  are zero or nonzero, and is independent of  $s$ . Hence there are at most  $2^{r-1}$  choices for  $\varepsilon(r, s, p)$  as  $s$ , with  $s \geq r$ , and the prime  $p$  vary.  $\square$

Theorem 6 has computational implications for the construction of Table 1. Theorem 7, and its proof, gives insight into the complexity of extending the  $r$ -values in Table 1.

**Proof of Theorem 7.** Fix  $r$ , and consider separately the cases:  $p < r$ , and  $p \geq r$ .

CASE  $p < r$ . By periodicity, we may assume that  $s$  equals one of  $r, r+1, \dots, r+p^m-1$ . This gives  $p^m$  choices for  $s$ . Multiplying  $p^{m-1} < r$  and  $p < r$  gives  $p^m < r^2$ . This case involves considering less than  $r^2$  values of  $s$ , and less than  $r$  values for the prime  $p$ . Thus we must compute a finite number of (less than  $r^3$ ) deviation vectors  $\varepsilon(r, s, p)$  in this case.

CASE  $p \geq r$ . By periodicity, we may assume that  $s \in \{r, r+1, \dots, r+p-1\}$  as  $m = 1$ . Indeed, we may assume that  $s \in \{r, r+1, \dots, r+(r-2)\}$  by Theorem 2, and by duality. For each of these  $r-1$  choices for  $s$ , compute the  $r-1$  integer determinants  $\delta_1, \dots, \delta_{r-1}$ , and factor  $\Delta := \delta_1 \cdots \delta_{r-1}$  using (6). Since the integers  $\delta_1, \dots, \delta_{r-1}$  depend only on  $r$  and  $s$  and  $r \leq s \leq 2r-2$ , the number of primes  $p$  dividing  $\Delta$  is bounded by a function of  $r$ . Thus the number of steps required to compute  $\varepsilon(r, s, p)$  for  $r \leq s \leq 2r-2$  and prime divisors  $p$  of  $\Delta$  is bounded by a function of  $r$ . For the remaining primes  $p$ , the partition  $\lambda(r, s, p)$  is standard, and  $\varepsilon(r, s, p) = (r-1, r-3, \dots, -(r-3), -(r-1))$ . Thus a finite computation suffices to determine the values of  $\varepsilon(r, s, p)$  with  $r \leq \min\{p, s\}$ .  $\square$

We mention a subtle point: it is not a finite computation to determine a mapping from the triples  $(r, s, p)$  where  $r$  is fixed and  $s \geq r$  and  $p$  vary, to the set of allowable values of  $\varepsilon(r, s, p)$ . The latter requires the computation of  $s$  modulo  $p$  for infinitely many  $s$  (and primes  $p > r$ ), and this is an infinite computation.



We conclude by stating an open problem.

**Problem 16.** Determine the asymptotic size as  $r \rightarrow \infty$  of the number  $n_r$  of different vectors  $\varepsilon(r, s, p)$  where  $s \geq r$  and  $p$  vary. (Does the  $\lim_{r \rightarrow \infty} n_r/2^{r-1}$  exist? Cf. Theorem 6.)

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## References

- [1] M.J.J. Barry, Generators for decompositions of tensor products of modules, *Arch. Math. (Basel)* 97 (2011) 503–512.
- [2] M.J.J. Barry, On a question of Glasby, Praeger, and Xia, *Comm. Algebra* 43 (10) (2015) 4231–4246.
- [3] I.S. Cohen, I. Kaplansky, Rings for which every module is a direct sum of cyclic modules, *Math. Z.* 54 (1951) 97–101.
- [4] S.P. Glasby, C.E. Praeger, B. Xia, Decomposing modular tensor products: ‘Jordan partitions’, their parts and  $p$ -parts, *Israel J. Math.* 209 (1) (2015) 215–233.
- [5] S.P. Glasby, C.E. Praeger, B. Xia, Jordan decompositions and ‘Jordan permutations’, in preparation.
- [6] S.P. Glasby, MAGMA computer code, available at <http://www.maths.uwa.edu.au/~glasby/>.
- [7] A. Granville, Arithmetic Properties of Binomial Coefficients I: Binomial Coefficients Modulo Prime Powers, *Canadian Mathematical Society Conference Proceedings*, vol. 20, 1997, pp. 253–275.
- [8] J.A. Green, The modular representation algebra of a finite group, *Illinois J. Math.* 6 (1962) 607–619.
- [9] B. Hartley, T.O. Hawkes, *Rings, Modules and Linear Algebra*, Chapman & Hall, London–New York, ISBN 0-412-09810-5, 1970.
- [10] X.D. Hou, Elementary divisors of tensor products and  $p$ -ranks of binomial matrices, *Linear Algebra Appl.* 374 (2003) 255–274.
- [11] K.-i. Iima, R. Iwamatsu, On the Jordan decomposition of tensored matrices of Jordan canonical forms, *Math. J. Okayama Univ.* 51 (2009) 133–148.
- [12] R. Lawther, Jordan block sizes of unipotent elements in exceptional algebraic groups, *Comm. Algebra* 23 (1995) 4125–4156.
- [13] J.H. Lindsey II, Groups with a T.I. cyclic Sylow subgroup, *J. Algebra* 30 (1974) 181–235.
- [14] M.W. Liebeck, G.M. Seitz, *Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras*, *Math. Surveys Monogr.*, vol. 180, American Mathematical Society, Providence, RI, ISBN 978-0-8218-6920-8, 2012.
- [15] J.D. McFall, How to compute the elementary divisors of the tensor product of two matrices, *Linear Multilinear Algebra* 7 (1979) 193–201.
- [16] J.D. McFall, On elementary divisors of the tensor product of two matrices, *Linear Algebra Appl.* 33 (1980) 67–86.
- [17] C.W. Norman, On Jordan bases for two related linear mappings, *J. Math. Anal. Appl.* 175 (1993) 96–104.
- [18] C.W. Norman, On Jordan bases for the tensor product and Kronecker sum and their elementary divisors over fields of prime characteristic, *Linear Multilinear Algebra* 56 (2008) 415–451.

<sup>5</sup> <http://mathoverflow.net/questions/134773/homocyclic-primary-module-over-pid>.

- [19] C.E. Praeger, Note on primitive permutation groups of prime power degree, *J. Lond. Math. Soc.* (2) 13 (1976) 191–192.
- [20] T. Ralley, Decomposition of products of modular representations, *J. Lond. Math. Soc.* 44 (1969) 480–484.
- [21] J.-C. Renaud, The decomposition of products in the modular representation ring of a cyclic group of prime power order, *J. Algebra* 58 (1979) 1–11.
- [22] J.-C. Renaud, Recurrence relations in a modular representation algebra, *Bull. Aust. Math. Soc.* 26 (1982) 215–219.
- [23] B. Srinivasan, The modular representation ring of a cyclic  $p$ -group, *Proc. Lond. Math. Soc.* 14 (1964) 677–688.
- [24] A. Trampus, A canonical basis for the matrix transformation  $X \rightarrow AXB$ , *J. Math. Anal. Appl.* 14 (1966) 153–160.
- [25] A. Trampus, A canonical basis for the matrix transformation  $X \rightarrow AX + XB$ , *J. Math. Anal. Appl.* 14 (1966) 242–252.