Journal of Algebra 465 (2016) 62–80

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Symmetric algebras in categories of corepresentations and smash products

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ARTICLE INFO

Article history:
Received 14 March 2016
Available online 21 July 2016
Communicated by Nicolás Andruskiewitsch

MSC:
16T05
18D10
16S40

Keywords:
Hopf algebra
Comodule
Monoidal category
Frobenius algebra
Symmetric algebra
Smash product

ABSTRACT

We investigate Frobenius algebras and symmetric algebras in the monoidal category of right comodules over a Hopf algebra H; for the symmetric property H is assumed to be cosovereign. If H is finite dimensional and A is an H-comodule algebra, we uncover the connection between A and the smash product $A\#H^*$ with respect to the Frobenius and symmetric properties.

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http://dx.doi.org/10.1016/j.jalgebra.2016.07.012

1. Introduction and preliminaries

We work over a basic field k. A finite dimensional algebra A is called Frobenius if A and its dual A^* are isomorphic as left (or equivalently, as right) A-modules. Frobenius algebras arise in representation theory, Hopf algebra theory, quantum groups, cohomology of compact oriented manifolds, topological quantum field theory, the theory of subfactors and of extensions of C^*-algebras, the quantum Yang–Baxter equation, etc., see [13]. A rich representation theory has been uncovered for such algebras, see [14,19]. It was showed in [1,2] that A is Frobenius if and only if it also has a coalgebra structure whose comultiplication is a morphism of A, A-bimodules. This equivalent definition of the Frobenius property has the advantage that it makes sense in any monoidal category. The study of Frobenius algebras in monoidal categories was initiated in [15,20,21], and such objects have occurred in several contexts, for example in the theory of Morita equivalences of tensor categories, in conformal quantum field theory, in reconstruction theorems for modular tensor categories, see more details and references in [11,12,16]; recent developments can be also found in [5].

The representation theory of Frobenius algebras uncovers several symmetry features, for example there is a duality between the categories of left and right finitely generated modules, and the lattices of left and right ideals are anti-isomorphic. Among Frobenius algebras there is a class of objects having even more symmetry. These are the symmetric algebras A, for which A and A^* are isomorphic as A, A-bimodules. The category of commutative symmetric algebras is equivalent to the category of 2-dimensional topological quantum field theories, see [1]. Symmetric algebras appear in block theory of group algebras in positive characteristic, see [19, Chapter IV].

It is not clear how one could define symmetric algebras in an arbitrary monoidal category. Symmetric algebras in monoidal categories with certain properties were first considered in [10], as structures related to correlation functions in conformal field theories. In [11] symmetric algebras are discussed in sovereign monoidal categories. In this paper we consider the monoidal category $\mathcal{M}^H$ of right comodules (or corepresentations) over a Hopf algebra H. If A is an algebra in this category, i.e. a right H-comodule algebra, then $A \in {}_A\mathcal{M}_A^H$, i.e. A is a left (A, H)-Doi–Hopf module and a right (A, H)-Doi–Hopf module. On the other hand, A^* is a right (A, H)-Doi–Hopf module, but not necessarily a left (A, H)-Doi–Hopf module; however A^* has a natural structure of a left $(A^{(S^2)}, H)$-Doi–Hopf module, where $A^{(S^2)}$ is the algebra A with the coaction shifted by S^2, where S is the antipode of H. If H is cosovereign, i.e. there exists a character u on H such that $S^2(h) = \sum u^{-1}(h_1)u(h_3)h_2$ for any $h \in H$, then $A \simeq A^{(S^2)}$ as comodule algebras, and this induces a structure of A^* as an object in ${}_A\mathcal{M}_A^H$, where the left A-action is a deformation of the usual one by u. Then it makes sense to consider when A and A^* are isomorphic in this category; in this case we say that A is symmetric in $\mathcal{M}^H$ with respect to u, or shortly that A is (H, u)-symmetric. As $\mathcal{M}^H$ is a sovereign monoidal category for such H, this is a special case of the general concept of symmetric algebra considered in [11]. In Section 2 we give explicit characterizations of this property in $\mathcal{M}^H$.

We show that the definition of symmetry depends on the character (i.e. on the associated sovereign structure of $\mathcal{M}^H$). Also, we use a modified version of the trivial extension construction to give examples of (H,u)-symmetric algebras of corepresentations. In the case where H is involutory, i.e. $S^2 = Id$, H is cosovereign if we take $u = \varepsilon$, the counit of H, and in this case it is clear that an (H,ε)-symmetric algebra is also symmetric as a k-algebra. However, we show that in general A may be (H,u)-symmetric, without being symmetric as a k-algebra.

Given a finite dimensional algebra A in the category $\mathcal{M}^H$, where H is a finite dimensional Hopf algebra, one can construct the smash product $A\#H^*$. Smash products are also called semidirect products, since the group algebra of a semidirect product of groups is just a smash product. Smash product constructions are of great relevance since they describe the algebra structure in a process of bosonization, which associates for instance a Hopf algebra to a Hopf superalgebra. It is proved in [4] that A is Frobenius if and only if so is $A\#H^*$. On the other hand, we show in an example that such a good connection does not hold for the symmetric property. In Section 3 we show that if A is a Frobenius algebra in $\mathcal{M}^H$, then $A\#H^*$ is a Frobenius algebra in $\mathcal{M}^{H^*}$, but the converse does not hold. In Section 4 we uncover a good transfer of the symmetry property between A and $A\#H^*$, more precisely we show that A is (H,α)-symmetric if and only if $A\#H^*$ is (H^*,g)-symmetric, where g and α are the distinguished grouplike (or modular) elements of H and H^*, provided that H is cosovereign by α, and H^* is cosovereign by g.

For basic concepts and notation on Hopf algebras we refer to [8,18].

2. Frobenius algebras and symmetric algebras of corepresentations

Let H be a Hopf algebra, and let A be a finite dimensional right H-comodule algebra, with H-coaction $a \mapsto \sum a_0 \otimes a_1$. Then there exists an element $\sum_i a_i \otimes h_i \otimes a_i^* \in A \otimes H \otimes A^*$ such that $\sum a_0 \otimes a_1 = \sum_i a_i^*(a) a_i \otimes h_i$ for any $a \in A$; this element corresponds to the H-comodule structure map of A through the natural isomorphism $A \otimes H \otimes A^* \simeq Hom(A, A \otimes H)$. A right H-comodule structure is induced on A^* by

$$a^* \mapsto \sum_i a^*(a_i) a_i^* \otimes S(h_i), \quad \text{for any } a^* \in A^*.$$

If we consider the left H^*-actions on A and A^* associated with these right H-comodule structures, denoted by $h^* \cdot a$ and $h^* \cdot a^*$ for $h^* \in H^*$, $a \in A$, $a^* \in A^*$, we have

$$\begin{aligned}(h^* \cdot a^*)(a) &= \sum_i h^*(S(h_i)) a^*(a_i) a_i^*(a) \\ &= a^*(\sum_i h^*(S(h_i)) a_i^*(a) a_i) \\ &= a^*((h^* S) \cdot a)\end{aligned}$$

so

$$(h^* \cdot a^*)(a) = a^*((h^*S) \cdot a) \tag{1}$$

Moreover, $A^* \in \mathcal{M}_A^H$, with the usual right A-action; this means that the A-module structure of A^* is right H-colinear. It is known (see [6, Theorem 2.4]) that the following are equivalent: (1) $A \simeq A^*$ in $\mathcal{M}_A^H$; (2) There exists a nondegenerate associative bilinear form $B : A \times A \to k$ such that $B(h^* \cdot a, b) = B(a, (h^*S) \cdot b)$ for any $a, b \in A$, $h^* \in H^*$; (3) There exists a linear map $\lambda : A \to k$ such that $\lambda(h^* \cdot a) = h^*(1)\lambda(a)$ for any $a \in A$, $h^* \in H^*$, and $\operatorname{Ker}\lambda$ does not contain a non-zero right ideal of A; (4) There exists a linear map $\lambda : A \to k$ such that $\lambda(h^* \cdot a) = h^*(1)\lambda(a)$ for any $a \in A$, $h^* \in H^*$, and $\operatorname{Ker}\lambda$ does not contain a non-zero subobject of A in $\mathcal{M}_A^H$; (5) A is a Frobenius algebra in the category $\mathcal{M}^H$. The connections between an isomorphism $\theta : A \to A^*$ as in (1), a bilinear map B as in (2) and a linear map λ as in (3), (4) are given by $\theta(a)(b) = B(a, b)$, $\lambda(a) = B(1, a)$, $B(a, b) = \lambda(ab)$.

On the other hand, A^* is also a left A-module in a natural way, but in general A^* is not an object of ${}_A\mathcal{M}^H$ (with a similar compatibility condition for the A-action and H-coaction). However, A^* is an object in ${}_{A^{(S^2)}}\mathcal{M}^H$, where $A^{(S^2)}$ is just the algebra A, with the H-coaction shifted by S^2, i.e. $a \mapsto \sum a_0 \otimes S^2(a_1)$.

Assume now that H is a cosovereign Hopf algebra in the sense of [3], i.e. there exists a character u on H (in other words, u is a grouplike element of the dual Hopf algebra H^*, or equivalently, an algebra morphism from H to k) such that $S^2(h) = \sum u^{-1}(h_1)u(h_3)h_2$ for any $h \in H$; this is the same with $(S^2)^*$ being an inner algebra automorphism of H^* via u. Following [3], we say that u is a sovereign character of H. Then $f : A \to A^{(S^2)}$, $f(a) = u^{-1} \cdot a = \sum u^{-1}(a_1)a_0$, is an isomorphism of right H-comodule algebras, and it induces an isomorphism of categories

$$F : {}_{A^{(S^2)}}\mathcal{M}^H \to {}_A\mathcal{M}^H$$

where for $M \in {}_{A^{(S^2)}}\mathcal{M}^H$, $F(M)$ is just M, with the same H-coaction, and A-action $*$ given by $a * m = f(a)m$, for any $a \in A$ and $m \in M$. By restriction, this induces an isomorphism of categories (and we denote it by F, too)

$$F : {}_{A^{(S^2)}}\mathcal{M}_A^H \to {}_A\mathcal{M}_A^H$$

Now $A^* \in {}_{A^{(S^2)}}\mathcal{M}_A^H$, so then $F(A^*) \in {}_A\mathcal{M}_A^H$.

Definition 2.1. Let H be a cosovereign Hopf algebra with u as a sovereign character. A finite dimensional right H-comodule algebra A is a symmetric algebra in the category $\mathcal{M}^H$ with respect to u if $F(A^*) \simeq A$ in the category ${}_A\mathcal{M}_A^H$. In this case we simply say that A is (H, u)-symmetric.

Now we give equivalent characterizations of this property. The next result can be derived from [11, Proposition 4.6], using the structure of duals in a category of corepresentations. In our sketch of proof, explicit description is given for several ways to describe symmetry of algebras of corepresentations.

Proposition 2.2. *Let A be a right H-comodule algebra, where H is a cosovereign Hopf algebra. Keeping the above notation, the following are equivalent.*

(1) *A is (H,u)-symmetric.*
(2) *There exists a nondegenerate bilinear form $B : A \times A \to k$ such that $B(b, ca) = B(bf(c), a)$, $B(b, a) = B(f(a), b)$, and $B(b, h^* \cdot a) = B((h^*S) \cdot b, a)$ for any $a, b, c \in A$, $h^* \in H^*$.*
(3) *There exists a linear map $\lambda : A \to k$ such that $\lambda(ba) = \lambda(af(b))$ and $\lambda(h^* \cdot a) = h^*(1)\lambda(a)$ for any $a, b \in A$, $h^* \in H^*$, and also* $\operatorname{Ker}\lambda$ *does not contain a non-zero right ideal of A.*
(4) *There exists a linear map $\lambda : A \to k$ such that $\lambda(ba) = \lambda(af(b))$ and $\lambda(h^* \cdot a) = h^*(1)\lambda(a)$ for any $a, b \in A$, $h^* \in H^*$, and also* $\operatorname{Ker}\lambda$ *does not contain a non-zero subobject of A in $\mathcal{M}_A^H$.*

More equivalent conditions can be added if we change right ideal with left ideal in (3), and $\mathcal{M}_A^H$ with ${}_A\mathcal{M}^H$ in (4).

Proof. We combine the proof of the equivalent characterizations of a symmetric algebra in the category of vector spaces, see [14, Theorem 16.54], and [6, Theorem 2.4], recalled above. Thus for (1) ⇔ (2), if $\theta : A \to F(A^*)$ is a linear map, then let $B : A \times A \to k$ be the bilinear map defined by $B(a, b) = \theta(b)(a)$. Then it is straightforward to check that θ is left A-linear if and only if

$$B(b, ca) = B(bf(c), a) \quad \text{for any } a, b, c \in A, \tag{2}$$

and θ is right A-linear if and only if

$$B(b, ac) = B(cb, a) \quad \text{for any } a, b, c \in A. \tag{3}$$

We see that if (2) and (3) hold, then $B(b, a) = B(b, a1) = B(bf(a), 1) = B(f(a), b)$, thus

$$B(b, a) = B(f(a), b) \quad \text{for any } a, b \in A. \tag{4}$$

Moreover, if (2) and (4) hold, then $B(b, ac) = B(f(ac), b) = B(f(a)f(c), b) = B(f(a), cb) = B(cb, a)$, so (3) holds.

We have that θ is H-colinear if and only if $B(b, h^* \cdot a) = B((h^*S) \cdot b, a)$ for any $a, b \in A$, $h^* \in H^*$, and θ is bijective if and only if B is non-degenerate, thus (1) ⇔ (2) is clear.

For (1) ⇔ (3), λ and B determine each other by the relations $\lambda(a) = B(1, a)$ for any $a \in A$, respectively $B(a, b) = \lambda(ba)$ for any $a, b \in A$. □

Example 2.3. 1) If $H = kG$, the group Hopf algebra of a group G, then $S^2 = Id$, so H is cosovereign with ε as a sovereign character. A right H-comodule algebra is just a G-graded algebra A, and A is (H, ε)-symmetric if and only if A is graded symmetric in the sense of [6, Section 5].

2) More generally, if H is an involutory Hopf algebra, i.e. $S^2 = Id$, then H is obviously cosovereign with ε as a sovereign character. In this case, if A is a finite dimensional algebra in $\mathcal{M}^H$, then $A^* \in {}_A\mathcal{M}_A^H$, so $F(A^*)$ is just A^*, with the usual left and right A-actions. Thus A is (H, ε)-symmetric if and only if $A^* \simeq A$ in ${}_A\mathcal{M}_A^H$.

Remark 2.4. The definition of symmetry depends on the cosovereign character. Thus it is possible that a cosovereign Hopf algebra H has two sovereign characters u and v, and a right H-comodule algebra A is (H, u)-symmetric, but not (H, v)-symmetric. Indeed, let $H = kC_2$, where $C_2 = \{e, g\}$ is a group of order 2 (e is the neutral element), and the characteristic of k is $\neq 2$. Let A be a commutative C_2-graded division algebra with support C_2; for example one can take $A = kC_2$. Then H is involutory, so it is a cosovereign Hopf algebra with two possible sovereign characters $\varepsilon = p_e + p_g$ and $u = p_e - p_g$, where $\{p_e, p_g\}$ is the basis of H^* dual to the basis $\{e, g\}$ of H. We have $u^2 = \varepsilon$, so $u^{-1} = u$.

It is easy to see that A is (H, ε)-symmetric, i.e. graded symmetric in the terminology of [6], for example by using the results of [7], where the question whether any graded division algebra is graded symmetric is addressed.

On the other hand, A is not (H, u)-symmetric. Indeed, let $\lambda : A \to k$ be a linear map such that $\lambda(ab) = \lambda(b(u \cdot a))$ for any $a, b \in A$. We have $u \cdot a = a$ for any $a \in A_e$ (the homogeneous component of degree e of A), and $u \cdot a = -a$ for any $a \in A_g$. Then for $b = 1$ and $a \in A_g$ we get $\lambda(a) = 0$, thus $\lambda(A_g) = 0$. Also, for $a, b \in A_g$ we obtain $\lambda(ab) = 0$, and since $A_g A_g = A_e$, this shows that $\lambda(A_e) = 0$. Thus λ must be zero.

Remark 2.5. It is obvious that a graded symmetric algebra is symmetric as a k-algebra. More general, if H is involutory, then a (H, ε)-symmetric algebra is symmetric as a k-algebra. However, for an arbitrary cosovereign Hopf algebra H with sovereign character u, if A is (H, u)-symmetric, then A is not necessarily symmetric as a k-algebra, as we show in the following example.

Let $H = H_4$, the 4-dimensional Sweedler's Hopf algebra. It is presented by algebra generators c and x, subject to relations

$$c^2 = 1, x^2 = 0, xc = -cx$$

The coalgebra structure is defined by

$$\Delta(c) = c \otimes c, \Delta(x) = c \otimes x + x \otimes 1, \varepsilon(c) = 1, \varepsilon(x) = 0.$$

The antipode S satisfies $S(c) = c$, $S(x) = -cx$, thus $S^2(x) = -x$. Apart from ε, H has just one more character α, given by $\alpha(c) = -1$, $\alpha(x) = 0$; it acts on the basis elements of H by

$$\alpha \cdot 1 = 1,\ \alpha \cdot c = -c,\ \alpha \cdot x = x,\ \alpha \cdot (cx) = -cx \tag{5}$$

H is cosovereign, and the only sovereign character is α. We show that the linear map $\lambda : H \to k$, $\lambda(1) = \lambda(c) = \lambda(cx) = 0$, $\lambda(x) = 1$ makes H an (H, α)-symmetric algebra. Indeed, we first see by a straightforward checking that λ is right H-colinear (or equivalently, left H^*-linear). Next, an easy computation using (5) shows that any element of the form $ba - a(\alpha \cdot b)$ lies in the span of 1, c and cx, thus $\lambda(ba) = \lambda(a(\alpha \cdot b))$ for any $a, b \in H$.

Finally, let I be a left ideal of H contained in Kerλ. Let $z = \delta 1 + \beta c + \gamma cx \in I$, where $\delta, \beta, \gamma \in k$. Then $cz = \delta c + \beta 1 + \gamma x \in I \subseteq < 1, c, cx >$, so $\gamma = 0$. Then $xz = \delta x - \beta cx \in I \subseteq < 1, c, cx >$, so $\delta = 0$. Now $cxz = -\beta x \in I \subseteq < 1, c, cx >$, so β must be zero, too. Thus $z = 0$, and H is (H, α)-symmetric. This will also follow from Proposition 4.4.

On the other hand, H is not symmetric as a k-algebra. Indeed, if $\lambda : H \to k$ is a linear map such that $\lambda(ab) = \lambda(ba)$ for any $a, b \in H$, then $\lambda(cx) = \lambda(xc) = -\lambda(cx)$, so $\lambda(cx) = 0$, and $\lambda(x) = \lambda(ccx) = \lambda(cxc) = -\lambda(x)$, so $\lambda(x) = 0$. Thus the two-sided ideal $< x, cx >$ of H is contained in Kerλ, showing that H is not symmetric. This can also be seen from a general result saying that a Hopf algebra is symmetric as an algebra if and only if it is unimodular (i.e. the spaces of left integrals and right integrals in H coincide) and S^2 is inner, see [17]; for H_4 the square of the antipode is inner, but the unimodularity condition fails.

Now we explain how examples of (H, u)-symmetric algebras in the category $\mathcal{M}^H$ can be constructed, where H is a cosovereign Hopf algebra with sovereign character u. We recall that for any algebra A (in the category of vector spaces), and any left A, right A-bimodule M, one can construct an algebra structure on the space $A \oplus M$ with the multiplication defined by $(a, m)(a', m') = (aa', am' + ma')$; this is called the trivial extension of A and M. The unit of this algebra is $(1, 0)$. If $M = A^*$ with the usual A, A-bimodule structure, the trivial extension of A and A^* is simply called the trivial extension of A, and it is a symmetric algebra, see [14, Example 16.60].

Proposition 2.6. *Let A be a right H-comodule algebra, where H is cosovereign with sovereign character u. Then $\mathcal{E}(A) = A \oplus F(A^*)$, with the direct sum structure of a right H-comodule, and the algebra structure obtained by the trivial extension of A and the A, A-bimodule $F(A^*)$, is a right H-comodule algebra which is (H, u)-symmetric.*

Proof. The multiplication of $\mathcal{E}(A)$ is given by

$$(a, a^*)(b, b^*) = (ab, a * b^* + a^* b) = (ab, (u^{-1} \cdot a)b^* + a^* b)$$

for any $a, b \in A$ and any $a^*, b^* \in A^*$.

We first see that $\mathcal{E}(A)$ is a right H-comodule algebra. Indeed, the H-coaction on $\mathcal{E}(A)$ is $\rho : \mathcal{E}(A) \to \mathcal{E}(A) \otimes H$, given by

$$\rho(a, a^*) = \sum (a_0, 0) \otimes a_1 + \sum (0, a_0^*) \otimes a_1^*$$

Then

$$\begin{aligned}
\rho((a, a^*)(b, b^*)) &= \rho(ab, a * b^* + a^* b) \\
&= \sum ((ab)_0, 0) \otimes (ab)_1 + \sum (0, (a * b^*)_0) \otimes (a * b^*)_1 \\
&\quad + \sum (0, (a^* b)_0) \otimes (a^* b)_1 \\
&= \sum (a_0 b_0, 0) \otimes a_1 b_1 + \sum (0, a_0 * b_0^*) \otimes a_1 b_1^* + \sum (0, a_0^* b_0) \otimes a_1^* b_1 \\
&= (\sum (a_0, 0) \otimes a_1 + \sum (0, a_0^*) \otimes a_1^*)(\sum (b_0, 0) \otimes b_1 + \sum (0, b_0^*) \otimes b_1^*) \\
&= \rho(a, a^*)\rho(b, b^*)
\end{aligned}$$

Let $\lambda : \mathcal{E}(A) \to k$ be the linear map defined by $\lambda(a, a^*) = a^*(1)$ for any $a \in A$, $a^* \in A^*$. Then

$$\begin{aligned}
\lambda(h^* \cdot (a, a^*)) &= (h^* \cdot a^*)(1) \\
&= a^*((h^* S) \cdot 1) \quad \text{(by (1))} \\
&= a^*((h^* S)(1)1) \\
&= h^*(1)a^*(1) \\
&= h^*(1)\lambda(a, a^*)
\end{aligned}$$

Now

$$\begin{aligned}
\lambda((a, a^*)(u^{-1} \cdot (b, b^*))) &= \lambda((a, a^*)(u^{-1} \cdot b, u^{-1} \cdot b^*)) \\
&= \lambda(a(u^{-1} \cdot b), (u^{-1} \cdot a)(u^{-1} \cdot b^*) + a^*(u^{-1} \cdot b)) \\
&= (u^{-1} \cdot b^*)(u^{-1} \cdot a) + a^*(u^{-1} \cdot b) \\
&= b^*((u^{-1} S) \cdot (u^{-1} \cdot a)) + a^*(u^{-1} \cdot b) \quad \text{(by (1))} \\
&= b^*(u \cdot (u^{-1} \cdot a)) + a^*(u^{-1} \cdot b) \\
&= b^*(a) + a^*(u^{-1} \cdot b) \\
&= \lambda(ba, (u^{-1} \cdot b)a^* + b^* a) \\
&= \lambda((b, b^*)(a, a^*))
\end{aligned}$$

Finally, we see that $\mathrm{Ker}\lambda$ does not contain non-zero right ideals. Indeed, if $\lambda((a, a^*)\mathcal{E}(A)) = 0$, then $b^*(u^{-1} \cdot a) + a^*(b) = 0$ for any $b \in A$, $b^* \in A^*$. If we take

$b^* = 0$, we get that $a^*(b) = 0$ for any b, so $a^* = 0$. Then $b^*(u^{-1} \cdot a) = 0$ for any b^*, so $u^{-1} \cdot a = 0$, showing that $a = 0$.

We conclude that λ makes $\mathcal{E}(A)$ an (H, u)-symmetric algebra. □

We note that the previous result shows that any finite dimensional algebra in the category $\mathcal{M}^H$, where H is cosovereign via u, is a subalgebra of an (H, u)-symmetric algebra, and also a quotient of an (H, u)-symmetric algebra. The construction in Proposition 2.6 helps us to provide more examples of algebras that are symmetric in categories of corepresentations with respect to certain characters, but which are not symmetric as k-algebras.

Example 2.7. Assume that k has characteristic $\neq 2$, and let $H = H_4$ be Sweedler's Hopf algebra. Let $A = k[X]/(X^2)$, a 2-dimensional algebra, with basis $\{1, X\}$, and relation $X^2 = 0$. Let c and x be the endomorphisms of the space A such that $c(1) = 1$, $c(X) = -X$, $x(1) = 0$, $x(X) = 1$. Then $c^2 = Id$, $x^2 = 0$ and $xc = -cx$. Moreover, c is an algebra automorphism of A, and it is easy to check that $x(ab) = c(a)x(b) + x(a)b$ for any $a, b \in A$, thus A is a left H-module algebra with the actions of c and x given by the endomorphisms above, i.e.

$$c \cdot 1 = 1,\ c \cdot X = -X,\ x \cdot 1 = 0,\ x \cdot X = 1.$$

The dual space A^* has a left H-action given by $(h \cdot a^*)(a) = a^*(S(h) \cdot a)$ for any $h \in H$, $a^* \in A^*$ and $a \in A$. If we consider the basis $\{p_1, p_X\}$ of A^* dual to the basis $\{1, X\}$, this action explicitly writes

$$c \cdot p_1 = p_1,\ c \cdot p_X = -p_X,\ x \cdot p_1 = -p_X,\ x \cdot p_X = 0.$$

On the other hand, A^* has usual left A-module structure given by

$$1p_1 = p_1,\ 1p_X = p_X,\ Xp_1 = 0,\ Xp_X = p_1,$$

and usual right A-module structure given by

$$p_1 1 = p_1,\ p_X 1 = p_X,\ p_1 X = 0,\ p_X X = p_1.$$

A is also a right comodule algebra over the dual Hopf algebra H^*. Since H^* is cosovereign with sovereign character c (via the isomorphism $H \simeq H^{**}$), the associated left A-module structure on $F(A^*)$ is given by $a * a^* = (c \cdot a)a^*$ for any $a \in A$, $a^* \in A^*$. Thus

$$1 * p_1 = p_1,\ 1 * p_X = p_X,\ X * p_1 = 0,\ X * p_X = -p_1$$

Then we can consider the algebra $\mathcal{E}(A) = A \oplus F(A^*)$, whose basis is $\{1, X, p_1, p_X\}$, and multiplication induced by

$$X^2 = p_1^2 = p_X^2 = p_1 p_X = p_X p_1 = 0$$
$$X * p_1 = 0,\ X * p_X = -p_1,\ p_1 X = 0,\ p_X X = p_1$$

If we denote $u = X$ and $v = p_X$, we can present $\mathcal{E}(A)$ by generators u, v, subject to relations $u^2 = v^2 = 0$, $vu = -uv$. The left H-module structure of $\mathcal{E}(A)$ is given by $c \cdot u = -u$, $c \cdot v = -v$, $x \cdot u = 1$, $x \cdot v = 0$. If we denote by $\{P_1, P_c, P_x, P_{cx}\}$ the basis of H^* dual to the standard basis $\{1, c, x, cx\}$ of H, the right H^*-comodule structure of $\mathcal{E}(A)$ is given by

$$\begin{aligned} u &\mapsto u \otimes P_1 + c \cdot u \otimes P_c + x \cdot u \otimes P_x + (cx) \cdot u \otimes P_{cx} \\ &= u \otimes (P_1 - P_c) + 1 \otimes (P_x + P_{cx}) \\ v &\mapsto v \otimes (P_1 - P_c) \end{aligned}$$

Since the Hopf algebra H is selfdual, a Hopf algebra isomorphism being given by $1 \mapsto P_1 + P_c$, $c \mapsto P_1 - P_c$, $x \mapsto P_x - P_{cx}$, $cx \mapsto -P_x - P_{cx}$, we can regard A as a right H-comodule algebra. Summarizing, $\mathcal{E}(A)$ is the algebra with generators u, v, relations

$$u^2 = v^2 = 0, vu = -uv$$

and H-comodule structure given by

$$u \mapsto u \otimes c - 1 \otimes cx,\ v \mapsto v \otimes c$$

By Proposition 2.6, $\mathcal{E}(A)$ is (H, α)-symmetric, where $\alpha = P_1 - P_c$ is the distinguished grouplike element of H^*.

On the other hand, $\mathcal{E}(A)$ is not symmetric as a k-algebra. Indeed, if $\lambda : \mathcal{E}(A) \to k$ is a linear map such that $\lambda(zz') = \lambda(z'z)$ for any $z, z' \in \mathcal{E}(A)$, then $\lambda(uv) = \lambda(vu) = -\lambda(uv)$, thus $\lambda(uv) = 0$. But the 1-dimensional space spanned by uv is a two-sided ideal of $\mathcal{E}(A)$, so Kerλ contains a non-zero ideal.

3. Frobenius smash products

Let A be an algebra in $\mathcal{M}^H$, where H is a finite dimensional Hopf algebra. Then A is a left H^*-module algebra and we can consider the smash product $A\#H^*$, which is an algebra with multiplication given by

$$(a\#h^*)(b\#g^*) = \sum a(h_1^* \cdot b)\#h_2^* g^*$$

It is known that A is a Frobenius algebra if and only if so is $A\#H^*$, see [4].

On the other hand, $A\#H^*$ is an algebra in the category $\mathcal{M}^{H^*}$, with the H^*-coaction induced by the comultiplication of H^*, i.e. $a\#h^* \mapsto \sum a\#h_1^* \otimes h_2^*$. The aim of this section is to discuss the connection between A being a Frobenius algebra in $\mathcal{M}^H$, and $A\#H^*$ being a Frobenius algebra in $\mathcal{M}^{H^*}$.

We consider the usual left and right actions of H^* on H, $h^* \rightharpoonup h = \sum h^*(h_2)h_1$ and $h \leftharpoonup h^* = \sum h^*(h_1)h_2$, and the usual left and right actions of H on H^*, denoted by $h \rightharpoonup h^*$ and $h^* \leftharpoonup h$, where $h \in H$ and $h^* \in H^*$. H also acts on $A\#H^*$ by $h \rightharpoonup (a\#h^*) = a\#(h \rightharpoonup h^*)$.

We recall that a right (respectively left) integral in H is an element $t \in H$ such that $th = \varepsilon(h)t$ (respectively $ht = \varepsilon(h)t$) for any $h \in H$, and a left integral on H is an element $T \in H^*$ such that $h^*T = h^*(1)T$ for any $h^* \in H^*$.

Theorem 3.1. *Let H be a finite dimensional Hopf algebra, and let A be a finite dimensional right H-comodule algebra which is a Frobenius algebra in the category $\mathcal{M}^H$. Then the smash product $A\#H^*$ is a Frobenius algebra in the category $\mathcal{M}^{H^*}$.*

Proof. Let $\lambda : A \to k$ be a linear map whose kernel does not contain non-zero right ideals of A, and such that $\lambda(h^* \cdot a) = h^*(1)\lambda(a)$ for any $h^* \in H^*$ and $a \in A$. Let t be a non-zero right integral in H. Define a linear map $\overline{\lambda} : A\#H^* \to k$ such that

$$\overline{\lambda}(a\#h^*) = \lambda(a)h^*(t) \text{ for any } a \in A, h^* \in H^*$$

We have that $\overline{\lambda}(h \cdot z) = \varepsilon(h)\overline{\lambda}(z)$ for any $h \in H$ and $z \in A\#H^*$. Indeed

$$\begin{aligned}\overline{\lambda}(h \cdot (a\#h^*)) &= \sum \overline{\lambda}(a\#h_2^*(h)h_1^*)\\ &= \sum h_2^*(h)\lambda(a)h_1^*(t)\\ &= \lambda(a)h^*(th)\\ &= \lambda(a)\varepsilon(h)h^*(t)\\ &= \varepsilon(h)\overline{\lambda}(a\#h^*)\end{aligned}$$

We show that $Ker(\overline{\lambda})$ does not contain non-zero subobjects of $A\#H^*$ in the category $\mathcal{M}^{H^*}_{A\#H^*}$. Let I be a right ideal of $A\#H^*$ which is also a right H^*-subcomodule (or equivalently, invariant with respect to the induced left H-action on $A\#H^*$) such that $I \subset Ker(\overline{\lambda})$. We know that $(A\#H^*)^{co\ H^*} = A\#1 \simeq A$ and $A\#H^*/A$ is a right H^*-Galois extension, see [8, Example 6.4.8]. Then $J = I^{co\ H^*}$ is a right ideal of A and the Weak Structure Theorem for Hopf–Galois extensions shows that the map

$$J \otimes_A (A\#H^*) \to I, \quad m \otimes z \mapsto mz$$

is an isomorphism in the category $\mathcal{M}^{H^*}_{A\#H^*}$, see [8, Theorem 6.4.4]. In particular $I = (J\#1)(A\#H^*) = J\#H^*$. Since $\overline{\lambda}(I) = 0$, we see that $\lambda(J) = 0$, so $J = 0$. We conclude that I must be zero. $\square$

Corollary 3.2. *A finite dimensional Hopf algebra H is Frobenius in the category $\mathcal{M}^H$.*

Proof. k is a right H^*-comodule algebra in a trivial way, and it is clear that k is Frobenius in $\mathcal{M}^{H^*}$. By Theorem 3.1 we get that $k\#H^{**}$ is Frobenius in $\mathcal{M}^{H^{**}}$. Since $H^{**} \simeq H$ as Hopf algebras, we see that H is Frobenius in $\mathcal{M}^H$. $\square$

The following example shows that the converse of Theorem 3.1 is not true.

Example 3.3. Let $A = A_0 \oplus A_1$ be a superalgebra, i.e. a C_2-graded algebra, which is Frobenius as an algebra, but not graded Frobenius (i.e. it is not a Frobenius algebra in the category $\mathcal{M}^{kC_2}$ of supervector spaces). In this example we use the additive notation for the operation of C_2. Examples of such A are given in [6, Section 6]; the trivial extension associated to a finite dimensional algebra is one such example. Let $\mu : A \to k$ be a linear map whose kernel does not contain non-zero left ideals of A. We define

$$\lambda : A\#(kC_2)^* \to k, \ \lambda(a\#p_x) = \mu(a) \quad \text{for any } a \in A, x \in C_2$$

Then $\lambda(y \rightharpoonup (a\#p_x)) = \lambda(a\#p_{x-y}) = \mu(a) = \varepsilon(y)\lambda(a\#p_x)$.

On the other hand, Kerλ does not contain non-zero left ideals of $A\#(kC_2)^*$. Indeed, assume that $\lambda((A\#(kC_2)^*)z) = 0$, where $z = a\#p_0 + b\#p_1$. Since $(c\#p_0)z = ca_0\#p_0 + cb_1\#p_1$, we have $0 = \mu(ca_0)+\mu(cb_1) = \mu(c(a_0+b_1))$ for any $c \in A$. Thus $\mu(A(a_0+b_1)) = 0$, showing that $a_0 + b_1 = 0$, and then $a_0 = b_1 = 0$. Similarly, since $(c\#p_1)z = ca_1\#p_0 + cb_0\#p_1$, we obtain $a_1 = b_0 = 0$. Thus $z = 0$. We conclude that λ makes $A\#(kC_2)^*$ a Frobenius algebra in the category $\mathcal{M}^{(kC_2)^*}$.

4. Symmetric smash products

The following example shows that the good connection between a finite dimensional right H-comodule algebra A and the smash product $A\#H^*$ being Frobenius does not work anymore for the symmetric property.

Example 4.1. Let $C_2 =< c >= \{e, g\}$ be the cyclic group of order 2, and let A be a C_2-graded algebra which is symmetric and such that the homogeneous component A_e is not symmetric. For example one can take the trivial extension $A = R \oplus R^*$ of a non-symmetric algebra R, with the grading $A_e = R$, $A_g = R^*$. Then A is a right kC_2-comodule algebra, so we can consider the smash product $A\#(kC_2)^*$. Denote by $\{p_e, p_g\}$ the basis of $(kC_2)^*$ dual to the basis $\{e, g\}$ of kC_2. Then $1\#p_e$ is an idempotent in $A\#(kC_2)^*$ and it is easy to check that

$$(1\#p_e)(A\#(kC_2)^*)(1\#p_e) = A_e\#p_e \simeq A_e = R$$

Then $(1\#p_e)(A\#(kC_2)^*)(1\#p_e)$ is not a symmetric algebra, so neither is $A\#(kC_2)^*$ by [14, Exercise 16.25].

In this section we discuss the connection between A being a symmetric algebra in $\mathcal{M}^H$ with respect to some character of H, and $A\#H^*$ being a symmetric algebra in $\mathcal{M}^{H^*}$ with respect to some character of H^* (i.e. a grouplike element of H).

Let H be a finite dimensional Hopf algebra. Then there exists a character $\alpha \in H^*$ such that $th = \alpha(h)t$ for any left integral t in H and any $h \in H$; α is called the distinguished grouplike element of H^*, and it also satisfies $ht' = \alpha^{-1}(h)t'$ for any right integral t' in H and any $h \in H$, see [18, Section 10.5] or [8, Section 5.5].

Similarly, there exists a distinguished grouplike element g of H, such that $Th^* = h^*(g)T$ for any left integral T on H and any $h^* \in H^*$. We note that in [18], g^{-1} is called the distinguished grouplike element of H; we prefer the way we defined because g will play the same role for H as α does for H^*. It is showed in [18, Theorem 10.5.4] that for any left integral t in H

$$\Delta(t) = \sum S^2(t_2)g^{-1} \otimes t_1 \tag{6}$$

Applying this for H^* we see that for any left integral T on H one has

$$\Delta(T) = \sum (T_2S^2)\alpha^{-1} \otimes T_1 \tag{7}$$

and then for any $h \in H$

$$\begin{aligned} T \leftharpoonup h &= \sum T_1(h)T_2 \\ &= ((T_2S^2)\alpha^{-1})(h)T_1 \\ &= \sum T_2(S^2(h_1))\alpha^{-1}(h_2)T_1 \\ &= \sum \alpha^{-1}(h_2)(S^2(h_1) \rightharpoonup T) \end{aligned}$$

Thus for any left integral T on H and any $h \in H$

$$T \leftharpoonup h = \sum \alpha^{-1}(h_2)(S^2(h_1) \rightharpoonup T) \tag{8}$$

Now if t is a left integral in H, then $S(t)$ is a right integral in H and

$$\begin{aligned} \Delta(S(t)) &= \sum S(t_2) \otimes S(t_1) \\ &= \sum S(t_1) \otimes S(S^2(t_2)g^{-1}) \quad (\text{ by (6)}) \\ &= \sum S(t_1) \otimes gS^2(S(t_2)) \\ &= S(t)_2 \otimes gS^2(S(t)_1) \end{aligned}$$

We conclude that for any right integral t in H

$$\Delta(t) = \sum t_2 \otimes gS^2(t_1) \tag{9}$$

We also see that for a right integral t in H and $h \in H$

$$\begin{aligned}\sum t_1 \otimes ht_2 &= \sum \varepsilon(h_1)t_1 \otimes h_2t_2 \\ &= \sum S(h_1)h_2t_1 \otimes h_3t_2 \\ &= \sum S(h_1)(h_2t)_1 \otimes (h_2t)_2 \\ &= \sum \alpha^{-1}(h_2)S(h_1)t_1 \otimes t_2\end{aligned}$$

Thus we showed that

$$\sum t_1 \otimes ht_2 = \sum \alpha^{-1}(h_2)S(h_1)t_1 \otimes t_2 \tag{10}$$

Theorem 4.2. *Let H be a finite dimensional Hopf algebra, and let g and α be the distinguished grouplike elements of H and H^*. We assume that $S^2(h) = g^{-1}hg = \sum \alpha^{-1}(h_1)\alpha(h_3)h_2$ for any $h \in H$. Then a right H-comodule algebra A is (H,α)-symmetric if and only if $A\#H^*$ is (H^*, g)-symmetric.*

Proof. We note that

$$S^{-2}(h) = \sum \alpha(h_1)\alpha^{-1}(h_3)h_2 \tag{11}$$

for any $h \in H$.

Assume that A is (H,α)-symmetric, and let $\lambda : A \to k$ such that $\lambda(ba) = \lambda(a(\alpha^{-1} \cdot b)) = \lambda(a\alpha^{-1}(b_1)b_0)$, $\lambda(h^* \cdot a) = h^*(1)\lambda(a)$ for any $a, b \in A$, $h^* \in H^*$, and also $\operatorname{Ker}\lambda$ does not contain a non-zero right ideal of A. Let t be a non-zero right integral in H and define

$$\bar{\lambda} : A\#H^* \to k, \quad \bar{\lambda}(a\#h^*) = \lambda(a)h^*(t)$$

as in the proof of Theorem 3.1. This makes $A\#H^*$ a Frobenius algebra in the category $\mathcal{M}^{H^*}$. In order to see that $A\#H^*$ is symmetric in $\mathcal{M}^{H^*}$, it remains to show that $\bar{\lambda}(zz') = \bar{\lambda}(z'(g^{-1} \rightharpoonup z))$ for any $z, z' \in A\#H^*$, where $g^{-1} \rightharpoonup (a\#h^*) = a\#(g^{-1} \rightharpoonup h^*)$. Indeed, we see that

$$\begin{aligned}\bar{\lambda}((b\#g^*)(g^{-1} \rightharpoonup (a\#h^*)) &= \bar{\lambda}((b\#g^*)(a\#(g^{-1} \rightharpoonup h^*))) \\ &= \sum \bar{\lambda}(b(g_1^* \cdot a)\#g_2^*(g^{-1} \rightharpoonup h^*)) \\ &= \sum \lambda(bg_1^*(a_1)a_0)g_2^*(t_1)h^*(t_2g^{-1}) \\ &= \sum \lambda(ba_0)g^*(a_1t_1)h^*(t_2g^{-1}) \\ &= \sum \lambda(a_0\alpha^{-1}(b_1)b_0)g^*(a_1t_1)h^*(t_2g^{-1})\end{aligned}$$

$$
\begin{aligned}
&= \sum \lambda(a_0 b_0) g^*(a_1 b_1 S(b_2) t_1) \alpha^{-1}(b_3) h^*(t_2 g^{-1}) \\
&= \sum \lambda(((S(b_1)t_1) \rightharpoonup g^*) \cdot (ab_0)) \alpha^{-1}(b_2) h^*(t_2 g^{-1}) \\
&= \sum ((S(b_1)t_1) \rightharpoonup g^*)(1) \lambda(ab_0) \alpha^{-1}(b_2) h^*(t_2 g^{-1}) \\
&= \sum g^*(S(b_1)t_1) \lambda(ab_0) \alpha^{-1}(b_2) h^*(t_2 g^{-1}) \\
&= \sum \lambda(ab_0) g^*(S(b_1)t_2) \alpha^{-1}(b_2) h^*(g S^2(t_1) g^{-1}) \quad \text{by (9)} \\
&= \sum \lambda(ab_0) g^*(S(b_1)t_2) \alpha^{-1}(b_2) h^*(t_1) \\
&= \sum \lambda(ab_0) g^*(t_2) \alpha^{-1}(b_3) h^*(\alpha^{-1}(S(b_1)) S(S(b_2)) t_1) \quad \text{by (10)} \\
&= \sum \lambda(ab_0) g^*(t_2) \alpha^{-1}(b_3) h^*(\alpha(b_1) S^2(b_2) t_1) \\
&= \sum \lambda(ab_0) g^*(t_2) h^*(S^2(\alpha(b_1) \alpha^{-1}(b_3) b_2) t_1) \\
&= \sum \lambda(ab_0) g^*(t_2) h^*(S^2(S^{-2}(b_1)) t_1) \quad \text{by (11)} \\
&= \sum \lambda(ab_0) g^*(t_2) h^*(b_1 t_1) \\
&= \sum \lambda(ab_0) g^*(t_2) h_1^*(b_1) h_2^*(t_1) \\
&= \sum \lambda(a(h_1^* \cdot b))(h_2^* g^*)(t) \\
&= \sum \bar{\lambda}(a(h_1^* \cdot b) \# h_2^* g^*) \\
&= \bar{\lambda}((a \# h^*)(b \# g^*))
\end{aligned}
$$

Conversely, assume that $A\#H^*$ is (H^*, g)-symmetric, and let $\mu : A\#H^* \to k$ be a linear map whose kernel does not contain non-zero right ideals of $A\#H^*$, and such that $\mu(h \rightharpoonup z) = \varepsilon(h)\mu(z)$ and $\mu(zz') = \mu(z'(g^{-1} \rightharpoonup z))$ for any $h \in H$ and any $z, z' \in A\#H^*$. Let T be a left integral on H and define

$$\tilde{\mu} : A \to k, \quad \tilde{\mu}(a) = \mu(a\#T).$$

Let $a \in A$ and $h^* \in H^*$. We note that $g \rightharpoonup T$ is a right integral on H, see [8, Proposition 5.5.4]. Let $z = a\#(g \rightharpoonup T)$ and $z' = 1\#h^*$. Then

$$
\begin{aligned}
zz' - z'(g^{-1} \rightharpoonup z) &= (a\#(g \rightharpoonup T))(1\#h^*) - (1\#h^*)(a\#T) \\
&= a\#(g \rightharpoonup T)h^* - \sum (h_1^* \cdot a)\#h_2^* T \\
&= a\#h^*(1)(g \rightharpoonup T) - \sum (h_1^* \cdot a)\#h_2^*(1) T \\
&= h^*(1) a\#(g \rightharpoonup T) - (h^* \cdot a)\#T
\end{aligned}
$$

Since $\mu(zz') = \mu(z'(g^{-1} \rightharpoonup z))$, we get

$$\begin{aligned}\tilde{\mu}(h^* \cdot a) &= \mu((h^* \cdot a)\#T)\\ &= h^*(1)\mu(a\#(g \rightharpoonup T))\\ &= h^*(1)\varepsilon(g)\mu(a\#T)\\ &= h^*(1)\tilde{\mu}(a)\end{aligned}$$

If I is a subobject of A in ${}_A\mathcal{M}^H$ contained in $\mathrm{Ker}\tilde{\mu}$, then $\mu(I\#T) = 0$. But $I\#T$ is a left ideal of $A\#H^*$, so it must be zero. Then I must be zero, too.

To show that A is symmetric in $\mathcal{M}^H$ it only remains to check that $\tilde{\mu}(ba) = \tilde{\mu}(a(\alpha^{-1} \cdot b))$ for any $a, b \in A$. This holds true since

$$\begin{aligned}\tilde{\mu}(ba) &= \mu(ba\#T)\\ &= \mu((b\#\varepsilon)(a\#T))\\ &= \mu((a\#T)(g^{-1} \rightharpoonup (b\#\varepsilon)))\\ &= \mu((a\#T)(b\#(g^{-1} \rightharpoonup \varepsilon)))\\ &= \mu((a\#T)(b\#\varepsilon))\\ &= \sum \mu(a(T_1 \cdot b)\#T_2)\\ &= \sum \mu(aT_1(b_1)b_0\#T_2)\\ &= \sum \mu(ab_0\#(T \leftharpoonup b_1))\\ &= \sum \mu(ab_0\#\alpha^{-1}(b_2)(S^2(b_1) \rightharpoonup T)) \quad \text{by (8)}\\ &= \sum \varepsilon(S^2(b_1))\alpha^{-1}(b_2)\mu(ab_0\#T)\\ &= \sum \varepsilon(b_1)\alpha^{-1}(b_2)\mu(ab_0\#T)\\ &= \sum \alpha^{-1}(b_1)\mu(ab_0\#T)\\ &= \mu(a(\alpha^{-1} \cdot b)\#T)\\ &= \tilde{\mu}(a(\alpha^{-1} \cdot b)) \quad \square\end{aligned}$$

Remark 4.3. (1) The conditions on H in Theorem 4.2 are satisfied if H is involutory and unimodular, and H^* is unimodular. Indeed, in this case the distinguished grouplike elements are trivial, i.e. $\alpha = \varepsilon$ and $g = 1$.

For example, this happens if $H = kG$, where G is a finite group. Thus a finite dimensional G-graded algebra is graded symmetric if and only if the smash product $A\#(kG)^*$ is symmetric in $\mathcal{M}^{(kG)^*}$ with respect to 1.

(2) In the case where the characteristic of k is 0, it is known that H is involutory if and only if H is semisimple, if and only if H is cosemisimple, see [18, Theorem 16.1.2], and in this situation H and H^* are always unimodular. Thus Theorem 4.2 applies to any semisimple Hopf algebra in characteristic 0.

(3) If k has positive characteristic, Theorem 4.2 applies to any semisimple cosemisimple Hopf algebra H. Indeed, it is known that any such H is involutory, see [9, Theorem 3.1].

(4) A Hopf algebra satisfying the conditions of Theorem 4.2 is not necessarily involutory, and it may be not unimodular; take for example Sweedler's 4-dimensional Hopf algebra.

As a consequence of Theorem 4.2 we obtain that if a Hopf algebra H is cosovereign by α and H^* is cosovereign by g, then H is (H,α)-symmetric. In fact, we can prove that H is (H,α)-symmetric with less assumptions.

Proposition 4.4. *Let H be a finite dimensional Hopf algebra which is cosovereign with sovereign element α, the distinguished grouplike element of H^*. Then H is (H,α)-symmetric.*

Proof. In order to use the notation we have already developed, it is more convenient to show that if H^* is cosovereign by g, then H^* is (H^*,g)-symmetric. If t is a right integral in H, by the proof of Theorem 3.1 (when we take $A = k$ and identify $A\#H^*$ with H^*) we have that the linear map $\lambda : H^* \to k$, $\lambda(h^*) = h^*(t)$ is H-linear, and its kernel does not contain nonzero subobjects of H^* in $\mathcal{M}_{H^*}^{H^*}$. On the other hand, for any $h^*, g^* \in H^*$

$$\begin{aligned}
\lambda(g^*(g^{-1} \rightharpoonup h^*)) &= \sum g^*(t_1)h^*(t_2g^{-1}) \\
&= g^*(t_2)h^*(gS^2(t_1)g^{-1}) \quad \text{by (9)} \\
&= \sum h^*(t_1)g^*(t_2) \\
&= (h^*g^*)(t) \\
&= \lambda(h^*g^*)
\end{aligned}$$

so λ makes H^* an (H^*,g)-symmetric algebra. □

5. Passing to coinvariants

If A is a right H-comodule algebra which is Frobenius (respectively symmetric) as an algebra, it is a natural question to ask whether this property transfers to the subalgebra of coinvariants A^{coH}. It is easy to see that such a transfer does not hold. Indeed, let A be the algebra from Example 4.1, which is symmetric. A is a kC_2-comodule algebra, and its subalgebra of coinvariants is just A_e, which is not even Frobenius. If the field k has characteristic $\neq 2$, the C_2-grading on A is equivalent to an action of C_2 on the algebra A, and we have that A is a symmetric algebra, while the subalgebra A^{C_2} of invariants (which is just A_e) is not even Frobenius; this is [14, Exercise 32, page 457].

A positive result in this direction is [14, Exercise 33, page 457], which says that if A is a Frobenius algebra with a nondegenerate associative bilinear form $B : A \times A \to k$, and G is a finite group of automorphisms of A, whose order is not divisible by the characteristic of k, and such that $B(g(a), g(b)) = B(a, b)$ for any $g \in G$, $a, b \in A$, then the subalgebra A^G of invariants is Frobenius. Moreover, if A is symmetric (and B is also symmetric), then A^G is symmetric. These results can be reformulated as follows: if the order of G is not divisible by the characteristic of k, and A is Frobenius (ε-symmetric) in the category $\mathcal{M}^{(kG)^*}$, then the subalgebra $A^{co\ (kG)^*}$ is Frobenius (symmetric). The following proposition generalizes this result, by showing that a good transfer occurs if A is Frobenius in the category $\mathcal{M}^H$, provided H is cosemisimple.

Proposition 5.1. *Let H be a cosemisimple Hopf algebra. If A is a right H-comodule algebra which is Frobenius in the category $\mathcal{M}^H$, then A^{coH} is a Frobenius algebra. If moreover, H is involutory and A is (H, ε)-symmetric, then A^{coH} is symmetric.*

Proof. Let $i : A^{coH} \to A$ be the inclusion map, and let $i^* : A^* \to (A^{coH})^*$ be its dual. Since i is a morphism of A^{coH}, A^{coH}-bimodules, then so is i^*. If A is Frobenius in $\mathcal{M}^H$, let $\theta : A \to A^*$ be an isomorphism in the category $\mathcal{M}_A^H$. We show that $i^*\theta i : A^{coH} \to (A^{coH})^*$ is an isomorphism of right A^{coH}-modules, i.e. A^{coH} is Frobenius. In fact it is enough to show that $i^*\theta i$ is injective; since $Im\ \theta i = (A^*)^{coH}$, this is the same with showing that $i^*_{|(A^*)^{coH}}$ is injective.

The left H^*-action on A^* induced by the right H-coaction is $(h^* \cdot a^*)(a) = \sum h^* S(a_1) a^*(a_0)$, for any $A \in A$. Then $a^* \in (A^*)^{coH}$ if and only if $h^* \cdot a^* = h^*(1) a^*$ for any $h^* \in H^*$, and this means that $a^*(\sum h^* S(a_1) a_0 - h^*(1) a) = 0$ for any $a \in A$ and any $h^* \in H^*$. Since S is bijective (H is cosemisimple), we get that $a^* \in (A^*)^{coH}$ if and only if a^* vanishes on the subspace

$$V = < h^*(a_1) a_0 - h^*(1) a \mid h^* \in H^*, a \in A \ >$$

Since

$$Ker\ i^*_{|(A^*)^{coH}} = \{a^* \in (A^*)^{coH} \mid a^*(A^{coH}) = 0\ \}$$

we see that $i^*_{|(A^*)^{coH}}$ is injective if and only if $V + A^{coH} = A$. But this is indeed true, since for a left integral T on H such that $T(1) = 1$, one has $a = T \cdot a - (T \cdot a - T(1)a)$. Moreover, $T \cdot a \in A^{coH}$, since $h^* \cdot (T \cdot a) = (h^* T) \cdot a = h^*(1) T a$ for any $h^* \in H^*$, and obviously $T \cdot a - T(1) a \in V$.

For the second part we just have to note that $i^*\theta i$ is a morphism of A^{coH}, A^{coH}-bimodules since θ is an isomorphism of A, A-bimodules; now the proof of the first part works also in this case. $\square$

Acknowledgments

The research was supported by the UEFISCDI Grant PN-II-ID-PCE-2011-3-0635, contract no. 253/5.10.2011 of CNCSIS. We thank Daniel Bulacu for valuable discussions about Frobenius algebras. We also thank the referee for the useful suggestions.

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