Journal of Algebra 470 (2017) 160–171

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A bocs theoretic characterization of gendo-symmetric algebras

René Marczinzik

Institute of algebra and number theory, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

ARTICLE INFO

Article history:
Received 28 April 2016
Available online 15 September 2016
Communicated by Volodymyr Mazorchuk

MSC:
16G10
16E10

Keywords:
Representation theory of finite dimensional algebras
Corings
Dominant dimension

ABSTRACT

Gendo-symmetric algebras were recently introduced by Fang and König in [7]. An algebra is called gendo-symmetric in case it is isomorphic to the endomorphism ring of a generator over a finite dimensional symmetric algebra. We show that a finite dimensional algebra A over a field K is gendo-symmetric if and only if there is a bocs-structure on $(A, D(A))$, where $D = Hom_K(-, K)$ is the natural duality. Assuming that A is gendo-symmetric, we show that the module category of the bocs $(A, D(A))$ is equivalent to the module category of the algebra eAe, when e is an idempotent such that eA is the unique minimal faithful projective-injective right A-module. We also prove some new results about gendo-symmetric algebras using the theory of bocses.

Introduction

A bocs is a generalization of the notion of coalgebra over a field. Bocses are also known under the name coring (see the book [4]). A famous application of bocses has been the proof of the tame and wild dichtomy theorem by Drozd for finite dimensional

E-mail address: marczire@mathematik.uni-stuttgart.de.

http://dx.doi.org/10.1016/j.jalgebra.2016.08.041
0021-8693/

algebras over an algebraically closed field (see [5] and the book [3]). For any given bocs (A, W) over a finite dimensional algebra, one can define a corresponding module category and analyze it. Given a finite dimensional algebra A over a field K, it is an interesting question whether for a given A-bimodule W, there exists a bocs structure on (A, W). The easiest example to consider is the case $W = A$ and in this case the module category one gets is just the module category of the algebra A. Every finite dimensional algebra has a duality $D = Hom_K(-, K)$ and so the next example of an A-bimodule to consider is perhaps $W = D(A)$. We will characterize all finite dimensional algebras A such that there is a bocs structure on $(A, D(A))$ and find a surprising connection to a recently introduced class of algebras generalizing symmetric algebras (see [8]). Those algebras are called gendo-symmetric and are defined as endomorphism rings of generators of symmetric algebras. Alternatively these are the algebras A, where there exists an idempotent e such that eA is a minimal faithful injective-projective module and $D(Ae) \cong eA$ as (eAe, A)-bimodules. Then eAe is the symmetric algebra such that $A \cong End_{eAe}(M)$, for an eAe-module M that is a generator of mod-eAe. Famous examples of non-symmetric gendo-symmetric algebras are Schur algebras $S(n, r)$ with $n \geq r$ and blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of a complex semisimple Lie algebra (for a proof of this, using methods close to ours, see [11] and for applications see [9]). The first section provides the necessary background on bocses and algebras with dominant dimension larger than or equal to 2. The second section proves our main theorem:

A. Theorem *(Theorem 2.2). A finite dimensional algebra A is gendo-symmetric if and only if $(A, D(A))$ has a bocs-structure.*

We also provide some new structural results about gendo-symmetric algebras in this section. For example we show, using bocs-theoretic methods, that the tensor product over the field K of two gendo-symmetric algebras is again gendo-symmetric and we proof that $Hom_{A^e}(D(A), A)$ is isomorphic to the center of A, where A^e denotes the enveloping algebra of A.

In the final section, we describe the module category $\mathcal{B}$ of the bocs $(A, D(A))$ in case A is gendo-symmetric. The following is our second main result:

B. Theorem *(Theorem 3.3). Let A be a gendo-symmetric algebra with a minimal faithful projective-injective module eA. Then the module category of the bocs $(A, D(A))$ is equivalent to eAe-mod as K-linear categories.*

I thank Steffen König for useful comments and proofreading. I thank Julian Külshammer for providing me with an early copy of his article [12]. I also thank the two anonymous reviewers for corrections and the final remark in section 2.

1. Preliminaries

We collect here all needed definitions and lemmas to prove the main theorems. Let an algebra always be a finite dimensional algebra over a field K and a module over such an algebra is always a finite dimensional right module, unless otherwise stated. $D = Hom_K(-, K)$ denotes the duality for a given finite dimensional algebra A. $mod - A$ denotes the category of finite dimensional right A-modules and $proj$ (inj) denotes the subcategory of finitely generated projective (injective) A-modules. We note that we often omit the index in a tensor product, when we calculate with elements. We often identify $A \otimes_A X \cong X$ for an A-module X without explicitly mentioning the natural isomorphism. The Nakayama functor $\nu : mod - A \to mod - A$ is defined as $DHom_A(-, A)$ and is isomorphic to the functor $(-) \otimes_A D(A)$. The inverse Nakayama functor $\nu^{-1} : mod - A \to mod - A$ is defined as $Hom_{A^{op}}(-, A)D$ and is isomorphic to the functor $Hom_A(D(A), -)$ (see [14] Chapter III section 5 for details). The Nakayama functors play a prominent role in the representation theory of finite dimensional algebras, since $\nu : proj \to inj$ is an equivalence with quasi-inverse ν^{-1}. For example they appear in the definition of the Auslander–Reiten translates τ and τ^{-1} (see [14] Chapter III. for the definitions):

1.1. Proposition. *Let M be an A-module with a minimal injective presentation* $0 \to M \to I_0 \to I_1$. *Then the following sequence is exact:* $0 \to \nu^{-1}(M) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1) \to \tau^{-1}(M) \to 0$.

Proof. See [14], Chapter III. Proposition 5.3. (ii). □

The *dominant dimension* domdim(M) of a module M with a minimal injective resolution $(I_i) : 0 \to M \to I_0 \to I_1 \to \ldots$ is defined as:

domdim$(M) := \sup\{n | I_i$ is projective for $i = 0, 1, ..., n\} + 1$, if I_0 is projective, and
domdim$(M) := 0$, if I_0 is not projective.

The dominant dimension of a finite dimensional algebra is defined as the dominant dimension of the regular module A_A. It is well-known that an algebra A has dominant dimension larger than or equal to 1 iff there is an idempotent e such that eA is a minimal faithful projective-injective module. The Morita–Tachikawa correspondence (see [15] for details) says that the algebras, which are endomorphism rings of generator-cogenerators are exactly the algebras with dominant dimension at least 2. The full subcategory of modules of dominant dimension at least $i \geq 1$ is denoted by Dom_i. A is called a *Morita algebra* iff it has dominant dimension larger than or equal to 2 and $D(Ae) \cong eA$ as A-right modules. This is equivalent to A being isomorphic to $End_B(M)$, where B is a selfinjective algebra and M a generator of mod-B (see [10]). A is called a *gendo-symmetric algebra* iff it has dominant dimension larger than or equal to 2 and $D(Ae) \cong eA$ as (eAe, A)-bimodules iff it has dominant dimension larger than or equal to 2 and $D(eA) \cong$

Ae as (A, eAe)-bimodules. This is equivalent to A being isomorphic to $End_B(M)$, where B is a symmetric algebra and M a generator of mod-B and in this case $B = eAe$ (see [7]).

1.2. Proposition. *Let A be a gendo-symmetric algebra and M an A-module. Then M has dominant dimension larger than or equal to two iff $\nu^{-1}(M) \cong M$.*

Proof. See [8], proposition 3.3. □

The following result gives a formula for the dominant dimension of Morita algebras:

1.3. Proposition. *Let A be a Morita algebra with a minimal faithful projective-injective module eA and M an A-module. Then $domdim(M) = \inf\{i \geq 0 | Ext^i(A/AeA, M) \neq 0\}$. Especially, $Hom_A(A/AeA, A) = 0$ for every Morita algebra, since they always have dominant dimension at least 2.*

Proof. This is a special case of [1], Proposition 2.6. □

The following lemma gives another characterization of gendo-symmetric algebras, which is used in the proof of the main theorem.

1.4. Lemma. *Let A be a finite dimensional algebra. Then A is a gendo-symmetric algebra iff $D(A) \otimes_A D(A) \cong D(A)$ as A-bimodules. Assume eA is the minimal faithful projective-injective module. In case A is gendo-symmetric, $D(A) \cong Ae \otimes_{eAe} eA$ as A-bimodules.*

Proof. See [8] Theorem 3.2. and [7] in the construction of the comultiplication following Definition 2.3. □

1.5. Lemma. *An A-module P is projective iff there are elements $p_1, p_2, ..., p_n \in P$ and elements $\pi_1, \pi_2, ..., \pi_n \in Hom_A(P, A)$ such that the following condition holds:*

$$x = \sum_{i=1}^{n} p_i \pi_i(x) \text{ for every } x \in P.$$

We then call the $p_1, ..., p_n$ a projective basis and $\pi_1, ..., \pi_n$ a dual projective basis of P.

Proof. See [13] Propostion 3.10. □

1.6. Example. Let $P = eA$, for an idempotent e. Then a projective basis is given by $p_1 = e$ and the dual projective basis is given by $\pi_1 = l_e \in Hom_A(eA, A)$, which is left multiplication by e. l_e can be identified with e under the (A, eAe)-bimodule isomorphism $Ae \cong Hom_A(eA, A)$.

1.7. Proposition. *1. $Hom_A(D(A), A)$ is a faithful right A-module iff there is an idempotent e, such that eA and Ae are faithful and injective.*

2. Let A be an algebra with $Hom_A(D(A), A) \cong A$ as right A-modules, then A is a Morita algebra.

Proof. 1. See [10], Theorem 1.

2. See [10], Theorem 3. □

1.8. Lemma. *Let Y and Z be A-bimodules. Then the following is an isomorphism of A-bimodules:*

$$Hom_A(Y, D(Z)) \cong D(Y \otimes_A Z).$$

Proof. See [2] Appendix 4, Proposition 4.11. □

1.9. Definition. Let A be a finite dimensional algebra and W an A-bimodule and let $c_r : W \to A \otimes_A W$ and $c_l : W \to W \otimes_A A$ be the canonical isomorphisms. Then the pair $\mathcal{B} := (A, W)$ is called a *bocs* (see [12]) or the module W is called an A-coring (see [4]) if there are A-bimodule maps $\mu : W \to W \otimes_A W$ (the comultiplication) and $\epsilon : W \to A$ (the counit) with the following properties: $(1_W \otimes_A \epsilon)\mu = c_l$, $(\epsilon \otimes_A 1_W)\mu = c_r$ and $(\mu \otimes_A 1_W)\mu = (1_W \otimes_A \mu)\mu$. We often say for short that W is a bocs, if A (and μ and ϵ) are clear from the context. The category of the finite dimensional bocs modules is defined as follows:

Objects are the finite dimensional right A-modules.

Homomorphism spaces are $Hom_{\mathcal{B}}(M, N) := Hom_A(M, Hom_A(W, N))$ with the following composition $*$ and units:

Let $g : M \to Hom_A(W, N) \in Hom_{\mathcal{B}}(M, N)$ and $f : L \to Hom_A(W, M) \in Hom_{\mathcal{B}}(L, M)$. Then $g * f := Hom_A(\mu, N)\psi Hom_A(W, g)f$, where ψ is the adjunction isomorphism $Hom_A(W, Hom_A(W, N)) \to Hom_A(W \otimes_A W, N)$. The units $1_M \in Hom_{\mathcal{B}}(M, M)$ are defined as follows: $1_M := Hom_A(\epsilon, M)\xi$, where $\xi : M \to Hom_A(A, M)$ is the canonical isomorphism. Note that the module category of a bocs is K-linear. We refer to [12] for other equivalent descriptions of the bocs module category and more information.

1.10. Examples. 1. (A, A) is always a bocs with the obvious multiplication and comultiplication. The next natural bimodule to look for a bocs-structure is $D(A)$. We will see that $(A, D(A))$ is not a bocs for arbitrary finite dimensional algebras.

2. The next example can be found in 17.6. in [4], to which we refer for more details. Let P be a (B, A)-bimodule for two finite dimensional algebras B and A such that P is projective as a right A-module and let $P^* := Hom(P, A)$, which is then a (A, B)-bimodule. Let $p_1, p_2, ..., p_n$ be a projective basis for P and $\pi_1, \pi_2, ..., \pi_n$ a dual projective basis of the projective A-module P. Denote the A-bimodule $P^* \otimes_B P$ by W and define the comultiplication $\mu : W \to W \otimes_A W$ as follows: Let $f \in P^*$ and $p \in P$, then $\mu(f \otimes p) = \sum_{i=1}^{n} (f \otimes p_i) \otimes (\pi_i \otimes p)$. Define the counit $\epsilon : W \to A$ as follows: $\epsilon(f \otimes p) = f(p)$.

Now specialise to $P = eA$, for an idempotent e and identify $Hom_A(eA, A) = Ae$. Then $\mu(ae \otimes eb) = (ae \otimes e) \otimes (e \otimes eb)$ and $\epsilon(ae \otimes eb) = aeb$. We will use this special case in the next section to show that $(A, D(A))$ is always a bocs for a gendo-symmetric algebra.

3. Let (A_1, W_1) and (A_2, W_2) be bocses, then $(A_1 \otimes_K A_2, W_1 \otimes_K W_2)$ is again a bocs. See [4] 24.1. for a proof.

2. Characterization of gendo-symmetric algebras

The following lemma, will be important for proving the main theorem.

2.1. Lemma. *Assume that $Hom_A(D(A), A) \cong A \oplus X$ as right A-modules for some right A-module X, then $domdim(A) \geq 2$ and $X = 0$.*

Proof. By assumption $Hom_A(D(A), A)$ is faithful and so there is an idempotent e with eA and Ae faithful and injective by 1.7 1., which implies that A has dominant dimension at least 1. Choose e minimal such that those properties hold. Now look at the minimal injective presentation $0 \to A \to I_0 \to I_1$ of A and note that $I_0 \in add(eA)$. Using 1.1, there is the following exact sequence: $0 \to \nu^{-1}(A) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1) \to \tau^{-1}(A) \to 0$. But $\nu^{-1}(A) \cong Hom_A(D(A), A) \cong A \oplus X$ and so there is the embedding: $0 \to A \oplus X \to \nu^{-1}(I_0)$. Note that $\nu^{-1}(I_0) \in add(eA)$ is the injective hull of $A \oplus X$, since $\nu^{-1} : inj \to proj$ is an equivalence and eA is the minimal faithful projective injective module. Thus $\nu^{-1}(I_0)$ has the same number of indecomposable direct summands as I_0. Therefore $soc(X) = 0$ and so $X = 0$, since every indecomposable summand of the socle of the module provides an indecomposable direct summand of the injective hull of that module. Thus $Hom_A(D(A), A) \cong A$ and A is a Morita algebra by 1.7 2. and so A has dominant dimension at least 2. □

We now give a bocs-theoretic characterization of gendo-symmetric algebras.

2.2. Theorem. *Let A be a finite dimensional algebra. Then the following are equivalent:*

1. *A is gendo-symmetric.*
2. *There is a comultiplication and counit such that $\mathcal{B} = (A, D(A))$ is a bocs.*

Proof. We first show that 1. implies 2.:

Assume that A is gendo-symmetric with a minimal faithful projective-injective module eA. Set $P := eA$ and apply the second example in 1.10, with $B := eAe$, to see that $\mathcal{B} := (A, Ae \otimes_{eAe} eA)$ has the structure of a bocs. Now note that by 1.4 $D(A) \cong Ae \otimes_{eAe} eA$ as A-bimodules and one can use this to get a bocs structure for $(A, D(A))$.

Now we show that 2. implies 1.:

Assume that $(A, D(A))$ is a bocs with comultiplication μ and counit ϵ. Note first that the comultiplication μ always has to be injective because in the identity $(\epsilon \otimes_A 1_W)\mu = c_r$ appearing the definition of a bocs, c_r is an isomorphism. So there is an injection μ :

$D(A) \to D(A) \otimes_A D(A)$ which gives a surjection $D(\mu) : D(D(A) \otimes_A D(A)) \to A$. Now using 1.8 we see that $D(D(A) \otimes_A D(A)) \cong Hom_A(D(A), A)$ as A-bimodules.

Since A is projective, $D(\mu)$ is split and $Hom_A(D(A), A) \cong A \oplus X$ for some A-right module X. By 2.1, this implies $Hom_A(D(A), A) \cong A$ and comparing dimensions, $D(\mu)$ and thus also μ have to be isomorphisms. By 1.4, A is gendo-symmetric. □

We give an interesting consequence of the previous theorem, where we need the definition of comonads from [4], 38.26.

2.3. Definition. Let C be a category and $I_C : C \to C$ the identity functor. Then a *comonad* is a functor $F : C \to C$ such that there exist natural transformations $\delta : F \to F \circ F$ and $\psi : F \to I_C$ with $\delta_{F(N)} \circ \delta_N = F(\delta_N) \circ \delta_N$ and $\psi_{F(N)} \circ \delta_N = F(\psi_N) \circ \delta_N = id_{F(N)}$.

2.4. Corollary. *Let A be a finite dimensional algebra. Then the following two conditions are equivalent:*

1. *A is gendo-symmetric.*
2. *ν is a comonad.*

Proof. In [4] 18.28. it is proven that an A-bimodule W is a bocs iff the functor $(-)\otimes_A W$ is a comonad. Applying this with $W = D(A)$ and using the previous theorem, the corollary follows. □

2.5. Remark. Theorem 2.2 also shows that the comultiplication of the bocs $(A, D(A))$ is always an A-bimodule isomorphism for a gendo-symmetric algebra A. In [7], section 2.2., it is noted that such an isomorphism is unique up to multiples of invertible central elements in A. Thus the comultiplication of the bocs is also unique in that sense.

The following proposition gives an application:

2.6. Proposition. *Let A and B be gendo-symmetric K-algebras. Then $A \otimes_K B$ is again a gendo-symmetric K-algebra. In particular, let F be a field extension of K and A a gendo-symmetric K-algebra. Then $A \otimes_K F$ is again gendo-symmetric.*

Proof. Let A and B two gendo-symmetric algebras. Then $\mathcal{B}_1 = (A, D(A))$ and $\mathcal{B}_2 = (B, D(A))$ are bocses. By example 3 of 1.10 also the tensor product of $\mathcal{B}_1$ and $\mathcal{B}_2$ are bocses, it is the bocs $\mathcal{C} = (A \otimes_K B, D(A) \otimes_K D(B))$. Recall the well known formula $(D(A) \otimes_K D(B)) \cong D(A \otimes_K B)$, which can be found as exercise 12. of chapter II. in [14]. Using this isomorphism one can find a bocs structure on $(A \otimes_K B, D(A \otimes_K B))$ using the bocs structure on $\mathcal{C}$. Thus by our bocs-theoretic characterization of gendo-symmetric algebras, also $A \otimes_K B$ is gendo-symmetric. The second part follows since every field is a symmetric and thus gendo-symmetric algebra. □

Let $A^e := A^{op} \otimes_K A$ denote the enveloping algebra of a given algebra A. The following proposition can be found in [4], 17.8.

2.7. Proposition. *Let (A, W) be a bocs and $c \in W$ with $\mu(c) = \sum_{i=1}^{n} c_{1,i} \otimes c_{2,i}$.*

1. $Hom_A(W, A)$ has a ring structure with unit ϵ and product $*^r$, given as follows for $f, g \in Hom_A(W, A)$:

$$f *^r g = g(f \otimes_A id_W)\mu.$$

There is a ring anti-morphism $\zeta : A \to Hom_A(W, A)$, given by $\zeta(a) = \epsilon(a(-))$.

2. $Hom_{A^e}(W, A)$ has a ring structure with unit ϵ and multiplication $*$ given as follows for $f, g \in Hom_{A^e}(W, A)$:

$$f * g(c) = \sum_{i=1}^{n} f(c_{1,i})g(c_{2,i}).$$

We now describe the ring structures on $Hom_{A^e}(D(A), A)$ and $Hom_A(D(A), A)$.

2.8. Proposition. *Let A be gendo-symmetric.*

1. ζ, as defined in the previous proposition, is a ring anti-isomorphism $\zeta : A \to Hom_A(D(A), A)$.

2. With the ring structure on $Hom_{A^e}(D(A), A)$ as defined in the previous proposition, $Hom_{A^e}(D(A), A)$ is isomorphic to the center $Z(A)$ of A.

Proof. We use the isomorphism of A-bimodule $D(A) \cong Ae \otimes_{eAe} eA$.

1. Since A and $Hom_A(D(A), A)$ have the same K-dimension, the only thing left to show is that ζ is injective. So assume that $\zeta(a) = \epsilon(a(-)) = 0$, for some $a \in A$. This is equivalent to $\epsilon(ax) = 0$ for every $x = ce \otimes ed \in Ae \otimes eA$. Now $\epsilon(a(ce \otimes ed)) = \epsilon(ace \otimes ed) = aced$. Thus, since c, d were arbitrary, $aAeA = 0$. This means that a is in the left annihilator $L(AeA)$ of the two-sided ideal AeA. But $L(AeA) = 0$, since $Hom_A(A/AeA, A) = 0$, by 1.3 and thus $a = 0$. Therefore ζ is injective.

2. Define $\psi : Hom_{A^e}(D(A), A) \to Z(eAe)$ by $\psi(f) = f(e \otimes e)$, for $f \in Hom_{A^e}(D(A), A)$. First, we show that this is well-defined, that is $f(e \otimes e)$ is really in the center of $Z(eAe)$. Let $x \in eAe$. Then $xf(e \otimes e) = f(xe \otimes e) = f(e \otimes ex) = f(e \otimes e)x$ and therefore $f(e \otimes e) \in Z(eAe)$. Clearly, ψ is K-linear. Now we show that the map is injective: Assume $\psi(f) = 0$, which is equivalent to $f(e \otimes e) = 0$. Then for any $a, b \in A : f(ae \otimes eb) = 0$, and thus $f = 0$.

Now we show that ψ is surjective. Let $z \in Z(eAe)$ be given. Then define a map $f_z \in Hom_{A^e}(D(A), A)$ by $f_z(ae \otimes eb) = zaeb$. Then, since z is in the center of eAe, f is A-bilinear and obviously $\psi(f_z) = f_z(e \otimes e) = ze = z$. ψ also preserves the unit and multiplication: $\psi(\epsilon) = \epsilon(e \otimes e) = e^2 = e$ and for two given $f, g \in Hom_{A^e}(D(A), A)$: $\phi(f * g) = (f * g)(e \otimes e) = (f * g)(e \otimes e) = f(e \otimes e)g(e \otimes e)$, by the definition of $*$. To finish the proof, we use the result from [7], Lemma 2.2., that the map $\phi : Z(A) \to Z(eAe)$, $\phi(z) = eze$ is a ring isomorphism in case A is gendo-symmetric. □

2.9. Remark. Let $\mathcal{B} = (A, W)$ be a bocs and take the natural isomorphism $\xi : End_{\mathcal{B}}(A) = Hom_A(A, Hom_A(W, A)) \to Hom_A(W, A)$. For $f, g \in Hom_A(W, A)$ one can define another multiplication $\times$ in $Hom_A(W, A)$ as follows: $g \times f(c) = \sum_{i=1}^{n} g(f(c_{1,i})c_{2,i})$, when $\mu(c) = \sum_{i=1}^{n} c_{1,i} \otimes c_{2,i}$. Note that this is the opposite multiplication as in 2.7 1. Then $Hom_{A^e}(W, A)$ is a K-subalgebra of $Hom_A(W, A)$ with respect to $\times$ and because of $\xi(uv) = \xi(u) \times \xi(v)$ for $u, v \in End_{\mathcal{B}}(A)$ and the fact that ξ preserves units, $\xi^{-1}(Hom_{A^e}(W, A))$ is a K-subalgebra of $End_{\mathcal{B}}(A)$.

3. Description of the module category of the bocs $(A, D(A))$ for a gendo-symmetric algebra

Let A be a gendo-symmetric algebra. In this section we describe the module category of the bocs $\mathcal{B} = (A, D(A))$ as a K-linear category. We will use the A-bimodule isomorphism $Ae \otimes_{eAe} eA \cong D(A)$ often without mentioning. Let M be an arbitrary A-module. Define for a given M the map $I_M : M \to Hom_A(D(A), M)$ by $I_M(m) = u_m$ for any $m \in M$, where $u_m : D(A) \to M$ is the map $u_m(ae \otimes eb) = maeb$ for any $a, b \in A$. Before we get into explicit calculation, let us recall how $*$ is defined in this special case. Let $f \in Hom_{\mathcal{B}}(L, M)$ and $g \in Hom_{\mathcal{B}}(M, N)$, then for $l \in L$ and $a, b \in A$: $(g * f)(l)(ae \otimes eb) = g(f(l)(ae \otimes e))(e \otimes eb)$.

3.1. Proposition. *1. I_M is well defined.*

2. I_M is injective, iff M has dominant dimension larger than or equal to 1.

3. I_M is bijective, iff M has dominant dimension larger than or equal to 2.

Proof. 1. We have to show two things: First, u_m is A-linear for any $m \in M$: $u_m((ae \otimes eb)c) = u_m(ae \otimes ebc) = maebc = (maeb)c = u_m(ae \otimes eb)c$. Second, I_M is also A-linear: $I_M(mc)(ae \otimes eb) = u_{mc}(ae \otimes eb) = mcaeb = u_m(cae \otimes eb) = (u_m c)(ae \otimes eb) = (I_M(m)c)(ae \otimes eb)$.

2. I_M is injective iff $(m = 0 \Leftrightarrow u_m = 0)$. Now $u_m = 0$ is equivalent to $maeb = 0$ for any $a, b \in A$. This is equivalent to the condition that the two-sided ideal AeA annihilates m. Thus there is a nonzero m with $u_m = 0$ iff $Hom_A(A/AeA, M) \neq 0$ iff M has dominant dimension zero by 1.3.

3. By 1.2 M has dominant dimension larger than or equal to two iff $M \cong \nu^{-1}(M)$.

Thus 3. follows by 2. since an injective map between modules of the same dimension is a bijective map. □

3.2. Lemma. *For any module M, there is an isomorphism $Hom_A(\mu, M)\psi : Hom_A(D(A), Hom_A(D(A), M)) \rightarrow Hom(D(A), M)$ and thus $\nu^{-1}(M) \cong \nu^{-2}(M)$. It follows that every module of the form $\nu^{-1}(M)$ has dominant dimension at least two.*

Proof. The result follows, since ψ is the canonical isomorphism $\psi : Hom_A(D(A), Hom_A(D(A), M)) \rightarrow Hom_A(D(A) \otimes_A D(A), M)$ and since μ is an isomorphism also $Hom_A(\mu, M)$ is an isomorphism. That $\nu^{-1}(M)$ has dominant dimension at least two, follows now from 1.2. $\square$

We define a functor $\phi : mod - A \rightarrow mod - \mathcal{B}$ by $\phi(M) = M$ and $\phi(f) = I_N f$ for an A-homomorphism $f : M \rightarrow N$. ϕ is obviously K-linear. The next result shows that it really is a functor and calculates its kernel on objects.

3.3. Theorem. *1. ϕ is a K-linear functor.*

2. $\phi(M) = 0$ iff the two-sided ideal AeA annihilates M, that is M is an A/AeA-module. All modules M that are annihilated by AeA have dominant dimension zero.

3. By restricting ϕ to Dom_2, one gets an equivalence of K-linear categories $Dom_2 \rightarrow Dom_2^{\mathcal{B}}$, where $Dom_2^{\mathcal{B}}$ denotes the full subcategory of $mod - \mathcal{B}$ having objects all modules of dominant dimension at least 2.

4. Any module A-module M is isomorphic to $\nu^{-1}(M)$ in $\mathcal{B}$-mod and thus $\mathcal{B}$-mod is equivalent to Dom_2 as K-linear categories, which is equivalent to the module category mod-eAe.

Proof. 1. It was noted above that ϕ is K-linear. We have to show $\phi(id_M) = Hom(\epsilon, M)\zeta$, where $\zeta : M \rightarrow Hom_A(A, M)$ is the canonical isomorphism, and $\phi(g \circ f) = I_N(g) * I_M(f)$, where $f : L \rightarrow M$ and $g : M \rightarrow N$ are A-module homomorphisms. To show the first equality $\phi(id_M) = Hom(\epsilon, M)\zeta$, just note that $Hom(\epsilon, M)\zeta(m)(ae \otimes eb) = l_m(\epsilon(ae \otimes eb)) = maeb = I_M(m)(ae \otimes eb)$, where $l_m : A \rightarrow M$ is left multiplication by m.

Next we show the above equality $\phi(g \circ f) = I_N(g) * I_M(f)$:

Let $l \in L$ and $a, b \in A$. First, we calculate $\phi(g \circ f)(l)(ae \otimes eb) = g(f(l))aeb$.

Second, $I_N(g) * I_M(f)(l)(ae \otimes eb) = I_N(g)(I_M(f)(l)(ae \otimes e))(e \otimes eb) = I_N(g)(u_{f(l)}(ae \otimes e))(e \otimes eb) = I_N(g)(f(l)(ae))(e \otimes eb) = g(f(l))aeb$.

Thus $\phi(g \circ f) = I_N(g) * I_M(f)$ is shown.

2. A module M is zero in the K-category mod-$\mathcal{B}$ iff its endomorphism ring $End_{\mathcal{B}}(M)$ is zero iff the identity of $End_{\mathcal{B}}(M)$ is zero. Thus M is zero in mod-$\mathcal{B}$ iff $I_M(m) = 0$ for every $m \in M$. But $I_M(m) = 0$ iff $mAeA = 0$ and so $\phi(M) = 0$ iff $MAeA = 0$. To see that such an M must have dominant dimension zero, note that AeA annihilates no element of M iff M has dominant dimension larger than or equal to 1 by 1.3.

3. Restricting ϕ to Dom_2, ϕ is obviously still dense by the definition of $Dom_2^{\mathcal{B}}$. Now recall that by the previous proposition a module M has dominant dimension at least two iff I_M is an isomorphism. Let now $h \in Hom_{\mathcal{B}}(M, N)$ be given with $M, N \in Dom_2^{\mathcal{B}}$. Then $\phi(I_N^{-1}h) = I_N(I_N^{-1}h)) = h$ and ϕ is full. Assume $\phi(h) = I_N h = 0$, then $h = 0$, since I_N is an isomorphism, and so ϕ is faithful.

4. Define $f \in Hom_{\mathcal{B}}(M, \nu^{-1}(M))$ as $f = (Hom_A(\mu, M)\psi)^{-1}I_M$ and $g \in Hom_{\mathcal{B}}(\nu^{-1}(M), M)$ as $g = id_{\nu^{-1}(M)}$. We show that $f * g = I_{\nu^{-1}}(M)$ and $g * f = I_M$, which by 1. are the identities of $Hom_{\mathcal{B}}(\nu^{-1}(M), \nu^{-1}(M))$ and $Hom_{\mathcal{B}}(M, M)$. This shows that any module M is isomorphic to $\nu^{-1}(M)$ in $\mathcal{B}$-mod.

Let $m \in M$ and $a, b \in A$.

Then $(g*f)(m)(ae\otimes eb) = g(f(m)(ae\otimes e))(e\otimes eb) = ((Hom_A(\mu, M)\psi)^{-1}I_M(m))((ae\otimes e))(e\otimes eb)) = maeb = I_M(m)(ae\otimes eb)$, where we used that g is the identity on $\nu^{-1}(M)$. Next we show that $f * g = I_{\nu^{-1}}(M)$: Let $l \in \nu^{-1}(M) = Hom_A(D(A), M)$.

First, note that by definition $I_{\nu^{-1}}(M)(l)(ae \otimes eb)(a'e \otimes eb') = (laeb)(a'e \otimes eb') = l(aeba'e \otimes eb')$. Next $(f * g)(l)(ae \otimes eb)(a'e \otimes eb') = f(g(l)(ae \otimes eb)(a'e \otimes eb') = f(l(ae \otimes eb))(a'e\otimes eb') = (Hom_A(\mu, M)\psi)^{-1}I_M(l(ae\otimes eb)(a'e\otimes eb') = l(ae\otimes eba'eb') = l(aeba'e\otimes eb')$, where we used in the last step that we tensor over eAe.

Now we use 3.2, to show that every module of the form $\nu^{-1}(M)$ has dominant dimension at least two. Since every module M is isomorphic to $\nu^{-1}(M)$, $\mathcal{B} - mod$ is isomorphic to $Dom_2^{\mathcal{B}}$, which is isomorphic to Dom_2 by 3. Now recall that there is an equivalence of categories mod-$eAe \to Dom_2$ (this is a special case of [1] Lemma 3.1.). Combining all those equivalences, we get that $\mathcal{B} - mod$ is equivalent to the module category mod-eAe. □

3.4. Corollary. *In case an A-module M has dominant dimension larger than or equal to 2, the map $Hom_A(M, I_M) : End_A(M) \to End_{\mathcal{B}}(M)$ is a K-algebra isomorphism. In particular $A \cong End_A(A) \cong End_{\mathcal{B}}(A)$, since A has dominant dimension at least 2.*

Proof. This follows since I_M is an isomorphism, in case M has dominant dimension at least two by 3.1 3. □

3.5. Example. Let $n \geq 2$ and $A := K[x]/(x^n)$ and J the Jacobson radical of A. Let $M := A \oplus \bigoplus_{k=1}^{n-1} J^k$ and $B := End_A(M)$. Then B is the Auslander algebra of A and B has n simple modules. The idempotent e is in this case primitive and corresponds to the unique indecomposable projective-injective module $Hom_A(M, A)$. By the previous theorem, the kernel of ϕ is isomorphic to the module category $mod - (A/AeA)$. Here A/AeA is isomorphic to the preprojective algebra of type A_{n-1} by [6] chapter 7.

We describe the bocs module category $\mathcal{B}$-mod of $(B, D(B))$ for $n = 2$ explicitly. In this case B is isomorphic to the Nakayama algebra with Kupisch series $[2, 3]$. Then B has five indecomposable modules. Let e_0 be the primitive idempotent corresponding to the indecomposable projective module with dimension two and e_1 the primitive idempotent corresponding to the indecomposable projective module with dimension three. Then e_1A is the unique minimal faithful indecomposable projective-injective module. Let S_i denote the simple B-modules. The only indecomposable module annihilated by Be_1B is S_0, which is therefore isomorphic to zero in the bocs module category. The two indecomposable projective modules $P_0 = e_0B$ and $P_1 = e_1B$ have dominant dimension at least two and thus are not isomorphic. The only indecomposable module of dominant dimension 1 is S_1 and the only indecomposable module of dominant dimension zero, which is not isomorphic to zero in $\mathcal{B}$-mod, is $D(Be_0)$. Now let $X = S_1$ or $X = D(Be_0)$, then $\nu^{-1}(X) = Hom_B(D(B), X) \cong e_0B$. Thus in $\mathcal{B}$-mod $S_1 \cong e_0B \cong D(Be_0)$ and e_1B are up to isomorphism the unique indecomposable objects.

References

[1] Maurice Auslander, Maria Ines Platzeck, Todorov Gordana, Homological theory of idempotent ideals, Trans. Amer. Math. Soc. 332 (2) (August 1992) 667–692.
[2] Ibrahim Assem, Daniel Simson, Andrzej Skowronski, Elements of the Representation Theory of Associative Algebras, vol. 1: Representation-Infinite Tilted Algebras, London Math. Soc. Stud. Texts, vol. 72, 2007.
[3] Raymundo Bautista, Leonardo Salmeron, Rita Zuazua Vega, Differential Tensor Algebras and Their Module Categories, London Math. Soc. Lecture Note Ser., vol. 362, 2009.
[4] Tomasz Brzezinski, Robert Wisbauer, Corings and Comodules, London Math. Soc. Lecture Note Ser., vol. 309, 2003.
[5] Yuriy Drozd, Tame and wild matrix problems, in: Representation Theory, II, Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979, in: Lecture Notes in Math., vol. 832, Springer, Berlin–New York, 1980, pp. 242–258.
[6] Vlastimil Dlab, Claus Michael Ringel, The module theoretical approach to quasi-hereditary algebras, in: Representations of Algebras and Related Topics, in: London Math. Soc. Lecture Note Ser., vol. 168, 1990, pp. 200–224.
[7] Ming Fang, Steffen Koenig, Gendo-symmetric algebras, canonical comultiplication, bar cocomplex and dominant dimension, Trans. Amer. Math. Soc. 368 (7) (2016) 5037–5055.
[8] Ming Fang, Steffen Koenig, Endomorphism algebras of generators over symmetric algebras, J. Algebra 332 (2011) 428–433.
[9] Ming Fang, Steffen Koenig, Schur functors and dominant dimension, Trans. Amer. Math. Soc. 363 (2011) 1555–1576.
[10] Otto Kerner, Kunio Yamagata, Morita algebras, J. Algebra 382 (2013) 185–202.
[11] Steffen Koenig, Inger Heidi Slungård, Changchang Xi, Double centralizer properties, dominant dimension, and tilting modules, J. Algebra 240 (1) (2001) 393–412.
[12] Julian Külshammer, In the bocs seat: quasi-hereditary algebras and representation type, SPP 1388 Conference Proceedings, 2016, in press; arXiv version: http://arxiv.org/abs/1601.03899.
[13] Joseph Rotman, An Introduction to Homological Algebra (Universitext), 2nd edition, Springer, 2009.
[14] Andrzej Skowronski, Kunio Yamagata, Frobenius Algebras I: Basic Representation Theory, EMS Textbk. Math., 2011.
[15] Hiroyuki Tachikawa, Quasi-Frobenius Rings and Generalizations: QF-3 and QF-1 Rings, Lecture Notes in Math., vol. 351, Springer, 1973.