

The center conjecture for non-exceptional buildings

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Abstract

We prove the center conjecture for spherical buildings of non-exceptional type. Our proof uses the point-line spaces associated with these buildings.

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1. Introduction

The main purpose of this paper is to provide a proof of the *center-conjecture* for irreducible spherical buildings of non-exceptional type. With the definitions and notation of [Ti74] this conjecture reads as follows.

Conjecture. *Let $\Delta = (\Delta, \subset)$ be a spherical building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Then (at least) one of the following holds.*

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- (a) For each simplex $A \in \tilde{\Delta}$ there is a simplex $B \in \tilde{\Delta}$ which is opposite to A in Δ .
- (b) The group $\text{Stab}_{\text{Aut}(\tilde{\Delta})}(\tilde{\Delta})$ fixes a non-trivial simplex of $\tilde{\Delta}$.

In view of Théorème 2.1. in [Se04] the conjecture above is equivalent to Conjecture 2.8 stated by J.-P. Serre in [Se04]. Following his terminology we call a convex subcomplex of a spherical building *completely reducible* if assertion (a) of the center-conjecture holds. If assertion (b) holds for a simplex \tilde{A} contained in a convex subcomplex $\tilde{\Delta}$ which is not completely reducible, then there are good reasons to call \tilde{A} a *center* of this subcomplex. Note, that for certain convex subcomplexes both assertions may be valid.

Here is the statement of the main result of this paper:

Main result. *If Δ is an irreducible spherical building which is not of type E_6 , E_7 , E_8 or F_4 , then the center-conjecture holds for Δ .*

The proof of this result is based on combinatorial properties of the incidence geometries associated with the buildings in question. Accordingly, we have to consider generalized polygons, projective spaces and polar spaces. It should be mentioned that for all those types of incidence geometries, the principal ideas are already in the literature. For instance, the case of generalized polygons is covered by part (a) of Proposition 2.10 in [Se04]. It is also treated in [Ti96/97, Section II.2]. In Sections II.3 and II.4 of [Ti96/97] the ideas for treating projective spaces and polar spaces are outlined. In this paper we give a detailed proof of the result above which is based on these ideas.

1.1. Buildings of type D_4

Buildings of type D_n can be identified with the flag-complexes of the oriflamme geometries of certain polar-spaces. If $n \geq 5$, the automorphism groups of the building and the polar space coincide, whereas there may exist trialities when $n = 4$. Hence in the latter case the automorphism group of the building might be bigger. This phenomenon requires some additional arguments. We shall just outline the proof in this case in the final paragraph of Section 6. We mention however, that a proof of the center-conjecture for buildings of type F_4 would also cover the D_4 case in view of [Ti74] 10.14.

1.2. A consequence of the main result

It is not difficult to prove that the center-conjecture holds for a spherical building if and only if it holds for each of its irreducible components. Taking this into account the main result implies the following:

Corollary. *Let Δ be a spherical building whose type has no direct factor E_6 , E_7 , E_8 or F_4 . Then the center-conjecture holds for Δ .*

2. Preliminaries

2.1. Incidence structures and flag complexes

A binary relation on a set V is called an *incidence relation* if it is reflexive and symmetric. An *incidence structure* (or *incidence geometry*) is a pair $\mathcal{G} = (V, \star)$ where V is a set and \star is an incidence relation on V .

Let $\mathcal{G} = (V, \star)$ be an incidence structure. A *flag* of \mathcal{G} is a subset of V whose elements are pairwise incident. We denote the set of flags of \mathcal{G} by $\text{flag } \mathcal{G}$ and $\text{Flag } \mathcal{G} := (\text{flag } \mathcal{G}, \subset)$ denotes the *flag complex* of \mathcal{G} . For any subset Ω of V , $\text{flag } \Omega$ denotes the set of all flags contained in Ω and $\text{Flag } \Omega$ the corresponding subcomplex of $\text{Flag } \mathcal{G}$. It readily follows from the definitions that the automorphism groups of \mathcal{G} and $\text{Flag } \mathcal{G}$ are the same and that the stabilizer of a subset $\Omega \subset V$ in $\text{Aut}(\mathcal{G})$ corresponds to the stabilizer of $\text{flag } \Omega$ in $\text{Aut}(\text{Flag } \mathcal{G})$.

2.2. Buildings

Throughout this paper we essentially adopt the definitions and notation of [Ti74] concerning buildings. Hence we view buildings as simplicial complexes. The buildings in this paper are not required to be thick which means that we use the expression ‘building’ for a structure which corresponds to the term ‘weak building whose Weyl complexes are Coxeter complexes’ in [Ti74, 3.1]. Since we are only interested in spherical buildings, all buildings considered in this paper are of finite rank.

We recall that an automorphism of a building $\Delta = (\Delta, \subset)$ is not assumed to be type-preserving, that the automorphism group of Δ is denoted by $\text{Aut}(\Delta)$ and that the group of type-preserving automorphism of Δ is denoted by $\text{Spe}(\Delta)$.

We also recall the definition of a *convex subcomplex* of a building $\Delta = (\Delta, \subset)$ (see [Ti74, 1.5]): A subset $\tilde{\Delta}$ of Δ is called a *chamber subcomplex* of Δ if it is a subcomplex and if its maximal elements are chambers of Δ . A chamber subcomplex of Δ is called *convex* if the set of chambers contained in it is a convex subset of the chamber graph of Δ . An arbitrary subcomplex of a building Δ is called *convex* if it is the intersection of a set of convex chamber subcomplexes of Δ .

The following fact is an immediate consequence of 3.19.1 in [Ti74]:

Lemma 2.1. *Let $\Delta = (\Delta, \subset)$ be a building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . For $A, B \in \tilde{\Delta}$, the projection $\text{proj}_{\tilde{\Delta}} B$, as defined in [Ti74, 3.19], is contained in $\tilde{\Delta}$. In particular, if $A, B \in \tilde{\Delta}$ have a common upper bound, then $A \cup B$ is contained in $\tilde{\Delta}$.*

Let $\Delta = (\Delta, \subset)$ be a spherical building. The following definition is due to J.-P. Serre [Se04, Définition 2.2.1].

Definition 2.2. Let $\Delta = (\Delta, \subset)$ be a spherical building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Then $\tilde{\Delta}$ is called *completely reducible* if for each simplex $A \in \tilde{\Delta}$ there exists a simplex $B \in \tilde{\Delta}$ which is opposite to A in Δ .

The following proposition is a restatement of Théorème 2.2 of [Se04]:

Proposition 2.3. *Let $\Delta = (\Delta, \subset)$ be a spherical building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Then the following are equivalent:*

- (a) $\tilde{\Delta}$ is completely reducible.
- (b) For each vertex $A \in \tilde{\Delta}$ there exists a vertex $B \in \tilde{\Delta}$ which is opposite to A in Δ .

Remark. It is known by 3.16 of [Ti74] that buildings of finite rank are flag complexes. In this paper we will identify the buildings we are interested in as flag complexes of certain classes of incidence structures, namely generalized polygons, projective geometries and polar geometries. For this purpose we make use of the identification Theorems 3.5, 4.9, 5.16 and 6.4. We will not give a detailed proof of these theorems because they summarize well-known facts. Most of them can be found in [Ti74] (see 3.34, 6.3, 7.4, 7.12) or easily deduced from the information given there. Only statements (c) of these theorems are less obvious from the literature. Their proofs, using simple properties of the corresponding Coxeter complexes, are straightforward and will be omitted.

2.3. Graphs

In this paper, a *graph* is a pair $\Gamma = (V, E)$ where V is a set and E is a subset of the set of 2-element subsets of V . The elements of V and E are called the *vertices* and the *edges* of Γ , respectively.

Let $\Gamma = (V, E)$ be a graph. For $v \in V$ we put $\Gamma_v := \{w \in V \mid \{v, w\} \in E\}$ and $v^\perp := \Gamma_v \cup \{v\}$. A *path* of Γ is a sequence $\gamma = (v_0, \dots, v_k)$ of vertices such that $v_i \in \Gamma_{v_{i-1}}$ for all $1 \leq i \leq k$; the number k is called its *length*. The path γ is called *simple* if $v_{i-1} \neq v_{i+1}$ for all $1 \leq i \leq k-1$. The path $\gamma = (v_0, \dots, v_k)$ is called a *circuit* if it is simple, of positive length and $v_0 = v_k$. For $v, v' \in V$, a path γ as above is a path from v to v' if $v = v_0$ and $v' = v_k$. The *distance* between two vertices x, y of Γ , denoted by $l(x, y)$, is the length of a shortest path from one to the other if there exists such a path; if not, their distance is defined to be infinite. The graph Γ is *connected*, if $l(x, y) < \infty$ for all $x, y \in V$. The *diameter* of Γ is defined to be the supremum taken over all $l(x, y)$ where $x, y \in V$. The *girth* is defined to be the length of a circuit of minimal length, if there exists a circuit; if not, the girth is defined to be infinite. For any subset W of V , Γ_W denotes the graph $(W, \{e \in E \mid e \subseteq W\})$; it is called *the subgraph spanned by W* .

A graph is called *bipartite* if all its circuits have even length. A *tree* is a connected graph without circuits.

The following lemma is easy.

Lemma 2.4. *Let $\Gamma = (V, E)$ be a tree of finite diameter and G its automorphism group. Then G fixes an element in $V \cup E$.*

2.4. Point-line-spaces

If S is a set, $\mathcal{P}(S)$ denotes the set of its subsets and $\mathcal{P}_2(S)$ denotes the set of subsets of S having cardinality at least 2.

A *point-line-space* is a pair $\mathcal{S} = (P, L)$ consisting of a set P (whose elements will be called *points*) and a subset L of $\mathcal{P}_2(S)$ (whose elements will be called *lines*).

Let $\mathcal{S} = (P, L)$ be a point-line-space. Two points are said to be *collinear* if there exists a line containing both or if they are equal. If p is a point, p^\perp denotes the set of all points which are collinear with p . For a subset S of P we put $S^\perp = \bigcap_{s \in S} s^\perp$. A subset S of P is called *singular* if $S \subset S^\perp$. A subset U of P is called a *subspace* of \mathcal{S} if the relation $|l \cap U| \geq 2$ implies $l \subset U$ for each $l \in L$. For a subspace U of \mathcal{S} we have a point-line-structure $\mathcal{S}_U = (U, L_U)$ on U induced from \mathcal{S} where $L_U = \{l \in L \mid l \subset U\}$. A subspace U will be called *singular*, if the set U is singular. For a subset S of P the intersection of all subspaces of \mathcal{S} containing S is denoted by $\langle S \rangle$; it is easily verified that $\langle S \rangle$ is a subspace of \mathcal{S} . If S_1, S_2 are two subsets of P , then $\langle S_1, S_2 \rangle$ will mean $\langle S_1 \cup S_2 \rangle$.

A point-line-space will be called *linear* if any two distinct points are contained in precisely one line. A point-line-space will be called *partially linear* if any two distinct points are contained in at most one line. Note that the partially linear point-line spaces are precisely those in which each line is a subspace.

3. Generalized polygons

3.1. Definitions and properties

Let $n \geq 2$ be a natural number. A *generalized n -gon* is a bipartite graph of diameter n and girth $2n$.

The following result is a summary of some well-known facts about generalized polygons (see, for instance, [TW02, Sections (3.1)–(3.4)]).

Lemma 3.1. *Let $n \geq 2$ and let $\Gamma = (V, E)$ be a generalized n -gon.*

- (a) *Given two vertices x and y at distance $k < n$, there exists a unique path of length k joining them.*
- (b) *Each simple path of length $n + 1$ is contained in a unique circuit of length $2n$.*
- (c) *For each $x \in V$ there exists $y \in V$ such that $l(x, y) = n$.*

3.2. Closed subsets

Definition 3.2. Let $n \geq 2$ and let $\Gamma = (V, E)$ be a generalized n -gon. A subset Ω of V is called *closed* if for any two $x, y \in \Omega$ at distance $k < n$, all the vertices on the unique path joining them are contained in Ω .

Proposition 3.3. *Let $n \geq 2$ and let $\Gamma = (V, E)$ be a generalized n -gon. Let $\Omega \subset V$ be a closed set. Then one of the following holds:*

- (i) Γ_Ω is a generalized n -gon and the distance between any two vertices of Γ_Ω coincides with their distance in Γ .

- (ii) Γ_Ω is a tree of diameter $k \leq n$.
- (iii) $|\Omega| > 1$ and the distance between any two distinct vertices of Ω is n .

Proof. Suppose that assumption (iii) does not hold. If $|\Omega| = 1$, Γ_Ω is a tree, hence (ii) holds; we may therefore also assume that $|\Omega| > 1$.

Now there exist $x, y \in \Omega$ with $x \neq y$ and $l(x, y) = k < n$. As Ω is closed it follows that the unique path $(x = v_0, v_1, \dots, v_k = y)$ is contained in Ω . Replacing y by v_1 we conclude that there is an edge $\{x, y\} \subset \Omega$. Let $z \in \Omega$. We have $l(z, x) = l(z, y) + 1$ or $l(z, x) = l(z, y) - 1$, since Γ is bipartite, and as $\text{diam } \Gamma = n$ we conclude that either $l(z, x) < n$ or $l(z, y) < n$. As Ω is closed there is a path from z to x or to y which is contained in Ω and we conclude that Γ_Ω is connected. It follows in particular that each $x \in \Omega$ is on an edge of Γ_Ω .

For $x, y \in \Omega$ let $l_\Omega(x, y)$ denote their distance in Γ_Ω . We claim that $l(x, y) = l_\Omega(x, y)$ for all $x, y \in \Omega$. Indeed, if $x, y \in \Omega$ are such that $l(x, y) < n$, then $l_\Omega(x, y) = l(x, y)$ because Ω is closed; if $l(x, y) = n$ we can choose $z \in \Omega$ such that $\{x, z\}$ is an edge of Γ_Ω . As Γ is bipartite and $\text{diam } \Gamma = n$ we have $l(z, y) = n - 1$. By the consideration above it follows that $l_\Omega(z, y) = n - 1$ and hence $l_\Omega(x, y) \leq n = l(x, y) \leq l_\Omega(x, y)$ which implies $l(x, y) = l_\Omega(x, y)$ also in this case.

Suppose now that there is a simple path γ of length $n + 1$ contained in Ω . Then the unique circuit containing γ is also contained in Ω because Ω is closed. This shows that $\text{girth } \Gamma_\Omega \leq 2n \leq \text{girth } \Gamma \leq \text{girth } \Gamma_\Omega$ and we conclude that $\text{girth } \Gamma_\Omega = 2n$. As $l_\Omega(x, y) = l(x, y)$ for all $x, y \in \Omega$ it follows that $\text{diam } \Gamma_\Omega = n$. Moreover, as a subgraph of a bipartite graph, Γ_Ω is bipartite and therefore Γ_Ω is a generalized n -gon. Hence, in this case alternative (i) holds.

Suppose now, that there is no simple path of length $n + 1$. Then Γ_Ω contains no circuit. As Γ_Ω is connected it follows that $\text{diam } \Gamma_\Omega \leq n$ and therefore alternative (ii) holds in this case. \square

Using Lemma 2.4 the previous proposition yields the following theorem:

Theorem 3.4. *Let $n \geq 2$ and let $\Gamma = (V, E)$ be a generalized n -gon. Let $\Omega \subset V$ be a closed set. Then one of the following holds.*

- (a) *For each $x \in \Omega$, there exists $y \in \Omega$ at distance n from x .*
- (b) *$\text{Aut}(\Gamma_\Omega)$ fixes an element in $\Omega \cup \{e \in E \mid e \subset \Omega\}$.*

3.3. Generalized polygons and buildings of rank 2

Let $\Gamma = (V, E)$ be a graph. We call $x, y \in V$ *incident*, if there is an edge containing x and y . In this way we obtain an incidence structure $\mathcal{G}(\Gamma) = (V, \star)$ associated with Γ . For some indications concerning the proof of the following theorem, we refer to the remark in the subsection about buildings in Section 2.

Theorem 3.5. *Let $n \geq 2$ be a natural number and let $\Gamma = (V, E)$ be a generalized n -gon. Then the following hold.*

- (a) $\text{Flag}(\mathcal{G}(\Gamma))$ is a building of type $I_2(n)$.
- (b) For $x, y \in V$ the singletons $\{x\}$ and $\{y\}$ are opposite in the building $\text{Flag}(\mathcal{G}(\Gamma))$ if and only if their distance in Γ is equal to n .
- (c) Let $\Omega \subset V$. Then $\text{flag}(\Omega)$ is a convex subcomplex of the building $\text{Flag}(\mathcal{G}(\Gamma))$ if and only if Ω is closed.

Conversely, if Δ is a building of type $I_2(n)$ then there exists a generalized polygon Γ' such that Δ is isomorphic to $\text{Flag}(\mathcal{G}(\Gamma'))$.

3.4. Proof of the center conjecture for buildings of type $I_2(n)$

Let $n \geq 2$ be a natural number, let $\Delta = (\Delta, \subset)$ be a building of type $I_2(n)$ and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Let $\Gamma = (V, E)$ be a generalized n -gon and let $\alpha: \text{flag}(\mathcal{G}(\Gamma)) \rightarrow \Delta$ be an isomorphism from $\text{Flag}(\mathcal{G}(\Gamma))$ onto Δ . Put $\Omega := \{v \in V \mid \alpha(\{v\}) \in \tilde{\Delta}\}$. As $\tilde{\Delta}$ is a convex subcomplex it follows from Lemma 2.1 that $\alpha(F) \in \tilde{\Delta}$ for all $F \in \text{flag}(\Omega)$ and we conclude that α maps the subcomplex $\text{flag}(\Omega)$ bijectively onto the subcomplex $\tilde{\Delta}$. Since $\tilde{\Delta}$ is convex, it follows that Ω is a closed subset of vertices of Γ by part (c) of Theorem 3.5. If for each element x in Ω , there is an element y in Ω at distance n in Γ , then each vertex in $\tilde{\Delta}$ has an opposite in $\tilde{\Delta}$ (by part (b) of Theorem 3.5) and it follows that $\tilde{\Delta}$ is completely reducible by Proposition 2.3. If this is not the case, then $\text{Aut}(\Gamma_\Omega)$ fixes a vertex v or an edge e of Γ_Ω by Theorem 3.4. It follows that $\text{Aut}(\text{Flag}(\mathcal{G}(\Gamma_\Omega)))$ fixes a non-trivial element F in $\text{flag}(\Omega)$. Hence $\text{Aut}(\tilde{\Delta})$ fixes the non-trivial simplex $\alpha(F)$. As each element of $\text{Stab}_{\text{Aut}(\Delta)}(\tilde{\Delta})$ induces an automorphism of the subcomplex $\tilde{\Delta}$ it follows that $\alpha(F)$ is a fixed point of this group and we are done.

4. Projective spaces

4.1. Definitions and preliminary facts

A *projective space* is a linear point-line-space $\mathcal{S} = (P, L)$ such that for any 5-tuple of pairwise distinct points a, b, c, p, q such that a, b, p and a, c, q are collinear on distinct lines, the lines $\langle b, c \rangle$ and $\langle p, q \rangle$ have a common point.

Let $\mathcal{S} = (P, L)$ be a projective space. We put $\text{rk } \mathcal{S} := \min\{|S| \mid S \subset P, \langle S \rangle = P\} - 1$ and this number will be called the *rank* of \mathcal{S} . We recall that for any subset X of P the point-line-space induced by \mathcal{S} on X is denoted by \mathcal{S}_X . Let U be a subspace of \mathcal{S} . Then the point-line-space \mathcal{S}_U is a projective space and we put $\dim U = \text{rk } \mathcal{S}_U$.

Let $\mathcal{S} = (P, L)$ be a projective space. We let $\mathcal{V}(\mathcal{S})$ denote the set of all non-trivial subspaces of \mathcal{S} . Two elements of $\mathcal{V}(\mathcal{S})$ are defined to be *incident* if one of them contains the other. In this way we obtain an incidence structure which we will denote by $\mathcal{G}(\mathcal{S})$; it is called the *projective geometry* associated with \mathcal{S} .

Two subspaces U, V of a projective space $\mathcal{S} = (P, L)$ are called *complementary* if $U \cap V = \emptyset$ and $\langle U, V \rangle = P$. We also say that V is a *complement* of U in \mathcal{S} .

The following lemma summarizes some well-known facts about complementary subspaces:

Lemma 4.1. *Let $\mathcal{S} = (P, L)$ be a projective space and let U, V, V_1 and V_2 be subspaces of \mathcal{S} . Then the following hold.*

- (a) *If V_1, V_2 are both complements of U in \mathcal{S} , then $\dim V_1 = \dim V_2$.*
- (b) *If V is a complement of U in \mathcal{S} , then $Y := V \cap X \neq \emptyset$ for each subspace X containing U properly. Moreover, Y is a complement of U in \mathcal{S}_X .*
- (c) *If U, V are subspaces with $U \cap V = \emptyset$, then U is a complement of V in $\mathcal{S}_{\langle U, V \rangle}$.*

4.2. Closed subsets

Throughout this subsection let $\mathcal{S} = (P, L)$ be a projective space.

Definition 4.2. A subset Ω of $\mathcal{V}(\mathcal{S})$ is called *closed* if the following hold for all $U, V \in \Omega$:

(PrC 1) $\langle U, V \rangle \neq P$ implies $\langle U, V \rangle \in \Omega$.

(PrC 2) $U \cap V \neq \emptyset$ implies $U \cap V \in \Omega$.

For the remainder of this subsection let $\Omega \subset \mathcal{V}(\mathcal{S})$ be a closed set. We let P_Ω (respectively H_Ω) denote the set of all minimal (respectively maximal) elements in Ω . For an element $Z \in \Omega$ we put $\Omega_Z := \{X \in \Omega \mid Z \neq X \subset Z\}$ and $P_\Omega(Z) := P_\Omega \cap \Omega_Z$; similarly we define $\Omega^Z := \{X \in \Omega \mid Z \neq X \supset Z\}$ and $H_\Omega(Z) := H_\Omega \cap \Omega^Z$.

The following lemma is immediate from the definitions:

Lemma 4.3. *Let $Z \in \Omega$. Then Ω_Z is a closed subset of $\mathcal{V}(\mathcal{S}_Z)$ and $P_\Omega(Z)$ is precisely the set of all minimal elements in Ω_Z .*

Proposition 4.4. *Suppose that \mathcal{S} has finite rank k and that for each $X \in P_\Omega$ there exists a complement $Y \in \Omega$ of X in \mathcal{S} . Then the following hold for all $Z \in \Omega$:*

- (a) *either $Z = \langle X \mid X \in P_\Omega(Z) \rangle$ or $Z \in P_\Omega$;*
- (b) *there is $W \in \Omega$ which is a complement of Z in \mathcal{S} .*

Moreover, if $\Omega \neq \emptyset$, then $P = \langle X \mid X \in P_\Omega \rangle$.

Proof. The proof is by induction on k . If $k \leq 1$, then $P_\Omega = \Omega$ and thus assertion (a) is trivial and assertion (b) follows directly from the assumptions of the proposition. If $\Omega \neq \emptyset$, then there are $p \neq q \in P$ such that $\{p\}, \{q\} \in \Omega$ and as \mathcal{S} has rank 1 we conclude that $P = \langle p, q \rangle$.

Let $k > 1$. We first prove the following.

Claim. *Let $Z \in \Omega$. Then for each $X \in P_\Omega(Z)$ there exists $Y \in \Omega_Z$ which is a complement of X in \mathcal{S}_Z .*

Indeed, let $Z \in \Omega$ and $X \in P_\Omega(Z)$. It follows that $Z \neq X \subset Z$ and that $X \in P_\Omega$. By the second fact and the assumption of the Proposition there is $Y' \in \Omega$ which is a complement

of X in \mathcal{S} . As Z contains X properly, it follows from part (b) of Lemma 4.1 that $Y := Z \cap Y' \neq \emptyset$ and that Y is a complement of X in \mathcal{S}_Z . As Ω is closed and $Y \neq \emptyset$, we have $Y \in \Omega$. As $Y \subset Y'$ it follows that $X \cap Y = \emptyset$ and therefore we have $Y \neq Z$ because $\emptyset \neq X \subset Z$. We conclude that Y is an element of Ω_Z . This completes the proof of the claim.

Let $Z \in \Omega$. Since the rank of \mathcal{S}_Z is strictly smaller than the rank of \mathcal{S} the claim above enables us to apply induction to \mathcal{S}_Z . Hence we have $Z = \langle X \mid X \in P_\Omega(Z) \rangle$ if $P_\Omega(Z) \neq \emptyset$. If $P_\Omega(Z) = \emptyset$, then $Z \in P_\Omega$ which finishes the proof of assertion (a). If $Z \in P_\Omega$, then there is $W \in \Omega$ which is a complement of Z in \mathcal{S} by the assumption of the proposition. If Z is not in P_Ω , then $P_\Omega(Z) \neq \emptyset$ and we can choose $X \in P_\Omega(Z)$. Let $Y' \in \Omega$ be a complement of X in \mathcal{S} , put $Y := Y' \cap Z$ and note that $Y \neq Y'$. It follows that $Y \in \Omega$ and that Y is a complement of X in \mathcal{S}_Z . Applying induction to $\mathcal{S}_{Y'}$ we see that there is $W \in \Omega$ which is a complement of Y in $\mathcal{S}_{Y'}$. Now $W \cap Z \subset Y' \cap Z = Y$ and $W \cap Z \subset W$ and as $W \cap Y = \emptyset$ we conclude that $W \cap Z = \emptyset$. On the other hand, we have $X, Y \subset Z$ and $\langle Y, W \rangle = Y'$ and we conclude that $P = \langle X, Y' \rangle = \langle Z, W \rangle$. This proves that W is a complement of Z in \mathcal{S} and finishes the proof of assertion (b) of the proposition.

It remains to show that $\Omega \neq \emptyset$ implies $P = \langle X \mid X \in P_\Omega \rangle$. Suppose $\Omega \neq \emptyset$ and that $Z := \langle X \mid X \in P_\Omega \rangle \neq P$. It follows that $P_\Omega \neq \emptyset$ and that $Z \in \Omega$. By part (b) of the proposition, we know that there is $W \in \Omega$ which is a complement of Z in \mathcal{S} . As there is an element W' in P_Ω which is contained in W and $W \cap Z = \emptyset$, we obtain a contradiction. \square

The following proposition is obtained by ‘dualizing’ the arguments in the proof of the previous one:

Proposition 4.5. *Suppose that \mathcal{S} has finite rank k and that for each $X \in H_\Omega$ there exists a complement $Y \in \Omega$ of X in \mathcal{S} . Then the following hold for all $Z \in \Omega$:*

- (a) $Z = \bigcap_{X \in H_\Omega(Z)} X$ or $Z \in H_\Omega$;
- (b) there is $W \in \Omega$ which is a complement of Z in \mathcal{S} .

Moreover, if $\Omega \neq \emptyset$, then $\bigcap_{X \in H_\Omega} X = \emptyset$.

Theorem 4.6. *Suppose that \mathcal{S} has finite rank. Let C_p (respectively C_h) denote the set of all $X \in P_\Omega$ (respectively $X \in H_\Omega$) for which there is no $Y \in \Omega$ which is a complement of X in \mathcal{S} . Put $C_p := \langle X \mid X \in C_p \rangle$ and $C_h := \bigcap_{X \in C_h} X$. Then one of the following holds.*

- (a) For each $X \in \Omega$ there is $Y \in \Omega$ which is a complement of X in \mathcal{S} .
- (b) C_p and C_h are incident elements which are both contained in Ω .

Proof. If $C_p = \emptyset$ or $C_h = \emptyset$, we are in case (a) by the two previous propositions. Thus we can assume that $C_p \neq \emptyset \neq C_h$. It follows that $C_p \neq \emptyset$ and $C_h \neq P$.

Let $X \in P_\Omega$ and $Y \in C_h$ and suppose that X is not contained in Y . As X is in P_Ω it follows that $X \cap Y = \emptyset$, because $X \cap Y$ is properly contained in X . On the other hand, $\langle X, Y \rangle$ contains $Y \in H_\Omega$ properly and hence $P = \langle X, Y \rangle$. We conclude that X is a complement

of Y in \mathcal{S} which provides a contradiction. Hence we have $X \subset Y$ for all $X \in P_\Omega$ and all $Y \in C_h$.

Put $C := \langle X \mid X \in P_\Omega \rangle$. It readily follows that $C_p \subset C \subset C_h$ and in particular that $C_p \neq P$ and $C_h \neq \emptyset$. Therefore C_p and C_h are incident elements in Ω . \square

4.3. Automorphisms of $\mathcal{G}(\mathcal{S})$

Throughout this subsection let $\mathcal{S} = (P, L)$ be a projective space and $\mathcal{G}(\mathcal{S}) = (\mathcal{V}(\mathcal{S}), \star)$ the incidence structure associated with \mathcal{S} .

An automorphism α of \mathcal{S} induces an automorphism of $\mathcal{G}(\mathcal{S})$ which we denote by $\bar{\alpha}$.

The following proposition is well known:

Proposition 4.7. *Let $\pi \in \text{Sym } \mathcal{V}(\mathcal{S})$ be an automorphism of $\mathcal{G}(\mathcal{S})$. Then π either preserves inclusion or it reverses inclusion. Moreover, if π preserves inclusion, then there exists a unique automorphism α of \mathcal{S} such that $\bar{\alpha} = \pi$.*

If π reverses inclusion, then the following holds for any set \mathcal{X} of subspaces of \mathcal{S} :

- (a) $\langle \pi(X) \mid X \in \mathcal{X} \rangle = \pi(\bigcap_{X \in \mathcal{X}} X)$;
- (b) $\bigcap_{X \in \mathcal{X}} \pi(X) = \pi(\langle X \mid X \in \mathcal{X} \rangle)$.

Theorem 4.8. *Let Ω be a closed subset of $\mathcal{V}(\mathcal{S})$. Then one of the following holds:*

- (a) *For each $X \in \Omega$ there is $Y \in \Omega$ which is a complement of X in \mathcal{S} .*
- (b) *The group $\text{Stab}_{\text{Aut}(\text{Flag}(\mathcal{G}(\mathcal{S})))}(\text{flag}(\Omega))$ fixes a non-trivial element in $\text{flag}(\Omega)$.*

Proof. Let $G := \text{Stab}_{\text{Aut}(\text{Flag}(\mathcal{G}(\mathcal{S})))}(\text{flag}(\Omega))$ and let G_1 be the inclusion-preserving subgroup of G . Define the sets C_p and C_h as in Theorem 4.6. It is clear that the group G_1 normalizes the sets C_p and C_h and therefore fixes the subspaces C_p and C_h ; moreover, an inclusion-reversing element of G just permutes these two subspaces. Suppose that Assumption (a) does not hold. Then we know by Theorem 4.6, that $\{C_p, C_h\}$ is an element in $\text{flag}(\Omega)$ which is fixed by the group G . \square

4.4. Projective spaces and buildings of type A_n

As before, we refer to the remark following Proposition 2.3 for indications concerning the proof of the theorem below.

Theorem 4.9. *Let $\mathcal{S} = (P, L)$ be a projective space of finite rank n and let $\mathcal{G} := \mathcal{G}(\mathcal{S}) = (\mathcal{V}(\mathcal{S}), \star)$ be the associated incidence structure. Then we have the following:*

- (a) *$\text{Flag}(\mathcal{G})$ is a building of type A_n .*
- (b) *For $X, Y \in \mathcal{V}(\mathcal{S})$ the flags $\{X\}$ and $\{Y\}$ are opposite in $\text{Flag}(\mathcal{G})$ if and only if Y is a complement of X in \mathcal{S} .*
- (c) *Let Ω be a subset of $\mathcal{V}(\mathcal{S})$. Then $\text{flag}(\Omega)$ is a convex subcomplex of $\text{Flag}(\mathcal{G})$ if and only if Ω is closed.*

Conversely, if $\Delta = (\Delta, \subset)$ is a building of type A_n , then there exists a projective space S' of rank n such that Δ is isomorphic to $\text{Flag}(\mathcal{G}(S'))$.

4.5. Proof of the center conjecture for buildings of type A_n

Let n be a natural number. Let $\Delta = (\Delta, \subset)$ be a building of type A_n and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Let $\mathcal{S} = (P, L)$ be a projective space and let $\alpha: \text{flag}(\mathcal{G}(\mathcal{S})) \rightarrow \Delta$ be an isomorphism from $\text{Flag}(\mathcal{G}(\mathcal{S}))$ onto Δ . Put $\Omega := \{X \in \mathcal{V}(\mathcal{S}) \mid \alpha(\{X\}) \in \tilde{\Delta}\}$. As $\tilde{\Delta}$ is a convex subcomplex it follows from Lemma 2.1 that $\alpha(F) \in \tilde{\Delta}$ for all $F \in \text{flag}(\Omega)$ and we conclude that α maps the subcomplex $\text{flag}(\Omega)$ bijectively onto the subcomplex $\tilde{\Delta}$. Since $\tilde{\Delta}$ is convex, it follows that Ω is a closed subset of $\mathcal{V}(\mathcal{S})$ by part (c) of Theorem 4.9. If for each element X in Ω , there is an element Y in Ω which is a complement of X in \mathcal{S} , then each vertex in $\tilde{\Delta}$ has an opposite in $\tilde{\Delta}$ (by part (b) of Theorem 4.9) and it follows that $\tilde{\Delta}$ is completely reducible by Proposition 2.3. If this is not the case, then $G := \text{Stab}_{\text{Aut}(\text{Flag}(\mathcal{G}(\mathcal{S})))}(\text{flag}(\Omega))$ fixes a non-trivial element F in $\text{flag}(\Omega)$ by Theorem 4.8. It follows that $\text{Stab}_{\text{Aut}(\Delta)}(\tilde{\Delta}) = \alpha G \alpha^{-1}$ fixes the non-trivial simplex $\alpha(F) \in \tilde{\Delta}$.

5. Polar spaces

5.1. Definitions and preliminary results

A point-line-space $\mathcal{S} = (P, L)$ is called a *polar space* if the following axioms are satisfied:

- (BS1) For any point p and any line l , one has $l \subset p^\perp$ or $|p^\perp \cap l| = 1$.
- (BS2) $p^\perp \neq P$ for all points p .

The standard references for polar spaces are Chapters 7–9 of [Ti74] and Chapters 9–12 of [BC]. The following proposition, essentially due to F. Buekenhout and E. Shult (see [BS74, Bu90]), summarizes several basic facts.

Proposition 5.1. *Let $\mathcal{S} = (P, L)$ be a polar space. Then the following hold.*

- (a) \mathcal{S} is a partially linear space.
- (b) For any set $X \subset P$, the set X^\perp is a subspace of \mathcal{S} .
- (c) If X is a singular subset of P , $\langle X \rangle$ is a singular subspace of \mathcal{S} .
- (d) A singular subspace of \mathcal{S} is a projective space.
- (e) All maximal singular subspaces have the same dimension.
- (f) If $x, y \in P$ are non-collinear, then $\mathcal{S}_{x^\perp \cap y^\perp}$ is a polar space.

Remark. The definition of a polar space above is the most recent one. Earlier definitions (as for instance the one given in Chapter 7 of [Ti74]) were based on the properties listed in the previous proposition. It is in fact the main result of [BS74] that Axioms (BS1) and

(BS2) are sufficient to characterize polar spaces. For further historical details about polar spaces we refer to the standard references given above.

For the remainder of this subsection let $\mathcal{S} = (P, L)$ be a polar space. We denote the set of all non-empty singular subspaces of \mathcal{S} by $\mathcal{V}(\mathcal{S})$, we define two elements of $\mathcal{V}(\mathcal{S})$ to be *incident* if one contains the other and we denote the corresponding incidence structure by $\mathcal{G}(\mathcal{S})$. It is called the *polar geometry* associated with \mathcal{S} .

For an element $X \in \mathcal{V}(\mathcal{S})$ we define its *type* by setting $\text{type } X := \text{rk } \mathcal{S}_X + 1$; as before, the dimension of X is defined to be the rank of \mathcal{S}_X . The *rank* of \mathcal{S} is defined to be the type of a maximal singular subspace.

The following lemma is immediate from the proposition above.

Lemma 5.2. *Let U, V be singular subspaces of \mathcal{S} . Then the following hold.*

- (a) $U \cap V, U \cap V^\perp$ and $\langle U \cap V^\perp, V \rangle$ are singular subspaces of \mathcal{S} .
- (b) If $U \subset V^\perp$ then $\langle U, V \rangle^\perp = U^\perp \cap V^\perp$.

5.2. Hyperbolic pairs

Throughout this subsection let $\mathcal{S} = (P, L)$ be a polar space of finite rank.

Definition 5.3. Let U, V be non-empty singular subspaces of \mathcal{S} . Then (U, V) is called a *hyperbolic pair* of \mathcal{S} if $U \cap V^\perp = \emptyset = V \cap U^\perp$. If (U, V) is a hyperbolic pair of \mathcal{S} then U (respectively V) is called a *hyperbolic complement* of V (respectively U) in \mathcal{S} .

The following lemma is proved by induction on the type of U using part (e) of Proposition 5.1.

Lemma 5.4. *If (U, V) is a hyperbolic pair of \mathcal{S} , the following hold.*

- (i) $\text{type } U = \text{type } V$.
- (ii) If X is a singular subspace of \mathcal{S} containing U , then $X = \langle X \cap V^\perp, U \rangle$ and $(V^\perp \cap X) \cap U = \emptyset$. In other words, the singular subspace $V^\perp \cap X$ is a complement of U in the projective space \mathcal{S}_X . In particular, if X contains U properly, then $V^\perp \cap X \neq \emptyset$.
- (iii) The space $\mathcal{S}_{U^\perp \cap V^\perp}$ is a polar space.

Definition 5.5. For a hyperbolic pair (X, Y) of \mathcal{S} we write $\mathcal{S}_{(X, Y)}$ for $\mathcal{S}_{X^\perp \cap Y^\perp}$.

Lemma 5.6. *Let (U, V) and (X, Y) be hyperbolic pairs of \mathcal{S} such that $X \cup Y \subset U^\perp \cap V^\perp$. Then $(\langle X, U \rangle, \langle Y, V \rangle)$ is a hyperbolic pair of \mathcal{S} .*

Proof. Note first that $X \cap U = \emptyset = Y \cap V$. As U is a complement of X in $\mathcal{S}_{\langle X, U \rangle}$, as $U \subset Y^\perp$ and as $\langle X, U \rangle \cap Y^\perp$ is a complement of X in $\mathcal{S}_{\langle X, U \rangle}$ (by part (ii) of the previous lemma) we conclude that $\langle X, U \rangle \cap Y^\perp = U$; similarly, we obtain $\langle X, U \rangle \cap V^\perp = X$.

Now it follows by part (b) of Lemma 5.2 that $\langle X, U \rangle \cap \langle Y, V \rangle^\perp = \langle X, U \rangle \cap Y^\perp \cap V^\perp = (\langle X, U \rangle \cap Y^\perp) \cap (\langle X, U \rangle \cap V^\perp) = X \cap U = \emptyset$. By symmetry we have also $\langle Y, V \rangle \cap \langle X, U \rangle^\perp = \emptyset$ which yields the claim. \square

Lemma 5.7. *Let $X, Y, Z \in \mathcal{V}(\mathcal{S})$ be such that (X, Y) is a hyperbolic pair in \mathcal{S} and such that $Z \subset X^\perp \cap Y^\perp$. If W is a hyperbolic complement of Z in \mathcal{S} , then $W^\perp \cap \langle Z, Y \rangle$ is a hyperbolic complement of X .*

Proof. Since $Z \subset X^\perp$ and $X^\perp \cap Y = \emptyset$ we have $Z \cap Y = \emptyset$. Therefore, Y is a complement of Z in the projective space $\mathcal{S}_{\langle Z, Y \rangle}$. Let $Y' := \langle Z, Y \rangle \cap W^\perp$. By part (ii) of Lemma 5.4 Y' is a complement of Z in $\mathcal{S}_{\langle Z, Y \rangle}$.

Again by part (ii) of Lemma 5.4 $X^\perp \cap \langle Z, Y \rangle$ is a complement of Y in $\mathcal{S}_{\langle Z, Y \rangle}$. As Z is contained in $X^\perp \cap \langle Z, Y \rangle$, we conclude that $Z = X^\perp \cap \langle Z, Y \rangle$. It follows that $Y' \cap X^\perp = W^\perp \cap \langle Z, Y \rangle \cap X^\perp = W^\perp \cap Z = \emptyset$.

Suppose now that $Y'^\perp \cap X \neq \emptyset$ and let $x \in Y'^\perp \cap X$. We have $x \in Z^\perp$ and hence $x \in Y'^\perp \cap Z^\perp$. As $Y' \subset Z^\perp$, it follows that $x \in \langle Y', Z \rangle^\perp = \langle Y, Z \rangle^\perp$ which means that $x \in Y^\perp$. As $X \cap Y^\perp = \emptyset$ we obtain a contradiction and conclude that $Y'^\perp \cap X = \emptyset$. It follows that (X, Y') is a hyperbolic pair and we are done. \square

Proposition 5.8. *Let $X, Y, Z \in \mathcal{V}(\mathcal{S})$ be such that (X, Y) is a hyperbolic pair in \mathcal{S} and such that $Z \subset X^\perp \cap Y^\perp$. Let W be a hyperbolic complement of Z in \mathcal{S} . Put $Y_1 := W^\perp \cap \langle Z, Y \rangle$, $W_1 := X^\perp \cap \langle Y_1, W \rangle$ and $W_2 := Y^\perp \cap \langle X, W_1 \rangle$. Then $W_2 \subset X^\perp \cap Y^\perp$ and W_2 is a hyperbolic complement of Z in \mathcal{S} .*

Proof. By the previous lemma, (Y_1, X) is a hyperbolic pair. As $Y_1 \subset Z^\perp \cap W^\perp$ (by the definition of Y_1) we can apply the previous lemma with $X := Z, Y := W, Z := Y_1$ and $W = X$ in order to conclude that (W_1, Z) is a hyperbolic pair. As $X \subset Z^\perp \cap W_1^\perp$ (by the definition of W_1) we can apply the previous lemma with $X := Z, Y := W_1, Z := X$ and $W := Y$ in order to conclude that (W_2, Z) is a hyperbolic pair and it follows by the definition of W_2 that $W_2 \subset X^\perp \cap Y^\perp$. This completes the proof. \square

5.3. Closed subsets

Throughout this subsection let $\mathcal{S} = (P, L)$ be a polar space of finite rank and let $\mathcal{V}(\mathcal{S})$ be the set of all non-empty singular subspaces of \mathcal{S} . For a singular subspace X of \mathcal{S} we let \mathcal{V}_X denote the set of all non-empty singular subspaces which are properly contained in X . Let (X, Y) be a hyperbolic pair of \mathcal{S} . Then $\mathcal{V}_{\langle X, Y \rangle}$ denotes the set of all non-empty singular subspaces of \mathcal{S} which are contained in $X^\perp \cap Y^\perp$.

Definition 5.9. A set $\Omega \subset \mathcal{V}(\mathcal{S})$ is called *closed* if it satisfies the following conditions for any two $V, U \in \Omega$:

- (PoC1) $U \cap V \neq \emptyset$ implies $U \cap V \in \Omega$,
- (PoC2) $U \cap V^\perp \neq \emptyset$ implies $U \cap V^\perp \in \Omega$,
- (PoC3) $\langle U \cap V^\perp, V \rangle \in \Omega$,

For the remainder of this subsection we assume that Ω is a closed subset of $\mathcal{V}(\mathcal{S})$ and we denote the set of all minimal elements in Ω by P_Ω . For $X \in \Omega$ we put $\Omega_X := \mathcal{V}_X \cap \Omega$ and $P_\Omega(X) := P_\Omega \cap \mathcal{V}_X$. For a hyperbolic pair (X, Y) such that X, Y are both in Ω we put $\Omega_{(X,Y)} := \mathcal{V}_{(X,Y)} \cap \Omega$ and $P_\Omega(X, Y) := \mathcal{V}_{(X,Y)} \cap P_\Omega$.

The following lemma is immediate from the definitions.

Lemma 5.10. *Let $X \in \Omega$, then Ω_X is a closed subset of \mathcal{V}_X (with respect to the projective space \mathcal{S}_X).*

Let (X, Y) be a hyperbolic pair such that X, Y are in Ω . Then $\Omega_{(X,Y)}$ is a closed subset of $\mathcal{V}_{(X,Y)}$ (with respect to the polar space $\mathcal{S}_{(X,Y)}$).

Lemma 5.11. *Suppose that for each $X \in P_\Omega$ there exists $Y \in \Omega$ such that (X, Y) is a hyperbolic pair. Then, for any hyperbolic pair (X, Y) and any $Z \in P_\Omega(X, Y)$, there exists $U \in \Omega_{(X,Y)}$ such that (U, Z) is a hyperbolic pair.*

Proof. By our assumption there is $W \in \Omega$ such that (W, Z) is a hyperbolic pair. We put $Y_1 := W^\perp \cap \langle Z, Y \rangle$, $W_1 := X^\perp \cap \langle Y_1, W \rangle$ and $U := Y^\perp \cap \langle X, W_1 \rangle$. Then $U \subset X^\perp \cap Y^\perp$ and U is a hyperbolic complement of Z by Proposition 5.8. Since Ω is closed, it follows that Y_1, W_1 and U are in Ω and therefore $U \in \Omega_{(X,Y)}$. \square

Proposition 5.12. *Suppose that for each $X \in P_\Omega$ there exists $Y \in \Omega$ such that (X, Y) is a hyperbolic pair. For any $Z \in \Omega$ there exists $W \in \Omega$ such that (Z, W) is a hyperbolic pair.*

Proof. The proof is by induction on the rank k of \mathcal{S} .

If $k \leq 1$ then $P_\Omega = \Omega$ and the assertion follows by our assumption.

Let $k > 1$ and let $Z \in \Omega$. Choose $X \in P_\Omega$ such that $X \subset Z$. If $X = Z$ we are done by the assumption. Hence we may suppose that X is properly contained in Z . Choose $Y \in \Omega$ such that (X, Y) is a hyperbolic pair. By the previous lemma we know that for each $X' \in P_\Omega(X, Y)$ there is $Y' \in \Omega_{(X,Y)}$ such that (X', Y') is a hyperbolic pair. Applying induction to the polar space $\mathcal{S}_{(X,Y)}$ we see that for each $Z' \in \Omega_{(X,Y)}$ there is a subspace $U' \in \Omega_{(X,Y)}$ such that (Z', U') is a hyperbolic pair.

Put $Z_0 := Y^\perp \cap Z$. Then $Z_0 \in \Omega_{(X,Y)}$ and we can choose $U_0 \in \Omega_{(X,Y)}$ such that (Z_0, U_0) is a hyperbolic pair. It follows from Lemma 5.6 that $(\langle X, Z_0 \rangle, \langle Y, U_0 \rangle)$ is a hyperbolic pair; by Lemma 5.4 we know that Z_0 is a complement of X in \mathcal{S}_Z and therefore $Z = \langle X, Z_0 \rangle$; finally, as Y, U_0 are both in Ω , it follows that $W := \langle Y, U_0 \rangle$ is contained in Ω and we are done. \square

Let $C_p(\Omega)$ denote the set of all elements in P_Ω for which there is no hyperbolic complement in Ω and put $\mathcal{C}_p := \langle X \mid X \in C_p(\Omega) \rangle$.

Lemma 5.13. *The subspace $\mathcal{C}_p(\Omega)$ is a singular subspace. If $C_p(\Omega) \neq \emptyset$, then $\mathcal{C}_p(\Omega) \in \Omega$ and there is no element $D \in \Omega$ such that $(\mathcal{C}_p(\Omega), D)$ is a hyperbolic pair.*

Proof. Let $X, Y \in C_p(\Omega)$. Suppose that $X^\perp \cap Y = \emptyset$ and put $Z := Y^\perp \cap X$. If $Z = \emptyset$ then (X, Y) is a hyperbolic pair and we obtain a contradiction. Hence $Z \neq \emptyset$ and $Z \in \Omega$.

As $X \in P_\Omega$ and $Z \subset X$ we conclude that $Z = X$ which implies that $X \subset Y^\perp$ and finally $X^\perp \cap Y = Y$. Again, we obtain a contradiction and conclude that $X^\perp \cap Y \neq \emptyset$. As $Y \in P_\Omega$ and $X^\perp \cap Y \in \Omega$, we finally obtain $X^\perp \cap Y = Y$ and hence $X \perp Y$.

It follows that $\bigcup_{X \in C_p(\Omega)} X$ is a singular subset of P and therefore $C_p(\Omega)$ is a singular subspace by part (c) of Proposition 5.1.

Suppose now that $C_p(\Omega) \neq \emptyset$. Since S has finite rank it follows that $C_p(\Omega) \in \Omega$. Let $D \in \Omega$ and let $Y \in P_\Omega$ be contained in D . It follows that the pair (X, Y) is not hyperbolic for each $X \in C_p(\Omega)$. Hence we have $X \perp Y$ for each such pair and therefore $Y \subset C_p(\Omega)^\perp \cap D$, which implies that the pair $(C_p(\Omega), D)$ is not hyperbolic. \square

5.4. Automorphisms of $\mathcal{G}(S)$

Throughout this subsection let $S = (P, L)$ be a polar space of finite rank $n \geq 3$ and $\mathcal{G}(S) = (\mathcal{V}(S), \star)$ be the polar geometry associated with S .

An automorphism α of S induces an automorphism of $\mathcal{G}(S)$ which we denote by $\bar{\alpha}$.

Proposition 5.14. *Let $\pi \in \text{Sym } \mathcal{V}(S)$ be an automorphism of $\mathcal{G}(S)$. Then π preserves inclusion and there exists a unique automorphism α of S such that $\bar{\alpha} = \pi$.*

Proof. The application from $\mathcal{V}(S)$ to the set $\{0, 1, \dots, n-1\}$ defined by $X \mapsto \dim X$ is a type function in the sense of [Ti81]. Endowed with this type function $\mathcal{G}(S)$ is a geometry of type C_n and the permutation π induces an automorphism of the diagram C_n . As $n \geq 3$, the diagram C_n admits no non-trivial automorphism and it follows that the permutation π maps singletons onto singletons and lines onto lines. It readily follows that $\pi = \bar{\alpha}$ for some $\alpha \in \text{Aut}(S)$. \square

Theorem 5.15. *Let $\Omega \subset \mathcal{V}(S)$ be a closed set and let G be the stabilizer of $\text{flag}(\Omega)$ in the group $\text{Aut}(\text{Flag}(\mathcal{G}(S)))$. Then one of the following holds.*

- (a) *For each $X \in \Omega$ there exists an element $Y \in \Omega$ such that (X, Y) is a hyperbolic pair.*
- (b) *There is a element C in Ω which is fixed by G and for which there exists no $D \in \Omega$ such that (C, D) is a hyperbolic pair. In particular G fixes a non-trivial element of $\text{flag}(\Omega)$.*

Proof. By the previous proposition we can identify the group G with the stabilizer \bar{G} of Ω in $\text{Aut}(S)$. It is clear that \bar{G} normalizes the set $C_p(\Omega)$ and therefore it fixes the subspace $C_p(\Omega)$. If assertion (a) of the theorem does not hold, then $C_p(\Omega) \neq \emptyset$ by Proposition 5.12 and it is an element of Ω by Lemma 5.13. This finishes the proof. \square

5.5. Polar spaces and buildings of type C_n

As before, we refer to the remark following Proposition 2.3 for indications concerning the proof of the theorem below.

Theorem 5.16. *Let $\mathcal{S} = (P, L)$ be a polar space of rank n and let $\mathcal{G} := \mathcal{G}(\mathcal{S}) = (\mathcal{V}(\mathcal{S}), \star)$ be the associated polar geometry. Then we have the following:*

- (a) $\text{Flag}(\mathcal{G})$ is a building of type C_n .
- (b) For $X, Y \in \mathcal{V}(\mathcal{S})$, $\{X\}$ and $\{Y\}$ are opposite in $\text{Flag}(\mathcal{G})$ if and only if (X, Y) is a hyperbolic pair of \mathcal{S} .
- (c) Let Ω be a subset of $\mathcal{V}(\mathcal{S})$. Then $\text{flag}(\Omega)$ is a convex subcomplex of $\text{Flag}(\mathcal{G})$ if and only if Ω is closed.

Conversely, if $\Delta = (\Delta, \subset)$ is a building of type C_n , then there exists a polar space \mathcal{S}' of rank n such that Δ is isomorphic to $\text{Flag}(\mathcal{G}(\mathcal{S}'))$.

5.6. Proof of the center conjecture for buildings of type C_n

The case $n = 2$ is covered by Section 3. Let $n \geq 3$ be a natural number. Let $\Delta = (\Delta, \subset)$ be a building of type C_n and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Let $\mathcal{S} = (P, L)$ be a polar space and let $\alpha : \text{flag}(\mathcal{G}(\mathcal{S})) \rightarrow \Delta$ be an isomorphism from $\text{Flag}(\mathcal{G}(\mathcal{S}))$ onto Δ . Put $\Omega := \{X \in \mathcal{V}(\mathcal{S}) \mid \alpha(\{X\}) \in \tilde{\Delta}\}$. As $\tilde{\Delta}$ is a convex subcomplex it follows from Lemma 2.1 that $\alpha(F) \in \tilde{\Delta}$ for all $F \in \text{flag}(\Omega)$ and we conclude that α maps the subcomplex $\text{flag}(\Omega)$ bijectively onto the subcomplex $\tilde{\Delta}$. Since $\tilde{\Delta}$ is convex, it follows that Ω is a closed subset of $\mathcal{V}(\mathcal{S})$ by part (c) of Theorem 5.16. If for each element X in Ω , there is an element Y in Ω which is a hyperbolic complement of X in \mathcal{S} , then each vertex in $\tilde{\Delta}$ has an opposite in $\tilde{\Delta}$ (by part (b) of Theorem 5.16) and it follows that $\tilde{\Delta}$ is completely reducible by Proposition 2.3. If this is not the case, then $G := \text{Stab}_{\text{Aut}(\text{Flag}(\mathcal{G}(\mathcal{S})))}(\text{flag}(\Omega))$ fixes a non-trivial element F in $\text{flag}(\Omega)$ by Theorem 5.15. It follows that $\text{Stab}_{\text{Aut}(\Delta)}(\tilde{\Delta}) = \alpha G \alpha^{-1}$ fixes the non-trivial simplex $\alpha(F) \in \tilde{\Delta}$.

6. Polar spaces of type D

By Theorem 5.16 there is a bijective correspondence between the buildings of type C_n and the polar spaces of rank n . Polar spaces of type D are polar spaces of finite rank satisfying an additional axiom. Theorem 6.4 below will give us then a bijective correspondence between the polar spaces of type D of rank n and the buildings of type D_n . Thus, each building of type D_n can be viewed a building of type C_n (see also [Ti74, Chapter 7]). For $n \geq 5$ the proof of the center conjecture for D_n -buildings is almost the same as for C_n buildings as it relies essentially on Theorem 5.12 of the previous section. There are, however, a few additional subtleties in the D_n -case and because of them it is more convenient to treat this case in a different section.

6.1. Definitions and preliminary results

Let $\mathcal{S} = (P, L)$ be a polar space. A *submaximal* singular subspace of \mathcal{S} is a proper hyperplane in a maximal singular subspace. A polar space $\mathcal{S} = (P, L)$ is called of type D

if it is of finite rank and if each submaximal singular subspace of \mathcal{S} is contained in precisely two maximal ones. Throughout this section let $\mathcal{S} = (P, L)$ be a polar space of type D .

The *oriflamme geometry* associated with \mathcal{S} is the incidence structure $\mathcal{G}_o(\mathcal{S}) = (\mathcal{V}_o(\mathcal{S}), \star)$ which is defined as follows. The set $\mathcal{V}_o(\mathcal{S})$ is the set of all non-trivial singular subspaces which are not submaximal. Two elements of $\mathcal{V}_o(\mathcal{S})$ are defined to be incident if one contains the other or if their intersection is a submaximal singular subspace.

Let Ω be a subset of $\mathcal{V}_o(\mathcal{S})$. We put $\text{Submax}(\Omega) := \{X \cap Y \mid X, Y \in \Omega, \text{ type } X = \text{rk } \mathcal{S} = \text{type } Y, \text{ type}(X \cap Y) = \text{type } X - 1\}$ and $\bar{\Omega} := \Omega \cup \text{Submax}(\Omega)$. The set $\bar{\Omega}$ is called the *submaximal completion* of Ω .

Definition 6.1. A set $\Omega \subset \mathcal{V}_o(\mathcal{S})$ is called *closed* in $\mathcal{V}_o(\mathcal{S})$ if its submaximal completion is a closed subset of $\mathcal{V}(\mathcal{S})$.

6.2. Automorphisms of oriflamme geometries

Let $\mathcal{S} = (P, L)$ be a polar space of type D . The following proposition can be proved by arguments which are similar to those used in the proof of Proposition 5.14.

Proposition 6.2. Let $\mathcal{S} = (P, L)$ be a polar space of type D and suppose that its rank is at least 5. Then all automorphisms of $\mathcal{G}_o(\mathcal{S})$ are inclusion-preserving permutations of $\mathcal{V}_o(\mathcal{S})$. In particular, $\text{Aut } \mathcal{G}(\mathcal{S})$ stabilizes the set $\mathcal{V}_o(\mathcal{S})$ and the mapping $\pi \mapsto \pi|_{\mathcal{V}_o(\mathcal{S})}$ is an isomorphism from $\text{Aut } \mathcal{G}(\mathcal{S})$ onto $\text{Aut } \mathcal{G}_o(\mathcal{S})$.

The following theorem is a consequence of the previous proposition, the definition of a closed subset in a polar geometry of type D and Theorem 5.15.

Theorem 6.3. Let $\mathcal{S} = (P, L)$ be a polar space of type D and suppose that its rank is at least 5. Let $\Omega \subset \mathcal{V}_o(\mathcal{S})$ be a closed subset and let G be the stabilizer of Ω in $\text{Aut}(\mathcal{G}_o(\mathcal{S}))$. Then one of the following holds:

- (a) For each $X \in \Omega$ there is $Y \in \Omega$ such that (X, Y) is a hyperbolic pair.
- (b) G fixes a non-trivial element of $\text{flag}(\Omega)$.

6.3. Polar spaces of type D and buildings of type D_n

As before, we refer to the remark following Proposition 2.3 for indications concerning the proof of the theorem below.

Theorem 6.4. Let $\mathcal{S} = (P, L)$ be a polar space of type D and of rank $n \geq 2$ and let $\mathcal{G} := \mathcal{G}_o(\mathcal{S}) = (\mathcal{V}_o(\mathcal{S}), \star)$ be the associated oriflamme geometry. Then we have the following:

- (a) $\text{Flag}(\mathcal{G})$ is a building of type D_n .
- (b) For $X, Y \in \mathcal{V}_o(\mathcal{S})$, $\{X\}$ and $\{Y\}$ are opposite in $\text{Flag}(\mathcal{G})$ if and only if (X, Y) is a hyperbolic pair of \mathcal{S} .

- (c) Let Ω be a subset of $\mathcal{V}_o(\mathcal{S})$. Then $\text{flag}(\Omega)$ is a convex subcomplex of $\text{Flag}(\mathcal{G})$ if and only if Ω is a closed subset of $\mathcal{V}_o(\mathcal{S})$.

Conversely, if $\Delta = (\Delta, \subset)$ is a building of type D_n , then there exists a polar space \mathcal{S}' of type D and of rank n such that Δ is isomorphic to $\text{Flag}(\mathcal{G}(\mathcal{S}'))$.

6.4. Proof of the center conjecture for buildings of type D_n , $n \geq 5$

Let $n \geq 5$ be a natural number. Let $\Delta = (\Delta, \subset)$ be a building of type D_n and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ . Let $\mathcal{S} = (P, L)$ be a polar space of type D of rank n and let $\alpha : \text{flag}(\mathcal{G}_o(\mathcal{S})) \rightarrow \Delta$ be an isomorphism from $\text{Flag}(\mathcal{G}_o(\mathcal{S}))$ onto Δ . Put $\Omega := \{X \in \mathcal{V}_o(\mathcal{S}) \mid \alpha(\{X\}) \in \tilde{\Delta}\}$. As $\tilde{\Delta}$ is a convex subcomplex it follows from Lemma 2.1 that $\alpha(F) \in \tilde{\Delta}$ for all $F \in \text{flag}(\Omega)$ and we conclude that α maps the subcomplex $\text{flag}(\Omega)$ bijectively onto the subcomplex $\tilde{\Delta}$. Since $\tilde{\Delta}$ is convex, it follows that Ω is a closed subset of $\mathcal{V}_o(\mathcal{S})$ by part (c) of Theorem 6.4. Let G be the stabilizer of Ω in $\text{Aut } \mathcal{G}_o(\mathcal{S}) = \text{Aut } \mathcal{G}(\mathcal{S})$ (cf. Proposition 6.2). Let $\tilde{\Omega}$ be the submaximal completion of Ω and let \tilde{G} be its stabilizer in $\text{Aut } \mathcal{G}(\mathcal{S})$. As Ω is a closed subset of $\mathcal{V}_o(\mathcal{S})$ the set $\tilde{\Omega}$ is a closed subset of $\mathcal{V}(\mathcal{S})$ and the group G is a subgroup of \tilde{G} .

If for each element X in $\tilde{\Omega}$, there is an element Y in $\tilde{\Omega}$ which is a hyperbolic complement of X in \mathcal{S} , then each vertex in $\tilde{\Delta}$ has an opposite in $\tilde{\Delta}$; this follows from part (b) of Theorem 6.4 and part (i) of Lemma 5.4. Using Proposition 2.3 we conclude that $\tilde{\Delta}$ is completely reducible.

Suppose that this is not the case. Then \tilde{G} fixes an element X in $\tilde{\Omega}$ by Theorem 5.15. If $X \in \Omega$, then G fixes the non-trivial flag $\{X\}$. If X is not in Ω , then $X \in \text{Submax}(\Omega)$ and G fixes the non-trivial flag $F := \{X, Z\}$ whose elements are the two maximal singular subspaces whose intersection is X ; as X is in $\text{Submax}(\Omega)$ it follows that Y and Z are in Ω and we conclude that F is in $\text{flag}(\Omega)$.

Thus, G fixes a non-trivial element F in $\text{flag}(\Omega)$. It follows that $\text{Stab}_{\text{Aut}(\Delta)}(\tilde{\Delta}) = \alpha G \alpha^{-1}$ fixes the non-trivial simplex $\alpha(F) \in \tilde{\Delta}$.

6.5. An outline of the proof for the center-conjecture for buildings of type D_4

We label nodes of the diagram D_4 by 1, 2, 3 and 4 in such a way that the central node of valency 3 is labeled by 2. The nodes of the diagram F_4 are labeled by the same set in the linear order.

For a building $\Delta = (\Delta, \subset)$ of type D_4 , we denote by $\bar{\Delta} = (\bar{\Delta}, \subset)$ the building of type F_4 associated with Δ as described in [Ti74, 10.14]. The vertices of type 1 of $\bar{\Delta}$ are the vertices of type 2 of Δ . The vertices of type 4 of $\bar{\Delta}$ are the vertices of Δ whose type is different from 2. Each vertex of $\bar{\Delta}$ is a simplex of Δ . Two vertices of Δ are opposite in Δ if and only if they are opposite in $\bar{\Delta}$ and each convex subcomplex of Δ corresponds to a convex subcomplex of $\bar{\Delta}$. Finally, there is a canonical isomorphism from $\text{Aut}(\Delta)$ onto $\text{Spe}(\bar{\Delta})$.

Using the metasymplectic space associated with a building of type F_4 (see [Ti74, 10.13]) one proves the following proposition.

Proposition 6.5. *Let $\bar{\Delta}$ be a building of type F_4 and let $G \leq \text{Spe}(\bar{\Delta})$. Let V be a vertex of type 1 or 4 of $\bar{\Delta}$ and suppose that the length of the G -orbit $G(V)$ of V divides 3. Then one of the following holds.*

- (a) *The group G fixes a vertex in the full convex hull of $G(V)$.*
- (b) *There is a vertex V' in the full convex hull of $G(V)$ which is opposite to V .*

Let $\Delta = (\Delta, \subset)$ be a building of type D_4 . We have a natural homomorphism $\text{type} : \text{Aut}(\Delta) \rightarrow \text{Aut}(D_4) = \text{Sym}\{1, 3, 4\}$ whose kernel is the group $\text{Spe}(\Delta)$.

Let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of Δ , let G be the stabilizer of $\tilde{\Delta}$ in $\text{Aut}(\Delta)$ and put $G_s := G \cap \text{Spe}(\Delta)$. Theorem 5.15 has the following consequence.

Proposition 6.6. *In the situation above one of the following holds.*

- (a) *$\tilde{\Delta}$ is completely reducible.*
- (b) *There exists a vertex V in $\tilde{\Delta}$ which is stabilized by G_s such that there is no $V' \in \tilde{\Delta}$ opposite to V .*

Put $H := \{g \in G \mid \text{type}(g) \in \text{Alt}(\{1, 3, 4\})\}$ and let V be a vertex of $\tilde{\Delta}$ stabilized by G_s having no opposite in $\tilde{\Delta}$. The vertex V is of type 1 or 4 in the building $\bar{\Delta}$ described above, $H \leq \text{Spe}(\bar{\Delta})$ and the length of the H -orbit of V divides 3. Hence we can apply Proposition 6.5. In view of the fact that there is no vertex $V' \in \tilde{\Delta}$ opposite to V , assertion (a) of Proposition 6.5 must hold and therefore H stabilizes a vertex in the full convex hull of the H -orbit of V . To this vertex corresponds a non-trivial simplex A in $\tilde{\Delta}$ which is fixed by H . In order to finish the proof of the center conjecture for Δ one uses the fact that the index of H in G is at most 2 and the following proposition which we leave to the reader as a non-trivial exercise in the theory of buildings (of arbitrary types).

Proposition 6.7. *Let $\Delta = (\Delta, \subset)$ be a building, let $G \leq \text{Aut}(\Delta)$ and let $A \in \Delta$ be a non-trivial simplex. Suppose that the length of the G -orbit of A divides 2. Then one of the following holds.*

- (a) *G fixes a non-trivial simplex in the full convex hull of the G -orbit of A .*
- (b) *The building Δ is spherical and there exists a simplex in the G -orbit of A which is opposite to A .*

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