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Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices

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Abstract

We extend a theorem of D. Rees on the existence of Rees valuations of an ideal A of a Noetherian ring to Noetherian multiplicative lattices L. This result also extends a result of D.P. Brithinee. We then apply this to projective equivalence and asymptotic primes of rational powers of A. In particular, it is shown that if L is a Noetherian multiplicative lattice, $A \in L$, $\{P_1, \dots, P_r\}$ is the set of centers of the Rees valuations $v_1, \dots, v_r$ of A and e is the least common multiple of the Rees numbers $e_1(A), \dots, e_r(A)$ of A, then $\operatorname{Ass}(L/A_{n/e}) \subseteq \{P_1, \dots, P_r\}$, where $A_\beta = \bigvee\{x \in L \mid \bar{v}_A(x) \geqslant \beta\}$. Further, if $A \nleq q$ for each minimal prime $q \in L$, then $\operatorname{Ass}(L/A_{n/e}) \subseteq \operatorname{Ass}(L/A_{n/e+k/e})$ for each $n \in \mathbb{N}$, where k/e is in a certain additive subsemigroup of $\mathbb{Q}_+$ which is naturally associated to the set of members of L which are projectively equivalent to A. These latter results are new even in the case of rings and extend results of L.J. Ratliff who gave them for rings in the case that the n/e and k/e are integers.
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1. Introduction

Let I and J be regular ideals in a local (Noetherian) commutative ring R. Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the real numbers, the positive integers and the nonnegative integers, respectively. In [24] Samuel defined, for each $n \in \mathbb{N}$, $m(n)$ to be the largest $k \in \mathbb{N}$ such that $J^n \subseteq I^k$, showed that $\lim_{n\to\infty} m(n)/n = l_I(J)$ exists in $\mathbb{R} \cup \{\infty\}$ and used this to define the following equivalence relations. In the terminology of [21], ideals I and J are said to be **asymptoticly equivalent** if

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$l_I(J) = l_J(I) = 1$ and **projectively equivalent** if $l_I(J)l_J(I) = 1$. In [21,23] Rees reformulated this in terms of pseudo-valuations. Recall that a **pseudo-valuation** is a map $v : R \to \mathbb{R} \cup \{\infty\}$ satisfying (i) $v(0) = \infty$, $v(1) = 0$, (ii) $v(x - y) \geqslant \min\{v(x), v(y)\}$ and (iii) $v(xy) \geqslant v(x) + v(y)$ for all $x, y \in R$. If further, $v(x^n) = nv(x)$ for all $x \in R$ and $n \in \mathbb{N}$, the pseudo-valuation v is said to be **homogeneous**. If $v(xy) = v(x) + v(y)$ for all $x, y \in R$, v is called a **valuation**. In [21,23] Rees noted that if we take J in Samuel's definition to be a principal ideal xR, and define $v_I(x)$ to be $\sup\{n \in \mathbb{N}_0 \mid x \in I^n\}$, then v_I is a pseudo-valuation, $\bar{v}_I(x) = \lim_{n\to\infty} v_I(x^n)/n$ exists in $\mathbb{R} \cup \{\infty\}$ and $l_I(J) = \inf\{\bar{v}_I(x) \mid x \in J\}$. He further showed:

(1) $\bar{v}_I$ is the smallest homogeneous pseudo-valuation greater that v_I;
(2) the integral closure I_a of I is $I_s = \{x \in R \mid \bar{v}_I(x) \geqslant 1\}$;
(3) there exist unique normalized valuations $v_i : R \to \mathbb{N}_0 \cup \{\infty\}$ and integers e_i, $i = 1, \ldots, r$, such that $\bar{v}_I(x) = \min\{v_i(x)/e_i \mid i = 1, \ldots, r\}$ for all $x \in R$;
(4) ideals I and J are asymptotically equivalent if and only if $I_a = J_a$; and
(5) I and J are projectively equivalent if and only if $(I^r)_a = (J^s)_a$ for some $r, s \in \mathbb{N}$.

Since that time these results of Rees, especially (3), have become ubiquitous in Noetherian commutative ring theory. The object of this note is to extend the well known and useful result (3) to Noetherian multiplicative lattices, as developed in [9], and to give some consequences for asymptotic primes and projective equivalence. In [12], statement (2) was extended to Noetherian multiplicative lattices and in [5], statement (3) was shown to hold in those Noetherian multiplicative lattices which have no nonzero zero-divisors and also satisfy a certain finiteness condition. In the case of rings this finiteness condition corresponds to the finiteness condition used by Rees in [22] to give his second case of his valuation theorem before giving the result (3) in [23]. A key impediment to (3) in both [5,22], was the lack of the Mori–Nagata theorem. In this note, we obtain part (3) for all Noetherian multiplicative lattices (Theorem 6.12). We then use Theorem 6.12 to give generalizations of some well known results on asymptotic prime ideals, which are new, even in the case of commutative rings.

To describe these results, recall that in [14] it is shown that for each integrally closed ideal J of a Noetherian ring R that is projectively equivalent to a regular ideal I of R, there is a unique largest $\beta \in \mathbb{Q}_+$ such that $J = I_\beta = \{x \in R \mid \bar{v}_I(x) \geqslant \beta\}$, and that for such β and γ, $(I_\beta I_\gamma)_a = I_{\beta+\gamma}$. It follows that the set $\mathbf{P}(I)$ of integrally closed ideals that are projectively equivalent to I forms a semigroup that is order isomorphic to a discrete additive subsemigroup of the rational numbers. In their forthcoming book [10], Craig Huneke and Irena Swanson consider the ideals of the form I_β for any rational number $\beta \in \mathbb{Q}_+$, and call these ideals **the rational powers** of I. It is easy to see that these ideals are integrally closed, and that $I_n = (I^n)_a$ for $n \in \mathbb{N}$. In Theorem 8.1, we generalize Ratliff's theorem on the stability of the sets $\mathrm{Ass}_R(R/(I^n)_a)$, $n \in \mathbb{N}$, to the sets, $\mathrm{Ass}_R(R/I_\beta)$, $\beta \in \mathbb{Q}_+$. In the case that $\beta \in \mathbb{N}$, our proof of Theorem 8.1 shows directly that if L is a Noetherian multiplicative lattice, $A \in L$ and $P_1, \ldots, P_r$ are the centers of a certain subset of the set of Rees valuations of A, then $\mathrm{Ass}(L/(A^n)_a) \subseteq \{P_1, \ldots, P_r\}$ with equality for n large. In particular, this furnishes a short and apparently new proof of a result of L.J. Ratliff [20] on the finiteness of the set $\bigcup_{n=1}^{\infty} \mathrm{Ass}(L/(A^n)_a)$. Extensions to rational powers of the well known results $\mathrm{Ass}(L/(A^n)_a) \subseteq \mathrm{Ass}(L/(A^{n+1})_a)$ for each $n \in \mathbb{N}$ (Theorem 8.4), and $\bigcup_{n=1}^{\infty} \mathrm{Ass}(L/(A^n)_a) \subseteq \bigcup_{n=1}^{\infty} \mathrm{Ass}(L/A^n)$ (Theorem 8.7) are also given. These results on asymptotic primes also improve some results of Becerra on lattices [4].

Recall [9], that a **multiplicative lattice** is a complete modular lattice L with a commutative, associative multiplication which distributes over arbitrary joins and such that the largest element

R of L is the identity for the multiplication. An element $M \in L$ is said to be **meet principal** (respectively **join principal**) if $(B \wedge (A : M))M = A \wedge MB$ (respectively $(AM \vee B) : M = A \vee (B : M)$) for all $A, B \in L$. If M is both meet principal and join principal, then M is said to be **principal**. If each element B of L is the join of a family of principal elements of L, L is said to be **principally generated**. In [9], Dilworth defined a multiplicative lattice to be **Noetherian** if it is modular, principally generated and satisfies the ascending chain condition (ACC). He then obtained the primary decomposition theorems of E. Noether and also the Krull principal ideal theorem and Krull intersection theorem in Noetherian multiplicative lattices. In [16] the authors used [5,6], to extend the Mori–Nagata theorem [15, Theorem 33.10], from integral domains to lattices. (See Theorem 3.12 of this paper.) In this paper this Mori–Nagata theorem is used to extend a result [5, Theorem 4.45], on the existence of Rees valuations, to any Noetherian multiplicative lattice, and to give some consequences for asymptotic primes and projective equivalence.

In Sections 2–4 we review some of the basic properties of multiplicative lattices that we will use including the definition of the quotient field lattice, as developed in [5,6]. We also review some results on localization and integrality in multiplicative lattices and the A transform $\mathcal{R}(L, A)$. In Section 5, we give an explicit description of the quotient field lattice $\mathcal{QR}(L, A)$ of the A-transform $\mathcal{R}(L, A)$ in the case that L is Noetherian and without zero-divisors and use it to give a description of the a-closure of $\mathcal{R}(L, A)$ in this case. In Section 6, we prove the existence of Rees valuations in Noetherian multiplicative lattices and some uniqueness results on the Rees valuations. In Section 7, we extend to lattices some results in [14] on the set $\mathbf{P}(I)$ of ideals J which are projectively equivalent to a given regular ideal which are needed for our results on asymptotic primes. These results on $\mathbf{P}(I)$ improve some of the results in [14] by weakening the requirement that I be regular. The results mentioned above on asymptotic primes are given in Section 8.

2. Preliminary results on cl-monoids

In this section we review some of the basic results on cl-monoids from [5,6].

Definition 2.1. A **cl-monoid** is a complete lattice-ordered (multiplicative) monoid $\mathcal{L}$ such that (1) if 0 is the smallest element of $\mathcal{L}$, $0A = 0$ for each $A \in \mathcal{L}$, and (2) for any family $\{B_\lambda \in \mathcal{L} \mid \lambda \in \Lambda\}$, $A(\bigvee\{B_\lambda \mid \lambda \in \Lambda\}) = \bigvee\{AB_\lambda \mid \lambda \in \Lambda\}$. The multiplicative identity of $\mathcal{L}$ is denoted R. If R is the largest element of $\mathcal{L}$, the lattice is said to be **integral**. The set $\mathcal{I} = \{A \in \mathcal{L} \mid A \leqslant R\}$ is a subcl-monoid of $\mathcal{L}$ which is integral. **Residuation** is defined for $A, B \in \mathcal{L}$ as $A : B = \bigvee\{C \in \mathcal{L} \mid CB \leqslant A\}$. We denote $R : A$ by A^{-1}. (Thus in the case that $\mathcal{L}$ is integral, $A^{-1} = R$.) If $A \leqslant B$ in $\mathcal{L}$, we denote $\{C \in \mathcal{L} \mid A \leqslant C \leqslant B\}$ by $[A, B]$.

The following are examples of a cl-monoids.

Example 2.2. (1) If R is a subring of a commutative ring R', the set of R-submodules of R' is a cl-monoid.

(2) If M is a cancellative torsion-free abelian monoid and R is a graded subring of the M-graded commutative ring R', the set of graded R-submodules of R' is a cl-monoid.

(3) Let M is a multiplicative cancellative monoid with quotient group G, and let M_0, G_0 be the monoids obtained by adjoining a zero element 0 to each. Let $L(M_0) = \{A \subseteq G_0 \mid M_0 A \subseteq A\}$,

ordered by inclusion, and with multiplication defined by $AB = \{ab \mid a \in A,\ b \in B\}$. Then $\mathcal{L}$ is a cl-monoid.

If $\mathcal{L}$ is a cl-monoid with identity R, an element $A \in \mathcal{L}$ is said to be a **ring element** of $\mathcal{L}$ if $R \leqslant A$ and A is subidempotent, that is $AA \leqslant A$. If A is a ring element of $\mathcal{L}$, then from $R \leqslant A$, we get $A = RA \leqslant A^2$, so $A = A^2$. If R' is a ring element of $\mathcal{L}$, the elements of the set $\mathcal{I}' = \mathcal{I}(R') = \{AR' \mid A \in \mathcal{L},\ A \leqslant R'\}$ are called R'-**ideal elements of** $\mathcal{L}$, and the elements in the set $\mathcal{L}' = \mathcal{L}(R') = \{AR' \mid A \in \mathcal{L}\}$ are called R'-**module elements of** $\mathcal{L}$. Observe that if $A \in \mathcal{L}$, then $A = AR \leqslant AR'$. If R' is a ring element of $\mathcal{L}$, then $\mathcal{L}(R')$ and $\mathcal{I}(R')$ are subcl-monoids of $\mathcal{L}$ with multiplicative identity R', and R' is the greatest element of $\mathcal{I}'$ [6, Proposition 1.4]. Further, if $[A : B]_{\mathcal{L}(R')}$, and $[A : B]_{\mathcal{I}(R')}$ denote residuation in $\mathcal{L}(R')$ and $\mathcal{I}(R')$, respectively, then $[A : B]_{\mathcal{L}(R')} = (A : B)$ for $A, B \in \mathcal{L}(R')$, and $[A : B]_{\mathcal{I}(R')} = (A : B) \wedge R'$ for $A, B \in \mathcal{I}(R')$ [6, Lemma 1.5].

Definition 2.3. An element M of a cl-monoid $\mathcal{L}$ is said to be

(a) $\mathcal{L}$-**meet principal** if $(A \wedge (B : M))M \geqslant B \wedge MA$ for all $A, B \in \mathcal{L}$;
(b) $\mathcal{L}$-**join principal** if $(A \vee BM) : M \leqslant B \vee (A : M)$ for all $A, B \in \mathcal{L}$; and
(c) $\mathcal{L}$-**principal** if M is both $\mathcal{L}$-meet principal and $\mathcal{L}$-join principal.

We sometimes write "principal" for $\mathcal{L}$-principal if there is no danger of confusion, and similarly for $\mathcal{L}$-meet principal and $\mathcal{L}$-join principal. It follows that: (1) the opposite inequalities in (a) and (b) always hold, (2) the elements 0 and R of $\mathcal{L}$ are principal, (3) meet principal elements and join principal elements are closed under multiplication [6, 1.10]. We also use the following result several times.

Proposition 2.4. [6, Proposition 1.11] *Let $\mathcal{L}$ be a cl-monoid and let R' be a ring element of $\mathcal{L}$. If X is $\mathcal{L}$-meet or $\mathcal{L}$-join principal in $\mathcal{L}$ then XR' is $\mathcal{L}(R')$-meet or $\mathcal{L}(R')$-join principal, respectively.*

If each element of a cl-monoid $\mathcal{L}$ is a join of principal elements, $\mathcal{L}$ is said to be **principally generated**. An element $M \in \mathcal{L}$ is said to be **invertible** if $MN = R$ for some $N \in \mathcal{L}$. In this case it follows that $N = R : M = M^{-1}$. Also, if A, B and $C \in \mathcal{L}$ with A invertible, then $B : A = BA^{-1}$, and the distributive law $(B \wedge C)A = BA \wedge CA$ holds [6, Lemma 1.19].

Definition 2.5. A cl-monoid $\mathcal{L}$ is said to be a **q.f. lattice** if the following hold.

(1) $AK = K$ for every nonzero $A \in \mathcal{L}$, where $K = \bigvee \mathcal{L}$.
(2) $\mathcal{L}$ is principally generated.
(3) There is a compact invertible element in $\mathcal{L}$.
(4) For each $A \in \mathcal{L} \setminus \{0\}$, $A \wedge R \neq 0$.

An element $M \in \mathcal{L}$ is said to be a **zero-divisor** if $M \neq 0$ and $MN = 0$ for some nonzero element $N \in \mathcal{L}$. It is shown in [6, Theorems 5.19] that if L is a principally generated multiplicative lattice without zero-divisors, then L can be embedded into a q.f. lattice $\mathcal{L}$. Further, if each principal element of L is compact, then by [6, Theorem 5.22], there exists a one-to-one cl-monoid homomorphism $f : L \to \mathcal{L}$ such that $f(L) = \mathcal{I}(f(R))$. Further, by [6, Proposition 1.21] or [16, Proposition 2.3], each $\mathcal{I}$-principal element of $\mathcal{I}$ is $\mathcal{L}$-principal. There is also a uniqueness statement [6, Proposition 5.28], which states that if $\mathcal{L}'$ is a q.f. lattice with integral elements L and for any $X \in L$, X is L-principal if and only if it is $\mathcal{L}'$-principal, then there is a q.f. lattice isomorphism $\mathcal{L}' \to \mathcal{L}$ which extends f. Therefore, under these conditions, we call $\mathcal{L}$ **the quotient field lattice** of $\mathcal{I}$.

Example 2.6. Example (1) of 2.2 is a q.f. lattice in the case that R is an integral domain and R' is the quotient field K of R. Example (2) of 2.2 is a q.f. lattice in the case that $R = \bigoplus_{m\in M} R_m$ is an M-graded integral domain and R' is R_S, where S is the set of nonzero homogeneous elements of R. Example (3) of 2.2 is a q.f. lattice.

If $\mathcal{L}$ is a q.f. lattice, an element $A \in \mathcal{L}$ is said to be **fractionary** if there exists a $D \in \mathcal{I} \setminus \{0\}$ such that $DA \in \mathcal{I}$. An $M \in \mathcal{L}$ is said to be **finitely generated** if it is the join of finitely many principal elements of $\mathcal{L}$.

3. Localization and integral closure in q.f. lattices

In this section we review some of the basic properties of localization and integrality in q.f. lattices from [5,6]. In this section $\mathcal{L}$ denotes a q.f. lattice. If $A \in \mathcal{L}$, we denote $R \vee A \vee A^2 \vee \cdots$ by $R[A]$.

Definition 3.1. Let $\mathcal{L}$ be a q.f. lattice and let $\mathcal{S} \subseteq \mathcal{I}$. If each $s \in \mathcal{S}$ is $\mathcal{L}$-principal, $R \in \mathcal{S}$ and $ss' \in \mathcal{S}$ for each $s, s' \in \mathcal{S}$, then $\mathcal{S}$ is said to be **multiplicative subset for** R. If P is a prime element of $\mathcal{I}$, we let $\mathcal{S}(P) = \{s \in \mathcal{I} \mid s \text{ is } \mathcal{L}\text{-principal and } s \nleqslant P\}$.

Definition 3.2. Let $\mathcal{L}$ be a cl-monoid and let $\mathcal{S}$ be a multiplicative subset for R. If $A \in \mathcal{L}$, the **localization of** A **at** $\mathcal{S}$ is $A_{\mathcal{S}} = \bigvee\{A : s \mid s \in \mathcal{S}\}$. If $\mathcal{S}$ is $\mathcal{S}(P)$ for a prime element P of $\mathcal{I}$, we denote $A_{\mathcal{S}(P)}$ by A_P.

Localization satisfies the usual properties [6, Section 2]. (These are summarized in [16, Proposition 3.1].)

Definition 3.3. [5, Definition 4.8] If $\mathcal{L}$ is a q.f. lattice, $\mathcal{I}$ is a **valuation lattice** if for every pair of $\mathcal{L}$-principal elements $A, B \in \mathcal{I}$, either $A \leqslant B$ or $B \leqslant A$. If V is a ring element such that $\mathcal{I}(V)$ is a valuation lattice, then V is said to be a **V-ring element**. A V-ring elements V is said to be **Noetherian** if $\mathcal{I}(V)$ is Noetherian.

Definition 3.4. An $A \in \mathcal{I}$ is said to be a-**dependent on** $B \in \mathcal{I}$ if for some $n \in \mathbb{N}$, $A^{n+1} \leqslant B(A \vee B)^n$. Then $B_a = \bigvee\{A \in \mathcal{I} \mid A \text{ is } a\text{-dependent on } B\}$ is called the a-**closure of** B **in** $\mathcal{I}$ [5, Definition 3.14]. An element $C \in \mathcal{L}$ is said to be a-**dependent on** R if $C = A : y$ for $A \in \mathcal{I}$ and an invertible element $y \in \mathcal{I}$ where A is a-dependent on y [5, Definition 3.16]. Let $R_a = \bigvee\{C \in \mathcal{L} \mid C \text{ is } \mathcal{L}\text{-compact and } a\text{-dependent on } R\}$. Then R_a is called the a-**closure of** R **in** K [5, Definition 3.35].

Proposition 3.5. [5, Corollary 3.22] *Let $\mathcal{L}$ be a q.f. lattice. If a compact element $C \in \mathcal{L}$ is a-dependent on R and $C = A : y$ for $A, y \in \mathcal{I}$ with y invertible in $\mathcal{L}$, then A is a-dependent on y.*

If for every fractionary A in the q.f. lattice $\mathcal{L}$ and every $B \in \mathcal{I}$, $(A \vee B) \wedge R = (A \wedge R) \vee B$, then R is said to be an **M-element**. It follows that R is an M-element if and only if every $\mathcal{L}$-join principal element in $\mathcal{I}$ is $\mathcal{I}$-join principal [5, Proposition 2.25]. Further, if $\mathcal{I}$ is modular then, by [16, Proposition 3.4], $\mathcal{L}$ is modular, and thus each ring element of $\mathcal{L}$ is an M-element.

Theorem 3.6. [16, Theorem 4.6] *If $\mathcal{L}$ is a q.f. lattice and each ring element in $\mathcal{L}$ is an M-element, then $R_a = \bigwedge\{V \mid V$ is a V-ring element of $\mathcal{L}\} = \bigvee\{C \in \mathcal{L} \mid C$ is $\mathcal{L}$-principal and a-dependent on $R\}$. In particular, these equalities hold if $\mathcal{I}$ is modular.*

Corollary 3.7. [16, Corollary 4.8] *Let $\mathcal{L}$ be a q.f. lattice such that each ring element of $\mathcal{L}$ is an M-element, and let $A \in \mathcal{L}$ be $\mathcal{L}$-compact. Then the following are equivalent*:

(1) $A \leqslant R_a$.
(2) *There exists a compact element $D \in \mathcal{L}$ such that $AD \leqslant D$.*
(3) $R[A]$ *is compact.*

Remark 3.8. Let D be a domain with quotient field K. Then D_a is the integral closure of D [5, Remark 3.37]. If M is a torsion-free monoid with quotient group G, recall that $y \in G$ is said to be integral over M if $y^n \in M$ for some $n \in \mathbb{N}$. It follows that in Example 2.6(3) M_a is the integral closure of M in G [16, Remark following Corollary 4.8].

If L is a multiplicative lattice with identity R and $D \in L$, recall that $[D, R] = \{X \in L \mid D \leqslant X\}$ is a multiplicative lattice with multiplication $\circ$ defined by $A \circ B = AB \vee D$ for $A, B \in [D, R]$ [9, pp. 488–489]. The multiplicative lattice $[D, R]$ is also denoted by L/D.

Lemma 3.9. [5, Lemma 3.47] *Let R' be a ring element of $\mathcal{L}$, let $A \in \mathcal{I}(R')$, and assume that $[0, R']$ is modular. Then the map $f : [A, R \vee A] \to [A \wedge R, R]$ defined by $f(B) = B \wedge R$ for $B \in [A, R \vee A]$, is a lattice isomorphism with inverse given by $g(C) = C \vee A$ for $C \in [A \wedge R, R]$. Further $R \vee A$ is a ring element and $[A, R \vee A] \subseteq \mathcal{I}(R \vee A)$.*

In [6, Definition 4.26], we have the following.

Definition 3.10. Let $\mathcal{L}$ be a q.f. lattice and let $\mathcal{P}$ denote the set of prime elements of $\mathcal{I}$ of height one. Then $\mathcal{I}$ is called a **Krull lattice** if the following hold.

(1) If $P \in \mathcal{P}$, then $\mathcal{I}(R_P)$ is a Noetherian valuation lattice.
(2) If $M \in \mathcal{I}$ is $\mathcal{L}$-principal, then there are only finitely many $P \in \mathcal{P}$ such that $M \leqslant P$.
(3) $R = \bigwedge\{R_P \mid P \in \mathcal{P}\}$.

As in the ring case, we have the following two results.

Theorem 3.11. [16, Theorem 8.3] *If $\mathcal{L}$ be a q.f. lattice, then $\mathcal{I}$ is a Krull lattice if and only if there exists a locally finite family $\{V_\lambda \mid \lambda \in \Lambda\}$ of Noetherian valuation ring elements of $\mathcal{L}$ such that $R = \bigwedge\{V_\lambda \mid \lambda \in \Lambda\}$.*

Theorem 3.12 *(Mori–Nagata theorem for q.f. lattices).* [16, Theorem 8.4] *If $\mathcal{L}$ is a q.f. lattice with $\mathcal{I}$ Noetherian, then $\mathcal{I}(R_a)$ is a Krull lattice.*

4. The A-transform

We also need a few facts on the A-transform, as given in [11].

Definition 4.1. Let L be a Noether lattice and let $A \in L$. The A-transform $\mathcal{R} = \mathcal{R}(L, A)$ is defined to be the set of all formal sums $\sum_{i=-\infty}^{\infty} B_i$, $B_i \in L$, such that $A^i \geqslant B_i \geqslant B_{i+1} \geqslant AB_i$ for each i, where A^i is defined to be R when $i \leqslant 0$, along with the operations

(1) $\sum B_i \leqslant \sum C_i$ if and only if $B_i \leqslant C_i$ for all i.
(2) $\sum B_i \vee \sum C_i = \sum (B_i \vee C_i)$.
(3) $\sum B_i \wedge \sum C_i = \sum (B_i \wedge C_i)$.
(4) $(\sum B_i)(\sum C_i) = \sum_i (\bigvee \{B_r C_s \mid r + s = i\})$.

If L is a Noether lattice and $A \in L$, then by [11, Theorem 2.11], $\mathcal{R}(L, A)$ is a Noether lattice.

Definition 4.2. If L is a Noether lattice and $A, B \in L$ with $B \leqslant A^r$, then $B^{[r]}$ denotes the smallest element $D = \sum D_i \in \mathcal{R}(L, A)$ such that $B \leqslant D_r$, and B^* denotes the element $\sum (B \wedge A^i)$ of $\mathcal{R}(L, A)$.

Proposition 4.3. [11, Section 2] *Let L be a Noether lattice and $A \in L$. Then the following hold where, as noted in Definition* 4.1, *$A^i = R$ for $i \leqslant 0$.*

(1) *The multiplicative and meet identity of $\mathcal{R}(L, A)$ is $R^* = \sum A^i$.*
(2) *If $B \in L$ with $B \leqslant A^r$ then $B^{[r]} = \sum (BA^{i-r}) = \sum D_i$ where $D_i = BA^{i-r}$.*
(3) *If $B, C \in L$ with $B \leqslant A^r$ and $C \leqslant A^r$, then $B^{[r]} \vee C^{[r]} = (B \vee C)^{[r]}$.*
(4) *If $B, C \in L$ with $B \leqslant A^r$ and $C \leqslant A^s$, then $B^{[r]} C^{[s]} = (BC)^{[r+s]}$.*
(5) *If $B \in L$ is principal with $B \leqslant A^r$ then $B^{[r]}$ is principal.*
(6) *If $B = \sum B_i \in \mathcal{R}(L, A)$ and $C \in L$ with $C \leqslant A^r$, then $BC^{[r]} = \sum_i B_{i-r} C$. In particular $BR^{[-r]} = \sum_i B_{i+r}$.*

Examination of Definition 4.1 and the proof in [11] of the above result shows that the Noetherian hypothesis on L is not necessary. This hypothesis was included in [11] because the applications of the A-transform in [11] concerned Noetherian multiplicative lattices. In fact non-Noetherian multiplicative lattices were apparently first considered by D.D. Anderson in his 1974 dissertation under Irving Kaplansky [1]. See [3] for a good survey.

The following simple lemma is basic to what follows.

Lemma 4.4. *If L is a multiplicative lattice without zero-divisors and $A \in L$, then $\mathcal{R}(L, A)$ has no zero-divisors.*

Proof. Suppose $BC = 0$, $B = \sum B_i$, $C = \sum C_i \in \mathcal{R}(L, A)$, with $B \neq 0$. Then $B_j \neq 0$ for some j. Then $(\sum B_i)(\sum C_i) = \sum_i D_i$ where $D_i = \bigvee \{B_r C_s \mid r + s = i\} = 0$ for each i. In particular, $B_j C_s \leqslant D_{j+s} = 0$ for each $s \in \mathbb{Z}$. Thus $C = 0$. $\square$

5. The q.f. lattice of the A-transform

Let $\mathcal{L}$ be a quotient field lattice with $L = \mathcal{I}$ and let $A \in L$. In this section we describe the quotient field lattice of the A-transform $\mathcal{R}(L, A)$ and develop some of its properties. In this section we do not assume that L is Noetherian except where it is specified.

Definition 5.1. Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I} = L$. Let $\mathcal{QR} = \mathcal{QR}(L, A)$ denote the set of all formal sums $\sum_{i=-\infty}^{\infty} B_i$, $B_i \in \mathcal{L}$, such that $B_i \geqslant B_{i+1} \geqslant AB_i$ for each i, where A^i is defined to be R when $i \leqslant 0$, along with the operations

(1) $\sum B_i \leqslant \sum C_i$ if and only if $B_i \leqslant C_i$ for all i.
(2) $\sum B_i \vee \sum C_i = \sum (B_i \vee C_i)$.
(3) $\sum B_i \wedge \sum C_i = \sum (B_i \wedge C_i)$.
(4) $(\sum B_i)(\sum C_i) = \sum_i (\bigvee \{B_r C_s \mid r + s = i\})$.

Lemma 5.2. *With $\mathcal{L}$ and A as in the above definition, $\mathcal{QR} = \mathcal{QR}(L, A)$ is a cl-monoid with identity element $R^* = \sum_i A^i$. Further $\mathcal{QR}$ is modular if $\mathcal{L}$ is.*

Proof. It is easily seen that $\mathcal{QR}$ is a complete lattice which is modular if $\mathcal{L}$ is. It is also immediate that the multiplication (4) is a multiplication on $\mathcal{QR}$ which is commutative, associative and distributes over arbitrary joins of elements of $\mathcal{QR}$. To see that $R^* = \sum_i A^i$ is the identity of $\mathcal{QR}$, let $B = \sum_i B_i \in \mathcal{QR}$. Then $R^* B$ has ith coordinate $\bigvee_{r+s=i} A^r B_s$. But $A^0 B_i = RB_i = B_i$ and $A^r B_s \leqslant B_{r+s} = B_i$ if $r \geqslant 0$. If $r < 0$, then since $r + s = i$, $s > i$, and then $A^r B_s = B_s \leqslant B_i$. $\square$

Definition 5.3. Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $B \in \mathcal{L}$, then $B^{[r]}$ denotes the smallest element $D = \sum D_i \in \mathcal{QR}(L, A)$ such that $B \leqslant D_r$.

Lemma 5.4. *Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $B \in \mathcal{L}$, then $B^{[r]} = \sum_i BA^{i-r} = \sum_i D_i$ with $D_i = BA^{i-r}$. If also $C \in \mathcal{L}$, then $B^{[r]} \vee C^{[r]} = (B \vee C)^{[r]}$.*

Proof. Clearly $\sum_i BA^{i-r} = \sum_i D_i$ satisfies $D_i \geqslant D_{i+1} \geqslant AD_i$ for each i. That is $BA^{i-r} \geqslant BA^{i+1-r} \geqslant A(BA^{i-r})$. Thus $\sum_i BA^{i-r} \in \mathcal{QR}$. Since $B = BA^{r-r} = D_r$, we have $B \leqslant D_r$.

Now let $E \in \mathcal{QR}$ with $B \leqslant E_r$. Then $D_r \leqslant E_r$ and for each $i \geqslant r$, $E_i \geqslant E_r A^{i-r} \geqslant BA^{i-r} = D_i$. For $i < r$, $E_i \geqslant E_r \geqslant B = BA^{i-r} = D_i$. Thus $D \leqslant E$.

It is clear that $B^{[r]} \vee C^{[r]} = (B \vee C)^{[r]}$. $\square$

From Lemma 5.4, it is clear that $R^{[0]} = R^*$, the identity of $\mathcal{QR}(L, A)$. The fact that $R^{[0]}$ acts as the identity element in $\mathcal{QR}(L, A)$ is also a special case of the following lemma.

Lemma 5.5. *Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $B \in \mathcal{QR}(L, A)$ and $C \in \mathcal{L}$, then $BC^{[r]} = \sum_i B_{i-r} C$.*

Proof. The ith coordinate of $BC^{[r]}$ is $\bigvee_{m+n=i} B_m (CA^{n-r})$. Let $m + n = i$. If $m > i - r$, then $n = i - m < i - (i - r) = r$. So $B_m (C^{[r]})_n = B_m (CA^{n-r}) = B_m C \leqslant B_{i-r} C$. If $m < i - r$, then $n = i - m > i - (i - r) = r$. So $B_m (C^{[r]})_n = B_m CA^{n-r} \leqslant B_{m+n-r} C = B_{i-r} C$. If $m = i - r$, then $n = r$ and $B_m (C^{[r]})_n = B_{i-r} C$. So $B_{i-r} C \leqslant \bigvee_{m+n=i} B_m (C^{[r]})_n \leqslant B_{i-r} C$. $\square$

Corollary 5.6. *Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $C, D \in \mathcal{L}$, then $C^{[r]}D^{[s]} = (CD)^{[r+s]}$.*

Proof. By Lemmas 5.4 and 5.5, $C^{[r]}D^{[s]}$ has ith coordinate $(C^{[r]})_{i-s}D = CA^{i-s-r}D = CDA^{i-(s+r)}$, which is the ith coordinate of $(CD)^{[r+s]}$. □

Lemma 5.7. *Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $C \in \mathcal{QR}$ and $B \in \mathcal{L}$, then $C : B^{[r]} = \sum_i (C_{i+r} : B)$.*

Proof. Let $D \in \mathcal{QR}$ satisfy $DB^{[r]} \leqslant C$. Then $D_{i-r}(B^{[r]})_r \leqslant C_i$ for each i. Thus $D_{i-r}B \leqslant C_i$ for each i. Thus $D_{i-r} \leqslant C_i : B$ for each i. Let $E_i = C_{i+r} : B$ for each i. Then $E_i \geqslant E_{i+1} \geqslant AE_i$ for each i. So $E = \sum_i E_i \in \mathcal{QR}$. Also by Lemma 5.5, $(EB^{[r]})_i = E_{i-r}B$ which is $(C_i : B)B \leqslant C_i$ for all i. Thus $E = C : B^{[r]}$. □

Theorem 5.8. *Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $B \in \mathcal{L}$ is principal, then $B^{[r]}$ is a principal element of $\mathcal{QR}$.*

Proof. Let $C, D \in \mathcal{QR}$. By Lemmas 5.5 and 5.7, $((C \wedge (D : B^{[r]})B^{[r]}))_i = (C \wedge (D : B^{[r]}))_{i-r}B = (C_{i-r} \wedge (D_i : B))B$ and since B is principal, this is $C_{i-r}B \wedge D_i = (CB^{[r]} \wedge D)_i$ for all i.

Similarly, $(((C \vee DB^{[r]}) : B^{[r]}))_i = (C_{i+r} \vee D_iB) : B$, and since B is principal this is $(C_{i+r} : B) \vee D_i = ((C : B^{[r]}) \vee D)_i$ for all i. □

Lemma 5.9. *Let $\mathcal{L}$ be a q.f. lattice and let $A \in \mathcal{I}$. If $B \in \mathcal{QR}$ then B is a join of principal elements of the form $D^{[r]}$ with $D \in \mathcal{L}$ principal.*

Proof. Observe that if $i < j$, then $(B_i^{[i]})_j = B_iA^{j-i} = (B_iA)A^{j-i-1} \leqslant B_{i+1}A^{j-i-1} = (B_{i+1}^{[i+1]})_j$. If $i \geqslant j$, then $(B_i^{[i]})_j = B_iA^{j-i} = B_iA^{j-i-1} \geqslant B_{i+1}A^{j-i-1} = (B_{i+1}^{[i+1]})_j$. So for fixed j, the jth coordinate $(B_i^{[i]})_j$ of $B_i^{[i]}$ is an increasing function of i for $i < j$. For $i = j$, $(B_i^{[i]})_j = B_j$. For $i \geqslant j$, $(B_i^{[i]})_j$ is a decreasing function of i. Therefore we have $B = \bigvee_i B_i^{[i]}$. Thus it suffices to show that if $B \in \mathcal{L}$, then $B^{[r]}$ is a join of principal elements of the form $D^{[r]}$ with D principal in $\mathcal{L}$. But this follows from Theorem 5.8. □

Theorem 5.10. *If $\mathcal{L}$ is a q.f. lattice and $A \in \mathcal{I}$, then $\mathcal{QR} = \mathcal{QR}(L, A)$ is a q.f. lattice with identity element $R^* = \sum_i A^i = R^{[0]}$ and integral elements $\mathcal{R}(L, A)$.*

Proof. By Lemma 5.2 $\mathcal{QR}$ is a cl-monoid with identity element $R^* = \sum_i A^i$. To see that $\mathcal{QR}$ is a q.f. lattice, let $K = \bigvee \mathcal{L}$. Then clearly the largest element of $\mathcal{QR}$ is the element $\sum_i K_i$ where $K_i = K$ for each $i \in \mathbb{Z}$. Also it is immediate that for each $B = \sum_i B_i \in \mathcal{QR} \setminus \{0\}$, $B \wedge R^* \neq 0$. By Lemma 5.9 $\mathcal{QR}$ is principally generated. Further, since $\mathcal{L}$ is a q.f. lattice, $\mathcal{L}$ contains a compact invertible element. It follows that the identity element R of $\mathcal{L}$ is compact [6, Proposition 1.2.1 and Lemma 1.26]. (See [16, Proposition 2.3 and the paragraph following it].) If $R^{[0]} \leqslant \bigvee\{B_\lambda \mid \lambda \in \Lambda\}$ for some family of $B_\lambda = \sum_i B_{\lambda i} \in \mathcal{QR}$, then $R \leqslant \bigvee\{B_{\lambda 0} \mid \lambda \in \Lambda\}$. It follows easily from this that $R^{[0]}$ is compact. Thus $\mathcal{QR}(L, A)$ is a q.f. lattice with integral elements $\mathcal{R}(L, A)$. □

Lemma 5.11. *Let $\mathcal{L}$ be a q.f. lattice with integral elements $\mathcal{I} = L$ and let $A \in \mathcal{I}$. If L is Noetherian and each L-principal element of L is $\mathcal{L}$-principal, then each $\mathcal{R}(L, A)$-principal element of $\mathcal{R}(L, A)$ is $\mathcal{QR}(L, A)$-principal.*

Proof. Let $B \in \mathcal{R} = \mathcal{R}(L, A)$ be $\mathcal{R}$-principal. We first show that B is $\mathcal{QR}$-meet principal. That is $(C \wedge (D : B))B = CB \wedge D$ for each $C, D \in \mathcal{QR}$. Since B is compact, it suffices, by the usual properties of localization (summarized in [16, Proposition 3.1]), to show that $((C \wedge (D : B))B)_P = (CB \wedge D)_P$ for each maximal element $P \in \mathcal{R}$.

By [11, Theorem 2.10], $B = {B_1}^{[f(1)]} \vee {B_2}^{[f(2)]} \vee \cdots \vee {B_n}^{[f(n)]}$ for some $n \in \mathbb{N}$ where each B_i is $\mathcal{I}$-principal and thus, by hypothesis, $\mathcal{L}$-principal. Therefore by Lemma 5.9, each $B_i^{[f(i)]}$ is $\mathcal{QR}$-principal. Let P be a maximal element of $\mathcal{R}$. Since $B \in \mathcal{R}$ is $\mathcal{R}$-principal, B_P is $\mathcal{I}(R_P^*)$-principal by the usual properties of localization ([6, Propositions 2.4, 2.12 and 2.14] or [16, Proposition 3.1(3)]). Also each $(B_i^{[f(i)]})_P$ is $\mathcal{QR}(\mathcal{R}_P)$-principal by Proposition 2.4. Since $\mathcal{I}(R_P^*)$ is local, $B_P \in \mathcal{I}(R_P^*)$ is principal and $B_P = ({B_1}^{[f(1)]})_P \vee ({B_2}^{[f(2)]})_P \vee \cdots \vee ({B_n}^{[f(n)]})_P$, then by [11, proof of Theorem 2.10], $B_P = (B_i^{[f(i)]})_P$ for some i. Then $((C \wedge (D : B))B)_P = (C_P \wedge (D_P : B_P))B_P = (C_P \wedge (D_P : (B_i^{[f(i)]})_P))(B_i^{[f(i)]})_P$, and since $(B_i^{[f(i)]})_P$ is $\mathcal{QR}(\mathcal{R}_P)$-principal, this is equal to $C_P(B_i^{[f(i)]})_P \wedge D_P = C_P B_P \wedge D_P = (CB \wedge D)_P$. Since this holds for each maximal element P of $\mathcal{R}(L, A)$, B is $\mathcal{QR}$-meet principal. To see that B is $\mathcal{QR}$-join principal, we could use a similar argument, or use the fact that principal and meet principal are equivalent for cl-monoids. (See [6, Proposition 1.21] or [16, Proposition 2.3].) Thus each $\mathcal{R}(L, A)$-principal element is $\mathcal{QR}$-principal. □

If L is a Noetherian multiplicative lattice, then each element of L is compact. Thus if L has no zero-divisors, then, as noted prior to Example 2.6, there exists a q.f. lattice $\mathcal{L}$ such that L is isomorphic to the set $\mathcal{I}$ of integral elements of $\mathcal{L}$ and such that each $\mathcal{I}$-principal element of $\mathcal{I}$ is $\mathcal{L}$-principal. The following shows that the corresponding q.f. lattice for $\mathcal{R}(L, A)$ can be taken to be $\mathcal{QR}(L, A)$.

Theorem 5.12. *Let L be a Noetherian multiplicative lattice without zero-divisors, let $f : L \to \mathcal{L}$ be the embedding of L into a q.f. lattice $\mathcal{L}$ mentioned above, and let $f^* : \mathcal{R} \to \mathcal{L}^*$ be the corresponding embedding of $\mathcal{R}(L, A)$ into a quotient field lattice $\mathcal{L}^*$. Then f^* extends to an isomorphism $F : \mathcal{QR}(L, A) \to \mathcal{L}^*$.*

Proof. Since L is modular by hypothesis, $\mathcal{L}$ modular by [16, Proposition 3.4]. Thus $\mathcal{QR}$ is modular by Lemma 5.2. But this condition implies that each $\mathcal{QR}$-principal element of $\mathcal{R}$ is $\mathcal{R}$-principal, by [16, Proposition 3.3(1)]. Conversely, each $\mathcal{R}$-principal element of $\mathcal{R}$ is $\mathcal{QR}$-principal by Lemma 5.11. Thus by the uniqueness result [6, Proposition 5.28] referred to prior to Example 2.6, f^* extends to an isomorphism $F : \mathcal{QR}(L, A) \to \mathcal{L}^*$ of q.f. lattices. □

Lemma 5.13. *Let $\mathcal{L}$ be a q.f. lattice with $\mathcal{I}$ Noetherian and let $A \in \mathcal{I}$. Let $u = R^{[-1]} \in \mathcal{QR}(\mathcal{I}, A)$ and let $t = u^{-1}$. Then*

(i) $(R^{[0]})_a = ((R_a)^{[0]})_a = \sum_i (A^i R_a)_a = \sum_i D_i$, *where* $D_i = (A^i R_a)_a$.
(ii) $(R^{[0]})_a \wedge R^{[0]}[t] = \sum_i (A^i)_a = \sum_i D_i$, *where* $D_i = (A^i)_a$.

Proof. (i) For the first equality, observe that if $B \in \mathcal{L}$ is compact and $B \leqslant R_a$ then $B^n \leqslant R \vee B \vee \cdots \vee B^{n-1}$ for some n by Corollary 3.7. Then by Corollary 5.6 and Lemma 5.4 we have $(B^{[0]})^n = (B^n)^{[0]} \leqslant (R \vee B \vee B^2 \vee \cdots \vee B^{n-1})^{[0]} = R^{[0]} \vee B^{[0]} \vee (B^2)^{[0]} \vee \cdots \vee (B^{n-1})^{[0]} = R^{[0]} \vee B^{[0]} \vee (B^{[0]})^2 \vee \cdots \vee (B^{[0]})^{n-1}$. So $B^{[0]} \leqslant (R^{[0]})_a$. Since this holds for each compact $B \leqslant R_a$, we get $(R_a)^{[0]} \leqslant (R^{[0]})_a$, and therefore $((R_a)^{[0]})_a = (R^{[0]})_a$.

Let $C = AR_a$ and suppose $B \leqslant (A^k R_a)_a = (C^k)_a$ with $B \in \mathcal{L}$ compact. Then $B^n \leqslant C^k(B \vee C^k)^{n-1} = A^k(B \vee A^k)^{n-1} R_a$ for some $n \in \mathbb{N}$. So, using Corollary 5.6, we get $(B^{[k]})^n = (B^n)^{[nk]} \leqslant (A^k(B \vee A^k)^{n-1} R_a)^{[nk]} = (A^k)^{[k]}[(B \vee A^k)^{n-1}]^{[(n-1)k]}(R_a)^{[0]} = (A^k)^{[k]}[(B \vee A^k)^{[k]}]^{n-1}(R_a)^{[0]}$, and by Lemma 5.4 this is $(A^k)^{[k]}[(B^{[k]} \vee (A^k)^{[k]}]^{n-1}(R_a)^{[0]} = (A^{[1]})^k[(B^{[k]}) \vee (A^{[1]})^k)]^{n-1}(R_a)^{[0]} \leqslant (R_a)^{[0]} \vee (B^{[k]}) \vee (B^{[k]})^2 \vee \cdots \vee (B^{[k]})^{n-1}$. So $B^{[k]} \leqslant ((R_a)^{[0]})_a = (R^{[0]})_a$ for each $B \leqslant A^k R_a$. So if $B = \sum_i B_i \leqslant \sum_i (A^n R_a)_a$, then $B = \bigvee_{i \in \mathbb{Z}} B_i^{[i]} \leqslant (R^{[0]})_a$.

Conversely suppose $B^{[k]} \leqslant (R^{[0]})_a$ for $B \in \mathcal{L}$ compact. Then for some $n \in \mathbb{N}$,

$$\begin{aligned}\left(B^n\right)^{[kn]} = \left(B^{[k]}\right)^n &\leqslant R^{[0]} \vee \left(B^{[k]}\right)^1 \vee \left(B^{[k]}\right)^2 \vee \cdots \vee \left(B^{[k]}\right)^{n-1} \\ &= R^{[0]} \vee B^{[k]} \vee \left(B^2\right)^{[2k]} \vee \cdots \vee \left(B^{n-1}\right)^{[k(n-1)]}.\end{aligned}$$

Now we consider the knth components. In general, the jth component of $E^{[i]}$ is EA^{j-i}. So the knth component of $(B^s)^{[ks]}$ is $B^s A^{kn-ks} = B^s A^{k(n-s)}$. Thus comparing knth components, we get

$$\begin{aligned}B^n &\leqslant A^{nk} \vee BA^{(n-1)k} \vee B^2 A^{(n-2)k} \vee \cdots \vee B^{n-1} A^k \\ &= A^k\left(A^{(n-1)k} \vee BA^{(n-2)k} \vee B^2 A^{(n-3)k} \vee \cdots \vee B^{n-1}\right) \\ &= A^k\left(A^k \vee B\right)^{n-1} \leqslant (AR_a)^k\left((AR_a)^k \vee B\right)^{n-1}.\end{aligned}$$

Thus $B \leqslant ((AR_a)^k)_a$. Thus if $B = \sum_i B_i \in \mathcal{QR}$ with $B = \bigvee_{i \in \mathbb{Z}} B_i^{[i]} \leqslant (R^{[0]})_a$, then $B_i \leqslant (A^i R_a)_a$.

Part (ii) follows from a similar argument or by using part (i) and the fact that $(A^i R_a)_a \wedge R = (A^i)_a$. $\square$

6. Rees valuations

In this section, we extend the main results of [21,23] from rings to multiplicative lattices. We begin by recalling some definitions and results from [12]. Let L be a multiplicative lattice with ACC and with largest element R, the identity element of L. Let $G = \mathbb{R} \cup \{\infty\}$ with the conventions $y \leqslant \infty$, $y + \infty = \infty + y = y \cdot \infty = \infty \cdot y = \infty$ and $y/\infty = 0$ for each $y \in \mathbb{R}$.

Definition 6.1. Let $v : L \to G$ be a map. Consider the following properties.

(1) $v(0) = \infty$.
(2) $v(R) = 0$.
(3) $v(AB) \geqslant v(A) + v(B)$ for all $A, B \in L$.
(4) $v(A \vee B) \geqslant \min\{v(A), v(B)\}$ for all $A, B \in L$.
(5) $v(A^n) = nv(A)$ for all $A \in L$ and $n \in \mathbb{N}$.
(6) $v(AB) = v(A) + v(B)$ for all $A, B \in L$.

If v satisfies (1)–(4), then v is said to be a **pseudo-valuation** on L. A pseudo-valuation v on L which satisfies (5) is said to be **homogeneous**, and if v satisfies (6), v is said to be a **valuation** on L.

Definition 6.2. If $A \in L$, define $v_A : L \to G$ by $v_A(B) = \infty$ if $B \leqslant A^n$ for all $n \in \mathbb{N}_0$, and $v_A(B) = m$ if $B \leqslant A^m$, $B \nleqslant A^{m+1}$ (where $A^0 = R$).

Then, as noted in [12, p. 236], a straightforward generalization of the argument given in [21] shows that if L is a multiplicative lattice, then v_A is a pseudo-valuation and $\bar{v}_A(B) = \lim_{n\to\infty} v_A(B^n)/n$ exists in G for all $A, B \in L$. In the case that L is the set $\mathcal{I}$ of integral elements of a q.f. lattice $\mathcal{L}$, $L = \mathcal{I}$ is a Noetherian valuation lattice and $A < R$ is the unique maximal element of L, then v_A is a valuation called **the valuation associated to** L or to the V-ring element R.

Lemma 6.3. [12, Lemma 1] *If L is a multiplicative lattice with ACC and $A \in L$, then the function $\bar{v}_A : L \to G$ is a homogeneous pseudo-valuation on L.*

If L is a multiplicative lattice, a map $A \mapsto A_x$ on L is said to be a **semiprime operation** if it satisfies the following three conditions for all $A, B \in L$: (i) $A \leqslant A_x$, (ii) if $A \leqslant B_x$, then $A_x \leqslant B_x$, and (iii) $A_x B_x \leqslant (AB)_x$. Some immediate formal consequences of the above three conditions are (iv) $(A_x)_x = A_x$, (v) $R_x = R$, (vi) $(A_x B_x)_x = (AB)_x$, and for any family $\{A_i \mid i \in I\} \subseteq L$, (vii) $(\bigvee\{A_i \mid i \in I\})_x = (\bigvee\{(A_i)_x \mid i \in I\})_x$ and (viii) $\bigwedge\{(A_i)_x \mid i \in I\} = (\bigwedge\{(A_i)_x \mid i \in I\})_x$.

Definition 6.4. Let L be a multiplicative lattice with ACC, and let $A \in L$. If $A \neq R$, let $A_s = \bigvee\{B \in L \mid \bar{v}_A(B) \geqslant 1\}$ and let $R_s = R$. The mapping $A \mapsto A_s$ is called the **AC-operation** on L.

It is shown in [12, Lemma 3] that if L is a multiplicative lattice satisfying the ascending chain condition, then the AC-operation on L is a semiprime operation. It follows from this and the next theorem that if L is a Noetherian multiplicative lattice, then the map $A \mapsto A_a$ is a semiprime operation on L.

Theorem 6.5. [12, Theorem 3] *Let L be a Noetherian multiplicative lattice. For each $A \in L$, $A_a = A_s$. Thus $B \leqslant A_a$ if and only if $\bar{v}_A(B) \geqslant 1$.*

We will need the following stronger form of Theorem 6.5.

Corollary 6.6. *Let L be a Noetherian multiplicative lattice. For each $A, B \in L$ and $n \in \mathbb{N}$, $B \leqslant (A^n)_a$ if and only if $\bar{v}_A(B) \geqslant n$.*

Proof. By [12, Lemma 3(c)], $\bar{v}_{A^n}(B) = \bar{v}_A(B)/n$. Thus, by Theorem 6.5, we have $B \leqslant (A^n)_a$ if and only if $\bar{v}_{A^n}(B) \geqslant 1$ if and only if $\bar{v}_A(B)/n \geqslant 1$ if and only if $\bar{v}_A(B) \geqslant n$. $\square$

The following results 6.7 and 6.9 are lattice versions of [13, 11.3 and 11.4].

Lemma 6.7. *Let $\mathcal{L}$ be a q.f. lattice such that $\mathcal{I}$ is a Krull lattice, and let $u \in \mathcal{I} \setminus \{R\}$ be nonzero and $\mathcal{L}$-principal. Let $P_1, \ldots, P_r$ be the height one prime elements P of $\mathcal{I}$ such that $u \leqslant P$. For $i = 1, \ldots, r$, let v_i denote the valuation associated to the valuation lattice $\mathcal{I}(R_{P_i})$, and let $e_i = e_i(u)$ denote the positive integer $v_i(u)$. Then for each $B \in \mathcal{I}$, $\bar{v}_u(B) = \min\{v_i(B)/e_i \mid i = 1, \ldots, r\}$.*

Proof. By renumbering, we may assume $v_1(B)/e_1 \leqslant v_i(B)/e_i$ for each i. Let $k = v_u(B)$. From $B \leqslant u^k$, we get $v_1(B) \geqslant v_1(u^k) = ke_1 = v_u(B)e_1$, and hence $v_1(B)/e_1 \geqslant v_u(B)$. Applying this to B^n, we get $v_1(B^n)/e_1 \geqslant v_u(B^n)$. Dividing by n and letting n go to infinity we get $v_1(B)/e_1 \geqslant \bar{v}_u(B)$.

For the opposite inequality, let $m = \lfloor v_1(B)/e_1 \rfloor$ (= the greatest integer $\leqslant v_1(B)/e_1$). Then for each i we have $v_i(u^m) = me_i \leqslant (v_1(B)/e_1) \cdot e_i \leqslant v_i(B)$. Since u is principal and $P_1, \ldots, P_r$ are the height one prime elements P of the Krull lattice $\mathcal{I}$ such that $u \leqslant P$, this implies $B \leqslant u^m$ by [6, Proposition 4.24 and Theorem 4.27]. So $k = v_u(B) \geqslant v_u(u^m) = m > (v_1(B)/e_1) - 1$. That is

$$v_u(B) \geqslant \lfloor v_1(B)/e_1 \rfloor > \big(v_1(B)/e_1\big) - 1.$$

Applying this to B^n, we get

$$v_u\big(B^n\big) \geqslant \lfloor v_1\big(B^n\big)/e_1 \rfloor > \big(v_1\big(B^n\big)/e_1\big) - 1.$$

Dividing by n and letting n go to infinity, we get $\bar{v}_u(B) \geqslant v_1(B)/e_1$. $\square$

Lemma 6.8. *Let $\mathcal{L}$ be a q.f. lattice such that $\mathcal{I}$ is Noetherian. Then $A_a = R \wedge (\bigwedge\{AV \mid V$ is a V-ring element of $R\})$ for each $A \leqslant R$.*

Proof. Let $\widehat{A} = R \wedge (\bigwedge\{AV \mid V$ is a V-ring element of $\mathcal{L}\})$. To see that $A_a \leqslant \widehat{A}$, let $x \leqslant A_a$ with x principal. Then $(x \vee A)^n = A(x \vee A)^{n-1}$. Let V be a V-ring element of $\mathcal{L}$. If $xV \nleqslant AV$, then $AV \leqslant xV$, and $(xV \vee AV)^n = AV(xV \vee AV)^{n-1}$ gives $x^nV = Ax^{n-1}V$ and we may cancel $x^{n-1}V$ to get $xV = AV$ by [6, Proposition 1.21] or [16, Proposition 2.3]. So $x \leqslant xV = AV$.

Let $x \leqslant \widehat{A}$ be principal and nonzero. Then x is invertible by [6, Proposition 1.21] or [16, Proposition 2.3]. Let $T = R[Ax^{-1}]$ and let $B = (Ax^{-1})T$.

Claim: $B = T$. If not, then by [5, Proposition 4.11], there exists a V-ring element V of $\mathcal{L}(T)$ such that $BV < V$. Since $x \leqslant AV$, $x \leqslant a_1t_1 \vee \cdots \vee a_mt_m$, $a_i \leqslant A$ and $t_i \leqslant V$ with the a_i and t_i $\mathcal{L}$-principal. Then $R \leqslant (a_1x^{-1})t_1 \vee \cdots \vee (a_mx^{-1})t_m \leqslant BV$ and thus $V = RV \leqslant BV$, a contradiction. Thus $B = (Ax^{-1})T = T$. Thus, since R is $\mathcal{L}$-compact and $R \leqslant T$, $R \leqslant \bigvee_{j=1}^n (Ax^{-1})^j$ for some n. Multiplying by x^n, we get $x^n \leqslant \bigvee_{j=1}^n (A^jx^{n-j}) = A(x \vee A)^{n-1}$. So $x \leqslant A_a$. $\square$

Lemma 6.9. *Let $\mathcal{L}$ be a q.f. lattice such that $\mathcal{I}$ is Noetherian. Let T be a ring element of $\mathcal{L}$ such that $T \leqslant R_a$. Let $B \in \mathcal{I}$ and let $C = BT$. Then for $A \in \mathcal{I}$, we have $\bar{v}_B(A) = \bar{v}_C(A)$.*

Proof. Since $B^n \leqslant C^n$ for each $n \in \mathbb{N}$, $\bar{v}_B(A) \leqslant \bar{v}_C(A)$.

Claim: $C^n \wedge R \leqslant (B^n)_a$. Indeed we have $T \leqslant R_a \leqslant V$ for each V-ring element V of R by Theorem 3.6. Thus $B^nT \leqslant B^nV$ for each V-ring element V of R. So $C^n \wedge R = B^nT \wedge R \leqslant B^nV \wedge R$ for each V-ring element V of R. Thus $C^n \wedge R \leqslant (B^n)_a$ by Lemma 6.8.

Let $\beta < \bar{v}_C(A)$ be rational. Then there exist large $n \in \mathbb{N}$ such that $v_C(A^n)/n > \beta$ with $n\beta \in \mathbb{N}$. Then $A^n \leqslant C^{n\beta} \wedge R \leqslant (B^{n\beta})_a$. By Corollary 6.6, $\bar{v}_B(A^n) \geqslant n\beta$. Thus $\bar{v}_B(A) = (\bar{v}_B(A^n))/n \geqslant \beta$. Since this holds for any $\beta < \bar{v}_C(A)$, we get $\bar{v}_B(A) \geqslant \bar{v}_C(A)$. $\square$

As noted prior to Lemma 3.9, if L is a multiplicative lattice with maximal element R and $D \in L$ then $[D, R] = L/D$ is a multiplicative lattice with multiplication $A \circ B = AB \vee D$ [9]. In this case if $A \in [D, R]$, we denote $A \in L$, when considered as an element of the lattice $[D, R]$ by A/D.

Lemma 6.10. (See [13, Lemma 3.6].) *Let L be a Noetherian multiplicative lattice with minimal prime elements $q_1, \dots, q_t$ and let $A, B \in L$. Let $A_i = (A \vee q_i)/q_i$ and $B_i = (B \vee q_i)/q_i$ in the lattice $L/q_i = [q_i, R]$. Then $A \leqslant (B)_a$ if and only if $A_i \leqslant (B_i)_a$ for each i.*

Proof. Suppose $a \in A$ is principal and $(a \vee q_i)/q_i = a_i \leqslant (B_i)_a$ for each i. Then there exists an $n \in \mathbb{N}$, such that $a_i^{n+1} \leqslant B_i(a_i \vee B_i)^n$ for all i. Thus $(B_i \vee a_i)^{n+1} = B_i(a_i \vee B_i)^n$. Here the multiplication is the multiplication $\circ$ defined above. That is we have

$$(B \vee a)^{n+1} \vee q_i = B(a \vee B)^n \vee q_i.$$

Write D for $B \vee a$. So $B \leqslant D$ and we have $D^{n+1} \vee q_i = BD^n \vee q_i$ for each i. We thus have $BD^n \leqslant D^{n+1} \leqslant BD^n \vee q_i$ for each i. Thus

$$D^{n+1} \leqslant D^{n+1} \wedge \left(BD^n \vee q_i\right) = BD^n \vee \left(D^{n+1} \wedge q_i\right) \leqslant D^{n+1},$$

where the equality is by modularity. Let $m \in \mathbb{N}$ be such that $(q_1 q_2 \cdots q_t)^m = 0$. Let $X = BD^n$ and $Y_i = D^{n+1} \wedge q_i$ for each i. Consider

$$\left(D^{n+1}\right)^{mt} = \left(\prod_{i=1}^{t}\left(BD^n \vee \left(D^{n+1} \wedge q_i\right)\right)^m\right) = \left(\prod_{i=1}^{t}(X \vee Y_i)\right)^m.$$

The terms of $(\prod_{i=1}^t (X \vee Y_i))^m$ consist of $(Y_1 Y_2 \cdots Y_t)^m$, which equals 0 since $Y_i \leqslant q_i$, and terms of the form $X^j Y_1^{j_1} \cdots Y_t^{j_t}$ where $j + j_1 + \cdots + j_1 = mt$ with $j_1 + \cdots + j_1 = mt - j < mt$. Thus, since $Y_i \leqslant D^{n+1}$ for each i, a typical term is

$$\begin{aligned} &\leqslant \left(BD^n\right)^j \left(D^{n+1}\right)^{mt-j} = \left(B^j D^{nj}\right) D^{(n+1)(mt-j)} \leqslant \left(BD^{nj+j-1}\right) D^{(n+1)(mt-j)} \\ &= BD^{nj+j-1+(n+1)(mt-j)} = BD^{(n+1)j-1+(n+1)(mt-j)} = BD^{(n+1)mt-1} \leqslant D^{(n+1)mt}. \end{aligned}$$

Thus $(D^{n+1})^{mt} = BD^{(n+1)mt-1} \leqslant D^{(n+1)mt}$. Thus $a^{(n+1)mt} \leqslant B(a \vee B)^{(n+1)mt-1}$ and hence $a \leqslant B_a$. Therefore $A \leqslant B_a$. The converse is clear. $\square$

Proposition 6.11. (See [13, Proposition 11.7].) *Let L be a Noetherian multiplicative lattice with minimal prime elements $q_1, \dots, q_t$ and let $A, B \in L$. Let $A_i = (A \vee q_i)/q_i$ and $B_i = (B \vee q_i)/q_i$ in the lattice L/q_i. Then $\bar{v}_A(B) = \min\{\bar{v}_{A_i}(B_i) \mid i = 1, \dots, t\}$.*

Proof. Let $\beta \leqslant \min\{\bar{v}_{A_i}(B_i) \mid i = 1, \dots, s\}$ with $\beta \in \mathbb{Q}$, and let $n \in \mathbb{N}$ be such that $n\beta \in \mathbb{N}$. Since the $\bar{v}_{A_i}$ are homogeneous, $n\beta \leqslant \min\{\bar{v}_{A_i}(B_i^n) \mid i = 1, \dots, s\}$. Then $B_i^n \leqslant (A_i^{n\beta})_a$ for each i by Corollary 6.6. But this implies that $B^n \leqslant (A^{n\beta})_a$ by Lemma 6.10. By Corollary 6.6, this gives $\bar{v}_A(B^n) \geqslant n\beta$. Thus $\bar{v}_A(B) = \bar{v}_A(B^n)/n \geqslant \beta$. This gives $\bar{v}_A(B) \geqslant \min\{\bar{v}_{A_i}(B_i) \mid i = 1, \dots, s\}$.

Now suppose $\beta < \bar{v}_A(B)$ with $\beta \in \mathbb{Q}$. Then there exist infinitely many $n \in \mathbb{N}$ such that $n\beta \in \mathbb{N}$ and $v_A(B^n)/n > \beta$. Thus $B^n \leqslant A^{n\beta}$. Thus for $i = 1, \dots, t$, $B_i^n \leqslant A_i^{n\beta}$, and hence $v_{A_i}(B_i^n) > n\beta$. Letting n go to infinity, we get $\bar{v}_{A_i}(B_i) \geqslant \beta$. Thus $\min\{\bar{v}_{A_i}(B_i) \mid i = 1, \dots, s\} \geqslant \bar{v}_A(B)$. $\square$

Theorem 6.12. *Let L be a Noetherian multiplicative lattice and let $A \in L \setminus \{R\}$. Then there exist discrete valuations $v_i : L \to G$, and positive integers $e_i = e_i(A)$, $i = 1, \dots, r$, such that $\bar{v}_A(B) = \min\{v_i(B)/e_i \mid i \in \{1, \dots, r\}\}$ for all $B \in L$.*

Proof. First assume that L is a Noetherian multiplicative lattice with no zero-divisors. Then the A-transform $\mathcal{R}(L,A)$ is also a Noetherian multiplicative lattice by [11, Theorem 2.11], has no zero-divisors by Lemma 4.4 and $\mathcal{R}(L,A)=\mathcal{R}$ is the lattice of integral elements of the q.f. lattice $\mathcal{QR}(L,A)=\mathcal{QR}$ by Theorem 5.10. Further, by Lemma 5.11, each $\mathcal{R}$-principal element is $\mathcal{QR}$-principal.

Let $u=R^{[-1]}$. Then $u\neq R^{[0]}$ is nonzero and is $\mathcal{QR}$-principal by Theorem 5.8. Further, by Theorem 3.12, $\mathcal{I}((R^{[0]})_a)$ is a Krull lattice and $u(R^{[0]})_a$ is $\mathcal{QR}((R^{[0]})_a)$-principal by Proposition 2.4. Let $P_1,P_2,\dots,P_r$ be the height one prime elements of $\mathcal{I}((R^{[0]})_a)$ such that $u(R^{[0]})_a\leqslant P_i$, and let w_i denote the valuation associated to the valuation lattice $\mathcal{I}(((R^{[0]})_a)_{P_i})$ for $i=1,\dots,r$. (See the paragraph preceding Lemma 6.3.) Also let $e_i=e_i(A)$ denote the positive integer $w_i(u)$ for $i=1,\dots,r$. Define $v_i:L\to\mathbb{N}_0\cup\{\infty\}$ by $v_i(B)=w_i(B^{[0]}(R^{[0]})_a)$. If $B,C\in L$, then by Proposition 4.3, we have $v_i(B\vee C)=w_i((B\vee C)^{[0]}(R^{[0]})_a)=w_i((B^{[0]}\vee C^{[0]})(R^{[0]})_a)=w_i(B^{[0]}(R^{[0]})_a\vee C^{[0]}(R^{[0]})_a)\geqslant\min\{w_i(B^{[0]}(R^{[0]})_a),w_i(C^{[0]}(R^{[0]})_a)\}=\min\{v_i(B),v_i(C)\}$, and similarly $v_i(BC)=v_i(B)+v_i(C)$. It follows that the map $v_i:L\to\mathbb{N}_0\cup\{\infty\}$ is a valuation on L and $v_i(B)=\infty$ if and only if $B=0$. It also follows that if $B\in L$, then $B\leqslant A^n$ in L if and only if $B^{[0]}\leqslant u^n$ in $\mathcal{R}$. Indeed $B\leqslant A^n$ in L if and only if $BA^i\leqslant A^{n+i}$ in L for all $i\in\mathbb{Z}$, where $A^j=R$ if $j\leqslant 0$, if and only if $B^{[0]}=\sum BA^i\leqslant\sum_i A^{n+i}=u^n$ in $\mathcal{R}$. Thus $v_A(B)=v_u(B^{[0]})$ for each $B\in L$. Thus $\bar{v}_A(B)=\bar{v}_u(B^{[0]})$ for each $B\in L$. By Lemma 6.9, $\bar{v}_u(B^{[0]})=\bar{v}_{u(R^{[0]})_a}(B^{[0]}(R^{[0]})_a)$ for each $B\in L$. By Lemma 6.7, $\bar{v}_{u(R^{[0]})_a}(B^{[0]}(R^{[0]})_a)=\min\{w_i(B^{[0]}(R^{[0]})_a)/e_i\mid i=1,\dots,r\}=\min\{v_i(B)/e_i\mid i=1,\dots,r\}$. Thus $\bar{v}_A(B)=\min\{v_i(B)/e_i\mid i=1,\dots,r\}$ for each $B\in L$.

Now let L be a Noetherian multiplicative lattice, possibly containing zero-divisors, let $A\in L\setminus\{R\}$ and let $q_1,\dots,q_t$ be the minimal prime elements of L such that $A\vee q_i\neq R$. Let $A_i=(A\vee q_i)/q_i$ and for each $B\in L$ let $B_i=(B\vee q_i)/q_i$ in the lattice L/q_i. Let $\mathcal{R}_i$ be the A_i-transform $\mathcal{R}(L/q_i,A_i)$. By the above case, for each $i\in\{1,2,\dots,t\}$ there exists a set $\{w_{i1},w_{i2},\dots,w_{ir_i}\}$ of valuations on $\mathcal{R}_i$ such that if $e_{ij}=w_{ij}(u_i)$, $u_i=R_i^{[-1]}\in\mathcal{R}_i=\mathcal{R}(L/q_i,A_i)$, and $h_{ij}:L/q_i\to\mathbb{N}_0\cup\{\infty\}$ is defined by $h_{ij}(B)=w_{ij}(B^{[0]}(R_i^{[0]})_a)$, then $\bar{v}_{A_i}(B_i)=\min\{h_{ij}(B)/e_{ij}\mid j\in\{1,2,\dots,r_i\}\}$ for each $B_i\in L/q_i$. For each i and j define $v_{ij}:L\to\mathbb{N}_0\cup\{\infty\}$ by $v_{ij}(B)=h_{ij}(B_i)$. Then $v_{ij}(B)=\infty$ if and only if $B\leqslant q_i$, and by Proposition 6.11, $\bar{v}_A(B)=\min\{\bar{v}_{A_i}(B_i)\mid i=1,\dots,t\}$ for each $B\in L$. Thus by applying the above to the pseudo-valuations $\bar{v}_{A_i}$ on L/q_i for $i=1,\dots,t$, we see that $\bar{v}_A(B)=\min\{v_{ij}(B)/e_{ij}\mid i\in\{1,\dots,t\},\ j\in\{1,\dots,r_i\}\}$ for each $B\in L$. $\square$

Recall that if $\mathcal{L}$ is a q.f. lattice and (V,M) is a Noetherian V-ring element of $\mathcal{L}$, the valuation v on $\mathcal{L}$ **associated to** (V,M) is defined by $v(X)=n$ where $XV=M^n$, for $X\in\mathcal{L}$ and $n\in\mathbb{Z}$. The element $R\wedge M\in\mathcal{I}$ is called the **center of** V, or **of** v, **on** $\mathcal{I}$. It is clearly a prime element of $\mathcal{I}$.

In the following it will be useful to know that, with the notation of the above theorem and proof, if $A\nleqslant q_i$ for a minimal prime q_i, then $e_{ij}=v_{ij}(A)$ for $j\in\{1,2,\dots,r_i\}$. To show this we use the following lemma which is adapted from [19, Proposition 3.6].

Lemma 6.13. *Let $\mathcal{L}$ be a q.f. lattice such that $L=\mathcal{I}$ is Noetherian, let $A\in\mathcal{I}$ and let w be a discrete valuation on $\mathcal{QR}(L,A)$ which is nonnegative on $\mathcal{I}((R^{[0]})_a)=\{B(R^{[0]})_a\mid B\in\mathcal{QR},\ B\leqslant(R^{[0]})_a\}$. If the center P of w on $\mathcal{I}((R^{[0]})_a)$ is a height one prime element of $\mathcal{I}((R^{[0]})_a)$ and $w(u)>0$, then $w(A^{[1]})=0$.*

Proof. If $B \in \mathcal{L}$, then $u^n B^{[n]} = R^{[-n]} B^{[n]} = B^{[0]}$. So if we let $t = u^{-1}$ in $\mathcal{L}^*$, then $B^{[n]} = B^{[0]} t^n$. The valuation w induces a valuation v on L defined by $v(B) = w(B^{[0]})$ for $B \in \mathcal{L}$. We call v **the restriction** of w to $\mathcal{L}$. Then $w(B^{[n]}) = w(B^{[0]} t^n) = v(B) + nw(t)$. The lattice $\mathcal{I}((R^{[0]})_a)$ is a Krull lattice by Theorem 3.12. Let $\alpha = -v(A)$ and define a valuation $w' : \mathcal{I}((R^{[0]})_a) \to \mathbb{Z} \cup \{\infty\}$ by $w'(\sum_i B_i) = \inf\{v(B_i) + i\alpha \mid i \in \mathbb{Z}\} = \inf\{w(B_i^{[0]}) + i\alpha \mid i \in \mathbb{Z}\}$. Observe that if $B = \sum_i B_i \in \mathcal{I}((R^{[0]})_a)$, then $B_i \leqslant (A^i R_a)_a$, by Lemma 5.13, and thus $v(B_i) + i\alpha \geqslant v(A^i R_a)_a + i\alpha = iv(A) + i\alpha = i(v(A) + \alpha) = 0$. Therefore w' is well defined on $\mathcal{I}((R^{[0]})_a)$.

Suppose for the moment that w' is a valuation on $\mathcal{I}((R^{[0]})_a)$. Then w' extends to a valuation on $\mathcal{QR}$ and if $w(A^{[1]}) > 0$, we get $v(A) = w(A^{[0]}) = w(uA^{[1]}) = w(u) + w(A^{[1]}) > w(u) = -w(t)$. So if $n \geqslant 0$ and $B \in \mathcal{L}$, then $w'(B^{[n]}) = v(B) + n\alpha = w(B^{[0]}) - nv(A) < w(B^{[0]}) + nw(t) = w(B^{[n]})$. If $B \leqslant R_a$ and $n < 0$, then $w(B^{[n]}) > 0$. It follows that the center P' of w' on $\mathcal{I}((R^{[0]})_a) \leqslant$ the center P of w on $\mathcal{I}((R^{[0]})_a)$. But since $P' \leqslant P$ and these are height one prime elements of $\mathcal{I}((R^{[0]})_a)$, the corresponding V-ring elements V' and V of w' and w are equal. Since $w'(A^{[0]}) = w(A^{[0]}) = v(A) > 0$, it follows that $w' = w$. But since $w'(A^{[1]}) = w(A^{[0]}) - w(A^{[0]}) = 0$, this contradicts the assumption $w(A^{[1]}) > 0$.

It remains to show that w' is a valuation on $\mathcal{I}((R^{[0]})_a)$. To see this let $B = \sum_i B_i$ and $C = \sum_i C_i \in \mathcal{I}((R^{[0]})_a)$. To show that $w'(B \vee C) \geqslant \min\{w'(B), w'(C)\}$, observe that

$$
\begin{aligned}
w'(B \vee C) &= \min\bigl\{v(B_i \vee C_i) + i\alpha \mid i \in \mathbb{Z}\bigr\} \geqslant \min\bigl\{\min\bigl\{v(B_i), v(C_i)\bigr\} + i\alpha \mid i \in \mathbb{Z}\bigr\} \\
&= \min\bigl\{\min\bigl\{v(B_i) + i\alpha, v(C_i) + i\alpha\bigr\} \mid i \in \mathbb{Z}\bigr\} \\
&= \min\bigl\{\min\bigl\{v(B_i) + i\alpha \mid i \in \mathbb{Z}\bigr\}, \min\bigl\{v(C_i) + i\alpha \mid i \in \mathbb{Z}\bigr\} = \min\bigl\{w'(B), w'(C)\bigr\}\bigr\}.
\end{aligned}
$$

To show that $w'(BC) = w'(B) + w'(C)$, let $r, s \in \mathbb{Z}$ be such that

$$
\begin{aligned}
v(B_r) + r\alpha = w'(B) < v(B_i) + i\alpha \quad &\text{for } i \leqslant r - 1, \quad \text{and} \\
v(C_s) + s\alpha = w'(C) < v(C_i) + i\alpha \quad &\text{for } i \leqslant s - 1.
\end{aligned}
$$

Let $BC = D = \sum_i D_i$. So $D_n = \bigvee_i B_i C_{n-i}$. Then

$$
\begin{aligned}
v(D_{r+s}) + (r+s)\alpha &= v\Biggl(\bigvee_{i \leqslant r+s} B_i C_{r+s-i}\Biggr) + (r+s)\alpha \\
&= v(B_r) + v(C_s) + r\alpha + s\alpha = w'(B) + w'(C).
\end{aligned}
$$

Thus $w'(BC) \leqslant w'(B) + w'(C)$. But, using 6.1(4), for any $t \in \mathbb{Z}$ we have

$$
\begin{aligned}
v(D_t) + t\alpha &= v\Biggl(\bigvee_{i \leqslant t} B_i C_{t-i}\Biggr) + t\alpha \geqslant v(B_i C_{t-i}) + t\alpha = v(B_i) + v(C_{t-i}) + t\alpha \\
&= \bigl(v(B_i) + i\alpha\bigr) + \bigl(v(C_{t-i}) + (t-i)\alpha\bigr) \geqslant w'(B) + w'(C) \\
&= v(B_r) + v(C_s) + r\alpha + s\alpha.
\end{aligned}
$$

Thus $w'(BC) = w'(B) + w'(C)$. Therefore w' is a valuation on $\mathcal{I}((R^{[0]})_a)$. $\square$

Corollary 6.14. *Let L be a Noetherian multiplicative lattice, let $A \in L \setminus \{R\}$ and let $q_1, \ldots, q_t$ be the minimal prime elements of L such that $A \vee q_i \neq R$. For each $i \in \{1, 2, \ldots, t\}$ let $\{v_{i\,1}, v_{i\,2}, \ldots, v_{i\,r_i}\}$ be the family of those valuations, given in the proof of Theorem* 6.12, *satisfying ${v_{i\,j}}^{-1}(\infty) = q_i$. Let $\{e_{i\,1}, e_{i\,2}, \ldots, e_{i\,r_i}\}$ be the corresponding set of positive integers.*

(a) *If $A \nleqslant q_i$, then $e_{i\,j} = w_{i\,j}((A \vee q_i)/q_i) = v_{i\,j}(A)$ for $j = 1, 2, \ldots, r_i$.*
(b) *If $A \leqslant q_i$, then $r_i = 1$, $e_{i\,1} = 1$ and $w_{i\,1}$ is the trivial valuation defined by $w_{i\,1}(B) = 0$ if $B \nleqslant q_i$ and by $w_{i\,1}(B) = \infty$ if $B \leqslant q_i$. In particular $e_{i\,1} = 1 \neq \infty = v_{i\,1}(A)$ in this case.*

Proof. For (a), if $A \nleqslant q_i$, then in the notation of the proof of Theorem 6.12, $0 \neq v_{i\,j}(A) = h_{i\,j}(A_i) = w_{i\,j}(A_i^{[0]}) = w_{i\,j}(u_i A_i^{[1]}) = w_{i\,j}(u_i) + w_{i\,j}(A_i^{[1]})$, which by Lemma 6.13 is $w_{i\,j}(u_i) = e_{i\,j}(A)$.

For (b), if $A \leqslant q_i$, then the A-transform $\mathcal{R}_i = \mathcal{R}(L/q_i, A \vee q_i/q_i) = \mathcal{R}(L/q_i, 0)$ is defined to be the set of all formal sums $\sum_{j=-\infty}^{\infty} B_j$, $B_j \in L/q_i$, such that $B_j = 0$ for $j > 0$ and $B_j \geqslant B_{j+1}$ for each j. By Lemma 5.13, the a-closure $((R_i)^{[0]})_a$ of the identity element $(R_i)^{[0]} \in \mathcal{QR}(L/q_i, 0)$ is $\sum_i D_i$ where $D_i = (R_i)_a$ if $i \leqslant 0$ and $D_i = 0$ for $i > 0$. It follows that $u((R_i)^{[0]})_a = R^{[-1]}((R_i)^{[0]})_a = \sum_i E_i$ where $E_i = (R_i)_a$ if $i < 0$ and $D_i = 0$ for $i \geqslant 0$, is a principal prime element of $\mathcal{I}(((R_i)^{[0]})_a)$. So $r_i = 1$, $e_{i\,1} = 1$. Further, if $B \in L$ and $B_i = (B \vee q_i)/q_i$, then $v_{i\,1}(B) = w_{i\,1}(B_i^{[0]}) = 0$ if $B \nleqslant q_i$ and $v_{i\,1}(B) = w_{i\,1}(B_i^{[0]}) = \infty$ if $B \leqslant q_i$. $\square$

Theorem 6.15. *Let L be a Noetherian multiplicative lattice without zero-divisors and let $A \in L \setminus \{R\}$. Let $v_1, \ldots, v_r$ and $e_1, \ldots, e_r$ denote the sequence of valuations and corresponding sequence of integers given in Theorem* 6.12. *If $r > 1$, then for each $j \in \{1, \ldots, r\}$ there exists $B(j) \in L$ such that $\bar{v}_A(B(j)) < \inf\{v_i(B(j))/e_i \mid i \in \{1, \ldots, r\} \setminus \{j\}\}$.*

Proof. Since $\mathcal{I}((R^{[0]})_a) = \{B(R^{[0]})_a \mid B \in \mathcal{QR},\ B \leqslant (R^{[0]})_a\}$ is a Krull lattice by Theorem 3.12, and $u(R^{[0]})_a$ is $\mathcal{QR}((R^{[0]})_a)$-principal by Proposition 2.4, then there are only finitely many prime elements $P_1, \ldots, P_r$ in $\mathcal{I}((R^{[0]})_a)$ which are minimal over $u(R^{[0]})_a$ and the ring elements $((R^{[0]})_a)_{P_j} = W_j$ of $\mathcal{QR}$ are Noetherian V-ring elements. Let w_j be the normalized valuation associated with W_j for $j = 1, \ldots, r$. Then, as in the proof of Lemma 6.13, if $B \in \mathcal{L}$, then $w_j(B^{[i]}) = w_j(B^{[0]}) - i w_j(u) = v_j(B) - i w_j(u)$ where v_j is the restriction of w_j to $\mathcal{L}$. Let V_j be the V-ring element of $\mathcal{L}$ associated to v_j for each j. By [6, Proposition 4.24 and Theorem 4.27], each divisorial element D of a Krull lattice is uniquely represented as a meet of symbolic powers of the minimal primes P of D: $DR_P \wedge R = P^{(v_P(D))} = \bigvee\{x \leqslant R \mid v_P(x) \geqslant v_P(D)\}$, as in the case of a Krull domain. But for $j \in \{1, \ldots, r\}$, an element $B = \sum_i B_i = \bigvee B_i^{[i]} \in \mathcal{I}((R^{[0]})_a)$ satisfies $B \leqslant (R^{[0]})_a \wedge u((R^{[0]})_a)_{P_j} = {P_j}^{(w_j(u))}$ $(= {P_j}^{(e_j)})$ if and only if $w_j(B_i^{[i]}) \geqslant w_j(u)$ for each i if and only if $v_j(B_i) - i w_j(u) \geqslant w_j(u)$ for each i if and only if $v_j(B_i) \geqslant (i+1)e_j$ for each i if and only if $B_i \leqslant A^{i+1} V_j$ for each i.

Since $u((R^{[0]})_a) = (R^{[0]})_a \wedge u((R^{[0]})_a)_{P_1} \wedge \cdots \wedge u((R^{[0]})_a)_{P_r} = {P_1}^{(e_1)} \wedge \cdots \wedge {P_r}^{(e_r)} < {P_2}^{(e_2)} \wedge \cdots \wedge {P_r}^{(e_r)} := J$, we have $J_i \nleqslant A^{i+1} V_1$ for some i. But since $J \in \mathcal{I}((R^{[0]})_a)$, we have $J = \sum_i J_i$, $J_i \in \mathcal{L}$ with $J_i \leqslant (A^i R_a)_a$ for each i. Since $\mathcal{L}$ is principally generated, there exists an $\mathcal{L}$-principal element $B \in \mathcal{L}$ such that $B \leqslant J_i$ but $B \nleqslant (A^{i+1} R_a)_a$. So $v_j(B) \geqslant (i+1)e_j$ for $j = 2, \ldots, r$ but $v_1(B) < (i+1)e_1$. So $\min\{v_j(B)/e_j \mid j = 2, \ldots, r\} > \min\{v_j(B)/e_j \mid j = 1, \ldots, r\}$. Also for $n = i+1$, we have $B \leqslant A^n V_j$ for $j = 2, \ldots, r$, $B \leqslant A^{n-1} V_1$, but $B \nleqslant A^n V_1$.

We would be done except that we do not have $B \leqslant R$. Since B is $\mathcal{L}$-principal and thus $\mathcal{L}$-compact, and since $B \leqslant R_a$, $R[B] = R \vee B \vee \cdots \vee B^n$ for some n, and thus there exists $C \leqslant R$ compact such that $CB^m \leqslant R$ for each $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $C \leqslant A^{k-1}V_1$ and $C \nleqslant A^k V_1$. So $v_1(C) < ke_1$. Since $B \nleqslant A^n V_1$, we have $v_1(B) < ne_1$. So $v_1(B) \leqslant ne_1 - 1$. Choose $m = ke_1$. From $v_1(B) \leqslant ne_1 - 1$, we get $v_1(B^m) \leqslant m(ne_1 - 1) = mne_1 - ke_1$. So $v_1(CB^m) < ke_1 + mne_1 - ke_1 = mne_1$. But $v_j(CB^m) \geqslant mv_j(B) \geqslant mne_j$ for $j = 2, \ldots, r$. Thus $v_1(CB^m)/e_1 < mn$, but $v_j(CB^m)/e_j \geqslant mv_j(B)/e_j \geqslant mn$ for $j = 2, \ldots, r$. □

Definition 6.16. Let L be a Noetherian multiplicative lattice and let $\bar{v} : L \to \mathbb{R} \cup \{\infty\}$ be a pseudo-valuation. If there exist valuations $v_i : L \to \mathbb{R} \cup \{\infty\}$ such that $\bar{v} = \min\{v_i \mid i = 1, \ldots, r\}$, then $\bar{v}$ is called a **sub-valuation**. If also $\bar{v} \neq \min\{v_i \mid i \in \{1, \ldots, r\} \setminus \{j\}\}$ for each $j \in \{1, \ldots r\}$, then the representation $\bar{v} = \min\{v_i \mid i = 1, \ldots, r\}$ is said to be **minimal**.

Proposition 6.17. *If L be a Noetherian multiplicative lattice, a sub-valuation $\bar{v} : L \to \mathbb{R} \cup \{\infty\}$ has a unique minimal representation $\bar{v} = \min\{v_i \mid i = 1, \ldots, r\}$.*

Proof. The following proof is an adaptation of the one given for rings by John Petro in [17].

Let $\bar{v} = \min\{v_i \mid i = 1, \ldots, r\}$ be a minimal representation of $\bar{v}$. By the minimality, there exists a principal $y \in L$ such that $v_1(y) < v_i(y)$ for $i = 2, \ldots, r$. (If $r = 1$, choose $y \in L$ such that $v_1(y) < \infty$.) Let $w : L \to \mathbb{R} \cup \{\infty\}$ be a valuation such that (1) $\bar{v} \leqslant w$, and (2) $v_1(y) = w(y)$.

Claim: $v_1 \leqslant w$. To see this let $x \in L$. To see that $v_1(x) \leqslant w(x)$, we may assume $w(x) < \infty$. Then for each $k \in \mathbb{N}$ there exists $i_k \in \{1, \ldots, r\}$ such that

$$\infty > w(xy^k) \geqslant \bar{v}(xy^k) = v_{i_k}(xy^k).$$

Then there exists an index $i \in \{1, \ldots, r\}$ and an increasing sequence $\{h_k \mid k \in \mathbb{N}\}$ such that $w(xy^{h_k}) \geqslant v_i(xy^{h_k})$. So $w(x) + h_k v_1(y) = w(x) + h_k w(y) = w(xy^{h_k}) \geqslant v_i(xy^{h_k}) = v_i(x) + h_k v_i(y)$. So $w(x) - v_i(x) \geqslant h_k[v_i(y) - v_1(y)]$ for all $k \in \mathbb{N}$. Since if $i > 1$, $v_i(y) - v_1(y) > 0$, we must have $i = 1$ in the above equation. So $w(x) \geqslant v_1(x)$, establishing the claim.

Now let $\bar{v} = \min\{w_i \mid i = 1, \ldots, s\}$ be another minimal representation of $\bar{v}$. Then, after possibly renumbering, there exists $y \in L$ such that $w_1(y) = \bar{v}(y) = v_1(y)$. Since $\bar{v} \leqslant w_1$, the above gives $v_1 \leqslant w_1$. By symmetry we get $w_1 \leqslant v_j$ for some j, and since the representation $\bar{v} = \min\{v_i \mid i = 1, \ldots, r\}$ is minimal, we must have $j = 1$. Similarly, $w_2 = v_j$ for some $j \geqslant 2$, which after renumbering the v_i, we may take $j = 2$, and so on. □

Definition 6.18. If $A \in L \setminus \{R\}$, and $\bar{v}_A = \min\{v_i/e_i \mid i \in \{1, \ldots, r\}\}$ is the representation given in Theorem 6.12, we call the valuations $v_i : L \to \mathbb{Z} \cup \{\infty\}$ the **Rees valuations of** A and the numbers e_i, the **Rees numbers of** A, $i = 1, \ldots, r$. We denote the set of Rees valuations of A by Rees A. By definition the set Rees A is naturally partitioned into subsets $\{v_{i\,1}, v_{i\,2}, \ldots, v_{i\,r_i}\}$, $i = 1, \ldots, t$, where ${v_{i\,j}}^{-1}(\infty) = q_i$ with $v_{i\,j}(B) = h_{i\,j}(B_i)$ where $B_i = (B \vee q_i)/q_i$ in L/q_i and $\{h_{i\,1}, h_{i\,2}, \ldots, h_{i\,r_i}\}$ are the Rees valuations of B_i for $i = 1, \ldots, t$. Further, if $A \leqslant q_i$ for some i, then by Corollary 6.14, $r_i = 1$ and $v_{i\,1}$ is a trivial valuation and the other members of Rees A are nontrivial. Thus in the following, we will refer to the trivial and nontrivial Rees valuations of A.

7. Rational powers and projective equivalence

Elements A and B of a Noetherian multiplicative lattice L are said to be **projectively equivalent** if $(A^m)_a = (B^n)_a$ for some m and $n \in \mathbb{N}$. It is immediate that this is an equivalence relation on elements of L. In order to generalize Ratliff's theorem on the sets $\mathrm{Ass}_R(R/(I^n)_a)$ to the sets, $\mathrm{Ass}_R(R/I_{n/m})$, $n, m \in \mathbb{N}$, some needed results on this topic are presented in this section for lattices. If L is a Noetherian multiplicative lattice, $\beta \in \mathbb{R}_+$ and $A \in L$, we let $A_\beta = \bigvee\{X \in L \mid \bar{v}_A(X) \geqslant \beta\}$. Unlike [14], we do not assume that A is a regular element of L. Throughout this section, L is a Noetherian multiplicative lattice.

Lemma 7.1. (See [14, Lemma 2.1].) *Let $A, B \in L$.*

(1) *For each $k \in \mathbb{N}$, $A_k = (A^k)_a$.*
(2) *If $n \in \mathbb{N}$ then $\bar{v}_A = n\bar{v}_{A^n}$.*
(3) *If $m, n \in \mathbb{N}$, then $(A^n)_a = (B^m)_a$ if and only if $m\bar{v}_A = n\bar{v}_B$. Thus if $i, j, k, l \in \mathbb{N}$ with $\frac{i}{j} = \frac{k}{l}$, then $(A^i)_a = (B^j)_a$ if and only if $(A^k)_a = (B^l)_a$.*
(4) *$((A_\beta)^m)_a \leqslant A_{m\beta}$ for each $m \in \mathbb{N}$.*
(5) *$\bar{v}_A \geqslant \beta\bar{v}_{A_\beta}$.*
(6) *A_β is integrally closed.*

Proof. Part (1) is Corollary 6.6.

Part (2) is [12, Lemma 3(c)].

For part (3), first observe $\bar{v}_A = \bar{v}_{(A_a)}$. Indeed for $p/q \in \mathbb{Q}$ and $X \in L$, we have $\bar{v}_A(X^q) \geqslant p$ if and only if $X^q \leqslant (A^p)_a = ((A_a)^p)_a$ if and only if $\bar{v}_{A_a}(X^q) \geqslant p$. That is $\bar{v}_A(X) \geqslant p/q$ if and only if $\bar{v}_{A_a}(X) \geqslant p/q$.

Part (3) now follows from part (2). Indeed by part (2), if $(A^n)_a = (B^m)_a$, then $\bar{v}_A/n = \bar{v}_{A^n} = \bar{v}_{(A^n)_a} = \bar{v}_{(B^m)_a} = \bar{v}_B/m$. Conversely, if $\bar{v}_A/n = \bar{v}_B/m$, then $m\bar{v}_A = n\bar{v}_B$ and $X \leqslant (A^n)_a$ if and only if $m\bar{v}_A(X) \geqslant mn$ if and only if $n\bar{v}_B(X) \geqslant mn$ if and only if $X \leqslant (B^m)_a$. Thus $(A^n)_a = (B^m)_a$ if and only if $\bar{v}_A/n = \bar{v}_B/m$.

For part (4), if $X \leqslant ((A_\beta)^m)_a$, then $v(X) \geqslant v(((A_\beta)^m)_a) = mv(A_\beta)$ for any valuation v on L. In particular $v_i(X) \geqslant mv_i(A_\beta)$ for each of the Rees valuations v_i, $i = 1, \ldots, r$, of A. So $\bar{v}_A(X) = \min\{v_i(X)/e_i(A) \mid i = 1, \ldots, r\} \geqslant \min\{mv_i(A_\beta)/e_i(A) \mid i = 1, \ldots, r\} = m\bar{v}_A(A_\beta) \geqslant m\beta$. That is $X \leqslant A_{m\beta}$.

For part (5), since $\bar{v}_B$ is a homogeneous pseudo-valuation for each $B \in L$, if $\bar{v}_{A_\beta}(X) \geqslant p/q$ for $X \in L$ and $p, q \in \mathbb{N}$, then $\bar{v}_{A_\beta}(X^q) \geqslant p$. Thus by (1) and (4), $X^q \leqslant ((A_\beta)^p)_a \leqslant A_{p\beta}$. Thus $\bar{v}_A(X^q) \geqslant p\beta$, or equivalently, $\bar{v}_A(X)/\beta \geqslant p/q$. Thus $\bar{v}_A/\beta \geqslant \bar{v}_{A_\beta}$.

Part (6) follows from (4) by setting $m = 1$. $\square$

In the case that L is the set of ideals of a Noetherian ring R and A, B are regular, the only if part of Theorem 7.2 was first proven in [14, Proposition 2.10] and the converse in [8, Theorem 3.4]. The following result does not require A and B to be regular.

Theorem 7.2. *Elements $A, B \in L$ are projectively equivalent if and only if*

(a) Rees A = Rees B *and*
(b) *there exists a positive $\beta \in \mathbb{Q}$ such that $\beta(e_1(A), \ldots, e_r(A)) = (e_1(B), \ldots, e_r(B))$ in $\mathbb{Q}^r$, where $\{v_1, \ldots, v_r\}$ is the set of nontrivial members of* Rees A = Rees B *and the $e_i(A)$ and $e_i(B)$ are the Rees numbers of A and B for the $v_1, \ldots, v_r$, respectively.*

Proof. ($\Rightarrow$) Let q_j be a minimal prime of L such that $A \nleqslant q_j$ and $A \vee q_j \neq R$. Then for each $n \in \mathbb{N}$, $(A^n)_a \nleqslant q_j$ and $(A^n)_a \vee q_j \neq R$. Letting $A \vee q_j/q_j = A_j$ and so on as before, we have $\bar{v}_{A_j} = \min\{v_{j1}/e_{j1}, \ldots, v_{jr_j}/e_{jr_j}\}$ where the v_{ji} are the Rees valuations of A_j and the $e_{ji} = e_{ji}(A)$ are the Rees numbers of A_j. We have by Lemma 7.1(2), $\bar{v}_{A_j^n} = \bar{v}_{A_j}/n = \min\{v_{j1}/ne_{j1}, \dots, v_{jr_j}/ne_{jr_j}\}$. Further, this representation is irredundant. Indeed for example, since v_{j1} is not redundant in the representation $\bar{v}_{A_j} = \min\{v_{j1}/e_{j1}, \dots, v_{jr_j}/e_{jr_j}\}$, by Theorem 6.15, there exists $Y \in L_j$ such that $\bar{v}_{A_j}(Y) = v_{j1}(Y)/e_{j1} < \min\{v_{j2}(Y)/e_{j2}, \dots, v_{jr_j}(Y)/e_{jr_j}\}$. Then $\bar{v}_{A_j^n}(Y) = \bar{v}_{A_j}(Y)/n = v_{j1}(Y)/ne_{j1} < \min\{v_{j2}(Y)/ne_{j2}, \dots, v_{jr_j}(Y)/ne_{jr_j}\}$. Thus v_{j1} is not redundant in the representation $\bar{v}_{A_j^n} = \min\{v_{j1}/ne_{j1}, \dots, v_{jr_j}/ne_{jr_j}\}$. Thus by Theorem 6.15 and Proposition 6.17, the v_{ji} are the Rees valuations of A_j^n and, by Corollary 6.14(a), $ne_{ji} = v_{ji}(A^n)$, $i = 1, \dots, r_j$, are the Rees numbers of A_j^n. Thus A_j and A_j^n have the same Rees valuations and the corresponding vectors of Rees numbers are proportional by a factor of n. Since this holds for each j with $A \nleqslant q_j$, A and A^n have the same nontrivial Rees valuations and the Rees numbers for these nontrivial Rees valuations are proportional by a factor of n. Let us write $C_1 \sim C_2$ if C_1 and C_2 have the same Rees valuations and the Rees numbers of the nontrivial ones are proportional. Then if $(A^n)_a = (B^m)_a$ with $m, n \in \mathbb{N}$, we have $A \sim A^n \sim (A^n)_a \sim (B^m)_a \sim B^m \sim B$.

($\Leftarrow$) Suppose that A and B have the same Rees valuations and the Rees numbers $(e_1(A), \dots, e_r(A)) = (e_1, \dots, e_r)$ and $(e_1(B), \dots, e_r(B)) = (f_1, \dots, f_r)$ in $\mathbb{Z}^r$ for the nontrivial members $\{v_1, \dots, v_k\}$ of $\operatorname{Rees} A = \operatorname{Rees} B$ are proportional. So for some $m, n \in \mathbb{N}$, $mf_i = ne_i$ for each i. Since, for the trivial members $v \in \operatorname{Rees} A = \operatorname{Rees} B$, $\alpha v = v$ for any $\alpha \in \mathbb{R}$, it follows that $\bar{v}_A/n = \bar{v}_B/m$. Thus by Lemma 7.1(3), $(A^n)_a = (B^m)_a$. So A and B are projectively equivalent. □

Remark 7.3. It follows from Corollary 6.14(a) that the above proposition remains valid if statement (b) is replaced by:

(b') There exists a positive $\beta \in \mathbb{Q}$ such that $\beta(v_1(A), \dots, v_r(A)) = (v_1(B), \dots, v_r(B))$ in $(\mathbb{Q} \cup \{\infty\})^r$, where $\{v_1, \dots, v_r\}$ is the set $\operatorname{Rees} A = \operatorname{Rees} B$.

We adapt the following definitions from [14].

Definition 7.4. Let L be a Noetherian multiplicative lattice and let $A \in L \setminus \{R\}$. We use the following notation.

(7.4.1) $\mathbf{W} = \mathbf{W}(A) = \{\beta \in \mathbb{R}_+ \mid \bar{v}_A(x) = \beta \text{ for some } x \in L\}$.
(7.4.2) $\mathbf{U} = \mathbf{U}(A) = \{\beta \in \mathbf{W} \mid A_\beta \text{ is projectively equivalent to } A\}$.
(7.4.3) $\mathbf{P} = \mathbf{P}(A) = \{A_\beta \mid \beta \in \mathbf{U}\}$.

Lemma 7.5. (See [13, Lemma 11.27].) *Let $A \in L$ be such that $A \nleqslant q_i$ for each minimal prime q_i of L. then $A_{n+k} : A_k = A_n$ for all $n, k \in \mathbb{N}$.*

Proof. Since $A_{n+k} : A_k \geqslant A_n$, it suffices to show the opposite inequality. For this we first assume L has no zero-divisors. Recall that by [4, Lemma 3.1] and Theorem 6.5, if $X, Y, C \in L$ and there exists a principal element $C_1 \leqslant C_a$ such that $0 : C_1 = 0$, then $(XC)_a \leqslant (YC)_a$ implies that $X_a \leqslant Y_a$. Let $X \leqslant A_{n+k} : A_k$. Then $XA_k \leqslant A_{n+k} = (A^{n+k})_a$. Thus $(XA^k)_a \leqslant (A^{n+k})_a$. Since we have assumed that L has no zero-divisors and $A^k \neq 0$, the above mentioned result gives $X \leqslant (A^n)_a = A_n$.

If L has zero-divisors, we work in $L/q_i = [q_i, R]$ for the minimal prime elements $q_1, \dots, q_r$ of L. Consider $(B \vee q_i)/q_i$ in the lattice $L/q_i = [q_i, R]$ for $B \in L$ as in Lemma 6.10. Now if $X \leqslant A_{n+k} : A_k$, then for each i we have $((XA^k) \vee q_i)/q_i = [(X \vee q_i)/q_i][(A \vee q_i)/q_i]^k \leqslant [(A \vee q_i)/q_i]_{n+k}$. But L/q_j has no zero-divisors and $A_i \neq 0$ in L/q_j. Thus by the above case, $(X \vee q_i)/q_i \leqslant [(A \vee q_i)/q_i]_n$ for each i. Thus by the construction of the Rees valuations, or Lemma 6.10, $X \leqslant A_n$. □

Lemma 7.6. (See [14, Lemma 2.2].) *Let $A \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for some minimal prime q_i of L such that $A \vee q_i \neq R$ and let $\beta \in \mathbb{R}_+$.*

(1) *If $((A_\beta)^m)_a = A_{m\beta}$ for arbitrarily large $m \in \mathbb{N}$, then $\beta \in \mathbf{W}$.*
(2) *If $((A_{n/m})^m)_a = A_n$ for some m and $n \in \mathbb{N}$, then $n/m \in \mathbf{U} \subseteq \mathbf{W}$.*

Proof. (1) By Theorem 6.12, it is clear that $\mathbf{W}$ is a discrete subset of $\mathbb{Q}_+$. Let $r \in \mathbf{W}$ be minimal with $\beta \leqslant r$. Then $A_\beta = A_r$. If $\beta < r$, we may choose $m \in \mathbb{N}$ so that $((A_\beta)^m)_a = A_{m\beta}$ and $m\beta < k < k+1 < mr$ for some $k \in \mathbb{N}$. Then we have the following where the first inequality is by Lemma 7.1(4): $((A_\beta)^m)_a = ((A_r)^m)_a \leqslant A_{mr} \leqslant A_{k+1} \leqslant A_k \leqslant A_{m\beta} = ((A_\beta)^m)_a$. Thus $(A^k)_a = A_k = A_{k+1} = (A^{k+1})_a$. Let $q_i \leqslant L$ be a minimal prime of L with $A \nleqslant q_i$ and $A \vee q_i \neq R$. Let $A_i = A/q_i \in L/q_i$. From $(A^k)_a = (A^{k+1})_a$, we get that $(A_i^k)_a = (A_i^{k+1})_a$. But by Lemma 7.5, $(A_i^{k+1})_a : (A_i^k)_a = (A_i)_a \neq R_i = R/q_i$, contradicting $(A_i^k)_a = (A_i^{k+1})_a$. Thus $\beta = r \in \mathbf{W}$.

(2) Let $B = A_{n/m}$. By Lemma 7.1(1) we have $(B^m)_a = A_n = (A^n)_a$ and thus by Lemma 7.1(3), $m\bar{v}_A = n\bar{v}_B$. To show that $n/m \in \mathbf{W}$, we use part (1) with the set of integers $\{km \mid k \in \mathbb{N}\}$. We have $((A_{n/m})^{km})_a = (B^{km})_a$ for $k \in \mathbb{N}$. So $X \leqslant ((A_{n/m})^{km})_a$ if and only if $\bar{v}_B(X) \geqslant km$. Using $m\bar{v}_A = n\bar{v}_B$, this is the same as $(m/n)\bar{v}_A(X) \geqslant km$ if and only if $\bar{v}_A(X) \geqslant km(n/m)$ if and only if $X \leqslant A_{km(n/m)}$. Thus $((A_{n/m})^{km})_a = A_{km(n/m)}$ for all $k \in \mathbb{N}$. Thus $n/m \in \mathbf{W}$ by part (1). But since $A_{n/m} = B_a$ is projectively equivalent to A, $n/m \in \mathbf{U}$. □

Proposition 7.7. (See [14, Proposition 2.3].) *Let $A, B \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for some minimal prime q_i of L such that $A \vee q_i \neq R$ and let $m, n \in \mathbb{N}$. Then $(B^m)_a = (A^n)_a$ if and only if $B_a = A_{n/m}$ and $n/m \in \mathbf{U}$.*

Proof. ($\Rightarrow$) If $(B^m)_a = (A^n)_a$, then by Lemma 7.1(3), $m\bar{v}_A = n\bar{v}_B$. So $X \leqslant B_a$ if and only if $\bar{v}_B(X) \geqslant 1$ if and only if $(m/n)\bar{v}_A(X) \geqslant 1$ if and only if $\bar{v}_A(X) \geqslant n/m$ if and only if $X \leqslant A_{n/m}$. So $B_a = A_{n/m}$ and $((A_{n/m})^m)_a = A_n$. Thus $n/m \in \mathbf{U}$ by Lemma 7.6(2).

($\Leftarrow$) If $n/m \in \mathbf{U}$, $m, n \in \mathbb{N}$ and $B_a = A_{n/m}$, then $B_a = A_{n/m}$ is projectively equivalent to A. Thus $(B^k)_a = (A^h)_a$, for some $h, k \in \mathbb{N}$. Then by Lemma 7.1(3), $k\bar{v}_A = h\bar{v}_B$. So $X \leqslant B_a = A_{n/m}$ if and only if $\bar{v}_B(X) \geqslant 1$ if and only if $(k/h)\bar{v}_A(X) \geqslant 1$ if and only if $\bar{v}_A(X) \geqslant h/k$ if and only if $X \leqslant A_{h/k}$. So $A_{n/m} = B_a = A_{h/k}$. Also $((A_{h/k})^k)_a = (B^k)_a = (A^h)_a = A_h$. So $h/k \in \mathbf{W}$ by Lemma 7.6(2). So since n/m and $h/k \in \mathbf{W}$, $n/m = h/k$. Then by Lemma 7.1(3), $((A_{n/m})^m)_a = (A^n)_a$. □

Corollary 7.8. (See [14, Corollary 2.4].) *Let $A \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for some minimal prime q_i of L such that $A \vee q_i \neq R$. Then $\{A_\beta \mid \beta \in \mathbf{U}\} = \{B \mid B = B_a$ and B projectively equivalent to $A\}$. This is a linearly ordered subset of L.*

Proof. This is immediate from Proposition 7.7. □

Proposition 7.9. (See [14, Proposition 2.5].) *Let $A \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for some minimal prime q_i of L such that $A \vee q_i \neq R$ and let $\beta \in \mathbb{R}_+$. The following are equivalent.*

(1) $((A_\beta)^m)_a = A_{m\beta}$ *for all* $m \in \mathbb{N}$.
(2) $\bar{v}_A = \beta \bar{v}_{A_\beta}$.
(3) $\beta \in \mathbf{U}$.

Proof. ((1) $\Rightarrow$ (3)) This holds by Lemma 7.6.

((3) $\Rightarrow$ (2)) Say $\beta = n/m \in \mathbf{U}$, $m, n \in \mathbb{N}$. Then $B = A_{n/m}$ is projectively equivalent to A and $(B^m)_a = (A^n)_a$ by Proposition 7.7. Then by Lemma 7.1(3), $m\bar{v}_A = n\bar{v}_{A_{n/m}}$. That is $\bar{v}_A = (n/m)\bar{v}_{A_{n/m}} = \beta\bar{v}_{A_\beta}$.

((2) $\Rightarrow$ (1)) If $X \leqslant A_{m\beta}$ then $\bar{v}_A(X) \geqslant m\beta$, and then (2) implies $\beta\bar{v}_{A_\beta}(X) \geqslant m\beta$. This gives $\bar{v}_{A_\beta}(X) \geqslant m$, which gives $X \leqslant ((A_\beta)^m)_a$. So $((A_\beta)^m)_a \geqslant A_{m\beta}$. But the opposite inequality always holds by Lemma 7.1(5). $\square$

Proposition 7.10. (See [14, Proposition 2.6].) *Let* $A \in L \setminus \{R\}$ *be such that* $A \nleqslant q_i$ *for some minimal prime* q_i *of* L *such that* $A \vee q_i \neq R$. *Then* $\mathbf{U}$ *is an additive subsemigroup of* $\mathbb{Q}_+$ *containing* $\mathbb{N}$ *and if* $\beta, \gamma \in \mathbf{U}$, *then* $(A_\beta A_\gamma)_a = A_{\beta+\gamma}$.

Proof. Since $A_1 = A_a$, $1 \in \mathbf{U}$. Let $\beta = n/m$ and $\gamma = h/k$ be in $\mathbf{U}$. By Proposition 7.7, $((A_\beta)^m)_a = (A^n)_a$ and $((A_\gamma)^k)_a = (A^h)_a$. So $((A_\beta A_\gamma)^{mk})_a = [((A_\beta)^m)^k((A_\gamma)^k)^m]_a = ((A^n)^k(A^h)^m)_a = (A^{nk+mh})_a$. Applying Proposition 7.7 to $B = A_\beta A_\gamma$ gives $(nk+mh)/mk = \beta + \gamma \in \mathbf{U}$ and $(A_\beta A_\gamma)_a = A_{\beta+\gamma}$. $\square$

It follows as in [14, Theorem 2.8] that if $A \in L \setminus \{R\}$, is such that $A \nleqslant q_i$ for some minimal prime q_i of L such that $A \vee q_i \neq R$, then $\mathbb{N} \subseteq \mathbf{U}(A) = \mathbf{U}$ and there exists $N \in \mathbb{N}$ and a unique $d \in \mathbb{N}$ such that (a) $\{\alpha \in \mathbf{U} \mid \alpha \geqslant N\} = \{N + (h/d) \mid h \in \mathbb{N}_0\}$; (b) $d\alpha \in \mathbb{N}$ for all $\alpha \in \mathbf{U}$; and (c) d is a common divisor of the Rees integers $e_1, \ldots, e_n$ of A.

8. Asymptotic primes

Some of the main facts about the sets $\mathrm{Ass}_R(R/(I^n)_a)$ when I is an ideal of a Noetherian ring are the following results of L.J. Ratliff and M. Brodmann. (See [7,18,20].)

(1) $\mathrm{Ass}_R(R/(I^n)_a) \subseteq \mathrm{Ass}_R(R/(I^{n+1})_a)$ for each n and these sets are eventually constant;
(2) $\bigcup_{i=1}^\infty \mathrm{Ass}_R(R/(I^n)_a) \subseteq \bigcup_{i=1}^\infty \mathrm{Ass}_R(R/I^n)$; and
(3) the sets $\mathrm{Ass}_R(R/I^n)$ are eventually constant.

In this section we give some extensions of (1) and (2) to rational powers in Noetherian multiplicative lattices. These results are new even in the ring case.

If L is a Noetherian multiplicative lattice and $A \in L$, an **associated prime** element of A is a prime element of L of the form $P = A : B$ for some $B \in L$. It then follows that $P = A : C$ for some principal $C \in L$. Following the notation for rings, we denote the set of associated prime elements of A by $\mathrm{Ass}(L/A)$. Recall that if v is a valuation on L, the center of v on L is $P = \bigvee\{X \in L \mid v(X) > 0\}$. If $XY \in P$, then $v(X) + v(Y) = v(XY) > 0$ and then $v(X) > 0$ or $v(Y) > 0$. So the center P of v is prime.

Let L be a Noetherian multiplicative lattice and let $\bar{v}: L \to \mathbb{R} \cup \{\infty\}$ be a sub-valuation. It is clear that if the representation $\bar{v} = \min\{v_i \mid i = 1, \ldots, r\}$ is not minimal, one can obtain a minimal one by deleting some of the v_i, and then this set is unique by Proposition 6.17. In the following result we consider a minimal subset $\mathcal{T} = \{v_1, \dots, v_r\}$ of Rees A such that $\bar{v}_A = \min\{v_i/e_i \mid i = 1, \dots, r\}$. If L has no zero-divisors, then necessarily $\mathcal{T} = \operatorname{Rees} A$ by Theorem 6.15.

Theorem 8.1. *Let L be a Noetherian multiplicative lattice, let $A \in L$, let $\mathcal{T} = \{v_1, \dots, v_r\}$ be a minimal subset of* Rees A *such that $\bar{v}_A = \min\{v_i/e_i \mid i = 1, \dots, r\}$ and for each i let P_i be the center of v_i on L. Let e denote the least common multiple of $e_1(A), \dots, e_r(A)$ and for each $\beta \in \mathbb{R}$, let $A_\beta = \bigvee\{x \in L \mid \bar{v}_A(x) \geqslant \beta\}$. Then* $\operatorname{Ass}(L/A_{n/e}) \subseteq \{P_1, \dots, P_r\}$ *with equality for n large.*

Proof. Let $e_i = e_i(A)$ for $i = 1, \dots, r$. First we show that $\bigcup_{n=1}^{\infty} \operatorname{Ass}(L/A_{n/e}) \subseteq \{P_1, \dots, P_r\}$. Suppose $P = A_{n/e} : B \in \operatorname{Ass}(L/A_{n/e})$ for some n. Then $B \nleqslant A_{n/e}$. So $\bar{v}_A(B) = \min\{v_i(B)/e_i \mid i = 1, \dots, r\} < n/e$ and $\bar{v}_A(PB) = \min\{v_i(PB)/e_i \mid i = 1, \dots, r\} \geqslant n/e$. But $v_i(PB)/e_i = v_i(P)/e_i + v_i(B)/e_i$. So $v_i(P) > 0$ for each i such that $v_i(B)/e_i < n/e$. So $P \leqslant P_i$ for each such i. Suppose $v_i(B)/e_i < n/e$ for $i = 1, \dots, k$ and $v_i(B)/e_i \geqslant n/e$ for $i = k+1, \dots, r$. Then if $I = P_1 P_2 \cdots P_k$, we have for each $i \in \{1, \dots, r\}$, $v_i(I^m B) = m v_i(I) + v_i(B) \geqslant n/e$ for some $m \in \mathbb{N}$, and thus $I^m \leqslant A_{n/e} : B = P$ for some $m \in \mathbb{N}$. Thus $P_j \leqslant P$ for some $j \in \{1, \dots, k\}$. Thus $P = P_j$ for some $j \in \{1, \dots, k\}$.

To show that $\operatorname{Ass}(L/A_{n/e}) \supseteq \{P_1, \dots, P_r\}$ for large n, let us show that $P_1 \in \operatorname{Ass}(L/A_{n/e})$ for large n. Since $\{v_1, \dots, v_r\}$ is irredundant, $\bar{v}_A(B) = \min\{v_i(B)/e_i \mid i = 1, \dots, r\} \neq \min\{v_i(B)/e_i \mid i = 2, \dots, r\}$ for some B. So $\bar{v}_A(B) = v_1(B)/e_1 < v_i(B)/e_i$ for each $i \geqslant 2$. Choose $n, j \in \mathbb{N}$ such that

$$v_1(B)/e_1 < n/je < v_i(B)/e_i \quad \text{for each } i \geqslant 2.$$

The second inequality gives $n/e < v_i(B^j)/e_i \leqslant v_i(P_1^k B^j)/e_i$ for each $i \geqslant 2$ and each $k \in \mathbb{N}$. We may choose $k \in \mathbb{N}$ such that $v_1(P_1^k B^j)/e_1 \geqslant n/e$. Then $P_1^k B^j \leqslant A_{n/e}$. The first inequality gives $v_1(B^j)/e_1 < n/e$. So $B^j \nleqslant A_{n/e}$. Thus we may further choose k minimal such that $P_1^k B^j \leqslant A_{n/e}$. Thus $P_1 \leqslant A_{n/e} : (P_1^{k-1} B^j)$.

For the opposite inequality, if $X \in L$ satisfies $X \leqslant A_{n/e} : (P_1^{k-1} B^j)$, then $v_1(X P_1^{k-1} B^j) = v_1(X) + v_1(P_1^{k-1} B^j) \geqslant n/e$, and thus $v_1(X) > 0$. Thus $X \leqslant P_1$. Thus $P_1 = A_{n/e} : (P_1^{k-1} B^j) \in \operatorname{Ass}(L/A_{n/e})$.

We have shown that $P_1 \in \operatorname{Ass}(L/A_{n/e})$ for each n such that for some $j \in \mathbb{N}$ we have $w = v_1(B)/e_1 < n/je < v_i(B)/e_i = u$ for each $i \geqslant 2$. But $w < n/je < u$ implies $w < n/je + 1/(mje) = (mn+1)/mje < u$ for some $m \in \mathbb{N}$. Since also $w < mn/mje < u$, we have $w < a/k < b/k < u$ for some $a, b, k \in \mathbb{N}$ with a and b relatively prime. It is easy to see that for all large $n \in \mathbb{N}$, there exist $x, y \in \mathbb{N}$ such that $n = ax + by$. Then $w < ax/kx < (ax+by)/(kx+ky) < by/ky < u$. It follows that $P_1 \in \operatorname{Ass}(L/A_{n/e})$ for all n large. □

Lemma 8.2. *Let L be a Noetherian multiplicative lattice, let $A \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for each minimal prime q_i of L, let* Rees $A = \{v_1, \dots, v_r\}$*, let e denote the least common multiple of $e_1(A), \dots, e_r(A)$ and for each $\beta \in \mathbb{R}$, let $A_\beta = \bigvee\{x \in L \mid \bar{v}_A(x) \geqslant \beta\}$. Then $A_{n/e+k/e} : A_{k/e} = A_{n/e}$ for all $n \in \mathbb{N}$ and $k/e \in \mathbf{U}$.*

Proof. Since $A_{n/e+k/e} : A_{k/e} \geqslant A_{n/e}$, it suffices to show the opposite inequality. Let $X \leqslant A_{n/e+k/e} : A_{k/e}$. Then $X A_{k/e} \leqslant A_{n/e+k/e}$, which, using Proposition 7.9, implies $X^e(A_k) = X^e((A_{k/e})^e)_a \leqslant ((A_{n/e+k/e})^e)_a \leqslant A_{n+k} = (A^{n+k})_a$. Thus $X^e \leqslant A_{n+k} : A_k$, which by Lemma 7.5 is A_n. Thus $X \leqslant A_{n/e}$. □

Theorem 8.3. *Let L be a Noetherian multiplicative lattice and let $A \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for each minimal prime q_i of L. Then* $\mathrm{Ass}(L/A_{n/e}) \subseteq \mathrm{Ass}(L/A_{n/e+k/e})$ *for each $n \in \mathbb{N}$ and $k/e \in \mathbf{U}$.*

Proof. Since $A_{n/e+k/e} : A_{k/e} = A_{n/e}$ by Lemma 8.2, if $P = A_{n/e} : B \in \mathrm{Ass}(L/A_{n/e})$ for some n, then $P = A_{n/e} : B \leqslant (A_{k/e}A_{n/e}) : (A_{k/e}B) \leqslant A_{n/e+k/e} : A_{k/e}B = [A_{n/e+k/e} : A_{k/e}] : B = A_{n/e} : B = P$. Thus $P = A_{n/e+k/e} : A_{k/e}B \in \mathrm{Ass}(L/A_{n/k+k/e})$. $\square$

Since $(A^n)_a = A_n$ for $n \in \mathbb{N}$, the following case $e = k = 1$ of the above result improves [4, Theorem 3.4] which requires the additional hypothesis that A_a contains a principal element B with $(0 : B) = 0$ and obtains the weaker conclusion that if $P \in \mathrm{Ass}(L/(A^n)_a)$, then there exists an $m \in \mathbb{N}$ such that $P \in \mathrm{Ass}(L/(A^{m+k})_a)$ for all $k \in \mathbb{N}$.

Corollary 8.4. *Let L be a Noetherian multiplicative lattice and let $A \in L \setminus \{R\}$ be such that $A \nleqslant q_i$ for each minimal prime q_i of L. Then* $\mathrm{Ass}(L/(A^n)_a) \subseteq \mathrm{Ass}(L/(A^{n+1})_a)$ *for each $n \in \mathbb{N}$.*

For the next result we need a method of localization in an arbitrary Noetherian lattice where, unlike in the case of q.f. lattices, localization is not built in. A method of localization in Noetherian multiplicative lattices was furnished by Dilworth in [9]. We use a simplification of Dilworth's method given by D.D. Anderson [2]. For this recall that if $\mathcal{S}$ is a multiplicative subset of compact elements and $A, B \in L$, write $A \leqslant B$ $(\mathcal{S})$ if for each principal $X \leqslant A$, there exists $T \in \mathcal{S}$ such that $TX \leqslant B$. Write $A \equiv B$ $(\mathcal{S})$ if $A \leqslant B$ $(\mathcal{S})$ and $B \leqslant A$ $(\mathcal{S})$. Then $\equiv (\mathcal{S})$ is an equivalence relation on L and for $A \in L$ we let $A_{\mathcal{S}}$ denote the $\equiv (\mathcal{S})$ equivalence class of A. Then the set $L_{\mathcal{S}} = \{A_{\mathcal{S}} \mid A \leqslant L\}$ of equivalence classes under $\equiv (\mathcal{S})$ is a multiplicative lattice under the partial order $A_{\mathcal{S}} \leqslant B_{\mathcal{S}}$ if $A \leqslant B$ $(\mathcal{S})$ and multiplication $A_{\mathcal{S}}B_{\mathcal{S}} = (AB)_{\mathcal{S}}$.

Lemma 8.5. *Let L be a Noetherian multiplicative lattice. If $B \in L$ and $\mathcal{S}$ is a multiplicative subset of L, then $(B_{\mathcal{S}})_a = (B_a)_{\mathcal{S}}$ in $L_{\mathcal{S}}$.*

Proof. If a principal element x of L satisfies $x_{\mathcal{S}} \leqslant (B_a)_{\mathcal{S}}$ then there exists $s \in \mathcal{S}$ such that $sx \leqslant B_a$. That is $(sx)^n \leqslant B((sx) \vee B)^{n-1}$ for some n. Thus $x^n_{\mathcal{S}} = (sx)^n_{\mathcal{S}} \leqslant [B((sx) \vee B)^{n-1}]_{\mathcal{S}} = B_{\mathcal{S}}((sx)_{\mathcal{S}} \vee B_{\mathcal{S}})^{n-1} = B_{\mathcal{S}}(x_{\mathcal{S}} \vee B_{\mathcal{S}})^{n-1}$, and therefore $x_{\mathcal{S}} \leqslant (B_{\mathcal{S}})_a$.

Conversely, suppose $x \in L$ is principal and $x_{\mathcal{S}} \leqslant (B_{\mathcal{S}})_a$. Then $(x_{\mathcal{S}})^n \leqslant B_{\mathcal{S}}(x_{\mathcal{S}} \vee B_{\mathcal{S}})^{n-1}$ for some n. That is $x^n \leqslant B(x \vee B)^{n-1}$ $(\mathcal{S})$ for some n. Since x is compact, there exists a compact element $t \in \mathcal{S}$ such that $tx^n \leqslant B(x \vee B)^{n-1}$. Then $(tx)^n \leqslant B(tx \vee B)^{n-1}$. Thus $tx \leqslant B_a$, and hence $x \leqslant B_a$ $(\mathcal{S})$. That is $x_S \leqslant (B_a)_{\mathcal{S}}$. Since each element $X_{\mathcal{S}}$ is a supremum of elements $x_{\mathcal{S}}$ with x principal in L [2, Propositions 2.5(1) and 2.5(5)], $(B_{\mathcal{S}})_a = (B_a)_{\mathcal{S}}$. $\square$

Proposition 8.6. *Let L be a Noetherian multiplicative lattice, let $B \in L$ and let $\mathcal{S}$ be a multiplicative subset of L. Then* $\mathrm{Ass}(L_{\mathcal{S}}/B_{\mathcal{S}}) = \{Q_{\mathcal{S}} \mid Q \in \mathrm{Ass}(L/B)$ *and $s \nleqslant Q$ for each $s \in \mathcal{S}\}$.*

Proof. $(\subseteq)$ Each prime element of $L_{\mathcal{S}}$ is of the form $Q_{\mathcal{S}}$ with $Q \in L$ prime and $t \nleqslant Q$ for each $t \in \mathcal{S}$ [2, Theorem 2.7]. Let $Q_{\mathcal{S}} \in \mathrm{Ass}(L_{\mathcal{S}}/B_{\mathcal{S}})$. Let $Q_{\mathcal{S}} = (B_{\mathcal{S}} : y_{\mathcal{S}})$, $y \in L$ principal. Then $s \nleqslant Q$ for each $s \in \mathcal{S}$. Indeed suppose there exists $s \in \mathcal{S}$ with $s \leqslant Q$. Then $s_{\mathcal{S}}y_{\mathcal{S}} \leqslant B_{\mathcal{S}}$ implies that there exists $t \in \mathcal{S}$ such that $tsy \leqslant B$. But $ts \in \mathcal{S}$ then implies $y_{\mathcal{S}} \leqslant B_{\mathcal{S}}$, a contradiction.

Now $Q_{\mathcal{S}}y_{\mathcal{S}} \leqslant B_{\mathcal{S}}$ implies that $sQy \leqslant B$ for some $s \in \mathcal{S}$, and thus $s \nleqslant Q$. We claim that $(B : sy) = Q$. Clearly $(B : sy) \geqslant Q$. For the opposite inequality, let $r \leqslant (B : sy)$. Then $sry \leqslant B$.

Thus $r_{\mathcal{S}}y_{\mathcal{S}} \leqslant B_{\mathcal{S}}$. Thus $r_{\mathcal{S}} \leqslant Q_{\mathcal{S}}$, and hence there exists $t \in \mathcal{S}$ such that $tr \leqslant Q$. Then since Q prime and $t \nleqslant Q$, $r \leqslant Q$. Thus $Q = (B : sy) \in \operatorname{Ass}(L/B)$ and $s \nleqslant Q$ for each $s \in \mathcal{S}$.

($\supseteq$) Let $Q \in \operatorname{Ass}(L/B)$ with $s \nleqslant Q$ for each $s \in \mathcal{S}$. To show that $Q_{\mathcal{S}} \in \operatorname{Ass}(L_{\mathcal{S}}/B_{\mathcal{S}})$, let $Q = (B : y)$, $y \in L$. Then $y_{\mathcal{S}} \nleqslant B_{\mathcal{S}}$. We claim that $(B_{\mathcal{S}} : y_{\mathcal{S}}) = Q_{\mathcal{S}}$. Clearly $Q_{\mathcal{S}} \leqslant (B_{\mathcal{S}} : y_{\mathcal{S}})$. Suppose $r_{\mathcal{S}}y_{\mathcal{S}} \leqslant B_{\mathcal{S}}$. Then $sry \leqslant B$ for some $s \in \mathcal{S}$. But then since $sr \leqslant (B : y) = Q$ and $s \in \mathcal{S}$, $r \leqslant Q$. Thus $Q_{\mathcal{S}} = (B_{\mathcal{S}} : y_{\mathcal{S}}) \in \operatorname{Ass}(L_{\mathcal{S}}/B_{\mathcal{S}})$. □

Theorem 8.7. *Let L be a Noetherian multiplicative lattice and let $A \in L$. Then*

$$\bigcup_{n=1}^{\infty} \operatorname{Ass}(L/A_{n/e}) = \bigcup_{n=1}^{\infty} \operatorname{Ass}\bigl(L/\bigl(A^n\bigr)_a\bigr) \subseteq \bigcup_{n=1}^{\infty} \operatorname{Ass}\bigl(L/A^n\bigr).$$

Proof. By Theorem 8.1, it suffices to show $\bigcup_{n=1}^{\infty} \operatorname{Ass}(L/(A^n)_a) \subseteq \bigcup_{n=1}^{\infty} \operatorname{Ass}(L/A^n)$. Let $P \in \operatorname{Ass}(L/(A^n)_a)$. By Lemma 8.5 and Proposition 8.6, to show $P \in \operatorname{Ass}(L/A^n)$ we may first localize, and thus we assume that P is maximal.

Since $P \in \operatorname{Ass}(L/(A^n)_a)$ we have $P = ((A^n)_a :_R x)$ for some principal x. Thus $Px \leqslant (A^n)_a$. By [12, Theorems 2 and 3], $(A^n)_a = \bigvee\{Y \in L \mid \text{there exists } j \in \mathbb{N} \text{ such that } Y^{j+i} \leqslant A^{ni} \text{ for all } i \geqslant 0\}$. Thus there exists a positive integer j such that $(Px)^{j+i} \leqslant A^{ni}$ for all $i \geqslant 0$. Further, $x^{j+i} \nleqslant A^{ni}$ for some $i \geqslant 0$ since $x \nleqslant (A^n)_a$. Thus we may choose k minimal such that $P^k x^{j+i} \leqslant A^{ni}$. Then $P \leqslant (A^{ni} : P^{k-1}x^{j+i}) \neq R$. Then by maximality of P, $P = (A^{ni} : P^{k-1}x^{j+i}) \in \operatorname{Ass}(L/A^{ni})$. □

Since the finiteness of $\bigcup_{n=1}^{\infty} \operatorname{Ass}(L/A^n)$ is relatively straightforward (see [4,7]) (and in fact easily extends to modules), the above theorem furnishes an alternate proof of the finiteness of the set $\bigcup_{n=1}^{\infty} \operatorname{Ass}(L/(A^n)_a)$, but without identifying this set of primes.

The following corollary is an immediate consequence of Theorems 8.7 and 8.1.

Corollary 8.8. *Let L be a Noetherian multiplicative lattice, let $A \in L$, let $\mathcal{T} = \{v_1, \ldots, v_r\}$ be a minimal subset of* Rees A *such that $\bar{v}_A = \min\{v_i/e_i \mid i = 1, \ldots, r\}$ and for each i let P_i be the center of v_i on L. Then $\{P_1, \ldots, P_r\} \subseteq \bigcup_{n=1}^{\infty} \operatorname{Ass}(L/A^n)$.*

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