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Covering functors without groups ☆

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ABSTRACT

Coverings in the representation theory of algebras were introduced for the Auslander–Reiten quiver of a representation-finite algebra in [Ch. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurüch, Comment. Math. Helv. 55 (1980) 199–224] and later for finite-dimensional algebras in [K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (3) (1982) 331–378; P. Gabriel, The universal cover of a representation-finite algebra, in: Proc. Representation Theory I, Puebla, 1980, in: Lecture Notes in Math., vol. 903, Springer, 1981, pp. 68–105; R. Martínez-Villa, J.A. de la Peña, The universal cover of a quiver with relations, J. Pure Appl. Algebra 30 (3) (1983) 277–292]. The best understood class of *covering functors* is that of *Galois covering functors* $F: A \to B$ determined by the action of a group of automorphisms of A. In this work we introduce the *balanced covering functors* which include the Galois class and for which classical Galois covering-type results still hold. For instance, if $F: A \to B$ is a balanced covering functor, where A and B are linear categories over an algebraically closed field, and B is tame, then A is tame.

Introduction and notation

Let k be a field and A be a finite-dimensional (associative with 1) k-algebra. One of the main goals of the *representation theory of algebras* is the description of the category of finite-dimensional left modules ${}_A\mathrm{mod}$. For that purpose it is important to determine the representation type of A. The finite representation type (that is, when A accepts only finitely many indecomposable objects in ${}_A\mathrm{mod}$,

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up to isomorphism) is well understood. In that context, an important tool is the construction of Galois coverings $F:\tilde{A}\to A$ of A since $\tilde{A}$ is a locally representation-finite category if and only if A is representation-finite [6,10]. For a tame algebra A and a Galois covering $F:\tilde{A}\to A$, the category $\tilde{A}$ is also tame, but the converse does not hold [7,12].

Coverings were introduced in [13] for the Auslander–Reiten quiver of a representation-finite algebra. For algebras of the form $A=kQ/I$, where Q is a quiver and I an admissible ideal of the path algebra kQ, the notion of covering was introduced in [2,6,9]. Following [2], a functor $F:A\to B$, between two locally bounded k-categories A and B, is a *covering functor* if the following conditions are satisfied:

(a) F is a k-linear functor which is onto on objects;
(b) the induced morphisms

$$\bigoplus_{Fb'=j} A(a,b')\to B(Fa,j)\quad\text{and}\quad\bigoplus_{Fa'=i} A(a',b)\to B(i,Fb)$$

are bijective for all i, j in B and a, b in A.

We denote by $({}_{b'}f^{\bullet}_{a})_{b'}\mapsto f$ and $({}^{\bullet}_{b}f_{a'})_{a'}\mapsto f$ the corresponding bijections. We shall consider $F_\lambda: {}_A\mathrm{mod}\to {}_B\mathrm{mod}$ the left adjoint to the pull-up functor $F_\bullet: {}_B\mathrm{mod}\to {}_A\mathrm{mod}$, $M\mapsto MF$, where ${}_C\mathrm{mod}$ denotes the category of left modules over the k-category C, consisting of covariant k-linear functors.

The best understood examples of covering functors are the *Galois covering functors* $A\to B$ given by the action of a group of automorphisms G of A acting freely on objects and where $F:A\to B=A/G$ is the quotient defined by the action. See [2,4,6,9,10] for results on Galois coverings. Examples of coverings which are not of Galois type will be exhibited in Section 1.

In this work we introduce *balanced coverings* as those coverings $F:A\to B$ where ${}_bf^{\bullet}_a={}^{\bullet}_bf_a$ for every $f\in B(Fa,Fb)$. Among many other examples, Galois coverings are balanced, see Section 2. We shall prove the following:

Theorem 0.1. *Let $F:A\to B$ be a balanced covering. Then every finitely generated A-module X is a direct summand of $F_\bullet F_\lambda X$.*

In fact, according to the notation in [1], we show that a balanced covering functor is a *cleaving* functor, see Section 3. This is essential for extending Galois covering-type results to more general situations. For instance, we show the following result.

Theorem 0.2. *Assume that k is an algebraically closed field and let $F:A\to B$ be a covering functor. Then the following hold*:

(a) *If F is induced from a map $f:(Q,I)\to(Q',I')$ of quivers with relations, where $A=kQ/I$ and $B=kQ'/I'$, then B is locally representation-finite if and only if so is A*;
(b) *If F is balanced and B is tame, then A is tame.*

More precise statements are shown in Section 4. For a discussion on the representation type of algebras we refer to [1,5,7,11,12].

1. Coverings: examples and basic properties

1.1. The pull-up and push-down functors

Following [2,6], consider a locally bounded k-category A, that is, A has a (possibly infinite) set of non-isomorphic objects A_0 such that

(a) $A(a,b)$ is a k-vector space and the composition corresponds to linear maps $A(a,b) \otimes_k A(b,c) \to A(a,c)$ for every a, b, c objects in A_0;
(b) $A(a,a)$ is a local ring for every a in A_0;
(c) $\sum_b A(a,b)$ and $\sum_b A(b,a)$ are finite-dimensional for every a in A_0.

For a locally bounded k-category A, we denote by ${}_A\mathrm{Mod}$ (resp. Mod_A) the category of covariant (resp. contravariant) functors $A \to \mathrm{Mod}_k$; by ${}_A\mathrm{mod}$ (resp. mod_A) we denote the full subcategory of locally finite-dimensional functors $A \to \mathrm{mod}_k$ of the category ${}_A\mathrm{Mod}$ (resp. Mod_A). In case A_0 is finite, A can be identified with the finite-dimensional k-algebra $\bigoplus_{a,b \in A_0} A(a,b)$; in this case the category ${}_A\mathrm{Mod}$ (resp. ${}_A\mathrm{mod}$) is equivalent to the category of left A-modules (resp. finitely generated left A-modules).

According to [5], in case k is algebraically closed, there exist a quiver Q and an ideal I of the path category kQ, such that A is equivalent to the quotient kQ/I. Then any module $M \in {}_A\mathrm{Mod}$ can be identified with a *representation* of the *quiver with relations* (Q, I). Usually our examples will be presented by means of quivers with relations.

Let $F : A \to B$ be a k-linear functor between two locally bounded k-categories. The *pull-up* functor $F_\bullet : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$, $M \mapsto MF$ admits a left adjoint $F_\lambda : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$, called the *push-down* functor, which is uniquely defined (up to isomorphism) by the following requirements:

(i) $F_\lambda A(a,-) = B(Fa,-)$;
(ii) F_λ commutes with direct limits.

In particular, F_λ preserves projective modules. Denote by $F_\rho : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$ the right adjoint to $F_\bullet$.

For covering functors $F : A \to B$ we get an explicit description of F_λ and F_ρ as follows:

Lemma 1.1. *(See [2].) Let $F : A \to B$ be a covering functor. Then:*

(a) *For any $X \in {}_A\mathrm{mod}$ and $f \in B(i,j)$,*

$$F_\lambda X(f) = \left(X\left({}_b f_a^\bullet\right)\right) : \bigoplus_{Fa=i} X(a) \to \bigoplus_{Fb=j} X(b), \quad \textit{with} \sum_{Fb=j} F\left({}_b f_a^\bullet\right) = f.$$

In particular, $F_\bullet(a,-) : F_\lambda A(a,-) \to B(Fa,-)$ is the natural isomorphism given by $({}_b f_a^\bullet)_b \mapsto f$.

(b) *For any $X \in {}_A\mathrm{mod}$ and $f \in B(i,j)$*

$$F_\rho X(f) = \left(X\left({}_b^\bullet f_a\right)\right) : \prod_{Fa=i} X(a) \to \prod_{Fb=j} X(b), \quad \textit{with} \sum_{Fa=i} F\left({}_b^\bullet f_a\right) = f.$$

In particular, $F_\bullet D(-,b) : F_\rho DA(-,b) \to DB(-,Fb)$ is the natural isomorphism induced by $({}_b^\bullet f_a)_a \mapsto f$.

1.2. The order of a covering

The following lemma allows us to introduce the notion of *order* of a covering.

Lemma 1.2. *Let $F : A \to B$ be a covering functor. Assume that B is connected and a fiber $F^{-1}(i)$ is finite, for some $i \in B_0$. Then the fibers have constant cardinality.*

Proof. Let $i \in B_0$ and $0 \neq f \in B(i,j)$. For $a \in F^{-1}(i)$, $\sum_{Fb=j} \dim_k A(a,b) = \dim_k B(i,j)$. Hence $|F^{-1}(i)| \dim_k B(i,j) = \sum_{Fa=i} \sum_{Fb=j} \dim_k A(a,b) = \sum_{Fb=j} \sum_{Fa=i} \dim_k A(a,b) = |F^{-1}(j)| \dim_k B(i,j)$ and $|F^{-1}(i)| = |F^{-1}(j)|$. Since B is connected, the claim follows. $\square$

In case $F : A \to B$ is a covering functor with B connected and A_0 is finite, we define the *order* of F as $\mathrm{ord}(F) = |F^{-1}(i)|$ for any $i \in B_0$. Thus $\mathrm{ord}(F)|B_0| = |A_0|$.

We recall from the Introduction that a covering functor $F : A \to B$ is *balanced* if ${}_b f_a^{\bullet} = {}_b^{\bullet} f_a$ for every couple of objects a, b in A.

Lemma 1.3. *Let $F : A \to B$ be a balanced covering functor, then $F_\lambda = F_\rho$ as functors ${}_A\mathrm{mod} \to {}_B\mathrm{mod}$.*

1.3. Examples

(a) Let A be a locally bounded k-category and let G be a group of k-linear automorphisms acting freely on A (that is, for $a \in A_0$ and $g \in G$ if $ga = a$, then $g = 1$). The quotient category A/G has as objects the G-orbits in the objects of A; a morphism $f : i \to j$ in A/G is a family $f : ({}_b f_a) \in \prod_{a,b} A(a,b)$, where a (resp. b) ranges in i (resp. j) and $g \cdot {}_b f_a = {}_{gb} f_{ga}$ for all $g \in G$. The canonical projection $F : A \to A/G$ is called a *Galois covering* defined by the action of G.

A particular situation is illustrated by the following algebras (given as quivers with relations):

$$A:\ \begin{tikzcd} \bullet \arrow[r, "\rho_0"] \arrow[dr, "\gamma_0"'] & \bullet \arrow[r, "\rho_1"] \arrow[dr, "\gamma_1"'] & \bullet \\ \bullet \arrow[r, "\sigma_0"'] \arrow[ur, "\nu_0"] & \bullet \arrow[r, "\sigma_1"'] \arrow[ur, "\nu_1"] & \bullet \end{tikzcd} \qquad B:\ \begin{tikzcd} \bullet \arrow[r, bend left, "\alpha_0"] \arrow[r, bend right, "\beta_0"'] & \bullet \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\beta_1"'] & \bullet \end{tikzcd}$$

$$\begin{cases} \rho_1\rho_0 = \nu_1\gamma_0, \\ \sigma_1\sigma_0 = \gamma_1\nu_0, \\ \rho_1\nu_0 = \nu_1\sigma_0, \\ \sigma_1\gamma_0 = \gamma_1\rho_0, \end{cases} \qquad \begin{cases} \alpha_1\alpha_0 = \beta_1\beta_0, \\ \beta_1\alpha_0 = \alpha_1\beta_0. \end{cases}$$

The algebra A is tame, but B is wild when char $k = 2$ [7]. The cyclic group C_2 acts freely on A and A/C_2 is isomorphic to B.

(b) Consider the algebras given by quivers with relations and the functor F as follows:

$$\begin{tikzcd} a_2 \arrow[r, bend left, "\alpha_2"] \arrow[r, bend right, "\beta_2"'] & b_2 \arrow[r, bend right, "\rho_2"'] & b_1 \arrow[l, bend right, "\rho_1"'] & a_1 \arrow[l, bend right, "\alpha_1"'] \arrow[l, bend left, "\beta_1"] \arrow[r, "F"] & a \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & b \arrow[loop right, "\rho"] \end{tikzcd}$$

both algebras with $\mathrm{rad}^2 = 0$ and $F\alpha_1 = \alpha,\ F\alpha_2 = \alpha + \beta,\ F\beta_i = \beta,\ F\rho_i = \rho,\ i = 1, 2$. It is a simple exercise to check that F is a balanced covering, but obviously it is not of Galois type.

(c) Consider the functor

$$\begin{tikzcd} & b_1 & & & \\ A:\ a_2 \arrow[ur, "\beta_2"] \arrow[dr, "\alpha_2"'] & & a_1 \arrow[ul, "\alpha_1"'] \arrow[dl, "\beta_1"] \arrow[r, "F"] & B:\ a \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & b \\ & b_2 & & & \end{tikzcd}$$

where $F\alpha_i = \alpha$, $i = 1, 2$, $F\beta_1 = \beta$, $F\beta_2 = \alpha + \beta$. Since $F(\beta_2 - \alpha_2) = \beta$ and $F(\beta_1) = \beta$, then ${}_{b_2}\beta^{\bullet}_{a_2} = -\alpha_2$ and ${}^{\bullet}_{b_2}\beta_{a_2} = 0$. Hence F is a non-balanced covering functor.

For the two-dimensional indecomposable A-module X given by $X(a_2) = k$, $X(b_2) = k$, $X(\alpha_2) = \mathrm{id}$ and zero otherwise, it follows that $F_{\bullet}F_{\lambda}X$ is indecomposable and hence X is not a direct summand of $F_{\bullet}F_{\lambda}X$.

(d) As a further example, consider the infinite category A and the balanced covering functor defined in the obvious way:

$$\begin{tikzcd} A: \cdots & \bullet \arrow[r, bend right, "\beta_2'"'] & \bullet \arrow[l, bend right, "\beta_2"'] \arrow[r, bend right, "\alpha_1'"'] & \bullet \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend right, "\beta_1'"'] & \bullet \arrow[l, bend right, "\beta_1"'] \arrow[loop right, "\alpha_0"] & \xrightarrow{F} & \bullet \arrow[loop left, "\beta"] \arrow[loop right, "\alpha"] \end{tikzcd}$$

where both categories A and B have $\mathrm{rad}^2 = 0$.

1.4. Coverings of schurian categories

We say that a locally bounded k-category B is *schurian* if for every $i, j \in B_0$, $\dim_k B(i, j) \leqslant 1$.

Lemma 1.4. *Let $F : A \to B$ be a covering functor and assume that B is schurian, then F is balanced.*

Proof. Let $0 \neq f \in B(i, j)$ and $Fa = i$, $Fb = j$. Since B is schurian, there is a unique $0 \neq {}_{b'}f^{\bullet}_a \in A(a, b')$ with $Fb' = j$ and a unique $0 \neq {}^{\bullet}_b f_{a'} \in A(a', b)$ with $Fa' = i$ satisfying $F_{b'} f^{\bullet}_a = f = F^{\bullet}_b f_{a'}$. In case $b = b'$, then $a = a'$ and ${}_b f^{\bullet}_a = {}^{\bullet}_b f_a$. Else $b \neq b'$ and ${}_b f^{\bullet}_a = 0$. In this situation $a \neq a'$ and ${}^{\bullet}_b f_a = 0$. $\square$

Proposition 1.5. *Let $F : A \to B$ be a covering functor with finite order and B schurian. Then for every $M \in {}_B\mathrm{mod}$, $F_{\lambda}F_{\bullet}M \cong M^{\mathrm{ord}(F)}$.*

Proof. For any $0 \neq f \in B(i, j)$ we get

$$\begin{tikzcd} F_{\lambda}F_{\bullet}M(i) = \bigoplus_{Fa=i} M(i) \arrow[r, "\sim"] \arrow[d, "(M(F_b f^{\bullet}_a))"'] & M^{\mathrm{ord}(F)}(i) \arrow[d, "\mathrm{diag}(M(f), \ldots, M(f))"] \\ F_{\lambda}F_{\bullet}M(j) = \bigoplus_{Fb=j} M(j) \arrow[r, "\sim"] & M^{\mathrm{ord}(F)}(j) \end{tikzcd}$$

Since for each a there is a unique b with ${}_b f^{\bullet}_a \neq 0$ such that $F_b f^{\bullet}_a = f$, then the square commutes. $\square$

Remark 1.6. If B is not schurian the result may not hold as shown in [7, (3.1)] for a Galois covering $F : B \to C$ with B as in example 1.3(a).

1.5. Coverings induced from a map of quivers

Let $q : Q' \to Q$ be a *covering map of quivers*, that is, q is an onto morphism of oriented graphs inducing bijections $i^+ \to q(i)^+$ and $i^- \to q(i)^-$ for every vertex i in Q', where x^+ (resp. x^-) denotes those arrows $x \to y$ (resp. $y \to x$). For the concept of covering and equitable partitions in graphs, see [8].

Assume that Q is a finite quiver. Let I be an *admissible ideal* of the path algebra kQ, that is, $J^n \subset I \subset J^2$ for J the ideal of kQ generated by the arrows of Q. We say that I is *admissible with respect* to q if there is an ideal I' of the path category kQ' such that the induced map $kq : kQ \to kQ'$ restricts to isomorphisms $\bigoplus_{q(a)=i} I'(a,b) \to I(i,j)$ for $q(b)=j$ and $\bigoplus_{q(b)=j} I'(a,b) \to I(i,j)$ for $q(a)=i$. Observe that most examples in Section 1.3 (not example (c)) are built according to the following proposition:

Proposition 1.7. *Let $q: Q' \to Q$ be a covering map of quivers, I an admissible ideal of kQ and I' an admissible ideal of kQ' making I admissible with respect to q as in the above definition. Then the induced functor $F: kQ'/I' \to kQ/I$ is a balanced covering functor.*

Proof. Since q is a covering of quivers, it has the unique lifting property of paths. Hence for any pair of vertices i in Q and a in Q' with $q(a)=i$, we have that

$$\begin{tikzcd}
\bigoplus_{q(b)=j} kQ'(a,b) \arrow[r, "k(q)", "\sim"'] \arrow[d] & kQ(i,j) \arrow[d] \\
\bigoplus_{F(b)=j} kQ'/I'(a,b) \arrow[r, "F"] & kQ/I(i,j)
\end{tikzcd}$$

is a commutative diagram with F an isomorphism. This shows that F is a covering functor.

For any arrow $i \xrightarrow{\alpha} j$ in Q and $q(a)=i$, there is a unique b in Q' and an arrow $a \xrightarrow{\alpha'} b$ with $q(\alpha')=\alpha$. Hence the class ${}_bf_a^\bullet$ of α' in $kQ'/I'(a,b)$ satisfies that $F({}_bf_a^\bullet)$ is the class $f=\bar{\alpha}$ of α in $kQ/I(i,j)$. By symmetry, ${}_bf_a^\bullet = {}_b^\bullet f_a$. For arbitrary $f \in kQ/I(i,j)$, f is the linear combination $\sum \lambda_i f_i$, where f_i is the product of classes of arrows in Q. Observe that for arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} m$ we have ${}_c(\bar{\beta}\bar{\alpha})_a^\bullet = ({}_c\bar{\beta}_b^\bullet)({}_b^\bullet\bar{\alpha}_a) = ({}_c\bar{\beta}_b^\bullet)({}_b\bar{\alpha}_a^\bullet)$. It follows that F is balanced. $\square$

In the above situation we shall say that the functor F is *induced* from a map $q:(Q',I') \to (Q,I)$ of quivers with relations.

2. On Galois coverings

2.1. Galois coverings are balanced

Proposition 2.1. *Let $F: A \to B$ be a Galois covering, then F is balanced.*

Proof. Assume F is determined by the action of a group G of automorphisms of A, acting freely on the objects A_0. Let i,j be objects of B and $f \in B(i,j)$. Consider a,b in A with $Fa=i$, $Fb=j$ and $({}_{b'}f_a^\bullet)_{b'} \in \bigoplus_{Fb'=j} A(a,b')$ with $\sum_{Fb'=j} F({}_{b'}f_a^\bullet) = f$.

For each object b' with $Fb'=j$, there is a unique $g_{b'} \in G$ with $g_{b'}(b')=b$. Then $(g_{b'}({}_{b'}f_a^\bullet))_{b'} \in \bigoplus_{b'} A(g_{b'}(a),b) = \bigoplus_{Fa'=i} A(a',b)$ with $\sum_{b'} F(g_{b'}({}_{b'}f_a^\bullet)) = \sum_{b'} F({}_{b'}f_a^\bullet) = f$. Hence $g_{b'}({}_{b'}f_a^\bullet) = {}_b^\bullet f_{g_{b'}(a)}$ for every $Fb'=j$. In particular, for $g_b=1$ we get ${}_bf_a^\bullet = {}_b^\bullet f_a$. $\square$

2.2. The smash-product

We say that a k-category B is *G-graded* with respect to the group G if for each pair of objects i,j there is a vector space decomposition $B(i,j) = \bigoplus_{g\in G} B^g(i,j)$ such that the composition induces linear maps

$$B^g(i,j) \otimes B^h(j,m) \to B^{gh}(i,m).$$

Then the *smash product* $B \# G$ is the k-category with objects $B_0 \times G$, and for pairs $(i,g),(j,h) \in B_0 \times G$, the set of morphisms is

$$(B \# G)\big((i,g),(j,h)\big) = B^{g^{-1}h}(i,j)$$

with compositions induced in natural way.

In [3] it was shown that $B \# G$ accepts a free action of G such that

$$(B \# G)/G \xrightarrow{\sim} B.$$

Moreover, if $B = A/G$ is a quotient, then B is a G-graded k-category and

$$(A/G) \# G \xrightarrow{\sim} A.$$

Proposition 2.2. *Let $F : A \to B$ be a covering functor and assume that B is a G-graded k-category. Then*

(i) *Assume A accepts a G-grading compatible with F, that is, $F(A^g(a,b)) \subseteq B^g(Fa, Fb)$, for every pair $a, b \in A_0$ and $g \in G$. Then there is a covering functor $F \# G : A \# G \to B \# G$ completing a commutative square*

$$\begin{tikzcd} A \# G \arrow[r, "F\#G"] \arrow[d] & B \# G \arrow[d] \\ A \arrow[r, "F"] & B \end{tikzcd}$$

where the vertical functors are the natural quotients. Moreover F is balanced if and only if $F \# G$ is balanced.

(ii) *In case B is a schurian algebra, then A accepts a G-grading compatible with F.*

Proof. (i) For each $a, b \in A_0$, consider the decomposition $A(a,b) = \bigoplus_{g\in G} A^g(a,b)$ and $B(Fa, Fb) = \bigoplus_{g\in G} B^g(Fa, Fb)$. Since these decompositions are compatible with F, then $A^g(a,b) = F^{-1}(B^g(Fa, Fb))$, for every $g \in G$.

For $\alpha \in (A \# G)((a,g),(b,h)) = A^{g^{-1}h}(a,b) = F^{-1}(B^{g^{-1}h}(Fa, Fb))$, we have

$$(F \# G)(\alpha) = F\alpha \in B^{g^{-1}h}(Fa, Fb) = (B \# G)\big((Fa, g), (Fb, h)\big).$$

(ii) Assume B is schurian and take $a, b \in A_0$ and $g \in G$. Either $B^g(Fa, Fb) = B(Fa, Fb) \neq 0$, if $A(a,b) \neq 0$ or $B^g(Fa, Fb) = 0$, correspondingly we set $A^g(a,b) = A(a,b)$ or $A^g(a,b) = 0$. Observe that the composition induces linear maps $A^g(a,b) \otimes A^h(b,c) \to A^{gh}(a,c)$, hence A accepts a G-grading compatible with F. $\square$

Remark. In the situation above, the fact that A and $B \# G$ are connected categories does not guaranty that $A \# G$ is connected. For instance, if $B = A/G$, then $A \# G = A \times G$.

The following result is a generalization of Proposition 2.2(ii).

Proposition 2.3. *Let $F : A \to B$ be a (balanced) covering functor induced from a map of quivers with relations. Let $F' : B' \to B$ be a Galois covering functor induced from a map of quivers with relations defined by the action of a group G. Assume moreover that B' is schurian. Then A accepts a G-grading compatible with F making the following diagram commutative.*

$$\begin{tikzcd} A \# G \arrow[r, "F\#G"] \arrow[d] & B' \arrow[d, "F'"] \\ A \arrow[r, "F"] & B \end{tikzcd}$$

Proof. Let $A = k\Delta/J$, $B = kQ/I$ and $B' = kQ'/I'$ be the corresponding presentations as quivers with relations, F induced from the map $\delta : \Delta \to Q$, while F' induced from the map $q : Q' \to Q$. For each vertex a in Δ fix a vertex a' in Q' such that $F'a' = Fa$.

Consider an arrow $a \xrightarrow{\alpha} b$ in Δ and $\overline{\alpha}$ the corresponding element of A. We claim that there exists an element $g_\alpha \in G$ such that $F(\overline{\alpha}) \in B^{g_\alpha}(Fa, Fb)$. Indeed, we get $F(\overline{\alpha}) = \overline{\beta} = F'(\overline{\beta'})$ for arrows $Fa \xrightarrow{\beta} Fb$ and $a' \xrightarrow{\beta'} g_\alpha b'$ for a unique $g_\alpha \in G$. Therefore $F(\overline{\alpha}) \in B^{g_\alpha}(Fa, Fb)$. We shall define $A^{g_\alpha}(a, b)$ as containing the space $k\overline{\alpha}$. For this purpose, consider $g \in G$ and any vertices a, b in Δ, then $A^g(a, b)$ is the space generated by the classes $\bar{u}$ of the paths $u : a \to b$ such that $F(\bar{u}) \in B^g(Fa, Fb)$. Since the classes of the arrows in Δ generate A, then $A(a, b) = \bigoplus_{g\in G} A^g(a, b)$. We shall prove that there are linear maps

$$A^g(a, b) \otimes A^h(b, c) \to A^{gh}(a, c).$$

Indeed, if $\bar{u} \in A^g(a, b)$ and $\bar{v} \in A^h(b, c)$ for paths $u : a \to b$ and $v : b \to c$ in Δ, let $F(\bar{u}) = F'(\overline{u'})$ and $F(\bar{v}) = F'(\overline{v'})$ for paths $u' : a' \to gb'$ and $v' : b' \to hc'$ in Q'. Since B' is schurian then the class of the lifting of $F(\overline{vu})$ to B' is $\overline{(gv')u'}$. Therefore

$$F(\bar{v})F(\bar{u}) = F'\big(\overline{(gv')u'}\big) \in B^{gh}(Fa, Fb).$$

By definition, the G-grading of A is compatible with F. We get the commutativity of the diagram from Proposition 2.2. □

2.3. Universal Galois covering

Let $B = kQ/I$ be a finite-dimensional k-algebra. According to [9] there is a k-category $\tilde{B} = k\tilde{Q}/\tilde{I}$ and a Galois covering functor $\tilde{F} : \tilde{B} \to B$ defined by the action of the fundamental group $\pi_1(Q, I)$ which is *universal* among all the Galois coverings of B, that is, for any Galois covering $F : A \to B$ there is a covering functor $F' : \tilde{B} \to A$ such that $\tilde{F} = FF'$. In fact, the following more general result is implicitly shown in [9]:

Proposition 2.4. *(See [9].) The universal Galois covering $\tilde{F} : \tilde{B} \to B$ is universal among all (balanced) covering functors $F : A \to B$ induced from a map $q : (Q', I') \to (Q, I)$ of quivers with relations, where $A = kQ'/I'$.*

3. Cleaving functors

3.1. Balanced coverings are cleaving functors

Consider the k-linear functor $F : A \to B$ and the natural transformation $F(a, b) : A(a, b) \to B(Fa, Fb)$ in two variables. The following is the main observation of this work.

Theorem 3.1. *Assume $F : A \to B$ is a balanced covering, then the natural transformation $F(a, b) : A(a, b) \to B(Fa, Fb)$ admits a retraction $E(a, b) : B(Fa, Fb) \to A(a, b)$ of functors in two variables a, b such that $E(a, b)F(a, b) = \mathbf{1}_{A(a,b)}$ for all $a, b \in A_0$.*

Proof. Set $E(a, b) : B(Fa, Fb) \to A(a, b)$, $f \mapsto {}^{\bullet}_{b}f_a$ which is a well-defined map. For any $\alpha \in A(a, a')$, $\beta \in A(b, b')$, we shall prove the commutativity of the diagrams:

$$\begin{array}{ccc} B(Fa, Fb) & \xrightarrow{E(a,b)} & A(a, b) \\ {\scriptstyle B(Fa, F\beta)}\downarrow & & \downarrow{\scriptstyle A(a,\beta)} \\ B(Fa, Fb') & \xrightarrow[E(a,b')]{} & A(a, b') \end{array} \qquad \begin{array}{ccc} B(Fa', Fb) & \xrightarrow{E(a',b)} & A(a', b) \\ {\scriptstyle B(F\alpha, Fb)}\downarrow & & \downarrow{\scriptstyle A(\alpha,b)} \\ B(Fa, Fb) & \xrightarrow[E(a,b)]{} & A(a, b) \end{array}$$

For the sake of clarity, let us denote by $\circ$ the composition of maps. Indeed, let $f \in B(Fa, Fb)$ and calculate $\sum_{Fa'=Fa} F(\beta \circ {}^{\bullet}_{b} f_{a'}) = F\beta \circ f$, hence

$$A(a,\beta) \circ E(a,b)(f) = \beta \circ {}^{\bullet}_{b} f_a = {}^{\bullet}_{b'}(F\beta \circ f)_a = E(a,b') \circ B(Fa, F\beta)(f),$$

and the first square commutes. Moreover, let $h \in B(Fa', Fb)$ and calculate $\sum_{Fb'=Fb} F({}_{b'}h^{\bullet}_{a'} \circ \alpha) = h \circ F\alpha$ and therefore ${}_{b}h^{\bullet}_{a'} \circ \alpha = {}_{b}(h \circ F\alpha)^{\bullet}_{a}$. Using that F is balanced we get that $E(a,b) \circ B(Fa, Fb)(h) = {}^{\bullet}_{b}(h \circ F\alpha)_a = {}^{\bullet}_{b}h_{a'} \circ \alpha = A(\alpha, b) \circ E(a', b)(h)$. $\square$

Given a k-linear functor $F: A \to B$ the composition $F_{\bullet}F_{\lambda}: {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$ is connected to the identity $\mathbf{1}$ of ${}_A\mathrm{Mod}$ by a *canonical transformation* $\varphi: F_{\bullet}F_{\lambda} \to \mathbf{1}$ determined by $F_{\bullet}F_{\lambda}A(a,-)(b) = \bigoplus_{Fb'=Fb} A(a,b') \to A(a,b)$, $(f_{b'}) \mapsto f_b$, see [1, p. 234]. Following [1], F is a *cleaving functor* if the canonical transformation φ admits a natural section $\varepsilon: \mathbf{1} \to F_{\bullet}F_{\lambda}$ such that $\varphi(X)\varepsilon(X) = \mathbf{1}_X$ for each $X \in {}_A\mathrm{Mod}$. The following statement, essentially from [1], yields Theorem 0.1 in the introduction.

Corollary 3.2. *Let $F: A \to B$ be a balanced covering, then F is a cleaving functor.*

Proof. Observe that $F_{\bullet}F_{\lambda}$ is exact, preserves direct sums and projectives (the last property holds since $F_{\bullet}B(i,-) = \bigoplus_{Fa=i} A(a,-)$). Hence to define $\varepsilon: \mathbf{1} \to F_{\bullet}F_{\lambda}$ it is enough to define $\varepsilon(A(a,-)): A(a,-) \to F_{\bullet}F_{\lambda}A(a,-)$ with the desired properties. For $b \in A_0$, consider $\varepsilon_b: A(a,b) \to \bigoplus_{Fb'=Fb} A(a,b') = F_{\bullet}F_{\lambda}A(a,-)(b)$ the canonical inclusion. For $h \in A(b,c)$ we shall prove the commutativity of the following diagram:

$$\begin{tikzcd} A(a,b) \arrow[r, "\varepsilon_b"] \arrow[d, "A(a,h)"'] & \bigoplus_{Fb'=Fb} A(a,b') \arrow[d, "(A(a, {}^{\bullet}_{c'}Fh_{b'})"] \\ A(a,c) \arrow[r, "\varepsilon_c"] & \bigoplus_{Fc'=Fc} A(a,c') \end{tikzcd}$$

Let $f \in A(a,b)$, since F is balanced $A(a, {}^{\bullet}_{c'}Fh_{b'}) \circ \varepsilon_b(f) = {}^{\bullet}_{c'}Fh_b \circ f = {}_{c'}Fh^{\bullet}_{b} \circ f = \varepsilon_c \circ A(a,h)(f)$, since ${}_{c'}Fh^{\bullet}_{b} = h$ if $c' = c$ and it is 0 otherwise. This is what we wanted to show. $\square$

4. On the representation type of categories

4.1. Representation-finite case

Recall that a k-category A is said to be *locally representation-finite* if for each object a of A there are only finitely many indecomposable A-modules X, up to isomorphism, such that $X(a) \neq 0$. For a cleaving functor $F: A \to B$ is was observed in [1] that in case B is of locally representation-finite then so is A. In particular this holds when F is a Galois covering by [6]. We shall generalize this result for covering functors.

Part (a) of Theorem 0.2 in the introduction is the following:

Theorem 4.1. *Assume that k is algebraically closed and let $F: A \to B$ be a covering induced from a map of quivers with relations. Then B is locally representation-finite if and only if so is A. Moreover in this case the functor $F_{\lambda}: {}_A\mathrm{mod} \to {}_B\mathrm{mod}$ preserves indecomposable modules and Auslander–Reiten sequences.*

Proof. Let $F: A \to B$ be induced from $q: (Q', I') \to (Q, I)$ where $A = kQ'/I'$ and $B = kQ/I$. Let $\tilde{B} = k\tilde{Q}/\tilde{I}$ be the universal cover of B and $\tilde{F}: \tilde{B} \to B$ the universal covering functor. By Proposition 2.4 there is a covering functor $F': \tilde{B} \to A$ such that $\tilde{F} = FF'$.

(1) Assume that B is a connected locally representation-finite category. Since F is induced by a map of quivers with relations, then Proposition 1.7 implies that F is balanced. Hence Corollary 3.2

implies that F is a cleaving functor. By [1, (3.1)], A is locally representation-finite; for the sake of completness, recall the simple argument: each indecomposable A-module $X \in {}_A\mathrm{mod}$ is a direct summand of $F_\bullet F_\lambda X = \bigoplus_{i=1}^n F_\bullet N_i^{n_i}$ for a finite family $N_1, \dots, N_n$ of representatives of the isoclasses N of the indecomposable B-modules with $N(i) \neq 0$ for some $i = F(a)$ with $X(a) \neq 0$.

(2) Assume that A is a locally representation-finite category. First we show that B is representation-finite. Indeed, by case (1), since $F' : \tilde{B} \to A$ is a covering induced by a map of a quiver with relations, then $\tilde{B}$ is locally representation-finite. By [10], B is representation-finite. In particular, [6] implies that $\tilde{F}_\lambda$ preserves indecomposable modules, hence F_λ and F'_λ also preserve indecomposable modules.

Let X be an indecomposable A-module. We shall prove that X is isomorphic to $F'_\lambda N$ for some indecomposable $\tilde{B}$-module N. Since indecomposable projective A-modules are of the form $A(a,-) = F'_\lambda \tilde{B}(x,-)$ for some x in $\tilde{B}$, using the connectedness of Γ_A, we may assume that there is an irreducible morphism $Y \xrightarrow{f} X$ such that $Y = F'_\lambda N$ for some indecomposable $\tilde{B}$-module N. If N is injective, say $N = D\tilde{B}(-, j)$, there is a surjective irreducible map $(h_i) : N \to \bigoplus_i N_i$ such that all N_i are indecomposable modules and

$$0 \longrightarrow S_j \longrightarrow N \xrightarrow{(h_i)} \bigoplus_i N_i \longrightarrow 0$$

is an exact sequence. Then $Y = DA(-, F'j)$ and the exact sequence

$$0 \longrightarrow S_{F'j} \longrightarrow Y \xrightarrow{(F'_\lambda(h_i))} \bigoplus_i F'_\lambda(N_i) \longrightarrow 0$$

yields the irreducible maps starting at Y (ending at the indecomposable modules $F'_\lambda(N_i)$). Therefore $X = F'_\lambda(N_r)$ for some r, as desired. Next, assume that N is not injective and consider the Auslander–Reiten sequence $\xi : 0 \to N \xrightarrow{g} N' \xrightarrow{g'} N'' \to 0$ in ${}_{\tilde{B}}\mathrm{mod}$. We shall prove that the push-down $F'_\lambda \xi : 0 \to F'_\lambda N \xrightarrow{F'_\lambda g} F'_\lambda N' \xrightarrow{F'_\lambda g'} F'_\lambda N'' \to 0$ is an Auslander–Reiten sequence in ${}_A\mathrm{mod}$. This implies that there exists a direct summand $\bar{N}$ of N' such that $X \xrightarrow{\sim} F'_\lambda \bar{N}$ which completes the proof of the claim.

To verify that $F'_\lambda \xi$ is an Auslander–Reiten sequence, let $h : F'_\lambda N \to Z$ be non-split mono in ${}_A\mathrm{mod}$. Consider $\mathrm{Hom}_A(F'_\lambda N, Z) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{B}}(N, F'_\bullet Z)$, $h \mapsto h'$ which is not a split mono (otherwise, then $\mathrm{Hom}_{\tilde{B}}(F'_\bullet Z, N) \xrightarrow{\sim} \mathrm{Hom}_A(Z, F'_\rho N)$, $\nu \mapsto \nu'$ with $\nu h' = 1_{F'_\bullet Z}$. By Lemma 1.3, $F'_\lambda = F'_\rho$ and $\nu' h = 1_Z$). Then there is a lifting $\bar{h} : N' \to F'_\bullet Z$ with $\bar{h} g = h'$. Hence $\mathrm{Hom}_{\tilde{B}}(N', F'_\bullet Z) \xrightarrow{\sim} \mathrm{Hom}_A(F'_\lambda N', Z)$, $\bar{h} \mapsto \bar{h}'$ with $\bar{h}' F'_\lambda g = h$.

We show that F_λ preserves Auslander–Reiten sequences. Let X be an indecomposable A-module of the form $X = F'_\lambda N$ for an indecomposable $\tilde{B}$-module N. Then $F_\lambda X = F_\lambda F'_\lambda N = \tilde{F}_\lambda N$. Since by [10], $\tilde{F}_\lambda$ preserves indecomposable modules, then $F_\lambda X$ is indecomposable. Finally, as above, we conclude that F_λ preserves Auslander–Reiten sequences. □

4.2. *Tame representation case*

Let k be an algebraically closed field. We recall that A is said to be of *tame representation type* if for each dimension $d \in \mathbb{N}$ and each object $a \in A_0$, there are finitely many $A - k[t]$-bimodules $M_1, \dots, M_s$ which satisfy:

(a) M_i is finitely generated free as right $k[t]$-module $i = 1, \dots, s$;
(b) each indecomposable $X \in {}_A\mathrm{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is isomorphic to some module of the form $M_i \otimes_{k[t]} (k[t]/(t-\lambda))$ for some $i \in \{1, \dots, s\}$ and $\lambda \in k$.

In fact, it is shown in [11] that A is tame if (a) and (b) are substituted by the weaker conditions:

(a′) M_i is finitely generated as right $k[t]$-module $i = 1, \dots, s$;
(b′) each indecomposable $X \in {}_A\mathrm{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is a direct summand of a module of the form $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \dots, s\}$ and $\lambda \in k$.

The following statement covers claim (b) of Theorem 0.2 in the introduction.

Theorem 4.2. *Let $F : A \to B$ be a balanced covering functor. If B is tame, then A is tame.*

Proof. Let $a \in A_0$ and $d \in \mathbb{N}$. Let $M_1, \dots, M_s$ be the $B - k[t]$-bimodules satisfying (a) and (b): each indecomposable $M \in {}_B\mathrm{mod}$ with $M(Fa) \neq 0$ and $\dim_k M \leqslant d$ is isomorphic to some $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \dots, s\}$ and $\lambda \in k$.

By Corollary 3.2 each indecomposable $X \in {}_A\mathrm{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is a direct summand of some $F_\bullet(M_i \otimes_{k[t]} (k[t]/(t - \lambda)))$, which is isomorphic to $F_\bullet M_i \otimes_{k[t]} (k[t]/(t - \lambda))$, for some $i \in \{1, \dots, s\}$ and $\lambda \in k$. Hence A satisfies conditions (a') and (b'). $\square$

References

[1] R. Bautista, P. Gabriel, A.V. Roiter, L. Salmerón, Representation-finite algebras and multiplicative bases, Invent. Math. 81 (1984) 217–285.
[2] K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (3) (1982) 331–378.
[3] C. Cibils, E. Marcos, Skew categories. Galois coverings and smash product of a category over a ring, Proc. Amer. Math. Soc. 134 (2006) 39–50.
[4] P. Dowbor, A. Skowroński, Galois coverings of representation infinite algebras, Comment. Math. Helv. 62 (1987) 311–337.
[5] P. Gabriel, Auslander–Reiten sequences and representation-finite algebras, in: Proc. ICRA II. Representation of Algebras, Ottawa, 1979, in: Lecture Notes in Math., vol. 831, Springer, 1980, pp. 1–71.
[6] P. Gabriel, The universal cover of a representation-finite algebra, in: Proc. Representation Theory I, Puebla, 1980, in: Lecture Notes in Math., vol. 903, Springer, 1981, pp. 68–105.
[7] Ch. Geiss, J.A. de la Peña, An interesting family of algebras, Arch. Math. 60 (1993) 25–35.
[8] C.D. Godsil, Algebraic Combinatorics, Chapman & Hall/CRC Math., Chapman and Hall, 1993.
[9] R. Martínez-Villa, J.A. de la Peña, The universal cover of a quiver with relations, J. Pure Appl. Algebra 30 (3) (1983) 277–292.
[10] R. Martínez-Villa, J.A. de la Peña, Automorphisms of representation-finite algebras, Invent. Math. 72 (1983) 359–362.
[11] J.A. de la Peña, Functors preserving tameness, Fund. Math. 137 (1991) 177–185.
[12] J.A. de la Peña, On the dimension of the module-varieties of tame and wild algebras, Comm. Algebra 19 (6) (1991) 1795–1807.
[13] Ch. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurüch, Comment. Math. Helv. 55 (1980) 199–224.