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On weak-injective modules over integral domains

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ABSTRACT

We show that a weak-injective module over an integral domain need not be pure-injective (Theorem 2.3). Equivalently, a torsion-free Enochs-cotorsion module over an integral domain is not necessarily pure-injective (Corollary 2.4). This solves a well-known open problem in the negative.

In addition, we establish a close relation between flat covers and weak-injective envelopes of a module (Theorem 3.1). This yields a method of constructing weak-injective envelopes from flat covers (and *vice versa*). Similar relation exists between the Enochs-cotorsion envelopes and the weak dimension $\leqslant 1$ covers of modules (Theorem 3.2).

1. Introduction

All modules are over a fixed integral domain R. For unexplained terminology and basic results we refer to Fuchs and Salce [5], Enochs and Jenda [4], and Göbel and Trlifaj [7].

Weak-injective modules have been introduced by Lee [9] as R-modules M satisfying $\operatorname{Ext}^1_R(W, M) = 0$ for all R-modules W of weak dimension $\leqslant 1$. The class $\mathcal{F}_1$ of modules of weak dimension $\leqslant 1$ and the class $\mathcal{W}$ of weak-injective modules form a cotorsion pair $\mathfrak{C} = (\mathcal{F}_1, \mathcal{W})$; for details we refer to Göbel and Trlifaj [7]. In Fuchs and Lee [6] it was shown that $\mathfrak{C}$ coincides with the cotorsion pair $\mathfrak{D} = (\mathcal{P}_1, \mathcal{D})$ if and only if R is an almost perfect domain in the sense of Bazzoni and Salce [3]. Here $\mathcal{P}_1$ denotes the class of R-modules of projective dimension $\leqslant 1$, while $\mathcal{D}$ stands for the class of divisible modules (see Bazzoni and Herbera [2]).

In [9] it was shown that h-divisible pure-injective R-modules are always weak-injective, but the converse implication remained an open problem. In view of the Matlis category equivalence, this

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problem turns out to be equivalent to the better known old open problem as to whether or not all torsion-free Enochs-cotorsion modules are pure-injective. We recall that an R-module M is said to be *cotorsion in the sense of Enochs*, or briefly *Enochs-cotorsion*, if it satisfies $\operatorname{Ext}^1_R(F, M) = 0$ for all flat R-modules F. The answer to these equivalent problems is in the positive if R happens to be a Prüfer domain, in which case Enochs-cotorsion modules are Warfield-cotorsion and the torsion-free ones are RD-injective; see e.g. Fuchs and Salce [5, Lemma 8.1, p. 458].

In the first part of this note we solve the two equivalent problems by showing that the answer is in the negative whenever R is an almost perfect, non-Dedekind domain; see Theorem 2.3 and Corollary 2.4. Though the cokernels of torsion-free Enochs-cotorsion modules in their injective hulls need not be pure-injective, over a coherent domain, the cokernel of an arbitrary Enochs-cotorsion module in its weak-injective envelope is always pure-injective (Proposition 2.6).

The second part of this note is devoted to a study of the weak-injective envelopes of modules, in particular, their relation to flat covers. Since the class of weak-injective R-modules is closed under extensions and contains all injective R-modules, the cotorsion pair $\mathfrak{C}$ mentioned above is perfect (see Göbel and Trlifaj [7, p. 106]), it follows that all R-modules admit weak-injective envelopes, i.e., every R-module can be embedded in a minimal weak-injective module with cokernel of weak dimension $\leqslant 1$. In Theorem 3.1 we show that there is a close relation between the flat cover (whose existence is guaranteed by the well-known theorem of Bican, El Bashir and Enochs [1]) and the weak-injective envelope of any R-module; this relation can be best illustrated by the first diagram in Section 3.

We also mention a sort of dual to Theorem 3.1. This is Theorem 3.2 that reveals a similar close connection between the so-called $\mathcal{F}_1$-cover and the Enochs-cotorsion envelope of a module.

2. Weak-injective modules and pure-injectivity

Before stating a main result of this note (Theorem 2.3) which gives an answer to the above mentioned open problem, we quote two lemmas that are crucial for the proof.

We recall a relevant definition: *almost perfect domains* were defined by Bazzoni and Salce [3] as domains all of whose proper quotients are perfect rings in the sense of H. Bass.

Lemma 2.1. *(See Fuchs and Lee [6].) A domain R has the property that h-divisibility is equivalent to weak-injectivity if and only if R is an almost perfect domain.*

Lee [8] called a domain *semi-Dedekind* if all of its h-divisible modules are pure-injective.

Lemma 2.2. *(See Salce [11, Corollary 2.5].) Semi-Dedekind domains are Dedekind domains.*

We are now able to prove:

Theorem 2.3. *Over an almost perfect domain R that is not a Dedekind domain, there exist weak-injective modules that fail to be pure-injective.*

Proof. Since R is not Dedekind, by Lemma 2.2 there is an h-divisible R-module D that is not pure-injective. The existence of such a D establishes our claim: by virtue of Lemma 2.1, R almost perfect implies that D is weak-injective. $\square$

Recall that there exist several examples of almost perfect domains that are not Dedekind. For instance, every noetherian domain of Krull dimension 1 is known to be almost perfect (these are exactly the almost perfect domains that are coherent). For examples of non-noetherian almost perfect domains we refer to the paper Bazzoni and Salce [3].

Lee [10] proved that the statement that all weak-injective R-modules are pure-injective is equivalent to saying that all torsion-free Enochs-cotorsion modules are pure-injective by showing that over a domain R, in the Matlis category equivalence, the weak-injective (resp. the h-divisible pure-injective) torsion modules D correspond to the Enochs-cotorsion (resp. pure-injective) torsion-free modules M. (Recall: corresponding modules D and M are connected by the exact sequence $0 \to M \to E \to D \to 0$ where E is torsion-free divisible, the injective hull of M.)

It has been an open question for a while whether or not torsion-free Enochs-cotorsion modules ought to be pure-injective. In view of the equivalence of this question with the one settled in our theorem above, we can conclude at once:

Corollary 2.4. *Let R be an almost perfect domain that is not Dedekind. Then there exists a torsion-free Enochs-cotorsion R-module M that is not pure-injective.*

Note that in view of the Matlis category equivalence such an M can be obtained from a weak-injective, not pure-injective R-module D by taking

$$M = \mathrm{Hom}_R(Q/R, D)$$

where Q denotes the field of quotients of R. (D can be recaptured from M as $D = Q/R \otimes_R M$.)

It is worthwhile pointing out that if the domain R is coherent, then we can claim something positive. In fact, we have the following result:

Proposition 2.5. *Over a coherent domain, we have*:

(i) *an h-divisible module of weak dimension $\leqslant 1$ is weak-injective if and only if it is pure-injective*; *equivalently,*
(ii) *a flat module is Enochs-cotorsion if and only if it is pure-injective.*

Proof. Claim (ii) is the same as [5, Lemma 6.3, p. 451]: over a coherent domain R, a flat module M is pure-injective if and only if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat R-modules F. □

Note that the preceding proposition fails to hold if 'flat' is replaced by 'torsion-free', as is shown by noetherian domains of Krull dimension 1 that are not Dedekind. Thus the cokernel of a torsion-free Enochs-cotorsion module in its injective hull need not be pure-injective. However, the following holds for all Enochs-cotorsion modules over a coherent domain.

Proposition 2.6. *Over a coherent domain, the cokernel of an Enochs-cotorsion module in its weak-injective envelope is pure-injective.*

Proof. We refer to Theorem 3.1 and its diagram. If A is an Enochs-cotorsion module, then its flat cover F is a flat Enochs-cotorsion module, so D is pure-injective. But D is the cokernel of A in its weak-injective envelope W. □

It remains an open problem to characterize the domains over which all torsion-free Enochs-cotorsion modules are pure-injective.

We still owe an example of a weak-injective module that is not pure-injective.

Example 2.7. Let R denote a non-noetherian almost perfect domain, and Q its field of quotients. Let J be a countably generated ideal of R, say, generated by the elements $r_n \in R$ with $n < \omega$, where we may assume without loss of generality that the ideals $J_n = R(r_0, \dots, r_n)$ $(n < \omega)$ form a properly ascending chain with union J. Define ϕ_n as the natural homomorphism $Q \to Q/J_n$ $(n < \omega)$, and let

$$A = \bigoplus_{n<\omega} \phi_n Q.$$

This is clearly an h-divisible R-module, and hence weak-injective, R being almost perfect. We show that it is not pure-injective by exhibiting a countable system of linear equations over A that is not solvable in A, though each of its finite subsystems admits a solution in A (see [5, Chapter XIII, Section 3]). x is the single unknown in the system of equations

$$r_n x = a_n \quad (n < \omega)$$

where

$$a_n = (\phi_0 r_n, \phi_1 r_n, \ldots, \phi_n r_n = 0,\ \phi_{n+1} r_n = 0, \ldots) \in A.$$

For every n, the subsystem consisting of the first n equations is solvable in A, e.g.

$$x = (\phi_0 1, \phi_1 1, \ldots, \phi_n 1, 0, 0, \ldots) \in A$$

is a solution. But the entire system cannot have a solution in A, since it would require an element with infinitely many non-zero coordinates.

The following much simpler example was suggested by the referee.

Example 2.8. Let R be as in the preceding example. As R is not noetherian, there is an infinite set $\{E_i \mid i \in I\}$ of injective (torsion) R-modules such that their direct sum

$$D = \bigoplus_{i \in I} E_i$$

is not injective. D is divisible, so weak-injective by Lemma 2.1. On the other hand, D is not pure-injective: it is a pure submodule of the direct product $P = \prod_{i \in I} E_i$, but it is not a summand in it, since otherwise it would be injective.

We were unable to find a noetherian example.

3. Weak-injective envelopes

We turn our attention to the weak-injective envelopes of R-modules. As mentioned before, their existence is ensured by general theorems on perfect cotorsion pairs. However, we do not know of any method that leads to their construction except for flat modules: if F is a flat module, then its injective hull E is at the same time its weak-injective envelope (observe that w.d. $E/F \leqslant 1$). However, in general, the weak-injective envelope is not even contained in the injective hull.

Let A be an R-module and $0 \to H \to F \xrightarrow{\alpha} A \to 0$ an exact sequence where (F, α) is the flat cover of A; by Bican, El Bashir and Enochs [1] flat covers always exist, and the kernels H are reduced Enochs-cotorsion modules. Let E be the injective hull of F, and define $W = E/H$. This leads to a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
0 \arrow[r] & F \arrow[r, "\epsilon"] \arrow[d, "\alpha"'] & E \arrow[r] \arrow[d, "\beta"'] & D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\gamma"] \arrow[d] & W \arrow[r, "\delta"] \arrow[d] & D \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. Here D is an h-divisible torsion module of weak dimension $\leqslant 1$. Evidently, W is also h-divisible as an epimorphic image of E, so we can write $W = V \oplus T$ where V is torsion-free and T is torsion. By the Matlis category equivalence, the h-divisible torsion T corresponds to the R-complete torsion-free module H. As H is Enochs-cotorsion, by Lee [10] T, and hence also W, is weak-injective. That E is a flat pre-cover of W follows from the exactness of the sequence

$$0 \to \mathrm{Hom}_R(G, H) \to \mathrm{Hom}_R(G, E) \to \mathrm{Hom}_R(G, W) \to \mathrm{Ext}^1_R(G, H) = 0$$

for any flat module G (the Ext vanishes, since G is flat and H is Enochs-cotorsion): every map $G \to W$ is a composite of some $G \to E$ and β. Since H – being reduced – cannot contain any non-zero summand of E, it follows that the pre-cover E is a cover.

Analogous argument shows that W is a weak-injective pre-envelope of A, since the cokernel D is of weak dimension $\leqslant 1$. In order to verify that it is actually the envelope, suppose that $W = W_1 \oplus W_2$ with $\mathrm{Im}\,\gamma \leqslant W_1$. E will have a corresponding decomposition $E = E_1 \oplus E_2$, i.e., β will carry E_i to W_i for $i = 1, 2$ (observe that covers respect direct sums). H decomposes accordingly, $H = H_1 \oplus H_2$. δ carries W_2 into D isomorphically, thus $D \cong \delta W_1 \oplus W_2$. The direct decompositions of E and D imply that $F = F_1 \oplus F_2$ with $F_i = \mathrm{Ker}(E_i \to \delta W_i)$. Then $\beta\epsilon F_2 = \gamma\alpha F_2 \leqslant \alpha A \leqslant W_1$, and $\beta\epsilon F_2 \leqslant \beta E_2 = W_2$; consequently, $\gamma\alpha F_2 = 0$, whence $\alpha F_2 = 0$ follows. This shows that F_2 is a summand of F contained in H, so $F_2 = 0$, F being a flat cover of A. Hence $E_2 = 0 = W_2$, and W is the weak-injective envelope of A.

Conversely, suppose that in the above diagram we start with the bottom exact sequence with (W, γ) as the weak-injective envelope of A; the existence of such an exact sequence is guaranteed by Lee [9] or Göbel and Trlifaj [7]. Next we take the flat cover (E, β) of W; in view of Lee [10], flat covers of weak-injectives are torsion-free injectives, so E is torsion-free injective. Let $H = \mathrm{Ker}\,\beta$, and define $\alpha : F \to A$ as the restriction of β. Since w.d. $D \leqslant 1$, F ought to be flat. H being reduced Enochs-cotorsion implies that F is a flat pre-cover of A. To show that it is a cover, suppose that $F = F_1 \oplus F_2$ with $H = H_1 \oplus F_2$. There is a corresponding decomposition $E = E_1 \oplus E_2$ with $F_i \leqslant E_i$, thus $W \cong E_1/H_1 \oplus E_2/F_2$. As $\alpha F_2 = 0$ implies $\epsilon F_2 \leqslant \mathrm{Ker}\,\beta$, we conclude that $\gamma A \leqslant E_1/H_1$, showing that $F_2 = 0$, i.e. F is the flat cover of A.

We have thus proved:

Theorem 3.1. *Suppose that in the above commutative diagram with exact rows and columns, A is an arbitrary R-module. Furthermore, let F be a flat module and E the injective hull (= weak-injective envelope) of F. Then F is the flat cover of A if and only if W is the weak-injective envelope of A.*

We thank the referee for pointing out that in the preceding proof the argument showing that the special pre-envelopes (pre-covers) are actually envelopes (covers) can be replaced by imitating

Xu's argument in [12, Theorem 3.4.8] that the maps involved are minimal. (There Xu proves that if a module M over a coherent ring has a flat cover, then it also has a cotorsion envelope.) Our diagram above is similar to Xu's.

The preceding theorem can be applied to find the weak-injective envelope (W, γ) of an arbitrary R-module A once its flat cover (F, α) is available (and *vice versa*). The torsion part of W will be the h-divisible module T corresponding to the torsion-free Enochs cotorsion module $\operatorname{Ker}\alpha$ in the Matlis category equivalence. The torsion-free part of W will be the direct sum of as many copies of Q as the torsion-free rank of A, since from the diagram it is clear that D torsion implies that the corank of H in F is the same as its corank in E.

It is worthwhile pointing out that there is an entirely analogous result that might be of independent interest. Given an arbitrary R-module A, consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & B \arrow[r, equal] \arrow[d] & B \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Theorem 3.2. *Assume that in the preceding commutative diagram with exact rows and columns, N has weak dimension $\leqslant 1$, and M is the Enochs-cotorsion envelope of N (so F is flat). Then N is the $\mathcal{F}_1$-cover of A if and only if C is the Enochs-cotorsion envelope of A.*

Proof. The proof is entirely similar to the proof of Theorem 3.1, so the details may be left for the reader. (It might be helpful to observe that in the implication $\Rightarrow$ one can argue that a summand N' of B contained in N will also be a summand of M, since $\operatorname{Ext}^1_R(F, N') = 0$, N' being Enochs-cotorsion.) $\square$

Finally, let us point out that the Matlis category equivalence yields a close relation between the $\mathcal{F}_1$-cover of an h-divisible torsion module D and the flat cover of the corresponding R-complete torsion-free module M. If $0 \to B \to A \xrightarrow{\alpha} D \to 0$ is an exact sequence with (A, α) the $\mathcal{F}_1$-cover of D (here B is weak-injective), then the induced exact sequence

$$0 \to \operatorname{Hom}_R(K, B) \to \operatorname{Hom}_R(K, A) \to \operatorname{Hom}_R(K, D) = M \to 0$$

(where $K = Q/R$ with Q the quotient field of R) provides a flat cover of M. In fact, the pre-cover property of $\operatorname{Hom}_R(K, A)$ is the consequence of $\operatorname{Hom}_R(K, B)$ being Enochs-cotorsion and $\operatorname{Hom}_R(K, A)$ being flat. Since tensoring this sequence with K brings us back to the original exact sequence, the first Hom cannot contain any non-zero summand of the second Hom, because this holds for A and B.

In a similar fashion, the weak-injective envelope of an h-divisible torsion module D is related to the Enochs-cotorsion envelope of the corresponding R-complete torsion-free module M.

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