A presentation of the trace algebra of three 3×3 matrices

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A R T I C L E   I N F O

A B S T R A C T

The trace algebra C_{nd} is generated by all traces of products of d generic $n \times n$ matrices. Minimal generating sets of C_{nd} and their defining relations are known for $n < 3$ and $n = 3$, $d = 2$. This paper states a minimal generating set and their defining relations for $n = d = 3$. Furthermore the computations yield a description of C_{33} as a free module over the ring generated by a homogeneous system of parameters.

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1. Introduction

Let $GL_n := GL_n(\mathbb{C})$ be the general linear group over $\mathbb{C}$ and $M_n := M_n(\mathbb{C})$ be the set of $n \times n$ matrices with entries in $\mathbb{C}$. Here GL_n acts on M_n^d by simultaneous conjugation, i.e. $g.(A_1, \ldots, A_d) = (gA_1g^{-1}, \ldots, gA_dg^{-1})$ for $g \in GL_n$ and $A_1, \ldots, A_d \in M_n$. This action extends to the coordinate ring $\mathbb{C}[M_n^d]$ of M_n^d, which is a polynomial ring generated by the projections $x_{ij}^{(k)} : M_n^d \to \mathbb{C}$, $x_{ij}^{(k)}(A_1, \ldots, A_d) = (A_k)_{ij}$. The action is given by $(g.x_{ij}^{(k)})(A) = x_{ij}^{(k)}(g^{-1}.A)$. The invariant ring

$$\mathbb{C}[M_n^d]^{GL_n} = \{f \in \mathbb{C}[M_n^d] \mid g.f = f \quad \text{for all } g \in GL_n\}$$

is the coordinate ring of the algebraic quotient of the above action of GL_n on M_n^d. We want to find a minimal presentation of this algebra, i.e. a minimal homogeneous generating set and a minimal set of defining relations between these generators.

The next theorem shows that the invariant ring is generated by the traces of products of generic matrices.

Theorem 1.1 *(First fundamental theorem). (See [16].)*

$$C_{nd} := \mathbb{C}\big[M_n^d\big]^{GL_n} = \mathbb{C}\big[\mathrm{tr}(X_{i_1} \cdots X_{i_k}) \mid i_1, \ldots, i_k \in \{1, \ldots, d\}, k \in \mathbb{N}\big],$$

where $X_k = (x_{ij}^{(k)})$ is the k-th generic matrix formed by the projections.

From this description it is obvious that the invariant ring is a (multi-)graded algebra. Here the degree is given by $\deg \mathrm{tr}(X_{i_1} \cdots X_{i_k}) = k$ which is compatible with the standard grading of $\mathbb{C}[M_n^d]$. The multi-grading counts how often the generic matrices occur, for example $\mathrm{mdeg}\,\mathrm{tr}(X_1 X_3^2 X_1 X_3 X_2) = (2, 1, 3)$.

This algebra is finitely generated (see [14, §2]) and there is a degree bound $N(n)$ for the generators which is sharp for large d [16]. The bound is given by the nilpotency class of certain (non-unitary) algebras in the Nagata–Higman theorem [16]. The values of this function are only known for $n < 5$ ($N(1) = 1$, $N(2) = 3$, $N(3) = 6$, $N(4) = 10$). The upper bound $N(n) \leqslant n^2$ for all n was given by Razmyslov (see [5, 6.2] and [9] for the details about $N(n)$). This yields a simple algorithm to compute a minimal homogeneous generating set. Namely we write down the traces mentioned in the first fundamental theorem up to degree $N(n)$ and check if they are generated by the other ones. There is a more sophisticated possibility given in [6] and [8] which uses a GL_d-action on C_{nd}. The minimal generating set for C_{33} given in Lemma 3.2 was computed in this way. The generators are grouped by the GL_d-action on C_{nd}. The first generator in each group is the highest weight vector with respect to this action. We refer to these generators by t_k where k is given in the list of Lemma 3.2. We fix this minimal generating set of C_{33} for the rest of the article.

If we fix a minimal homogeneous generating set $E = \{t_1, \ldots, t_k\}$ of C_{nd} we write $C_E = \mathbb{C}[T_1, \ldots, T_k]$ for the corresponding polynomial ring. The T_i correspond to the generators and we have a canonical map onto C_{nd} given by $T_i \mapsto t_i$.

Given such a minimal homogeneous generating set there is a minimal graded free resolution

$$0 \to \bigoplus_j C_E[-j]^{\beta_{kj}} \to \cdots \to \bigoplus_j C_E[-j]^{\beta_{1j}} \to C_E \to C_{nd} \to 0$$

of C_{nd} [2, Chapter 1.5]. Here $C_E[-j]$ denotes the shift of the grading of C_E by j so that $1 \in C_E[-j]$ has degree j. Furthermore the β_{ij} do not depend on the choice of the minimal homogeneous generating set. The map $\bigoplus_j C_E[-j]^{\beta_{1j}} \to C_E$ decodes the minimal generating set of the ideal of relations among the generators. Similar to the Nagata–Higman bound we define

$$N^i(n, d) := \max_j \{\beta_{ij} \neq 0\}.$$

There are two simplifications mentioned already in [8]. First

$$C_{nd} = \mathbb{C}\big[\mathrm{tr}(X_1), \ldots, \mathrm{tr}(X_d)\big] \otimes_{\mathbb{C}} \mathbb{C}[Q]$$

where Q is the set of all traces of words in matrices of trace 0. One gets this description by $x_i := X_i - n\,\mathrm{tr}(X_i)E$ where E is the $n \times n$ identity matrix. We will denote these generic traceless matrices with small letters. By the description above the relations of C_{nd} are given by the relations of $\mathbb{C}[Q]$. The second simplification is that we can assume that x_1 is a diagonal generic matrix, because the diagonalizable matrices form a dense subset of M_n. (For $A \in M_n(\mathbb{C})$ let $\Delta(\chi_A)$ be the discriminant of the characteristic polynomial of A. The map $\Delta(\chi_A)$ is polynomial in the entries of A and the set $\{A \in M_n(\mathbb{C}) \mid \Delta(\chi_A) \neq 0\} \subset M_n(\mathbb{C})$ is open and therefore dense in $M_n(\mathbb{C})$. This set is the set of matrices with n distinct eigenvalues and hence a subset of the diagonalizable matrices.) This is essential for the computations, because we get rid of $d + n(n - 1)$ variables. In case $d = 3$ we have only 3 generic matrices. We denote these by X, Y, Z and the traceless ones by x, y, z.

We have seen that C_{nd} is a graded algebra. But it also has the property of being a Cohen–Macaulay ring [7,10]. This means there is a sequence of homogeneous successive non-zero divisors of length $\dim C_{nd}$ called a maximal homogeneous regular sequence. For a positive graded Cohen–Macaulay ring maximal homogeneous regular sequences coincide with homogeneous systems of parameters (see [18]). This will be used to compare the Hilbert series of C_{nd} with the Hilbert series given by the generators and relations. The Hilbert series of C_{nd} was computed in [1] for small cases, in particular for $n = 3 = d$.

2. Algorithms

Given a minimal homogeneous generating set $\{t_1, \ldots, t_k\}$ of C_{nd} we get a morphism of affine varieties $\varphi : M_n^d \to \mathbb{A}^k$ with $\varphi(A) = (t_1(A), \ldots, t_k(A))$. The comorphism $\varphi^* : \mathbb{C}[T_1, \ldots, T_k] \to \mathbb{C}[M_n^d]$ sends $T_i \mapsto T_i \circ \varphi = t_i$. The kernel of this comorphism is the ideal of relations of the minimal generating set. If we fix the degrees of the T_i by $\deg T_i := \deg t_i$ the comorphism is a graded algebra homomorphism. So we can compute the kernel degree by degree. In each degree we have to solve one linear system (which is given by the coefficients of the images of the monomials of that degree in $\mathbb{C}[T_1, \ldots, T_k]$). The problem here is that these linear systems grow quite fast and furthermore it is not obvious how many degrees we have to consider. We will see how one could obtain a degree bound by a theorem of Harm Derksen [4] in Section 3, which uses the Cohen–Macaulay property of C_{nd}.

The second algorithm is a consequence of the second fundamental theorem. This theorem describes the relations of C_{nd} in terms of the generators which occur in the first fundamental theorem. The formal trace algebra C_∞ is the polynomial ring generated by formal traces, i.e. it is generated by formal traces $\text{Tr}(w)$ where w is a word in $X_1, X_2, \ldots$. But we have to consider the trivial relations between traces. Two formal traces $\text{Tr}(w)$ and $\text{Tr}(w')$ are equal if and only if w is a cyclic permutation of w'. We can also consider the formal trace algebra generated by d letters and denote this algebra by $C_{\infty,d}$. This algebra is graded in the same way as the grading defined on C_{nd}. We have a canonical map $\pi : C_\infty \to C_n$ which replaces Tr by tr and the letters by generic $n \times n$ matrices. Here

$$C_n := \mathbb{C}\big[\text{tr}(X_{i_1} \cdots X_{i_l}) \,\big|\, i_1, \ldots, i_l, \, l \in \mathbb{N}\big]$$

is given as in the first fundamental theorem but we allow arbitrary many generic matrices. We need the next two definitions to state the second fundamental theorem given by Procesi [16].

Definition 2.1. A *trace identity* for $n \times n$ matrices is an element $f \in C_\infty$ such that $\pi(f) = 0$.

Definition 2.2. The fundamental trace identity for $n \times n$ matrices is given by

$$F(X_1, \ldots, X_{n+1}) := \sum_{\sigma \in S_{n+1}} \text{Tr}_\sigma(X_1, \ldots, X_{n+1}).$$

Here S_{n+1} is the permutation group of $n + 1$ elements.

Example. $\text{Tr}_{(12)(3)}(X, Y, Z) = \text{Tr}(XY)\,\text{Tr}(Z)$.

Theorem 2.3 (*Second fundamental theorem*). (See [16].) *Every trace identity for $n \times n$ matrices is included in the ideal generated by the*

$$F(M_1, \ldots, M_{n+1}),$$

where the M_i are (non-constant) monomials in the X_i.

The advantage of the second fundamental theorem is that a relation can be given as a tuple of monomials. This is a very efficient way to describe the generators of the ideal. The relations of C_{nd} are quite complicated. The goal in this section is to describe the minimal generating set of the relations of C_{33} by some of these monomials and some extra data.

We define $I_{\infty,d}$ to be the ideal in $C_{\infty,d}$ which is generated by these $F(M_1, \ldots, M_{n+1})$ where only the letters $X_1, \ldots, X_d$ occur in the monomials. The ideal $I_{\infty,d}$ is generated by formal traces of arbitrary degree. On the other hand the relations according to the minimal homogeneous generating set of C_{nd} are also part of this ideal. The main problem is to rewrite the formal traces in terms of the minimal generating set. Here $C_E = \mathbb{C}[T_1, \ldots, T_k]$ can be seen as a subalgebra of $C_{\infty,d}$. So rewriting is given by an evaluation map $R : C_{\infty,d} \to \mathbb{C}[T_1, \ldots, T_k]$. Here we only consider evaluation maps which are constant on $\mathbb{C}[T_1, \ldots, T_k]$. Furthermore the diagram

$$
\begin{array}{ccc}
C_{\infty,d} & \xrightarrow{\ \pi\ } & C_{nd} \\
\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle id} \\
C_E & \xrightarrow{\ \pi|_{C_E}\ } & C_{nd}
\end{array}
$$

should commute and R should be a (multi-)graded map. These are quite natural assumptions. From the diagram we see that $R(x)$ and x may differ up to an element of $\ker \pi$.

The next lemma shows that we only have to fix an evaluation map and a set of tuples $M = (M_1, \ldots, M_{n+1})$ as in the second fundamental theorem.

Lemma 2.4. *Let $R : C_{\infty,d} \to C_E$ be a graded evaluation map which is a projection on C_E and commutes with π. Then there exist a set S of tuples of monomials such that the set $S_R := \{R(F(M)) \mid M \in S\}$ is a minimal generating set of $\ker(\pi|_{C_E})$.*

Proof. Let $r \in \ker(\pi|_{C_E})$. Then $r \in I_{\infty,d}$. From the second fundamental theorem follows that there are $c_M \in C_{\infty,d}$ with $r = \sum_M c_M F(M)$. Then $r = R(r) = \sum_M R(c_M) R(F(M))$. So we get a finite generating set of the relations by choosing all $R(F(M))$ up to degree $N^1(n, d)$. Since C_{nd} is a positive graded ring and the $R(F(M))$ are homogeneous we can choose a minimal generating set which consists of some of the $R(F(M))$. $\square$

There is one problem with this setup. Take an element $F(M)$ with $R(F(M)) \neq 0$. Because $R(F(M)) \in I$ we can define another evaluation R' by just changing the evaluation of the formal traces of maximum degree in $F(M)$ in the following way. Let $R'(c) := R(c) - \frac{1}{n!} R(F(M))$ for these formal traces. Then $R'(F(M)) = 0$ and this is not a part of a minimal generating set. That means the tuples M depend on the evaluation map.

So we have to define such an evaluation map. One possibility would be to compute the evaluation for every element by solving a system of linear equations. But then we could also use the first method and furthermore we would only fix it up to a given degree. The second method depends on the next definition.

Definition 2.5. A *trace reduction for $n \times n$ matrices* is a multi-homogeneous trace identity of $n \times n$ matrices

$$
\mathrm{Tr}(X_1 X_2 \cdots X_{\mathrm{N}(n)} X_{\mathrm{N}(n)+1}) = \sum_w \lambda_w \prod_w \mathrm{Tr}(w), \tag{1}
$$

where w are words in the letters $\{X_1, \ldots, X_{\mathrm{N}(n)+1}\}$ and $\lambda_w \in \mathbb{C}$. Trace identity means that

$$
\mathrm{Tr}(X_1 X_2 \cdots X_{\mathrm{N}(n)} X_{\mathrm{N}(n)+1}) - \sum_w \lambda_w \prod_w \mathrm{Tr}(w) \in \ker \pi. \tag{2}
$$

It is obvious from the definition how one could reduce a trace of degree $N(n) + 1$ in terms of traces of lower degrees. Since all the variables of C_∞ correspond to such traces we can reduce them in a unique way by the following algorithm. Here we write $\text{Tr}(Y_{i_1} \cdots Y_{i_k})$ instead of $\text{Tr}(X_{i_1} \cdots X_{i_k})$ for the elements we want to reduce to distinguish them from the trace reduction.

1. A generator of C_∞ corresponds to a formal trace $\text{Tr}(Y_{i_1} \cdots Y_{i_k})$.
2. If we permute these i_j by a cyclic permutation the image under π is the same. All the elements in that orbit should be reduced to the same element. So choose the one where the defining tuple $(i_1, \ldots, i_k)$ is minimal for the lexicographical order.
3. Replace this formal trace with the right-hand side of the trace reduction with $X_1 := Y_{i_1}$, ..., $X_{N(n)} := Y_{i_{N(n)}}$ and $X_{N(n)+1} := Y_{i_{N(n)+1}} \cdots Y_{i_k}$.
4. Repeat this until all given traces have degree $\leqslant N(n)$.
5. Reduce traces in terms of the chosen minimal generating set (unique for $n = 3$).

The next question is how one gets such a trace reduction. This can be done by the second fundamental theorem. We know that $\text{Tr}(X_1 \cdots X_{N(n)+1})$ only occurs in $F(M)$ if and only if each letter $X_1, \ldots, X_{N(n)+1}$ in the monomials occur exactly once. When one inserts such a monomial into the fundamental trace identity it is obvious that only the traces of full degree which are not $\text{Tr}(X_1 X_2 \cdots X_{N(n)} X_{N(n)+1})$ have to be eliminated. Let m be the number of tuples of monomials which fulfill the above assumptions. We get an $N(n)! \times m$-matrix A with entries in $\mathbb{Z}$ (evidently all entries are 0 or 1) where the rows are indexed by the formal traces of length $N(n) + 1$ and the columns are indexed by these tuples of monomials. If we order these traces such that $\text{Tr}(X_1 \cdots X_{N(n)+1})$ is the first one, we only have to solve the linear equation $Ax = e_1$. Then the entries of x are the coefficients of the corresponding trace reduction.

Example 2.6. We find the trace reduction for 2×2-matrices. Here $N(2) = 3$ so we need to find a reduction for $\text{Tr}(X_1 X_2 X_3 X_4)$. If we evaluate

$$
\begin{aligned}
F(X_1 X_2, X_3, X_4) = {} & \text{Tr}(X_1 X_2 X_3 X_4) + \text{Tr}(X_1 X_2 X_4 X_3) \\
& - \text{Tr}(X_1 X_2 X_3) \text{Tr}(X_4) - \text{Tr}(X_1 X_2 X_4) \text{Tr}(X_3) \\
& - \text{Tr}(X_1 X_2) \text{Tr}(X_3 X_4) + \text{Tr}(X_1 X_2) \text{Tr}(X_3) \text{Tr}(X_4).
\end{aligned}
$$

we see that only the first two terms on the right-hand side are relevant. Because the notation is quite clumsy we will denote $\text{Tr}(X_1 X_2) \text{Tr}(X_3 X_4)$ by $[12][34]$, which is not ambiguous as long as we restrict to less than 10 matrices. We will further write $F([12, 3, 4])$ for the left-hand side. Now we get some equations:

$$
F([12, 3, 4]) \mathrel{\hat{=}} [1234] + [1243],
$$

$$
F([41, 2, 3]) \mathrel{\hat{=}} [1234] + [1324],
$$

$$
F([24, 1, 3]) \mathrel{\hat{=}} [1324] + [1243].
$$

Here we write $\mathrel{\hat{=}}$ because we only consider the traces of maximal length. This gives rise to the following linear system

$$
\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

with solution $x = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. This means the trace reduction is given by

$$[1234] = -\frac{1}{2}\big([243][1] - [123][4] - [124][3] - [134][2] + [12][3][4]$$
$$+ [14][2][3] - [24][1][3] - [12][34] - [14][23] + [24][13]\big).$$

We have already seen that we may consider traceless matrices. So we may assume that X_1, X_2 and X_3 correspond to traceless matrices and denote them by x_1, x_2, x_3. In the reduction we will insert products of matrices into X_4, so we cannot assume X_4 to be traceless. In this case we get a more compact trace reduction

$$\mathrm{Tr}(x_1 x_2 x_3 X_4) = \frac{1}{2}\big(\mathrm{Tr}(x_1 x_2 x_3)Tr(X_4) + \mathrm{Tr}(x_1 x_2)\,\mathrm{Tr}(x_3 X_4)$$
$$+ \mathrm{Tr}(x_1 X_4)\,\mathrm{Tr}(x_2 x_3) - \mathrm{Tr}(x_2 X_4)\,\mathrm{Tr}(x_1 x_3)\big). \tag{3}$$

Using traceless matrices has the advantage that we always reduce the degree by at least 2 (at least if we are not in degree $N(n) + 1$). We could enlarge the linear systems with terms like $F([23, 1, 4])$ such that the matrix has a non-trivial kernel. But there is only one trace reduction if we restrict ourselves to terms of a minimal generating set of C_{2d}, i.e. $\mathrm{tr}(X_1), \ldots, \mathrm{tr}(X_d), \mathrm{tr}(X_i X_j)i \leqslant j, \mathrm{tr}(X_i X_j X_k)$ for $i < j < k$. With these assumptions there is only one trace reduction, because there are no non-trivial relations of degree 4 in C_{2d} see [5, Theorem 5.3.8] and an element of the kernel of the above matrix would give rise to such a non-trivial relation.

Remark 2.7. The example shows how one can get such a trace reduction for $n \times n$ matrices. Write down all coefficients of the cycles in $F(M)$ for every suitable M, i.e. all numbers from 1 to $N(n) + 1$ occur once in M. Pick a solution of the corresponding linear system. This always works since these formal traces of degree $N(n) + 1$ only occur for such M isolated and by the definition of $N(n)$ there has to be such a relation. Unfortunately such a trace reduction is not unique for $n \geqslant 3$.

The set of trace reductions depends on the kernel of a linear map, i.e. we can describe them as an affine variety. The reduction of $R(F(M))$ only depends on the coefficients of this affine variety. That such a set of $R(F(M))$ is a minimal generating set is equivalent to $\dim((R(F(M)))_k) = \dim(\ker(\pi|_{C_E})_k)$ for a finite number of degrees k (it is enough to check the degrees of the defining relations). If the set of $R(F(M))$ are not a generating set then a $\dim(\ker(\pi|_{C_E})_k)$ minor of the corresponding set of linear equations given by the $R(F(M))$ vanishes. Since we only have to check a finite number of degrees, the product of the corresponding minors gives us the equation, when such a trace reduction eliminates the minimal generating set given by the fixed set of tuples of monomials.

Lemma 2.8. *Let R be an evaluation map which is given by a trace reduction. Further let S be a (minimal) set of tuples of monomials such that $S_R = \{R(F(M)) \mid M \in S\}$ is a minimal generating set of $\ker(\pi|_{C_E})$. Then the set*

$$\big\{R' \mid R' \text{ given by trace reduction with } (S_{R'}) = \ker(\pi|_{C_E})\big\}$$

is generic.

Proof. The set of R' with $(S_{R'}) \neq \ker(\pi|_{C_E})$ is closed by the observation above. So the complement is a dense subset. □

Remark 2.9. The lemma tells us that it is unlikely to choose the wrong trace reduction. Therefore the main Theorem 4.1 only states the tuples of monomials. One can find the corresponding trace reduction in [11, pp. 95–103].

Conjecture 2.10. *If we choose R as in Definition 2.5 then the chosen monomials always form a minimal generating set.*

3. An upper bound for the relations

From [4] follows that the degree of the defining relations of C_{nd} is bounded by

$$d_1 + d_2 + \cdots + d_{\dim C_{nd}+1} + a(C_{nd}).$$

Here the d_i are the degrees of the elements of the minimal generating set ordered descending and $a(C_{nd})$ is the degree of the Hilbert series of C_{nd} (degree of nominator − degree of denominator). Here $a(C_{nd}) \leqslant -\dim C_{nd}$ holds due to [4] via [12]. Since $\dim C_{nd} = (d-1)n^2 + 1$ ([13]) for $d \geqslant 2$ and the generators are bounded by $N(n) \leqslant n^2$ one gets an upper bound for the defining relations

$$N^1(n,d) \leqslant \big((d-1)n^2 + 2\big)n^2 - \big((d-1)n^2 + 1\big) = (d-1)n^4 + (3-d)n^2 - 1.$$

For C_{33} follows $N^1(3,3) \leqslant 161$ which is quite a bad bound. With some more concrete values for C_{33}, i.e. $a(C_{33}) = -27$, $(d_i) = (6^{10}, 5^9, 4^9, 3^{11}, 2^6, 1^3)$ one gets

$$N^1(3,3) \leqslant 82.$$

This is also not a very sharp bound. The following lemma allows us to examine the whole setup modulo a homogeneous system of parameters.

Lemma 3.1. *let $R = \mathbb{C}[X_1, \ldots, X_k]$ be the polynomial ring in k variables and $\deg X_i > 0$ for all i. Further let I be a homogeneous ideal of R and $\{f_1, \ldots, f_n\}$ be a homogeneous minimal generating set of I with $\deg f_i > 0$ for all i. If $g \in R$ is a homogeneous non-zero divisor of R/I with $\deg g > 0$ then $\{f_1, \ldots, f_n, g\}$ is a minimal homogeneous generating set of the ideal (I, g).*

Proof. First $g \notin I$ because g is R/I-regular. It is enough to show that no f_i can be expressed by g and the other f_j. Assume

$$f_i = \sum_{j \neq i} \beta_j f_j + \beta g$$

where $\beta, \beta_j \in R$. So $\beta g \in I$ and therefore $\beta \in I$ because g is a non-zero divisor of R/I. So there exist $\alpha_k \in R$ with $\beta = \sum_{\deg(f_k) < \deg(f_i)} \alpha_k f_k$ because g has positive degree. But now

$$f_i = \sum_{j \neq i} \beta_j f_j + \sum_{\deg(f_k) < \deg(f_i)} g \alpha_k f_k$$

can be expressed by the other f_i. This is a contradiction to the minimality of $\{f_1, \ldots, f_n\}$. $\quad\square$

Using this lemma for the homogeneous parameter system H given in Theorem 5.1 we get

$$N^1(3,3) \leqslant 6 - 27 + 48 = 27.$$

Here $a(R/(I, H)) = 48 - 27$.

Lemma 3.2. *The following elements form a minimal generating set of C_{33}.*

$\underline{W_3(1):}$

(a) $\mathrm{tr}(X)$

(b) $\mathrm{tr}(Y)$

(c) $\mathrm{tr}(Z)$

$\underline{W_3(2):}$

(1) $\mathrm{tr}(x^2)$

(2) $\mathrm{tr}(y^2)$

(3) $\mathrm{tr}(z^2)$

(4) $\mathrm{tr}(xy)$

(5) $\mathrm{tr}(xz)$

(6) $\mathrm{tr}(yz)$

$\underline{W_3(3):}$

(7) $\mathrm{tr}(x^3)$

(8) $\mathrm{tr}(y^3)$

(9) $\mathrm{tr}(z^3)$

(10) $\mathrm{tr}(x^2 y)$

(11) $\mathrm{tr}(x^2 z)$

(12) $\mathrm{tr}(y^2 x)$

(13) $\mathrm{tr}(y^2 z)$

(14) $\mathrm{tr}(z^2 x)$

(15) $\mathrm{tr}(z^2 y)$

(16) $\mathrm{tr}(xyz) + \mathrm{tr}(xzy)$

$\underline{W_3(2^2):}$

(18) $\mathrm{tr}(x^2 y^2) - \mathrm{tr}(xyxy)$

(19) $\mathrm{tr}(x^2 z^2) - \mathrm{tr}(xzxz)$

(20) $\mathrm{tr}(y^2 z^2) - \mathrm{tr}(yzyz)$

(21) $\mathrm{tr}(x^2 yz) + \mathrm{tr}(x^2 zy) - 2\,\mathrm{tr}(xyxz)$

(22) $\mathrm{tr}(y^2 xz) + \mathrm{tr}(y^2 zx) - 2\,\mathrm{tr}(yxyz)$

(23) $\mathrm{tr}(z^2 xy) + \mathrm{tr}(z^2 yx) - 2\,\mathrm{tr}(zxzy)$

$\underline{W_3(2, 1^2):}$

(24) $\mathrm{tr}(x^2 yz) - \mathrm{tr}(x^2 zy)$

(25) $\mathrm{tr}(y^2 xz) - \mathrm{tr}(y^2 zx)$

(26) $\mathrm{tr}(z^2 xy) - \mathrm{tr}(z^2 yx)$

$\underline{W_3(3, 1^2):}$

(27) $\mathrm{tr}(x^2 yxz) - \mathrm{tr}(x^2 zxy)$

(28) $\mathrm{tr}(y^2 xyz) - \mathrm{tr}(y^2 zyx)$

(29) $\mathrm{tr}(z^2 yzx) - \mathrm{tr}(z^2 xzy)$

(30) $\mathrm{tr}(yxyxz) + \mathrm{tr}(x^2 y^2 z) - \mathrm{tr}(xyzxy) - \mathrm{tr}(x^2 zy^2)$

(31) $\mathrm{tr}(zxzxy) + \mathrm{tr}(x^2 z^2 y) - \mathrm{tr}(xzyxz) - \mathrm{tr}(x^2 yz^2)$

(32) $\mathrm{tr}(yzyzx) + \mathrm{tr}(z^2 y^2 x) - \mathrm{tr}(zyxzy) - \mathrm{tr}(z^2 xy^2)$

$\underline{W_3(1^3):}$

(17) $\mathrm{tr}(xyz) - \mathrm{tr}(xzy)$

$\underline{W_3(2^2, 1):}$

(33) $\mathrm{tr}(x^2 y^2 z) + \mathrm{tr}(x^2 zy^2) + \mathrm{tr}(xyxyz) + \mathrm{tr}(xyxzy) - 2\,\mathrm{tr}(x^2 yzy) - 2\,\mathrm{tr}(xy^2 xz)$

(34) $\mathrm{tr}(x^2 z^2 y) + \mathrm{tr}(x^2 yz^2) + \mathrm{tr}(xzxzy) + \mathrm{tr}(xzxyz) - 2\,\mathrm{tr}(x^2 zyz) - 2\,\mathrm{tr}(xz^2 xy)$

(35) $\mathrm{tr}(y^2 z^2 x) + \mathrm{tr}(y^2 xz^2) + \mathrm{tr}(yzyzx) + \mathrm{tr}(yzyxz) - 2\,\mathrm{tr}(y^2 zxz) - 2\,\mathrm{tr}(yz^2 yx)$

$\underline{W_3(3^2):}$

(36) $\mathrm{tr}(x^2 y^2 xy) - \mathrm{tr}(y^2 x^2 yx)$

(37) $\mathrm{tr}(x^2 z^2 xz) - \mathrm{tr}(z^2 x^2 zx)$

(38) $\mathrm{tr}(y^2 z^2 yz) - \mathrm{tr}(z^2 y^2 zy)$

(39) $\mathrm{tr}(x^2 yxyz) + \mathrm{tr}(x^2 yxzy) + \mathrm{tr}(x^2 zxy^2) - \mathrm{tr}(x^2 y^2 xz) - \mathrm{tr}(x^2 yzxy) - \mathrm{tr}(x^2 zyxy)$

(40) $\mathrm{tr}(y^2 xyxz) + \mathrm{tr}(y^2 xyzx) + \mathrm{tr}(y^2 zyx^2) - \mathrm{tr}(y^2 x^2 yz) - \mathrm{tr}(y^2 xzyx) - \mathrm{tr}(y^2 zxyx)$

(41) $\mathrm{tr}(z^2 yzyx) + \mathrm{tr}(z^2 yzxy) + \mathrm{tr}(z^2 xzy^2) - \mathrm{tr}(z^2 y^2 zx) - \mathrm{tr}(z^2 yxzy) - \mathrm{tr}(z^2 xyzy)$

(42) $\mathrm{tr}(x^2 zxzy) + \mathrm{tr}(x^2 zxyz) + \mathrm{tr}(x^2 yxz^2) - \mathrm{tr}(x^2 z^2 xy) - \mathrm{tr}(x^2 zyxz) - \mathrm{tr}(x^2 yzxz)$

(43) $\mathrm{tr}(y^2 zyzx) + \mathrm{tr}(y^2 zyxz) + \mathrm{tr}(y^2 xyz^2) - \mathrm{tr}(y^2 z^2 yx) - \mathrm{tr}(y^2 zxyz) - \mathrm{tr}(y^2 xzyz)$

(44) $\mathrm{tr}(z^2 xzxy) + \mathrm{tr}(z^2 xzyx) + \mathrm{tr}(z^2 yzx^2) - \mathrm{tr}(z^2 x^2 zy) - \mathrm{tr}(z^2 xyzx) - \mathrm{tr}(z^2 yxzx)$

(45) $\mathrm{tr}(x^2 y^2 z^2) + \mathrm{tr}(x^2 zyzy) + \mathrm{tr}(xyxz^2 y) + \mathrm{tr}(xyzxyz) + \mathrm{tr}(xzxzy^2)$
    $- \mathrm{tr}(x^2 yzyz) - \mathrm{tr}(x^2 z^2 y^2) - \mathrm{tr}(xyxyz^2) - \mathrm{tr}(xy^2 zxz) - \mathrm{tr}(xzyxzy)$

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4. The relations

Theorem 4.1. *The following tuples represent the minimal set of relations between the generators given in Lemma* 3.2.

degree 7	deg (3, 3, 3)	deg (4, 3, 3)	deg (4, 4, 3)
deg (3, 2, 2)	(1132, 223, 1, 3)	(1111, 222, 33, 3)	(1212, 2121, 33, 3)
(111, 22, 3, 3)	(1332, 223, 1, 1)	(1111, 222, 33, 3)	(1212, 2112, 33, 3)
	(3332, 221, 1, 1)	(1112, 122, 33, 3)	(1212, 2211, 33, 3)
degree 8	(3331, 221, 2, 1)	(1122, 112, 33, 3)	(1212, 1221, 33, 3)
deg (4, 3, 1)	(1112, 223, 3, 3)	(1322, 112, 13, 3)	(1212, 1212, 33, 3)
(1111, 22, 2, 3)	(2223, 331, 1, 1)	(22213, 11, 13, 3)	(1212, 1122, 33, 3)
deg (4, 2, 2)	(2213, 331, 2, 1)	(22123, 11, 13, 3)	(1122, 1122, 33, 3)
(1111, 22, 3, 3)	(111, 222, 33, 3)	(21223, 11, 13, 3)	(1122, 2211, 33, 3)
(1112, 21, 3, 3)	(112, 223, 33, 1)	(12223, 11, 13, 3)	(1122, 2121, 33, 3)
(1122, 11, 3, 3)	(122, 231, 33, 1)	(22231, 11, 13, 3)	
deg (3, 3, 2)	(122, 231, 33, 1)	(23221, 11, 13, 3)	**degree** 12
(1112, 22, 3, 3)	(123, 123, 12, 3)	(32221, 11, 13, 3)	**deg** (6, 6, 0)
(1122, 21, 3, 3)	(132, 123, 12, 3)	(22321, 11, 13, 3)	(2112, 121, 122, 12)
(1122, 23, 1, 3)	(132, 132, 12, 3)	(23123, 11, 12, 3)	**deg** (6, 5, 1)
(1322, 23, 1, 1)			(112212, 1123, 1, 2)
(3312, 11, 2, 2)	**degree** 10		**deg** (6, 4, 2)
	deg (6, 2, 2)	**degree** 11	(112122, 1133, 1, 2)
degree 9	(121132, 11, 1, 3)	**deg** (6, 4, 1)	(112312, 1123, 1, 2)
deg (5, 2, 2)	**deg** (5, 4, 1)	(1111, 1122, 22, 3)	**deg** (6, 3, 3)
(1111, 212, 3, 3)	(121122, 21, 1, 3)	**deg** (6, 3, 2)	(112212, 1133, 1, 3)
(1112, 112, 3, 3)	(122122, 11, 1, 3)	(1113, 1122, 21, 3)	(112312, 1132, 1, 3)
deg (4, 4, 1)	**deg** (5, 3, 2)	**deg** (5, 5, 1)	**deg** (5, 2, 2)
(1112, 122, 2, 3)	(122121, 11, 3, 3)	(1112, 1122, 22, 3)	(1122, 112, 122, 33)
(1132, 122, 2, 1)	(122111, 21, 3, 3)	(2112, 1122, 12, 3)	(123, 132, 112, 221)
deg (5, 3, 1)	(122311, 21, 1, 3)	**deg** (5, 4, 2)	**deg** (5, 4, 3)
(1112, 112, 2, 3)	(122131, 21, 1, 3)	(13112, 1222, 1, 3)	(33131, 121, 212, 2)
deg (4, 3, 2)	(121231, 21, 1, 3)	(1133, 1122, 22, 1)	(1231, 132, 321, 21)
(1111, 222, 3, 3)	**deg** (4, 4, 2)	(1133, 1212, 22, 1)	(123, 132, 321, 112)
(1112, 122, 3, 3)	(123212, 21, 1, 3)	(1311, 121, 222, 3)	**deg** (4, 4, 4)
(1122, 112, 3, 3)	(123212, 11, 2, 3)	(111, 112, 222, 33)	(11232, 123, 123, 3)
(1222, 113, 1, 3)	(323212, 11, 2, 1)	**deg** (5, 3, 3)	(13233, 11223, 1, 2)
(3222, 111, 1, 3)	(12211, 122, 3, 3)	(11223, 112, 13, 3)	(13232, 1123, 13, 2)
(3222, 111, 1, 3)	(12121, 122, 3, 3)	(11223, 121, 13, 3)	(1323, 123, 123, 12)
(3322, 111, 1, 2)	(11221, 122, 3, 3)	(11223, 221, 13, 3)	
(3322, 121, 1, 1)	(11122, 122, 3, 3)	(11232, 112, 13, 3)	
	(22111, 122, 3, 3)	(11232, 121, 13, 1)	
	(22113, 122, 1, 3)		
	(22311, 122, 1, 3)		

Remark 4.2. Observe that only the relations are given for the multi-degrees which are partitions. One gets the other ones by permuting the letters in the formal traces and reduce them afterwards in terms of the minimal generating set.

The degree bound of the relations in this case is 12. This allows us to state the following conjecture.

Conjecture 4.3. $N^1(n, d) \leqslant n(n + 1)$.

5. Proof

The idea of the proof is to compute the Hilbert series of the candidate ideal. For this one has to compute a Gröbner basis. In the initial setup there are too many variables, so the Gröbner basis computation is too expensive. But we can reduce the problem by the following homogeneous system of parameters.

Theorem 5.1. *(See [13].) The following elements form a homogeneous system of parameters of C_{33}.*

$\mathrm{tr}(X)$	$\mathrm{tr}(Y)$	$\mathrm{tr}(Z)$
$\mathrm{tr}(x^2)$	$\mathrm{tr}(xy)$	$\mathrm{tr}(xz)$
$\mathrm{tr}(y^2)$	$\mathrm{tr}(yz)$	$\mathrm{tr}(z^2)$
$\mathrm{tr}(x^3)$	$\mathrm{tr}(y^3)$	$\mathrm{tr}(z^3)$
$\mathrm{tr}(x^2 y) - \mathrm{tr}(y^2 z) - \mathrm{tr}(xz^2)$	$\mathrm{tr}(x^2 z) - \mathrm{tr}(z^2 y)$	$\mathrm{tr}(x^2 z) - \mathrm{tr}(y^2 x)$
	$\mathrm{tr}(xyz) - \mathrm{tr}(xzy)$	
$\mathrm{tr}(x^2 y^2)$	$\mathrm{tr}(x^2 z^2)$	$\mathrm{tr}(y^2 z^2)$

The theorem of Lopatin is much more general. Here we just picked one special system of parameters of C_{33}.

For all elements of degree $\leqslant 3$ is obvious, how they correspond to the generators. For those of degree 4 we get the relations

$$\mathrm{tr}(x^2 y^2) = \frac{1}{6} t_1 t_2 + \frac{1}{3} t_4^2 + \frac{1}{3} t_{18},$$

$$\mathrm{tr}(x^2 z^2) = \frac{1}{6} t_1 t_3 + \frac{1}{3} t_5^2 + \frac{1}{3} t_{19},$$

$$\mathrm{tr}(y^2 z^2) = \frac{1}{6} t_2 t_3 + \frac{1}{3} t_6^2 + \frac{1}{3} t_{20}.$$

Because $t_1, \ldots, t_6$ are also elements of the homogeneous system of parameters, we can replace the traces on the left by t_{18}, t_{19} and t_{20} and get the following system of parameters:

$$t_a, t_b, t_c,$$

$$t_1, \ldots, t_9,$$

$$t_{10} - t_{13} - t_{14},$$

$$t_{11} - t_{15},$$

$$t_{11} - t_{12},$$

$$t_{17}, \ldots, t_{20}.$$

If we divide out the homogeneous system of parameters we can eliminate some variables by the following reductions.

$$t_a, t_b, t_c, t_1, \ldots, t_9 \rightsquigarrow 0,$$

$$t_{10} \rightsquigarrow t_{13} + t_{14},$$

$$t_{15} \rightsquigarrow t_{11},$$

$$t_{12} \rightsquigarrow t_{11},$$

$$t_{17}, t_{18}, t_{19}, t_{20} \rightsquigarrow 0.$$

Let J be the ideal given by Theorem 4.1 and H be the homogeneous system of parameters above. Because the elements of this homogeneous system of parameters are given in terms of traces, choose the canonical preimage of these elements in $\mathbb{C}[T_a, \ldots, T_{45}]$ and denote the ideal generated by J and these elements by J_H. Then we get the following inequality of the Hilbert series due to [19]:

$$\frac{H(\mathbb{C}[T_a, \ldots, T_{45}]/J_H)}{\prod(1 - t^{d_i})} \geqslant H(\mathbb{C}[T_a, \ldots, T_{45}]/J) \geqslant H(C_{33}) = \frac{H(C_{33}/H)}{\prod(1 - t^{d_i})}.$$

Here the products in the denominator are determined by the degrees of the homogeneous parameter set. The inequality in the middle holds, because we have the surjective map $\pi : \mathbb{C}[T_a, \ldots, T_{45}]/J \to C_{33}$. Equality holds for our candidate and C_{33} if and only if the outer Hilbert series are equal. And these can be compared by comparing the nominators.

The Hilbert series of J_H can be computed by Singular [3] since dividing out the homogeneous system of parameters allows us to eliminate some variables. Therefore Singular can compute the Hilbert series of $\mathbb{C}[T_a, \ldots, T_{45}]/J_H$. Since the Hilbert series of C_{33} is known (see [8]) one gets that the elements given in Theorem 4.1 generate $\ker(\pi|_{C_E})$.

Additional due to [18] every $\mathbb{C}$-basis $\mathbb{C}[T_a, \ldots, T_{45}]/J_H$ coming from homogeneous elements gives rise to a generating set of C_{33} as a free $\mathbb{C}[H]$-module. From the Gröbner basis computation we get the following corollary.

Corollary 5.2. C_{33} *is a free* $\mathbb{C}[t_a, t_b, t_c, t_1, \ldots, t_9, t_{17}, t_{18}, t_{19}, t_{20}, t_{10} - t_{13} - t_{14}, t_{11} - t_{15}, t_{11} - t_{12}]$-*module. The following elements and their divisors form a basis of this module.*

$t_{11}^2 t_{45}$	$t_{11}t_{13}t_{45}$	$t_{11}t_{14}t_{44}$	$t_{11}t_{14}t_{45}$	$t_{11}t_{16}t_{44}$	$t_{11}t_{16}t_{45}$	$t_{11}t_{22}t_{44}$	$t_{11}t_{23}t_{44}$
$t_{11}t_{23}t_{45}$	$t_{11}t_{24}$	$t_{11}t_{25}t_{45}$	$t_{11}t_{26}t_{44}$	$t_{11}t_{26}t_{45}$	$t_{11}t_{30}$	$t_{11}t_{31}$	$t_{11}t_{32}t_{45}$
$t_{11}t_{33}$	$t_{11}t_{34}$	$t_{11}t_{35}t_{45}$	$t_{11}t_{40}$	$t_{11}t_{41}$	$t_{11}t_{42}$	$t_{11}t_{43}$	$t_{13}^2 t_{45}$
$t_{13}t_{14}t_{44}$	$t_{13}t_{14}t_{45}$	$t_{13}t_{16}t_{30}$	$t_{13}t_{16}t_{44}$	$t_{13}t_{16}t_{45}$	$t_{13}t_{22}$	$t_{13}t_{23}t_{35}$	$t_{13}t_{23}t_{45}$
$t_{13}t_{24}$	$t_{13}t_{25}t_{45}$	$t_{13}t_{26}t_{44}$	$t_{13}t_{26}t_{45}$	$t_{13}t_{31}$	$t_{13}t_{32}t_{45}$	$t_{13}t_{33}$	$t_{13}t_{34}$
$t_{13}t_{35}t_{45}$	$t_{13}t_{40}$	$t_{13}t_{42}$	$t_{14}^2 t_{44}$	$t_{14}^2 t_{45}$	$t_{14}t_{16}t_{31}$	$t_{14}t_{16}t_{32}$	$t_{14}t_{16}t_{43}$
$t_{14}t_{16}t_{44}$	$t_{14}t_{16}t_{45}$	$t_{14}t_{21}$	$t_{14}t_{22}t_{44}$	$t_{14}t_{22}t_{45}$	$t_{14}t_{23}t_{44}$	$t_{14}t_{23}t_{45}$	$t_{14}t_{24}t_{45}$
$t_{14}t_{25}t_{44}$	$t_{14}t_{25}t_{45}$	$t_{14}t_{26}t_{44}$	$t_{14}t_{26}t_{45}$	$t_{14}t_{30}$	$t_{14}t_{31}t_{45}$	$t_{14}t_{32}t_{45}$	$t_{14}t_{33}$
$t_{14}t_{34}t_{44}$	$t_{14}t_{34}t_{45}$	$t_{14}t_{35}t_{44}$	$t_{14}t_{35}t_{45}$	$t_{14}t_{41}t_{44}$	$t_{14}t_{41}t_{45}$	$t_{14}t_{42}t_{44}$	$t_{14}t_{42}t_{45}$
$t_{16}^2 t_{43}$	$t_{16}^2 t_{44}$	$t_{16}^2 t_{45}$	$t_{16}t_{22}t_{44}$	$t_{16}t_{22}t_{45}$	$t_{16}t_{23}t_{44}$	$t_{16}t_{23}t_{45}$	$t_{16}t_{24}$
$t_{16}t_{25}t_{44}$	$t_{16}t_{25}t_{45}$	$t_{16}t_{26}t_{44}$	$t_{16}t_{26}t_{45}$	$t_{16}t_{27}t_{44}$	$t_{16}t_{27}t_{45}$	$t_{16}t_{28}t_{45}$	$t_{16}t_{29}t_{43}$
$t_{16}t_{29}t_{45}$	$t_{16}t_{30}t_{40}$	$t_{16}t_{31}t_{45}$	$t_{16}t_{32}t_{45}$	$t_{16}t_{33}$	$t_{16}t_{34}t_{44}$	$t_{16}t_{34}t_{45}$	$t_{16}t_{35}t_{45}$
$t_{16}t_{39}$	$t_{16}t_{40}t_{45}$	$t_{16}t_{41}t_{45}$	$t_{16}t_{42}t_{45}^2$	$t_{16}t_{44}^2$	$t_{16}t_{44}t_{45}$	$t_{21}t_{33}t_{45}$	$t_{21}t_{34}t_{45}$
$t_{21}t_{35}t_{45}$	$t_{21}t_{38}$	$t_{21}t_{40}$	$t_{21}t_{42}t_{45}$	$t_{21}t_{43}$	$t_{21}t_{44}$	$t_{21}t_{45}^2$	$t_{22}t_{33}t_{45}$
$t_{22}t_{34}$	$t_{22}t_{35}t_{45}$	$t_{22}t_{37}$	$t_{22}t_{39}$	$t_{22}t_{40}$	$t_{22}t_{41}t_{45}$	$t_{22}t_{42}$	$t_{22}t_{43}$
$t_{22}t_{45}^2$	$t_{23}t_{33}$	$t_{23}t_{34}t_{45}$	$t_{23}t_{35}t_{45}$	$t_{23}t_{38}$	$t_{23}t_{40}$	$t_{23}t_{41}$	$t_{23}t_{42}t_{45}$
$t_{23}t_{43}$	$t_{23}t_{45}^2$	$t_{24}t_{27}t_{45}$	$t_{24}t_{30}$	$t_{24}t_{31}$	$t_{24}t_{32}$	$t_{24}t_{40}$	$t_{24}t_{42}$
$t_{24}t_{43}$	$t_{24}t_{44}$	$t_{24}t_{45}^2$	$t_{25}t_{28}t_{45}$	$t_{25}t_{30}$	$t_{25}t_{32}$	$t_{25}t_{39}$	$t_{25}t_{40}$
$t_{25}t_{41}$	$t_{25}t_{42}$	$t_{25}t_{43}$	$t_{25}t_{45}^2$	$t_{26}t_{29}t_{45}$	$t_{26}t_{31}$	$t_{26}t_{32}$	$t_{26}t_{40}$
$t_{26}t_{41}$	$t_{26}t_{42}$	$t_{26}t_{43}$	$t_{26}t_{45}^2$	$t_{27}t_{40}$	$t_{27}t_{43}$	$t_{28}t_{41}$	$t_{28}t_{44}$
$t_{29}t_{42}$	$t_{29}t_{44}$	$t_{30}t_{41}$	$t_{30}t_{44}$	$t_{30}t_{45}^2$	$t_{31}t_{42}$	$t_{31}t_{44}$	$t_{31}t_{45}^2$
$t_{32}t_{42}$	$t_{32}t_{44}$	$t_{32}t_{45}^2$	$t_{33}t_{39}$	$t_{33}t_{40}$	$t_{33}t_{41}$	$t_{33}t_{44}$	$t_{33}t_{45}^2$
$t_{34}t_{40}$	$t_{34}t_{42}$	$t_{34}t_{43}$	$t_{34}t_{45}^2$	$t_{35}t_{41}$	$t_{35}t_{42}$	$t_{35}t_{43}$	$t_{35}t_{45}^2$
$t_{36}t_{44}$	$t_{37}t_{42}$	$t_{37}t_{43}$	$t_{37}t_{44}$	$t_{37}t_{45}$	$t_{38}t_{41}$	$t_{38}t_{44}$	$t_{38}t_{45}$
$t_{40}t_{42}$	$t_{40}t_{44}$	$t_{41}t_{42}t_{45}$	$t_{42}^2 t_{45}$	$t_{43}t_{44}$	t_{44}^3		

Remark 5.3. The first algorithm (Section 2) was implemented first in Maple [15] and later in Sage [17]. That the relations given by this algorithm generate $\ker \pi|_{C_E}$ was confirmed using Singular [3] by comparing the Hilbert series (Section 5). The second algorithm was also implemented in Sage.

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