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The Witt ring of a curve with good reduction over a non-dyadic local field



Jeanne M. Funk^{a,*}, Raymond T. Hoobler^b

^a Department of Mathematics, Engineering and Computer Science,
LaGuardia Community College, 31-10 Thomson Ave., Room E218-E,
Long Island City, NY 11101, United States

^b Mathematics Program, The Graduate Center, CUNY, 365 5th Ave., New York,
NY 10016, United States

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ABSTRACT

In this work, we present a generalization to varieties and sheaves of the fundamental ideal of the Witt ring of a field by defining a sheaf of fundamental ideals \tilde{I} and a sheaf of Witt rings \tilde{W} in the obvious way. The Milnor conjecture then relates the associated graded of \tilde{W} to Milnor K-theory and so allows the classical invariants of a bilinear space over a field to be extended to our setting using étale cohomology. As an application of these results, we calculate the Witt ring of a smooth curve with good reduction over a non-dyadic local field.

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1. Introduction

The goals of this work are twofold. First, we present a generalization to the category of varieties and sheaves of the fundamental ideal of the Witt ring and of some techniques associated with this ideal which have proven useful in describing the Witt ring of a field.

* Corresponding author.

E-mail addresses: jfunk@lagcc.cuny.edu (J.M. Funk), rhoobler@ccny.cuny.edu (R.T. Hoobler).

Second, we calculate the Witt ring of a smooth projective curve with good reduction over a non-dyadic local field as an application of these techniques.

Much work has been done toward generalizing the Witt ring, its fundamental ideal, and the connections between the latter, K-theory, and cohomology. Calculations of Witt groups and rings using such connections are seen, for example, in Carmena's work on complex surfaces [3], Sujatha's on real projective surfaces [12], and Parimala's on affine three folds [11]. [1] is a good recent literature review on generalization of the Milnor conjecture to the categories of rings and schemes. Unless otherwise specified, we will use definitions presented by Knebusch [9], whose work forms much of the basis for study of Witt rings of varieties and schemes.

Our approach is to define a sheaf of fundamental ideals \tilde{I} and a sheaf of Witt rings \tilde{W} . For low dimensional cases, the global sections $\mathbb{I}^n := \Gamma(X, \tilde{I}^n)$ form a filtration of the Witt ring $W(X)$ and we can study the Witt ring via the quotients $\mathbb{I}^n/\mathbb{I}^{n+1}$. We will show that $\tilde{I}^n/\tilde{I}^{n+1}$ is isomorphic to the Zariski sheafification of étale cohomology with μ_2 coefficients (this follows from the Milnor conjecture and most of the work has already been done by Kerz, Milnor, Orlov, Vishik, and Voevodsky, among others). We then show that \tilde{W}/\tilde{I} , \tilde{I}/\tilde{I}^2 , and \tilde{I}^2/\tilde{I}^3 are isomorphic to the Zariski sheaves $\mathbb{Z}/2\mathbb{Z}$, $\mathcal{O}^\times/\mathcal{O}^{\times 2}$, and ${}_2Br$ respectively and that the classes of elements in $W(X)/\mathbb{I}$, \mathbb{I}/\mathbb{I}^2 , and $\mathbb{I}^2/\mathbb{I}^3$ are determined by rank, signed discriminant, and Witt invariant, respectively.

For a smooth geometrically connected projective curve C with good reduction C_k over a non-dyadic local field with uniformizing parameter π we can show that $\mathbb{I}^n/\mathbb{I}^{n+1}$ is trivial when $n > 2$. Thus, using the generalizations outlined above, we can describe each element of the Witt ring in terms of its rank, signed discriminant, and Witt invariant. We may equate each space of odd rank to $\langle -1 \rangle$ in $W(X)/\mathbb{I}$ and each even rank space to $\langle 1, \delta \rangle$ in \mathbb{I}/\mathbb{I}^2 where δ is the signed discriminant. It remains to describe the contribution from $\mathbb{I}^2/\mathbb{I}^3$, which is determined by the class of the Clifford algebra in the order 2 part of the Brauer group. We show that, over the function field of C , ${}_2Br(C)$ consists of distinct “quaternions” $(\frac{\xi, \pi}{k(C)})$, allowing us to equate each member of \mathbb{I}^2 to the four-dimensional form $\langle 1, -\xi, -\pi, \pi\xi \rangle$ in $\mathbb{I}^2/\mathbb{I}^3$. In this way, we can explicitly represent each element of the Witt ring of a sum of line bundles and prove that $W(C) \simeq W(C_k)[\mu_2]$, which nicely generalizes the classical result for the Witt ring of a local field [10].

We would like to thank Jean-Louis Colliot-Thelene for a very useful remark he made to us at the beginning of this work.

2. The fundamental ideal

Define \tilde{W} and \tilde{I} to be the sheaves associated to the presheaves $W(-) : U \mapsto W(\mathcal{O}_X(U))$ and $I(-) : U \mapsto I(W(U))$ respectively. The sheaves \tilde{I}^n form a filtration of \tilde{W} and the global sections $\mathbb{I}^n := \Gamma(X, \tilde{I}^n)$ form a filtration of $\Gamma(X, \tilde{W})$.

Assuming that X is proper, smooth, and geometrically connected and that $\dim(X) \leq 3$, we first recall a result [2] that establishes $W(X) = \Gamma(X, \tilde{W})$ and so the \mathbb{I}^n form a filtration of $W(X)$.

Proposition 1. *Let X be a proper, smooth, geometrically connected variety. $\Gamma(X, \tilde{W}) = W(X)$ if $\dim X \leq 3$.*

The following result (Proposition 2) shows that, in order to describe the Witt ring of a variety, it frequently suffices to study the quotients $\mathbb{I}^n/\mathbb{I}^{n+1}$. Theorem 1 then shows that these quotients may be described cohomologically.

Proposition 2. *If X is a proper, smooth, geometrically connected variety with $\dim(X) \leq 3$ and the rank of anisotropic forms over $k(X)$ is bounded then $[E] = 0 \in W(X)$ if and only if $[E] \in \mathbb{I}^n \forall n$.*

Proof. Let $E \in \mathbb{I}^n$ for all integers n . Then $E_\eta \in \tilde{I}_\eta^n = I^n(k(X))$. Hauptsatz X.5.1 of [10] shows that the $I^n(k(X))$ are eventually trivial. \square

Theorem 1 (Generalized Theorem of Kerz, Milnor, Orlov, Vishik, Voevodsky et al.). *If X is a smooth, quasi-projective variety over an infinite field k then there are isomorphisms*

$$\begin{array}{ccc} \mathcal{K}_M^n / 2\mathcal{K}_M^n & \longrightarrow & \mathcal{H}_{\text{et}}^n(X, \mu_2) \\ \downarrow & \nearrow & \\ \tilde{I}^n / \tilde{I}^{n+1} & & \end{array}$$

where \mathcal{K}_M^n is the Zariski sheafification of Milnor K -theory and $\mathcal{H}_{\text{et}}^n(X, \mu_2)$ is the Zariski sheafification of the étale cohomology presheaf $U \mapsto H_{\text{et}}^n(U, \mu_2)$.

Proof. These maps are defined on presheaves as follows:

$$\begin{aligned} K_M^n / 2K_M^n &\rightarrow H^n(\mu_2) : \{a_1, \dots, a_n\} \mapsto a_1 \cup \dots \cup a_n \\ K_M^n / 2K_M^n &\rightarrow I^n / I^{n+1} : \{a_1, \dots, a_n\} \mapsto \langle\langle -a_1, \dots, -a_n \rangle\rangle. \end{aligned}$$

That they are isomorphisms at each stalk follows from [8]. \square

We can use this theorem together with the injection $\mathbb{I}^n/\mathbb{I}^{n+1} \hookrightarrow \Gamma(X, \tilde{I}^n/\tilde{I}^{n+1})$ to obtain a description of the first three $\mathbb{I}^n/\mathbb{I}^{n+1}$ (and the images of bilinear spaces therein) in terms of classical invariants. We first describe the sheaves.

Proposition 3. *Let X be a connected scheme. Suppose 2 is a unit on X . Then we have the following.*

1. $\tilde{W}/\tilde{I} = \mathbb{Z}/2\mathbb{Z}$
2. $\tilde{I}/\tilde{I}^2 = \mathcal{O}^\times/\mathcal{O}^{\times 2}$
3. $\tilde{I}^2/\tilde{I}^3 = {}_2Br$

Proof. \tilde{W}/\tilde{I} is the sheaf defined on each open $U \subseteq X$ by the presheaf $W(\mathcal{O}_X(U))/I(\mathcal{O}_X(U))$, which is the ring $\mathbb{Z}/2$ for all connected U .

Since the Kummer sequence $0 \rightarrow \mu_2 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$ is exact in the étale setting, there is, for each open $U \subseteq X$, a long exact sequence

$$H_{et}^0(U, \mathbb{G}_m) \xrightarrow{2} H_{et}^0(U, \mathbb{G}_m) \rightarrow H_{et}^1(U, \mu_2) \rightarrow H_{et}^1(U, \mathbb{G}_m) \rightarrow H_{et}^1(U, \mathbb{G}_m)$$

After the identifications $H_{et}^0(U, \mathbb{G}_m) = \Gamma(U, \mathbb{G}_m)$ and $H_{et}^1(U, \mathbb{G}_m) = \text{Pic}(U)$, this yields a short exact sequence of presheaves

$$0 \rightarrow \Gamma(U, \mathbb{G}_m)/\Gamma(U, \mathbb{G}_m)^2 \rightarrow H_{et}^1(U, \mu_2) \rightarrow {}_2\text{Pic}(U) \rightarrow 0$$

on each open U so that there is an exact sequence of Zariski sheaves

$$0 \rightarrow \mathcal{O}_X^\times/\mathcal{O}_X^{\times 2} \rightarrow \mathcal{H}_{et}^1(\mu_2) \rightarrow \underline{{}_2\text{Pic}(X)} \rightarrow 0.$$

However, the Picard group of a local ring is trivial, and so, checking stalks, we get a sheaf isomorphism $\mathcal{O}^\times/\mathcal{O}^{\times 2} \cong \mathcal{H}_{et}^1(X, \mu_2)$.

In a similar way, we have a short exact sequence

$$0 \rightarrow \text{Pic}(U)/\text{Pic}(U)^2 \rightarrow \mathcal{H}_{et}^2(U, \mu_2) \rightarrow {}_2\text{Br}(U) \rightarrow 0$$

and, with no contribution from the Picard group at the stalks, we have a sheaf isomorphism $\mathcal{H}_{et}^2(X, \mu_2) \cong {}_2\text{Br}$. \square

The following proposition describes the first three quotients $\mathbb{I}^n/\mathbb{I}^{n+1}$ in terms of rank, signed discriminant, and Witt invariant.

Proposition 4.

1. The class of E in $W(X)/\mathbb{I}$ is determined by the parity of $n = \text{rank}(E)$.
2. Given an even rank element E of $W(X)$, the class of $E \in \mathbb{I}/\mathbb{I}^2$ is determined by the signed discriminant $d_\pm(E) = (-1)^{\frac{n(n+1)}{2}} \bigwedge^{\text{rk}(E)} E$.
3. Given an even rank $E \in W(X)$ whose signed discriminant is trivial, the class of E in $\mathbb{I}^2/\mathbb{I}^3$ is determined by the Witt invariant of E (the class of the Clifford algebra of E in the order 2 Brauer group).

Proof. At each stalk, the composition $\tilde{W}_x \rightarrow (\tilde{I}^n/\tilde{I}^{n+1})_x$ ($n \in \{0, 1, 2\}$) is determined by the invariant specified in the proposition. \square

Note that \mathbb{I} consists of the elements of $W(X)$ represented by spaces of even rank and that \mathbb{I}^2 consists of those elements of $W(X)$ that are represented by even rank symmetric spaces with trivial signed discriminant. Furthermore, every nonzero element of $W(X)/\mathbb{I}$

may be equated in $W(X)/\mathbb{I}$ to a rank 1 form and every element of \mathbb{I}/\mathbb{I}^2 may be equated to a rank 2 form which shares its signed discriminant and is an orthogonal sum of rank 1 forms.

3. Forms of rank 1

In order to describe the Witt ring, it will be useful to first address some results and notational issues concerning the part of the Witt ring that is additively generated by forms of rank 1. For the purpose of this discussion, we let X be a proper, smooth, geometrically connected variety over a field k such that $\dim(X) \leq 3$ and $W(X)$ be the Witt ring of X . $Q(X)$ will represent the multiplicative subgroup of $W(X)$ represented by rank 1 forms $(\mathcal{L}, \varphi : \mathcal{L} \rightarrow \mathcal{L}^\vee)$ ($(-)^{\vee}$ represents the dual).

Proposition 5. *Given a line bundle $\mathcal{L} \in {}_2\text{Pic}(X)$, the Witt classes (\mathcal{L}, φ) are in one-to-one correspondence with the square classes of units of the base field k .*

Proof. First, note that every line bundle \mathcal{L} of order 2 is equipped with at least one non-degenerate symmetric bilinear form.

Two forms $(\mathcal{L}, \varphi), (\mathcal{L}, \psi) \in Q(X)$ may differ at most by a global endomorphism of \mathcal{L} as in the following diagram:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\varphi} & \mathcal{L}^\vee \\ m_\ell = \varphi\psi^{-1} \downarrow & \nearrow \psi & \\ \mathcal{L} & & \end{array}$$

As $\mathcal{E}nd(\mathcal{L}) \cong \mathcal{O}_X$, these global endomorphisms are precisely the units of k . Thus, φ and ψ differ by multiplication m_ℓ by some unit $\ell \in k$.

φ and ψ represent the same Witt class precisely when $\varphi \cong \psi$ which occurs if and only if $\varphi = m_\ell \circ \psi \circ m_\ell^\vee$ so that $\varphi = \ell^2 \psi$.

This shows that the Witt classes of X associated to \mathcal{L} are in one-to-one correspondence with the square classes of the base field. Furthermore, the square classes of the base field act multiplicatively to permute the Witt classes associated to \mathcal{L} . \square

We fix a form on each \mathcal{L} so that the forms on $\mathcal{L}_1, \dots, \mathcal{L}_n$ multiply to the fixed form on $\mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_n$ and the fixed form on the structure sheaf is the identity. $\langle \mathcal{L} \rangle \in W(X)$ will denote the Witt class of a line bundle with such a fixed form and $\langle s\mathcal{L} \rangle$ will denote multiplication of $\langle \mathcal{L} \rangle$ by the Witt class $\langle s \rangle \in W(k)$. $\langle s_1\mathcal{L}_1, \dots, s_n\mathcal{L}_n \rangle$ will denote the orthogonal sum $\langle s_1\mathcal{L}_1 \rangle \perp \dots \perp \langle s_n\mathcal{L}_n \rangle$. The classes $\langle s \rangle \in Q(k)$ will be identified with the classes $\langle s\mathcal{O}_X \rangle \in Q(X)$.

We can now give an exact description of $Q(X)$ as a $W(k)$ -module.

Proposition 6. *There is a short exact sequence of commutative rings $0 \rightarrow Q(k) \rightarrow Q(X) \rightarrow {}_2\text{Pic}(X) \rightarrow 0$.*

Proof. Let $\langle s\mathcal{L} \rangle = \langle t\mathcal{M} \rangle \in Q(X)$. We will show that \mathcal{L} and \mathcal{M} are isomorphic as line bundles.

We know that $\langle s\mathcal{L} \rangle \perp M = \langle t\mathcal{M} \rangle \perp M' \in \text{Bil}(X)$ for metabolic spaces M and M' of the same rank, $2m$.

Taking determinants on both sides, we see that

$$\begin{aligned} \det(\langle s\mathcal{L} \rangle) \det(M) &= \det(\langle t\mathcal{M} \rangle) \det(M'), \\ \langle s\mathcal{L} \rangle (\langle -1 \rangle^m) &= \langle t\mathcal{M} \rangle (\langle -1 \rangle^m), \end{aligned}$$

and

$$\langle s\mathcal{L} \rangle = \langle t\mathcal{M} \rangle \in \text{Bil}(X). \quad \square$$

In fact, we have the following.

Proposition 7. *The sub- $W(k)$ -module $Q(X)$ of $W(X)$ consisting of classes represented by forms on line bundles is isomorphic to*

$$Q(k) \times {}_2\text{Pic}(X).$$

4. The Witt ring of a curve with good reduction over a non-dyadic local field

If $\dim(X) \leq 3$ then the natural map $W(X) \rightarrow W(k(X))$ to the Witt ring of the function field is injective [2]. If X contains a k -rational point, the map $W(k) \rightarrow W(X)$ induced by the structure map $X \rightarrow \text{Spec}(k)$ is also injective. This map gives $W(X)$ the structure of a $W(k)$ -module and identifies $W(k)$ with the subring of $W(X)$ generated (as a $W(k)$ module) by classes of forms on \mathcal{O}_X .

It is worth noting that the question of whether the composition $W(k) \rightarrow W(X)$ is injective is precisely the question of whether the composition $W(k) \rightarrow W(k(X))$ is injective, which is in turn equivalent to whether a non-square in the group of units k^\times can become a square in $k(X)^\times$.

We will now calculate the Witt ring of a smooth geometrically connected projective curve C with good reduction over a non-dyadic local field K . We assume that C contains a K -rational point.

We first observe that the elements of the Witt ring $W(C)$ are determined entirely by parity of rank, signed discriminant, and Witt invariant, reducing the calculation of the Witt ring to consideration of classical invariants.

Proposition 8. $\mathbb{I}^n(C)/\mathbb{I}^{n+1}(C)$ is trivial when $n > 2$.

Proof. $H^0(C, \mathcal{H}_{et}^3(C, \mu_2)) = H_{un}^3(C, \mu_2) = 0$ by [7, Proposition 5.2] and the fact that the 2-cohomological dimension of C is 3. \square

Note that the good reduction assumption is essential here, as examples are known of curves with bad reduction over local fields for which $\mathbb{I}^3/\mathbb{I}^4$ is nontrivial.

The current situation is shown in the following diagram

$$\begin{array}{ccccc} C_k & \longrightarrow & C_\nu & \longleftarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_\nu) & \longleftarrow & \mathrm{Spec}(K) \end{array}$$

where $C = C_\nu \otimes K$ is the generic fiber of the proper, smooth curve $C_\nu \rightarrow \mathrm{Spec} \mathcal{O}_\nu$ with special fiber $C_k = C_\nu \otimes k$ smooth over $\mathrm{Spec}(k)$. The K point of C determines an \mathcal{O}_ν rational point of C_ν by the valuative criterion of properness.

A key tool is the cohomological calculation of these invariants and so we need to understand the relationship between them for the curves C_ν and C_k .

Lemma 1. *Let $f : X \rightarrow \mathrm{Spec}(\mathcal{O}_\nu)$ be a proper morphism over the Hensel ring \mathcal{O}_ν with residue field k and let X_k denote the fiber of p over $\mathrm{Spec}(k)$. Then the base change map $H_{et}^p(X, \mu_n) \rightarrow H_{et}^p(X_k, \mu_n)$ is an isomorphism.*

Proof. If \mathcal{O}_ν is strictly local, this is just the proper base change isomorphism. Since finite, étale covers of $\mathrm{Spec}(k)$ and $\mathrm{Spec}(\mathcal{O}_\nu)$ are the same, the result follows immediately from the Hochschild–Serre spectral sequences. \square

Once again, the forms on an individual line bundle are in one-to-one correspondence with the rank 1 forms in the Witt ring of the base field, of which there are four. We will equip each line bundle with a fixed form as described in Section 3 which extends to a non-degenerate form on C_k .

Proposition 9. ${}_2\mathrm{Pic}(C_k) = {}_2\mathrm{Pic}(C_\nu) = {}_2\mathrm{Pic}(C)$.

Proof. Since C_ν is proper and smooth over \mathcal{O}_ν with a rational point, its Picard scheme has an abelian variety $\mathrm{Pic}_{C_\nu/\mathcal{O}_\nu}^0$ as its connected component and, for any \mathcal{O}_ν scheme T , $\mathrm{Pic}_{C_\nu/\mathcal{O}_\nu}^0(T) = \mathrm{Pic}^0(C \times_{\mathcal{O}_\nu} T)/\mathrm{Pic}(T)$ where $\mathrm{Pic}^0(C \times_{\mathcal{O}_\nu} T)$ consists of line bundles on $C \times_{\mathcal{O}_\nu} T$ of degree 0 on each fiber over T . In particular, ${}_2\mathrm{Pic}^0(C_k) = {}_2\mathrm{Pic}_{C_\nu/\mathcal{O}_\nu}^0(k)$, ${}_2\mathrm{Pic}^0(C_\nu) = {}_2\mathrm{Pic}_{C_\nu/\mathcal{O}_\nu}^0(\mathcal{O}_\nu)$ and ${}_2\mathrm{Pic}^0(C) = {}_2\mathrm{Pic}_{C_\nu/\mathcal{O}_\nu}^0(K)$. But ${}_2\mathrm{Pic}_{C_\nu/\mathcal{O}_\nu}^0$ is a finite étale group scheme since K is non-dyadic and \mathcal{O}_ν is henselian. Consequently ${}_2\mathrm{Pic}^0(C_k) = {}_2\mathrm{Pic}^0(C_\nu)$. Moreover, the valuative criterion for separation and properness shows that ${}_2\mathrm{Pic}^0(C_\nu) = {}_2\mathrm{Pic}^0(C)$. \square

Thus, associated to each line bundle $\mathcal{L} \in {}_2\text{Pic}(C)$, there is a unique line bundle $\mathcal{L}_\nu \in {}_2\text{Pic}(C_\nu)$. We select a Witt class represented by $(\mathcal{L}_\nu, \varphi)$ and note that $(\mathcal{L}, \varphi \otimes 1)$ represents a Witt class of C . Furthermore, $(\bar{\mathcal{L}}, \bar{\varphi})$ represents a Witt class of C_k and every rank 1 element of $W(C_k)$ may be obtained by base change from a form over C_ν . Note that the selected forms multiply as expected and we may take the form on \mathcal{O}_C to be $\langle 1 \rangle = \langle 1_{\mathcal{O}_C} \rangle$. There are four Witt classes associated to each \mathcal{L} . These are $\langle \mathcal{L} \rangle$, $\langle s\mathcal{L} \rangle$, $\langle \pi\mathcal{L} \rangle$, and $\langle s\pi\mathcal{L} \rangle$ where $\langle 1 \rangle \neq \langle s \rangle \in W(k)$ and π is the uniformizing parameter of \mathcal{O}_ν .

Since the forms $\langle \mathcal{L} \rangle$ and $\langle s\mathcal{L} \rangle$ arise from forms over C_k , these forms behave like forms over a curve over a finite field. In particular, we have relations

$$\langle u\mathcal{L}, v\mathcal{M} \rangle = \langle 1, uv\mathcal{L}\mathcal{M} \rangle$$

and

$$\langle \pi u\mathcal{L}, \pi v\mathcal{M} \rangle = \langle \pi, \pi uv\mathcal{L}\mathcal{M} \rangle$$

in $W(C)$ where $u, v \in k^\times/k^{\times 2}$ [5].

We can equate each form to a rank 1 form in $W(C)/\mathbb{I}$ and each even dimensional form to a sum of rank 1 forms in \mathbb{I}/\mathbb{I}^2 . It remains to describe $\mathbb{I}^2/\mathbb{I}^3$, which we do by describing the order 2 part of the Brauer group.

Given a square class, s , of K^\times and an order 2 line bundle, \mathcal{L} , we will construct an Azumaya algebra $(\frac{s\mathcal{L}, \pi}{\mathcal{O}_C})$ modeled after the quaternion algebras. Let $\mathcal{L} \in {}_2\text{Pic}(C)$, $\langle s \rangle \in W(k)$. Take $f : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_C$ to be the isomorphism corresponding to the form on $\langle s \rangle \mathcal{L}$. Let multiplication of graded rings on $\mathcal{O}_C \oplus \mathcal{L}$ be induced by f (locally $(a, \ell) \cdot (a', \ell') = (aa' + f(\ell, \ell'), a\ell' + a'\ell)$) and on $\mathcal{O}_C \oplus \mathcal{O}$ by the isomorphism $\mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}$ induced by the form $\langle \pi \rangle$ (locally $(a, b) \cdot (a', b') = (aa' + \pi^2 bb', ab' + a'b)$).

Define $(\frac{s\mathcal{L}, \pi}{\mathcal{O}_C})$ to be the graded tensor product $(\mathcal{O}_C \oplus \mathcal{L}) \hat{\otimes} (\mathcal{O}_C \oplus \mathcal{O})$.

Note that, as an \mathcal{O}_C -module, $(\frac{s\mathcal{L}, \pi}{\mathcal{O}_C}) = \mathcal{O}_C \oplus \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{O}_C$ and that multiplication on the second copy of \mathcal{L} is given by πf . Locally this is just the quaternion algebra $(\frac{s\xi, \pi}{\mathcal{O}_x})$ additively generated by i, j, k with $i^2 = \xi$, $j^2 = \pi$, and $ij = -ji = k$ where $\langle \xi \rangle$ is the image of \mathcal{L} in $W(\mathcal{O}_x)$. In particular, $(\frac{s\mathcal{L}, \pi}{\mathcal{O}_C})$ is a sheaf of Azumaya algebras on C . The norm form of $(\frac{\xi, \pi}{k(C)})$ is $\langle 1, -s\xi, -\pi, s\xi\pi \rangle$, which is the image in $W(k(C))$ of the form $\langle 1, -s\mathcal{L}, -\pi, s\pi\mathcal{L} \rangle$.

These $(\frac{s\mathcal{L}, \pi}{\mathcal{O}_C})$ will comprise the order 2 Brauer group, as shown in the following two theorems.

Theorem 2. *Let C be a curve with good reduction over a non-dyadic local field K that has a K rational point P . The order 2 Brauer group, ${}_2\text{Br}(C)$, fits into a short exact sequence $0 \rightarrow k^\times/k^{\times 2} \rightarrow {}_2\text{Br}(C) \rightarrow {}_2\text{Pic}(C) \rightarrow 0$.*

Proof. Gabber's proof of purity [4] extends Corollary 6.2 from [6] to our setting. The localization sequence, suitably interpreted, thus becomes the exact sequence

$$0 \rightarrow Br(C_\nu)(\ell) \rightarrow Br(C)(\ell) \rightarrow H_{et}^1(C_k, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow H_{et}^3(C_\nu, G_m)(\ell) \xrightarrow{i} H_{et}^3(C, G_m)(\ell)$$

for any prime $\ell \neq 2$. By Artin's result [6, Theorem 3.1] and the vanishing of $Br(C_k)$, $Br(C_\nu) = 0$. We will show that the map i is injective for $\ell \neq p = \text{char}(k)$. Then $Br(C)(2) \cong H_{et}^1(C_k, \mathbb{Q}_2/\mathbb{Z}_2)$ and so ${}_2Br(C) \cong H_{et}^1(C_k, \mathbb{Z}/2)$ as desired.

First we need to calculate $H_{et}^3(C_\nu, G_m)(\ell)$. So take the limit over the Kummer sequences for powers of ℓ and observe that $H_{et}^2(C_\nu, G_m) = 0$. Then we find that $H_{et}^3(C_\nu, \mu_{\ell^\infty}) \cong H_{et}^3(C_\nu, G_m)(\ell)$. Similarly $H_{et}^3(C_k, \mu_{\ell^\infty}) \cong H_{et}^3(C_k, G_m)(\ell)$. Now Lemma 1 establishes the base change isomorphism $H_{et}^3(C_\nu, \mu_{\ell^\infty}) \cong H_{et}^3(C_k, \mu_{\ell^\infty})$. But the Hochschild–Serre spectral sequence $H^p(G(k_s/k), H_{et}^q(C_{k,s}, G_m)) \implies H_{et}^n(C_k, G_m)$ and Tsen's theorem establish the isomorphism $H_{et}^2(G(k_s/k), H_{et}^1(C_{k,s}, G_m)) \cong H_{et}^3(C_k, G_m)$ where $C_{k,s}$ is the base extension of C_k to the separable closure of k . But C_k has a k rational point and so there is a $G(k_s/k)$ isomorphism $H_{et}^1(C_{k,s}, G_m) = \text{Pic}^0(C_{k,s}) \oplus \mathbb{Z}$ where \mathbb{Z} has trivial $G(k_s/k)$ action. $\text{Pic}^0(C_{k,s})$ is torsion and so has trivial $G(k_s/k)$ second cohomology while \mathbb{Z} fits into a short exact sequence of trivial $G(k_s/k)$ modules, $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$. So $H_{et}^2(C_k, G_m) = \text{Hom}(G(k_s/k), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. (For $\ell \neq p = \text{char}(k)$, we could also use the Kummer sequence for ℓ^n to identify $H_{et}^3(C_k, \mu_{\ell^n}) = \mathbb{Z}/\ell^n$ with $\ell^n H_{et}^3(C_k, G_m)$ and then observe that $\mathbb{Z}/\ell^n = \ell^n \mathbb{Q}/\mathbb{Z}$ maps into the summand of $H^2(G(k_s/k), H_{et}^1(C_{k,s}, G_m))$ defined by the summand $\mathbb{Z} \hookrightarrow \text{Pic}(C_{k,s})$ since the identification comes from $H_{et}^2(C_{k,s}, \mu_{\ell^n}) = \mathbb{Z}/\ell^n[P_k]$ where P_k , the k rational point on C_k , is the reduction of the \mathcal{O}_ν point P_ν which, in turn is the extension of the K rational point P on C to an \mathcal{O}_ν point on C_ν .)

Finally we show that i is injective by noting that the existence of P on C that reduces to P_k on C_k extends to an identification of the generator of $\mathbb{Z} = \text{Pic}(C)/\text{Pic}^0(C)$ with a generator of $\text{Pic}(C_k)/\text{Pic}^0(C_k)$. Consequently

$$\begin{aligned} H^2(G(k_s/k), H_{et}^1(C_{k,s}, G_m)) \\ = H^2(G(k_s/k), H_{et}^1(C_\nu^s, G_m)) \rightarrow H^2(G(K_s/K), H_{et}^1(C_s, G_m)) \end{aligned}$$

may be interpreted as $\text{Hom}(G(k_s/k), \mathbb{Q}/\mathbb{Z}) \hookrightarrow \text{Hom}(G(K_s/K), \mathbb{Q}/\mathbb{Z})$. (Here C_ν^s is the extension of C_ν to the strict henselization of \mathcal{O}_ν .) But, in the Hochschild–Serre spectral sequence over K , $H^p(G(K_s/K), H_{et}^q(C_s, G_m)) \implies H_{et}^n(C, G_m)$, $H^2(G(K_s/K), H_{et}^1(C_s, G_m)) \subseteq H_{et}^3(C, G_m)$ because $E_\infty^{2,1} = E_2^{2,1}$ as we observe by noting $cd(K) = 2$ and $Br(C_s) = 0$. \square

The final step is to explicitly identify the algebras that lie in ${}_2Br(C)$.

Theorem 3. *The order 2 Brauer group of a smooth curve with good reduction over a local field consists of the distinct quaternions $(\frac{s\mathcal{L}}{C}, \pi)$ where $s \in k^\times/k^{\times 2}$ and $\mathcal{L} \in {}_2\text{Pic}(C)$.*

Proof. It suffices to show that the quaternions are distinct. That they comprise the entire order 2 Brauer group follows by a cardinality argument. We work over the function field

via the injections ${}_2Br(C) \hookrightarrow {}_2Br(k(C))$ and $W(C) \hookrightarrow W(k(C))$, taking $\langle s\xi \rangle$ and $\langle t\zeta \rangle$ to be the images of $\langle s\mathcal{L} \rangle$ and $\langle t\mathcal{M} \rangle$ in $W(k(C))$.

If $(\frac{s\xi, \pi}{k(C)}) = (\frac{t\zeta, \pi}{k(C)})$ then their norm forms are equal in $W(k(C))$ and we have a relation

$$\langle 1, -\pi, -\xi, \pi\xi, -1, \pi, \zeta, -\pi\zeta \rangle = \langle 1, -\pi, -\xi\zeta, \pi\xi\zeta \rangle = 0 \in W(k(C)).$$

This is the norm form of $(\frac{st\xi\zeta, \pi}{k(C)})$.

Thus, it suffices to show that $(\frac{s\xi, \pi}{k(C)})$ is nontrivial when $\langle \xi \rangle \neq \langle 1 \rangle$.

If $(\frac{s\xi, \pi}{k(C)})$ is trivial in ${}_2Br(k(C))$ then there is an identity $a'^2 - b'^2\xi = \pi$ in $k(C)$ [10, Theorem III.2.7].

If we take p to be the extension of the K -rational point over the closed point of $\text{Spec}(\mathcal{O}_\nu)$ then the following are true of the local ring $\mathcal{O}_{C_\nu, p}$:

- $\mathcal{O}_{C_\nu, p}$ is a unique factorization domain
- π is not a unit of $\mathcal{O}_{C_\nu, p}$
- π is a prime/irreducible element of $\mathcal{O}_{C_\nu, p}$
- The field of fractions of $\mathcal{O}_{C_\nu, p}$ is $k(C)$

This means that we can write $a' = \frac{a}{d}$ and $b' = \frac{b}{e}$ where $a, b, d, e \in \mathcal{O}_{C_\nu, p}$ and $(a, d) = (b, e) = 1$, giving a relation $a^2e^2 - b^2d^2\xi = \pi d^2e^2$ in $\mathcal{O}_{C_\nu, p}$. We can then factor all copies of π from both sides of this equation and reduce modulo π to get a relation $\bar{a}^2 - \bar{b}^2s\xi = 0$ in the local ring $\mathcal{O}_{C_k, p}$. This means that $\langle s\xi \rangle = \langle 1 \rangle$ at the stalk $\mathcal{O}_{C_k, p}$ and in the function field of C_k . \square

We can now calculate the Witt ring of C by describing its unique representatives.

Theorem 4. *Each nontrivial element of $W(C)$ has a representative of one of the following forms: $\langle s\mathcal{L} \rangle$, $\langle t\pi\mathcal{M} \rangle$, $\langle 1, s\mathcal{L} \rangle$, $\langle s\mathcal{L}, t\pi\mathcal{M} \rangle$, $\langle \pi, t\pi\mathcal{M} \rangle$, $\langle 1, s\mathcal{L}, t\pi\mathcal{M} \rangle$, $\langle s\mathcal{L}, \pi, t\pi\mathcal{M} \rangle$, or $\langle 1, s\mathcal{L}, \pi, t\pi\mathcal{M} \rangle$ where $\langle s \rangle, \langle t \rangle \in W(k)$ and $\mathcal{L}, \mathcal{M} \in {}_2\text{Pic}(C)$.*

Furthermore, these forms represent distinct Witt classes provided the fewest possible line bundles are used to represent each class and even rank forms are nontrivial except in the obvious cases.

Proof. Let n be the cardinality $|{}_2\text{Pic}(C)|$.

$W(C)/\mathbb{I}$ is the two element group with nontrivial representative $\langle 1 \rangle$, \mathbb{I}/\mathbb{I}^2 has $4n$ elements represented by forms $\langle -1, \delta \rangle$, and $\mathbb{I}^2/\mathbb{I}^3$ has $2n$ elements represented by quaternion norm forms $\langle 1, -s\mathcal{L}, -\pi, s\pi\mathcal{L} \rangle$.

This gives us two important facts:

$W(C)$ is a ring with $16n^2$ elements.

Every nontrivial element of $W(C)$ may be written $\langle 1 \rangle^{d_1} \perp \langle -1, d \rangle^{d_2} \perp \langle 1, -s\mathcal{L}, -\pi, s\pi\mathcal{L} \rangle^{d_3}$, $d_i \in \{0, 1\}$ (using Proposition 8).

We will now perform the calculation to show that $\langle 1 \rangle \perp \langle -1, \delta \rangle \perp \langle 1, -s\mathcal{L}, -\pi, s\pi\mathcal{L} \rangle$ has a representative of one of the listed forms.

$$\begin{aligned} E &= \langle 1 \rangle \perp \langle -1, \delta \rangle \perp \langle 1, -s\mathcal{L}, -\pi s\mathcal{L}, \pi \rangle \\ &= \langle 1, -1, \delta, 1, -s\mathcal{L}, -\pi s\mathcal{L}, \pi \rangle \\ &= \langle 1, -s\mathcal{L}, \delta, \pi, -\pi s\mathcal{L} \rangle. \end{aligned}$$

If $\delta = \langle t\mathcal{M} \rangle$ is a form which extends to a non-degenerate form over C_ν then $\langle 1, -s\mathcal{L}, \delta, \pi, -\pi s\mathcal{L} \rangle = \langle st\mathcal{L}\mathcal{M}, \pi, -s\pi\mathcal{L} \rangle$.

If $\delta = \langle t\pi\mathcal{M} \rangle$ is a form which does not extend to a non-degenerate form over C_ν then $\langle 1, -s\mathcal{L}, \delta, \pi, -\pi s\mathcal{L} \rangle = \langle 1, -s\mathcal{L}, st\pi\mathcal{L}\mathcal{M} \rangle$.

This completes the case $d_1 = d_2 = d_3 = 1$. The calculations for other values of d_i are analogous or obvious.

It remains to show that, assuming the fewest number of line bundles possible are used to represent each form, the forms described are distinct.

We know that $W(C)$ has $16n^2 - 1$ nontrivial elements and count the forms that cannot be written in an obvious way as a sum of fewer line bundles. There are $2n$ forms each $\langle s\mathcal{L} \rangle$ and $\langle t\pi\mathcal{M} \rangle$, $2n - 1$ forms each $\langle 1, s\mathcal{L} \rangle$ and $\langle \pi, t\pi\mathcal{M} \rangle$, $(2n)^2$ forms $\langle s\mathcal{L}, t\pi\mathcal{M} \rangle$, $2n(2n - 1)$ forms each $\langle 1, s\mathcal{L}, t\pi\mathcal{M} \rangle$ and $\langle s\mathcal{L}, \pi, t\pi\mathcal{M} \rangle$, and $(2n)^2 - 2n - 2n + 1$ forms $\langle 1, s\mathcal{L}, \pi, t\pi\mathcal{M} \rangle$. This is a total of $16n^2 - 1$ forms.

A cardinality argument shows that the forms of this proposition are distinct. \square

We conclude this paper with two theorems describing the algebraic structure of $W(C)$ as it relates to $W(C_k)$. This generalizes the analogous result for the Witt ring of a local field [10, Section VI.1].

Theorem 5. $W(C)$ is an abelian group isomorphic to $W(C_k) \oplus W(C_k)$.

Proof. There is an inclusion $W(C_k) \rightarrow W(C) : \bar{\mathcal{L}} \mapsto \mathcal{L}$ which maps $W(C_k)$ onto the elements of the forms $\langle s\mathcal{L} \rangle$ or $\langle 1, s\mathcal{L} \rangle$ in $W(C)$. We identify $W(C_k)$ with its image under this inclusion. The cokernel of this map is $\langle \pi \rangle W(C_k)$ and $\langle \pi \rangle W(C_k) \cong W(C_k)$ by factoring out $\langle \pi \rangle$. This gives a short exact sequence.

$$0 \rightarrow W(C_k) \rightarrow W(C) \rightarrow W(C_k) \rightarrow 0$$

We will define a left splitting map $W(C) \rightarrow W(C_k)$.

$\langle 1, -\pi \rangle W(C_k)$ is an ideal in $W(C)$ as $\langle s\mathcal{L} \rangle \langle 1, -\pi \rangle \in \langle 1, -\pi \rangle W(C_k)$ and

$$\langle s\pi\mathcal{L} \rangle \langle 1, -\pi \rangle = \langle s\mathcal{L} \rangle \langle \pi, -1 \rangle = \langle -s\mathcal{L} \rangle \langle 1, -\pi \rangle,$$

which is also in $\langle 1, \pi \rangle W(C_k)$. Furthermore, the composition f in the following diagram is an isomorphism.

$$\begin{array}{ccc}
 W(C_k) & \xrightarrow{i} & W(C) \\
 & \searrow f & \downarrow p \\
 & & W(C)/\langle 1, -\pi \rangle W(C_k)
 \end{array}$$

is an isomorphism.

The desired splitting map is $f^{-1} \circ p$. \square

Note that the splitting map $f^{-1} \circ p$ is a map of commutative rings.

Theorem 6. $W(C)$ is isomorphic as a ring to the group ring $W(C_k)[G]$ where G is a two-element group.

Proof. Taking $\langle 1 \rangle$ and $\langle \pi \rangle$ as the representatives for G , this is a corollary of Theorem 5. \square

References

- [1] Asher Auel, Remarks on the Milnor conjecture over schemes, in: Galois–Teichmüller Theory and Arithmetic Geometry, in: Adv. Stud. Pure Math., vol. 63, Math. Soc. Japan, Tokyo, 2012, pp. 1–30.
- [2] P. Balmer, C. Walter, A Gersten–Witt spectral sequence for regular schemes, Ann. Sci. Éc. Norm. Supér. 35 (1) (2002) 127–152.
- [3] Fernando Fernández-Carmena, The Witt group of a smooth complex surface, Math. Ann. 277 (3) (1987) 469–481.
- [4] Kazuhiro Fujiwara, A proof of the absolute purity conjecture (after Gabber), in: Algebraic Geometry 2000, Azumino (Hotaka), in: Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 153–183.
- [5] Jeanne M. Funk, Raymond T. Hoobler, The Witt ring of a smooth projective curve over a finite field, Comm. Algebra 42 (9) (2014) 3731–3735.
- [6] Alexander Grothendieck, Le groupe de Brauer. III. Exemples et compléments, in: Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188.
- [7] Kazuya Kato, A Hasse principle for two-dimensional global fields, J. Reine Angew. Math. 366 (1986) 142–183, with an appendix by Jean-Louis Colliot-Thélène.
- [8] Moritz Kerz, The Gersten conjecture for Milnor K -theory, Invent. Math. 175 (1) (2009) 1–33.
- [9] Manfred Knebusch, Symmetric bilinear forms over algebraic varieties, in: Conference on Quadratic Forms—1976, Proc. Conf., Queen’s Univ., Kingston, ON, 1976, in: Queen’s Papers in Pure and Appl. Math., vol. 46, Queen’s Univ., Kingston, ON, 1977, pp. 103–283.
- [10] T.Y. Lam, Introduction to Quadratic Forms over Fields, Grad. Stud. Math., vol. 67, American Mathematical Society, Providence, RI, 2005.
- [11] R. Parimala, Witt groups of affine three-folds, Duke Math. J. 57 (3) (1988) 947–954.
- [12] R. Sujatha, Witt groups of real projective surfaces, Math. Ann. 288 (1) (1990) 89–101.