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Grade, dominant dimension and Gorenstein algebras

Nan Gao [a,1], Steffen Koenig [b,*]

[a] *Department of Mathematics, Shanghai University, Shanghai 200444, PR China*
[b] *Institut für Algebra und Zahlentheorie, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany*

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ABSTRACT

We first give precise connections between Auslander–Bridger's grade, double centraliser properties and dominant dimension, and apply these to homological conjectures. Then we introduce gendo-d-Gorenstein algebras as correspondents of Gorenstein algebras under a Morita–Tachikawa correspondence. We characterise these algebras by homological properties and derive several of their properties, including higher Auslander correspondence.

1. Introduction

A classical result of representation theory is the Morita–Tachikawa correspondence [23] providing a bijection between pairs (Λ an artin algebra, M a generator-cogenerator) and algebras Γ of dominant dimension at least two. A celebrated special case is Auslander's bijection relating algebras Λ of finite representation type with algebras Γ of global dimension at most two and dominant dimension at least two. This has been generalised by Iyama [16] to 'Higher Auslander correspondence'. More recently, other

* Corresponding author.
E-mail addresses: nangao@shu.edu.cn (N. Gao), skoenig@mathematik.uni-stuttgart.de (S. Koenig).
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interesting classes of algebras have been defined by specialising the Morita–Tachikawa correspondence: Gendo-symmetric algebras [11,12] correspond to pairs (Λ, M) where Λ is symmetric; Morita algebras [19] correspond to pairs where Λ is self-injective.

In this paper, another such class of artin algebras is introduced, corresponding to pairs where Λ is d-Gorenstein and M is a Gorenstein projective generator. These 'gendo-d-Gorenstein algebras' Γ will be given two different characterisations, in terms of homological properties (see Theorem 4.2). Both characterisations explain $d = \mathrm{injdim}_\Lambda \Lambda$ in terms of Γ. A number of properties of these algebras is established, concerning global dimension, Hochschild cohomology, higher Auslander correspondence, and more.

In order to prove these results, a close relation between dominant dimension and double centraliser properties on the one hand and Auslander–Bridger's concept of grade is being worked out, extending results of Buchweitz [8]. The results give precise connections between these concepts, valid for artin algebras in general. As a by-product, some new results about homological conjectures are obtained.

This article is organised as follows: Section 2 is devoted to clarifying the connections among grade, double centraliser properties and dominant dimension for artin algebras. The main results are Theorem 2.3 characterising the double centraliser property in terms of grade, and Theorem 2.14 characterising grade in terms of dominant dimension. Section 3 uses the techniques set up in Section 2 to prove some assertions about homological conjectures, in particular several sufficient criteria for a 'grade' version of the Strong Nakayama Conjecture to hold true.

In Section 4, gendo-d-Gorenstein algebras are defined as correspondents of Gorenstein algebras under a Morita–Tachikawa correspondence, and characterised (in Theorem 4.2) in terms of homological properties. The final Section 5 then provides properties of these algebras. The proofs strongly use the techniques provided in Section 2.

2. Grade and dominant dimension

In this section, we will relate grade to double centraliser properties and to dominant dimension. By building on [8] we clarify the connections between these concepts.

Let Λ be an artin algebra. Denote by Λ-mod the category of finitely generated left Λ-modules. Recall from [26] that the *dominant dimension* of a module M in Λ-mod, which we denote by $\mathrm{domdim}\, M$, is the maximal number t (or ∞) having the following property: let $0 \to M \to I^0 \to I^1 \to \cdots \to I^t \to \cdots$ be a minimal injective resolution of M, then I^j is projective for all $j < t$ (or ∞). Let e be an idempotent of Λ and Λe a faithful projective-injective left Λ-module. Then $\mathrm{domdim}\, \Lambda \geq 2$ if and only if the left Λ-module Λe has the double centraliser property which means that $\mathrm{End}_{e\Lambda e}(\Lambda e) \cong \Lambda$ (see [26, 7.7]). Buchweitz [8, Proposition 2.9] showed for any idempotent e that $\Lambda \cong \mathrm{End}_{e\Lambda e}(\Lambda e)$ if and only if Auslander–Bridger's grade of $\Lambda/\Lambda e\Lambda$ as a right Λ-module is greater than 2.

Let M and T be in Λ-mod. Extending the classical theory of dominant dimension, one defines the *dominant dimension of T relative to M*, M- $\mathrm{domdim}\, T$, as the supremum of all $n \in \mathbb{Z}$ such that there exists an exact sequence $0 \to T \to M^1 \to M^2 \to \cdots \to M^n$ with all

M^i in add M (see [24, Section 3], [20]). Let $\Gamma = \mathrm{End}_\Lambda(M)$. Auslander and Solberg showed that the natural homomorphism $\alpha_T : T \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(T, M), M)$ is an isomorphism in Λ-mod if and only if there exists an exact sequence $0 \to T \xrightarrow{f} M^n \to M^m$, where $f : T \to M^n$ is a left add M-approximation (see [4, Proposition 2.1]). The latter implies that $M\text{-}\operatorname{domdim} T \geq 2$. This allows us to give the following definition.

Definition 2.1. Let Λ be an artin algebra. Let M, T be in Λ-mod and $\Gamma = \mathrm{End}_\Lambda(M)$. We say M *has the double centraliser property with respect to* T, if the natural homomorphism $\alpha_T : T \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(T, M), M)$ is an isomorphism in Λ-mod.

We will give a relation between grade and double centraliser property, using the following definition of grade.

Definition 2.2. (See Auslander–Bridger [2].) Let Λ be an artin algebra and Λ-mod be the category of finitely generated left Λ-modules. Let $M, X \in \Lambda$-mod. The *grade* of X with respect to M, written $\mathrm{grade}_M X$, is defined by

$$\mathrm{grade}_M X = \inf\{i \geq 0 \mid \mathrm{Ext}^i_\Lambda(X, M) \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

Theorem 2.3. *Let Λ be an artin algebra, M and T be in Λ-mod with $\Gamma = \mathrm{End}_\Lambda(M)$. Assume there is an exact sequence $M^m \xrightarrow{f} M^n \to T \to 0$, and set $X = \mathrm{Coker}\mathrm{Hom}_\Lambda(f, M)$. Then M has the double centraliser property with respect to T if and only if* $\mathrm{grade}_M X \geq 3$.

Proof. There is an exact sequence

$$\begin{tikzcd} M^m \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & M^n \arrow[r] & T \arrow[r] & 0 . \\ & K \arrow[ur, hook, "i_1"'] & & & \end{tikzcd} \qquad (*)$$

Applying the functor $\mathrm{Hom}_\Lambda(-, M)$ to $(*)$ gives the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (T, M) \arrow[r] & (M^n, M) \arrow[rr, "{(f,M)}"] \arrow[dr, "\pi_2"'] & & (M^m, M) \arrow[r] & X \arrow[r] & 0 \\ & & & C \arrow[ur, hook, "i_2"'] & & & \end{tikzcd}$$

where $(-, -)$ denotes the functor $\mathrm{Hom}_\Lambda(-, -)$. Since $\mathrm{Hom}_\Lambda(M^m, M)$ is a projective left Γ-module, we have $\mathrm{Ext}^1_\Gamma(\mathrm{Hom}_\Lambda(M^m, M), M) = \mathrm{Ext}^2_\Gamma(\mathrm{Hom}_\Lambda(M^m, M), M) = 0$. Hence, there is a long exact sequence

$$0 \to \mathrm{Hom}_\Gamma(X,M) \to \mathrm{Hom}_\Gamma\bigl(\mathrm{Hom}_\Lambda\bigl(M^m,M\bigr),M\bigr) \to \mathrm{Hom}_\Gamma(C,M) \to \mathrm{Ext}^1_\Gamma(X,M)$$
$$\to 0 \to \mathrm{Ext}^1_\Gamma(C,M) \to \mathrm{Ext}^2_\Gamma(X,M) \to 0.$$

So $\mathrm{Ext}^1_\Gamma(C,M) \cong \mathrm{Ext}^2_\Gamma(X,M)$. The map $M^n \to T$ induces the commutative diagram

$$\begin{tikzcd} M^n \arrow[r] \arrow[d, "\alpha_{M^n}"] & T \arrow[d, "\alpha_T"] \\ (\mathrm{Hom}_\Lambda(M^n,M),M) \arrow[r] & (\mathrm{Hom}_\Lambda(T,M),M) \end{tikzcd}$$

which is part of the following exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i_1"] \arrow[d, "g"] & M^n \arrow[r] \arrow[d, "\alpha_{M^n}"] & T \arrow[r] \arrow[d, "\alpha_T"] & 0 \arrow[d] \\ 0 \arrow[r] & (C,M) \arrow[r, "(\pi_2,M)"] & (\mathrm{Hom}_\Lambda(M^n,M),M) \arrow[r] & (\mathrm{Hom}_\Lambda(T,M),M) \arrow[r] & \mathrm{Ext}^1_\Gamma(C,M) \end{tikzcd}$$

where $(-,-)$ denotes the functor $\mathrm{Hom}_\Gamma(-,-)$ and g is an induced homomorphism. Applying the snake lemma yields $\mathrm{Ker}\,\alpha_T \cong \mathrm{Coker}\,g$ and $\mathrm{Coker}\,\alpha_T \cong \mathrm{Ext}^1_\Gamma(C,M) \cong \mathrm{Ext}^2_\Gamma(X,M)$.

Consider the following diagram

$$\begin{tikzcd} M^m \arrow[r, "\alpha_{M^m}"] \arrow[d, "\pi_1"] & \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(M^m,M),M) \arrow[r, "\mathrm{Hom}_\Gamma(i_2,M)"] & \mathrm{Hom}_\Lambda(C,M) \arrow[d, "="] \\ K \arrow[rr, "g"] \arrow[d, "i_1"] & & \mathrm{Hom}_\Lambda(C,M) \arrow[d, "\mathrm{Hom}_\Gamma(\pi_2,M)"] \\ M^n \arrow[rr, "\alpha_{M^n}"] & & \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(M^n,M),M) \end{tikzcd}$$

which commutes because $\alpha_{M^n} \circ f = \mathrm{Hom}_\Gamma((f,M),M) \circ \alpha_{M^m}$ and $\mathrm{Hom}_\Gamma(\pi_2,M) \circ g = \alpha_{M^n} \circ i_1$. This induces the following commutative diagram

$$\begin{tikzcd} & M^m \arrow[r, "\mathrm{Hom}_\Gamma(i_2,M)\circ\alpha_{M^m}"] \arrow[d, "\pi_1"] & \mathrm{Hom}_\Lambda(C,M) \arrow[r] \arrow[d, "="] & \mathrm{Ext}^1_\Gamma(X,M) \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & K \arrow[r, "g"] & \mathrm{Hom}_\Lambda(C,M) \arrow[r] & \mathrm{Coker}\,g \arrow[r] & 0 \end{tikzcd}.$$

By the snake lemma again, h is an isomorphism. So $\mathrm{Ker}\,\alpha_T \cong \mathrm{Coker}\,g \cong \mathrm{Ext}^1_\Gamma(X,M)$ and we obtain the exact sequence

$$0 \to \mathrm{Ext}^1_\Gamma(X,M) \to T \xrightarrow{\alpha_T} \mathrm{Hom}_\Gamma\big(\mathrm{Hom}_\Lambda(T,M),M\big) \to \mathrm{Ext}^2_\Gamma(X,M) \to 0.$$

This completes the proof. $\square$

The next result is dual to Theorem 2.3.

Theorem 2.4. *Let Λ be an artin algebra and $M \in \Lambda\text{-mod}$ with $\Gamma = \mathrm{End}_\Lambda(M)$. Let $T \in \Lambda\text{-mod}$ and $0 \to T \to M^n \xrightarrow{f} M^m$ be an exact sequence such that $X = \mathrm{Coker}\,\mathrm{Hom}_\Lambda(M,f)$. Then the natural homomorphism $\alpha_T : \mathrm{Hom}_\Lambda(M,T) \otimes_\Gamma M \to T$ is an isomorphism in $\Lambda\text{-mod}$ if and only if $\mathrm{Tor}^1_\Gamma(X,M) = \mathrm{Tor}^2_\Gamma(X,M) = 0$.*

Proof. The sequence

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] & M^n \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & M^m \\ & & & K \arrow[ur, hook, "i_1"'] & \end{tikzcd} \tag{$**$}$$

is exact. Applying the functor $\mathrm{Hom}_\Lambda(M,-)$ to $(**)$ gives the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (M,T) \arrow[r] & (M,M^n) \arrow[rr, "{(M,f)}"] \arrow[dr, "\pi_2"'] & & (M,M^m) \arrow[r] & X \arrow[r] & 0 \\ & & & C \arrow[ur, hook, "i_2"'] & & & \end{tikzcd}$$

where $(-,-)$ denotes the functor $\mathrm{Hom}_\Lambda(-,-)$. Hence, there is a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Tor}^2_\Gamma(X,M) \to \mathrm{Tor}^1_\Gamma(C,M) \to 0 \to \mathrm{Tor}^1_\Gamma(X,M) \to C \otimes_\Gamma M \\ &\to \big(M,M^m\big) \otimes_\Gamma M \to X \otimes_\Gamma M \to 0. \end{aligned}$$

Thus, $\mathrm{Tor}^1_\Gamma(C,M) \cong \mathrm{Tor}^2_\Gamma(X,M)$. In the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}^1_\Gamma(C,M) \arrow[r] & (M,T) \otimes_\Gamma M \arrow[r] \arrow[d, "\alpha_T"] & (M,M^n) \otimes_\Gamma M \arrow[r] \arrow[d, "\alpha_{M^n}"] & C \otimes_\Gamma M \arrow[r] \arrow[d, "g"] & 0 \\ & 0 \arrow[r] & T \arrow[r] & M^n \arrow[r] & K \arrow[r] & 0 \end{tikzcd}$$

the map α_{M^n} is an isomorphism and g is an induced homomorphism. The snake lemma implies that $\mathrm{Ker}\,\alpha_T = \mathrm{Tor}^1_\Gamma(C,M) \cong \mathrm{Tor}^2_\Gamma(X,M)$ and $\mathrm{Coker}\,\alpha_T \cong \mathrm{Ker}\,g$.

By the induced commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Ker} g & \longrightarrow & C \otimes_\Gamma M & \xrightarrow{g} & K & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \|} & & \downarrow{\scriptstyle i_1} & & & & \\ 0 & \longrightarrow & \operatorname{Tor}^1_\Gamma(X,M) & \longrightarrow & C \otimes_\Gamma M & \longrightarrow & M^m & \longrightarrow & X \otimes_\Gamma M & \longrightarrow & 0 \end{array}$$

$\operatorname{Ker} g \cong \operatorname{Tor}^1_\Gamma(X,M)$, and the following sequence is exact:

$$0 \to \operatorname{Tor}^2_\Gamma(X,M) \to \operatorname{Hom}_\Lambda(M,T) \otimes_\Gamma M \xrightarrow{\alpha_T} T \to \operatorname{Tor}^1_\Gamma(X,M) \to 0.$$

This completes the proof. □

Corollary 2.5. *Let Λ be an artin algebra and e be an idempotent of Λ. Suppose there is an exact sequence $0 \to \Lambda \to (\Lambda e)^n \xrightarrow{f} (\Lambda e)^m$ in Λ-mod. Let $X = \operatorname{Coker} \operatorname{Hom}_\Lambda(\Lambda e, f)$. Then Λ and $e\Lambda e$ are Morita equivalent if and only if $\operatorname{Tor}^1_{e\Lambda e}(\Lambda e, X) = \operatorname{Tor}^2_{e\Lambda e}(\Lambda e, X) = 0$. In particular, if $(\Lambda, e\Lambda e)$-bimodule Λe has double centraliser property, then $\Lambda/\Lambda e\Lambda = 0$ if and only if $\operatorname{Tor}^1_{e\Lambda e}(\Lambda e, X) = \operatorname{Tor}^2_{e\Lambda e}(\Lambda e, X) = 0$.*

Proof. We take $T = \Lambda$ and $M = \Lambda e$ in Theorem 2.4. Then $\Gamma = e\Lambda e$ and $\Lambda e \otimes_{e\Lambda e} e\Lambda \cong \Lambda$ if and only if $\operatorname{Tor}^1_{e\Lambda e}(\Lambda e, X) = \operatorname{Tor}^2_{e\Lambda e}(\Lambda e, X) = 0$. However, the former implies that $\Lambda/\Lambda e\Lambda = 0$, which is equivalent to saying that Λ and $e\Lambda e$ are Morita equivalent by [8, Corollary 1.10]. By [4, Proposition 2.1], if $(\Lambda, e\Lambda e)$-bimodule Λe has the double centraliser property, then there is an exact sequence $0 \to \Lambda \to (\Lambda e)^n \to (\Lambda e)^m$. This completes the proof. □

Next we will connect grade with dominant dimension in the sense of Kato [18].

Definition 2.6. (See Kato [18].) Let Λ be an artin algebra and $T, M \in \Lambda$-mod. T is said to have *M-dominant dimension greater than or equal to n*, written $M\text{-}\operatorname{domdim} T \geq n$, if each of the first n terms in a minimal injective resolution of T is cogenerated by M.

Proposition 2.7. *Let Λ be an artin algebra, and M and T be in Λ-mod. Suppose $M\text{-}\operatorname{domdim} T \geq 1$. Then, for any $n \geq 2$, $M\text{-}\operatorname{domdim} T \geq n$ if and only if $\operatorname{grade}_T X \geq n$ for any $X \in \Lambda$-mod with $\operatorname{Hom}_\Lambda(X, M) = 0$.*

Proof. Let $0 \to T \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} I_i \xrightarrow{f_{i+1}} \cdots$ be a minimal injective resolution of ${}_\Lambda T$. For any $X \in \Lambda$-mod and $i \geq 1$, there is an exact sequence

$$\operatorname{Hom}_\Lambda(X, I_{i-1}) \to \operatorname{Hom}_\Lambda(X, \operatorname{Im} f_i) \to \operatorname{Ext}^i_\Lambda(X,T) \to 0 \qquad (***)$$

Suppose M-$\operatorname{domdim} T \geq n$. Then $I_i \in \operatorname{add} M$ for any $0 \leq i \leq n-1$. So, if $\operatorname{Hom}_\Lambda(X, M) = 0$, then $\operatorname{Hom}_\Lambda(X, I_i) = 0$ and $\operatorname{Hom}_\Lambda(X, \operatorname{Im} f_i) = 0$ for any $0 \leq i \leq n-1$. Then (***) implies $\operatorname{Ext}^i_\Lambda(X,T) = 0$ for any $0 \leq i \leq n-1$. This proves that $\operatorname{grade}_T X \geq n$.

For the converse, we first have that $I_0 \in \operatorname{add} M$ by assumption. We claim that $\mathrm{Hom}_\Lambda(X, M) \neq 0$ for $X = \Lambda x$ and $x \in \operatorname{Im} f_1$. Otherwise, $\mathrm{Ext}^i_\Lambda(X, T) = 0$ for any $0 \leq i \leq n-1$ by assumption. The vanishing of $\mathrm{Hom}_\Lambda(X, I_0) = 0$, implies that $\mathrm{Hom}_\Lambda(X, \operatorname{Im} f_1) = 0$ by $(\ast\ast\ast)$, which is a contradiction. It follows that $\operatorname{Im} f_1$ is cogenerated by M. This implies that $I_1 \in \operatorname{add} M$. By similar arguments, we have $I_{n-1} \in \operatorname{add} M$. This completes the proof. □

Corollary 2.8. *Let Λ be an artin algebra and T be in Λ-mod. Suppose* $\operatorname{domdim} T \geq 1$. *Then, for any $n \geq 2$,* $\operatorname{domdim} T \geq n$ *if and only if* $\mathrm{Ext}^i_\Lambda(X, T) = 0$ *for any $X \in \Lambda$-mod with* $\mathrm{Hom}_\Lambda(X, \Lambda) = 0$ *and* $i = 1, 2, \cdots, n-1$.

Proof. This is an immediate consequence of Proposition 2.7. □

In the following, A is an artin algebra and A-mod is the category of finitely generated left A-modules. We will relate the grade of A/AeA as a left A-module to $D(eA)$-dominant dimension of A, where e is an idempotent of A and A/AeA is the quotient algebra of A modulo the idempotent ideal generated by e.

Theorem 2.9. *(See Psaroudakis [25, Theorem 3.10].) Let N be in A-mod and n be an integer. Then* $\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_{eAe}(eM, eN)$ *for any $M \in A$-mod and $0 \leq i \leq n$ if and only if there exists an exact sequence* $0 \to N \to \mathrm{Hom}_{eAe}(eA, I^0) \to \cdots \to \mathrm{Hom}_{eAe}(eA, I^{n+1})$ *with* $I^i \in \operatorname{add}({}_{eAe}D(eAe))$ *for* $0 \leq i \leq n+1$.

Lemma 2.10. *Let N be in A-mod. If* $\mathrm{Ext}^i_{eAe}(eA, eN) = 0$ *for* $1 \leq i \leq n$, *then for any M in A-mod, there are canonical isomorphisms*

$$\mathrm{Ext}^i_A\big(M, \mathrm{Hom}_{eAe}(eA, eN)\big) \cong \mathrm{Ext}^i_{eAe}(eM, eN)$$

for $0 \leq i \leq n$.

Proof. Choose an injective resolution of eN as a left eAe-module

$$0 \to eN \to I^0 \to \cdots \to I^n \to \cdots$$

such that each $I^j \in \operatorname{add} D(eAe)$. Since $\mathrm{Ext}^i_{eAe}(eA, eN) = 0$ for $1 \leq i \leq n$, there is an exact sequence

$$0 \to \mathrm{Hom}_{eAe}(eA, eN) \to \mathrm{Hom}_{eAe}\big(eA, I^0\big) \to \cdots \to \mathrm{Hom}_{eAe}\big(eA, I^{n+1}\big)$$

Theorem 2.9 implies

$$\mathrm{Ext}^i_A\big(M, \mathrm{Hom}_{eAe}(eA, eN)\big) \cong \mathrm{Ext}^i_{eAe}(eM, eN)$$

for any $M \in A$-mod and $0 \leq i \leq n$. □

Proposition 2.11. *Let $n \geq 2$ be an integer. Then $A \cong \mathrm{End}_{eAe}(eA)^{\mathrm{op}}$ and $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq n-2$ if and only if $D(eA)$-$\mathrm{domdim}\, A \geq n$.*

Proof. If $A \cong \mathrm{Hom}_{eAe}\,(eA, eA)^{\mathrm{op}}$, and $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq n-2$, then by Lemma 2.10 and Theorem 2.9 there is an exact sequence $0 \to A \to \mathrm{Hom}_{eAe}(eA, I^0) \to \cdots \to \mathrm{Hom}_{eAe}(eA, I^{n-1})$ with $I^i \in \mathrm{add}\, D(eAe)$ for $0 \leq i \leq n-1$. Because of the isomorphism $\mathrm{Hom}_{eAe}(eA, D(eAe)) \cong D(eA)$, we get $D(eA)$-$\mathrm{domdim}\, A \geq n$.

If $D(eA)$-$\mathrm{domdim}\, A \geq n$, then there is an exact sequence $0 \to A \to \mathrm{Hom}_{eAe}(eA, I^0) \to \cdots \to \mathrm{Hom}_{eAe}(eA, I^{n-1})$ with $I^i \in \mathrm{add}\, D(eAe)$ for $0 \leq i \leq n-1$ by $\mathrm{Hom}_{eAe}(eA, D(eAe)) \cong D(eA)$. By Theorem 2.9, $A \cong \mathrm{End}_{eAe}(eA)^{\mathrm{op}}$ and $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq n-2$. □

Lemma 2.12. *Let M be in A-mod.*

(1) *Then $D(eA)$-$\mathrm{domdim}\, M \geq n+1$ if and only if $\mathrm{Ext}^i_A(X, M) = 0$ for any $X \in A/AeA$-mod and $0 \leq i \leq n$. In particular, $D(eA)$-$\mathrm{domdim}\, A \geq n+1$ if and only if $\mathrm{grade}_A X \geq n+1$ for any $X \in A/AeA$-mod if and only if $\mathrm{grade}_A A/AeA \geq n+1$.*

(2) *Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of A-modules. Let $n = D(eA)$-$\mathrm{domdim}\, M$ and $n_i = D(eA)$-$\mathrm{domdim}\, M_i$ for $i = 1, 2$. Then $n \geq \min\{n_1, n_2\}$. Moreover,*

(a) $n_1 < n \Rightarrow n_2 = n_1 - 1$.
(b) $n_1 = n \Rightarrow n_2 \geq n - 1$; $n_1 = n + 1 \Rightarrow n_2 \geq n$; $n_1 \geq n + 2 \Rightarrow n_2 = n$.
(c) $n < n_2 \Rightarrow n_1 = n$.
(d) $n = n_2 \Rightarrow n_1 \geq n_2$; $n = n_2 + 1 \Rightarrow n_1 \geq n_2 + 1$; $n \geq n_2 + 2 \Rightarrow n_1 = n_2 + 1$.

Proof. The isomorphism ${}_AD(eA) \cong {}_A\mathrm{Hom}_{eAe}(eA, D(eAe))$ and [25, Proposition 3.4] imply that $D(eA)$-$\mathrm{domdim}\, M \geq n+1$ if and only if $\mathrm{Ext}^i_A(X, M) = 0$ for any $X \in A/AeA$-mod and $0 \leq i \leq n$. This implies that $D(eA)$-$\mathrm{domdim}\, A \geq n+1$ if and only if $\mathrm{grade}_A X \geq n+1$ for any $X \in A/AeA$-mod. If $\mathrm{grade}_A A/AeA = n$, then $\mathrm{Ext}^i_A(X, A) = 0$ for any $X \in A/AeA$-mod and $0 \leq i \leq n-1$ by [8, Lemma 2.2]. This means that $\mathrm{grade}_A X \geq n = \mathrm{grade}_A A/AeA$ for any $X \in A/AeA$-mod. This completes the proof.

(2) Applying the functor $\mathrm{Hom}_A(X, -)$ to the exact sequence above for any $X \in A/AeA$-mod, yields a long exact sequence

$$\cdots \to \mathrm{Ext}^i_A(X, M) \to \mathrm{Ext}^i_A(X, M_2) \to \mathrm{Ext}^{i+1}_A(X, M_1) \to \mathrm{Ext}^{i+1}_A(X, M) \to \cdots$$

By (1), it is clear that $n \geq \min\{n_1, n_2\}$. If $n_1 < n$, then by the long exact sequence, $\mathrm{Ext}^i_A(X, M_2) = 0$ for $0 \leq i \leq n_1 - 2$ and $\mathrm{Ext}^{n_1-1}_A(X, M_2) \cong \mathrm{Ext}^{n_1}(X, M_1) \neq 0$. So $n_2 = n_1 - 1$. Similarly, (b), (c) and (d) hold. □

Remark 2.13. For a finite dimensional algebra A over a field k, Iyama has proved in [16, Proposition 3.5.1] that $D(eA)$-$\mathrm{domdim}\, A \geq n+1$ if and only if $\mathrm{grade}_A X \geq n+1$ for any $X \in A/AeA$-mod; this is one equivalence in Lemma 2.12(1).

Theorem 2.14. *Let n be a non-negative integer. Then* $\operatorname{grade}_A A/AeA = n$ *if and only if* $D(eA)\text{-}\operatorname{domdim} A = n$.

Proof. If $\operatorname{grade}_A A/AeA = n$, then by Lemma 2.12(1) $D(eA)\text{-}\operatorname{domdim} A \geq n$. We claim that $D(eA)\text{-}\operatorname{domdim} A \ngeq n+1$. Otherwise, by Lemma 2.12(1) again, $\operatorname{Ext}_A^n(A/AeA, A) = 0$. This is a contradiction.

If $D(eA)\text{-}\operatorname{domdim} A = n$, then by Lemma 2.12(1) $\operatorname{grade}_A A/AeA \geq n$. We claim that $\operatorname{grade}_A A/AeA \ngeq n+1$. Otherwise, by Lemma 2.12(1) again, $D(eA)\text{-}\operatorname{domdim} A \geq n+1$. This is a contradiction. This implies that $\operatorname{grade}_A A/AeA = n$. □

Appendix

The grade of the A-module A/AeA is closely related to another homological dimension, the faithful dimension (in our case, of Ae) defined by Buan and Solberg [7]. In this appendix we make this connection precise and explain how faithful dimension can be used in our context.

First we recall the definitions due to Buan and Solberg [7]. Let A be an artin algebra and M be an A-module. There is a complex $\eta : 0 \to A \xrightarrow{f^1} M^1 \xrightarrow{f^2} M^2 \to \cdots \xrightarrow{f^n} M^n \to \cdots$, with $K^i = \operatorname{Coker} f^i$ for $i \geq 1$ and $K^0 = A$, such that each $K^i \to M^{i+1}$ is a minimal left $\operatorname{add} M$-approximation. Let η^n denote the truncated complex ending in M^n obtained from η. Then M is said to have *faithful dimension* n if η^n is exact, but η^{n+1} is not. If η is exact, then M has infinite faithful dimension. This dimension is denoted by $\operatorname{fadim} M$.

There is also a complex $\theta : \cdots \to M'_n \xrightarrow{f'_n} \cdots \xrightarrow{f'_2} M'_1 \xrightarrow{f'_1} D(A^{\mathrm{op}}) \to 0$, with $K'_i = \operatorname{Im} f'_i$ such that each $M'_i \to K'_i$ is a minimal right $\operatorname{add} M$-approximation. Let θ_n denote the truncated complex starting in M'_n obtained from θ. Then M is said to have *cofaithful dimension* n if θ_n is exact, but θ_{n+1} is not. This dimension is denoted by $\operatorname{cofadim} M$.

Proposition 2.15. *(See Buan–Solberg [7, Proposition 2.1 and Proposition 2.2].) Let A be an artin algebra and n be a non-negative integer. Then:*

(1) $\operatorname{fadim} Ae = n$ *if and only if* $\operatorname{cofadim} Ae = n$.

(2) *The (A, eAe)-bimodule Ae has the double centraliser property if and only if Ae has faithful dimension at least 2.*

(3) *Suppose that the (A, eAe)-bimodule Ae has the double centraliser property. Then Ae has faithful dimension n if and only if* $\operatorname{Ext}^i_{eAe}(Ae, Ae) = 0$ *for* $1 \leq i \leq n-2$ *and* $\operatorname{Ext}^{n-1}_{eAe}(Ae, Ae) \neq 0$.

In the context of this paper, these results translate into the first two statements of the following corollary. Combining Proposition 2.15 with Theorem 2.14 then gives the connection between grade and faithful dimension.

Corollary 2.16. *Let A be an artin algebra and n be a non-negative integer.*

(1) *The (A, eAe)-bimodule Ae has double centraliser property if and only if* $D(Ae)\text{-}\operatorname{domdim} A \geq 2$*;*

(2) $A \cong \mathrm{End}_{eAe}(Ae)$, $\mathrm{Ext}^i_{eAe}(Ae, Ae) = 0$ *for* $1 \le i \le n-2$ *and* $\mathrm{Ext}^{n-1}_{eAe}(Ae, Ae) \ne 0$ *if and only if* $D(Ae)$-$\mathrm{domdim}\, A = n$*;*

(3) *Ae has faithful dimension n if and only if* $\mathrm{grade}_A\, A/AeA = n$.

Proof. By Proposition 2.15, Ae has faithful dimension n if and only if $D(Ae)$-domdim $A = n$, and the (A, eAe)-bimodule Ae has double centraliser property if and only if Ae has faithful dimension at least 2, and Ae has faithful dimension $n \ge 2$ if and only if $A \cong \mathrm{End}_{eAe}(Ae)$, $\mathrm{Ext}^i_{eAe}(Ae, Ae) = 0$ for $1 \le i \le n-2$ and $\mathrm{Ext}^{n-1}_{eAe}(Ae, Ae) \ne 0$. So the (A, eAe)-bimodule Ae has double centraliser property if and only if $D(Ae)$-$\mathrm{domdim}\, A \ge 2$, and also $A \cong \mathrm{End}_{eAe}(Ae)$, $\mathrm{Ext}^i_{eAe}(Ae, Ae) = 0$ for $1 \le i \le n-2$ and $\mathrm{Ext}^{n-1}_{eAe}(Ae, Ae) \ne 0$ if and only if $D(Ae)$-$\mathrm{domdim}\, A = n$. By Theorem 2.14, Ae has faithful dimension n if and only if $\mathrm{grade}_A\, A/AeA = n$. □

Remark 2.17. (1) and (2) in Corollary 2.16 imply Proposition 2.11.

3. Application to homological conjectures

In this section, we will give some consequences for homological conjectures.

We first recall two crucial definitions. Let Λ be an artin algebra. Following Jans [17], a Λ-module X is said to have *an ultimately closed injective resolution at* n if, in an injective resolution $(I^\bullet, \delta^i)$ of X, $\mathrm{Im}\,\delta^n = \bigoplus_{j=1}^m W^j$ with each W^j isomorphic to a direct summand of some $\mathrm{Im}\,\delta^j$, $0 \le j \le n-1$. A Λ-module M is called *self-orthogonal* if $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for any $i \ge 1$.

Lemma 3.1. *Let Λ, T, M and α_T be as in Definition 2.1. Let* $M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \to M_1 \xrightarrow{d_1} M_0 \to T \to 0$ *with* $M_i \in \mathrm{add}\, M$ *be an exact sequence in* Λ-mod. *Suppose that M is self-orthogonal and M has an ultimately closed injective resolution at $n-1$ as a left Γ-module. If* $\mathrm{Ext}^i_\Lambda(T, M) = 0$ *for any* $1 \le i \le n-1$*, then α_T is an isomorphism.*

Proof. We first claim that there is an exact sequence

$$0 \to \mathrm{Ext}^n_\Gamma(Y, M) \to T \xrightarrow{\alpha_T} \mathrm{Hom}_\Gamma\big(\mathrm{Hom}_\Lambda(T, M), M\big) \to \mathrm{Ext}^{n+1}_\Gamma(Y, M) \to 0 \qquad (\divideontimes)$$

where $Y = \mathrm{Coker}\,\mathrm{Hom}_\Lambda(d_n, M)$.

The case $n = 1$ follows from Theorem 2.3. Now suppose $n \ge 2$. Since M is self-orthogonal and $\mathrm{Ext}^i_\Lambda(T, M) = 0$ for $1 \le i \le n-1$, there is an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(T, M) \to \mathrm{Hom}_\Lambda(M_0, M) \to \cdots \to \mathrm{Hom}_\Lambda(M_n, M) \to {}_\Gamma Y \to 0$$

where $Y = \mathrm{Coker}\,\mathrm{Hom}_\Lambda(d_n, M)$. As $\mathrm{Hom}_\Lambda(M_j, M)$ is Γ-projective for each $j \ge 0$, we get $\mathrm{Ext}^i_\Gamma(X, M) \cong \mathrm{Ext}^{i+n-1}_\Gamma(Y, M)$ where $X = \mathrm{Coker}\,\mathrm{Hom}_\Lambda(d_1, M)$. This yields the exact sequence $(\divideontimes)$.

Exactness of the induced sequence $0 \to \mathrm{Hom}_\Gamma(Y, M) \to M_n \xrightarrow{d_n} M_{n-1} \to \cdots \xrightarrow{d_0} M_0 \to T \to 0$ implies $\mathrm{Ext}^i_\Gamma(Y, M) = 0$ for any $1 \le i \le n-1$.

Let $0 \to M \xrightarrow{\delta^0} I^0 \xrightarrow{\delta^1} I^1 \to \cdots \xrightarrow{\delta^i} I^i \to \cdots$ be an ultimately closed injective resolution of M at $n-1$. Then $\mathrm{Im}\,\delta^{n-1} = \bigoplus_{j=1}^m W^j$ such that each W^j is a direct summand of some $\mathrm{Im}\,\delta^j$ with $0 \le j \le n-2$. So $\mathrm{Ext}^n_\Gamma(Y, M) \cong \mathrm{Ext}^1_\Gamma(Y, \mathrm{Im}\,\delta^{n-1}) \cong \bigoplus_{j=1}^m \mathrm{Ext}^1_\Gamma(Y, W^j)$. Since $\bigoplus_{j=1}^m \mathrm{Ext}^1_\Gamma(Y, \mathrm{Im}\,\delta^j) \cong \bigoplus_{j=1}^m \mathrm{Ext}^{j+1}_\Gamma(Y, M) = 0$ as above, it follows that $\mathrm{Ext}^n_\Gamma(Y, M) = 0$. Similarly $\mathrm{Ext}^{n+1}_\Gamma(Y, M) = \bigoplus_{j=1}^m \mathrm{Ext}^{j+2}_\Gamma(Y, M) = 0$. Thus α_T is an isomorphism. □

Recall from [8] the *Strong Nakayama Conjecture* (**SNC**) for noetherian algebras: Let Λ be a noetherian algebra. If M is a finitely generated Λ-module, then $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for any $i \ge 0$ implies $M = 0$. Let (Λ, e) be a Wedderburn context [8, Definition 2.15] with Λ a noetherian algebra. Buchweitz proved that if **SNC** holds, then $\mathrm{Ext}^i_\Lambda(\Lambda/\Lambda e\Lambda, \Lambda) = 0$ for all $i \ge 0$ implies that $\Lambda/\Lambda e\Lambda = 0$, which he calls the Idempotent Nakayama Conjecture (**INC**). Inspired by this result, we formulate the following condition:

Condition (SNC′). Let Λ be a noetherian algebra. If T and M are two finitely generated Λ-modules, then $\mathrm{Ext}^i_\Lambda(T, M) = 0$ for any $i \ge 0$ implies $T = 0$.

Note that when choosing $T = M$ and $M = \Lambda$ in **SNC′**, then **SNC′** is exactly **SNC**.

Corollary 3.2. *Let Λ, T, M and α_T be as Lemma 3.1. Then* **SNC′** *holds.*

Proof. By Lemma 3.1, α_T is an isomorphism. Since $\mathrm{Hom}_\Lambda(T, M) = 0$, it follows that $T = 0$. □

The next proposition generalises [9, Theorem 2].

Proposition 3.3. *Under the assumption of Lemma 3.1, suppose ${}_\Lambda M$ is flat. If $\mathrm{Ext}^i_\Lambda(T, \Lambda) = 0$ for any $i \ge 0$, then $T = 0$. That is,* **SNC** *holds.*

Proof. By [2, Theorem 2.8], the following sequence is exact for any $i \ge 0$,

$$\mathrm{Ext}^i_\Lambda(T, \Lambda) \otimes_\Lambda M \to \mathrm{Ext}^i_\Lambda(T, M) \to \mathrm{Tor}^\Lambda_1(X, M),$$

where $X = \mathrm{Coker}(P_0^* \to P_1^*)$ with $P_1 \to P_0 \to \Omega^i(T) \to 0$ being an exact sequence in $\mathrm{mod}\,\Lambda$ where P_0 and P_1 are projective. Because ${}_\Lambda M$ is flat, $\mathrm{Tor}^\Lambda_1(X, M) = 0$. So $\mathrm{Ext}^i_\Lambda(T, M) = 0$ for any $i \ge 0$, which implies $T = 0$ by Corollary 3.2. □

Lemma 3.4. *Let Λ be an artin algebra and $M \in \Lambda\text{-mod}$ self-orthogonal with $\Gamma = \mathrm{End}_\Lambda(M)$. Let $T \in \Lambda\text{-mod}$ and $0 \to T \to M^0 \xrightarrow{f^1} M^1 \to \cdots \xrightarrow{f^n} M^n$ be an exact sequence with $M^i \in \mathrm{add}\,M$. If $\mathrm{Ext}^i_\Lambda(M, T) = 0$ for any $1 \le i \le n-1$, then there is the following exact sequence*

$$0 \to \operatorname{Tor}_\Gamma^{n+1}(Y, M) \to \operatorname{Hom}_\Lambda(M, T) \otimes_\Gamma M \xrightarrow{\alpha_T} T \to \operatorname{Tor}_\Gamma^n(Y, M) \to 0$$

where $Y = \operatorname{Coker} \operatorname{Hom}_\Lambda(M, f^n)$.

Proof. The case $n = 1$ follows from the proof of Theorem 2.4. Now suppose $n \geq 2$. Consider the exact sequence

$$0 \to T \to M^0 \to M^1 \to \cdots \xrightarrow{f^n} M^n.$$

Since M is self-orthogonal and $\operatorname{Ext}_\Lambda^i(M, T) = 0$ for any $1 \leq i \leq n-1$, the following sequence

$$0 \to (M, T) \to \big(M, M^0\big) \to \big(M, M^1\big) \to \big(M, M^2\big) \to \cdots \to \big(M, M^n\big) \to Y \to 0$$

is exact, where $(-,-)$ denotes the functor $\operatorname{Hom}_\Lambda(-,-)$ and $Y = \operatorname{Coker}(M, f^n)$. Note that $\operatorname{Tor}_\Gamma^{i+n-1}(Y, M) \cong \operatorname{Tor}_\Gamma^i(X, M)$ where $X = \operatorname{Coker}(M, f^1)$. Then we have the following exact sequence, again by Theorem 2.4

$$0 \to \operatorname{Tor}_\Gamma^{n+1}(Y, M) \to \operatorname{Hom}_\Lambda(M, T) \otimes_\Gamma M \xrightarrow{\alpha_T} T \to \operatorname{Tor}_\Gamma^n(Y, M) \to 0. \qquad \square$$

Corollary 3.5. *Let Λ be an artin algebra and $M \in \Lambda$-mod self-orthogonal with $\Gamma = \operatorname{End}_\Lambda(M)$. Let $T \in \Lambda$-mod and $0 \to T \to M^0 \xrightarrow{f^0} M' \to \cdots \xrightarrow{f^n} M^n$ be an exact sequence with $M^i \in \operatorname{add} M$. If M has an ultimately closed projective resolution at $n-1$ as a left Γ-module and $\operatorname{Ext}_\Lambda^i(M, T) = 0$ for any $1 \leq i \leq n-1$, then α_T is an isomorphism. In particular,* **SNC**$'$ *holds.*

Proof. Let $\cdots \to P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_0 \xrightarrow{d_0} M \to 0$ be an ultimately closed projective resolution of M at $n-1$ as a Γ-module. Then $\operatorname{Im} d_{n-1} = \bigoplus_{j=1}^m U_j$ such that each U_j is a direct summand of some $\operatorname{Im} d_j$ with $0 \leq j \leq n-2$. So, $\operatorname{Tor}_\Gamma^n(Y, M) \cong \operatorname{Tor}_\Gamma^1(Y, \operatorname{Im} d_{n-1}) \cong \bigoplus_{j=1}^m \operatorname{Tor}_\Gamma^1(Y, U_j)$. Since $\bigoplus_{j=1}^m \operatorname{Tor}_\Gamma^1(Y, \operatorname{Im} d_j) \cong \bigoplus_{j=1}^m \operatorname{Tor}_\Gamma^{j+1}(Y, M) = 0$ by assumption, it follows that $\operatorname{Tor}_\Gamma^n(Y, M) = 0$. Similarly, $\operatorname{Tor}_\Gamma^{n+1}(Y, M) = \bigoplus_{j=1}^m \operatorname{Tor}_\Gamma^{j+2}(Y, M) = 0$. Thus α_T is an isomorphism by Lemma 3.4. If $\operatorname{Hom}_\Lambda(M, T) = 0$, then $T = 0$ by the above arguments. $\square$

Proposition 3.6. *Under the assumption of Corollary 3.5, suppose ${}_\Lambda T$ is flat. If $\operatorname{Ext}_\Lambda^i(M, \Lambda) = 0$ for any $i \geq 0$, then $M = 0$. That is,* **SNC** *holds.*

Proof. The proof of Proposition 3.3 works here as well. $\square$

4. Morita–Tachikawa correspondence for Gorenstein algebras

In this section we characterise by intrinsic properties the algebras corresponding to Gorenstein algebras (paired with Gorenstein projective generators) under (an extended version of) Morita–Tachikawa correspondence.

Recall from [14,10] that a noetherian algebra A is Iwanaga–Gorenstein (for short, *Gorenstein*) if $\operatorname{injdim}_A A < \infty$ and $\operatorname{injdim} A_A < \infty$. A Gorenstein algebra A is *d-Gorenstein* if $\operatorname{injdim}_A A \leq d < \infty$. Let A be an artin algebra. Denote by A-mod (resp. mod-A) the category of finitely generated left (resp. right) A-modules, and A-proj (resp. proj-A) the full subcategory of finitely generated projective left (resp. right) A-modules. An A-module M is said to be *Gorenstein projective* in A-mod (resp. mod-A), if there is an exact sequence $P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow P^2 \longrightarrow \cdots$ in A-proj (resp. proj-A) with $\operatorname{Hom}_A(P^\bullet, Q)$ exact for any A-module Q in A-proj (resp. proj-A), such that $M \cong \ker d^0$. Denote by A-$\mathcal{G}$proj (resp. $\mathcal{G}$proj-A) the full subcategory of Gorenstein projective modules in A-mod (resp. mod-A). Note that the functor $\operatorname{Hom}_A(-, A) : A\text{-}\mathcal{G}\mathrm{proj} \to \mathcal{G}\mathrm{proj}\text{-}A$ is a duality of categories.

Lemma 4.1. *Let A be an artin R-algebra over a commutative artin ring R. Let e be an idempotent of A and d a non-negative integer. Suppose there is an exact sequence of finitely generated right A-modules $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ with each $I^j \in \operatorname{add}(D(Ae))$, and an exact sequence of finitely generated left A-modules $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \operatorname{add}(D(eA))$. Then Ae is a finitely generated Gorenstein projective right eAe-module and eAe is a d-Gorenstein artin R-algebra.*

Proof. Since A is an artin R-algebra and Ae is a finitely generated left A-module, $eAe = \operatorname{End}_A(Ae)^{\mathrm{op}}$ is an artin R-algebra and Ae is a finitely generated right eAe-module, and also A/AeA is an artin R-algebra. Consider the triangle

$$Ae \otimes^{\mathbb{L}}_{eAe} eA \xrightarrow{f} A \to X^\bullet \to Ae \otimes^{\mathbb{L}}_{eAe} eA[1] \tag{$\triangle$}$$

in $D(A^{\mathrm{op}} \otimes_R A)$, where f is the composition $Ae \otimes^{\mathbb{L}}_{eAe} eA \to Ae \otimes_{eAe} eA \xrightarrow{\text{mult.}} A$ of natural maps. Applying the functor $eA \otimes^{\mathbb{L}}_A -$ to ($\triangle$) provides us with an isomorphism $eA \otimes^{\mathbb{L}}_A f$. Thus $eA \otimes^{\mathbb{L}}_A X^\bullet = 0$ holds. This means that $e\mathrm{H}^i(X^\bullet) = 0$ and hence $\mathrm{H}^i(X^\bullet) \in A/AeA$-mod for any $i \in \mathbb{Z}$. Since Ae and eA are concentrated in degree 0, $\mathrm{H}^i(Ae \otimes^{\mathbb{L}}_{eAe} eA)$ vanishes for $i > 0$, and then $\mathrm{H}^i(X^\bullet) = 0$ for any $i > 0$.

Applying the functor $\operatorname{Hom}_A(Ae, -)$ to the exact sequence $0 \to {}_AAe \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \operatorname{add}({}_AD(eA))$ for $0 \leq j \leq d$ yields an exact sequence $0 \to {}_{eAe}eAe \to E'^0 \to \cdots \to E'^d \to 0$ with each $E'^j \in \operatorname{add}({}_{eAe}D(eAe))$ for $0 \leq j \leq d$. Therefore, $\operatorname{injdim}_{eAe} eAe \leq d$. Lemma 2.10(1) and $\operatorname{injdim}_A Ae \leq d$ imply $\operatorname{Ext}^i_A(X, Ae) = 0$ for any $X \in A/AeA$-mod and $i \in \mathbb{Z}$, and so $\mathbb{R}\operatorname{Hom}_A(X^\bullet, Ae) = 0$. Applying $\mathbb{R}\operatorname{Hom}_A(-, Ae)$ to ($\triangle$) yields a series of isomorphisms

$$\begin{aligned} Ae = \mathbb{R}\operatorname{Hom}_A(A, Ae) &\cong \mathbb{R}\operatorname{Hom}_A\big(Ae \otimes^{\mathbb{L}}_{eAe} eA, Ae\big) \\ &\cong \mathbb{R}\operatorname{Hom}_{eAe}\big(eA, \mathbb{R}\operatorname{Hom}_A(Ae, Ae)\big) \\ &\cong \mathbb{R}\operatorname{Hom}_{eAe}(eA, eAe) \end{aligned}$$

in $D((A)^{\mathrm{op}} \otimes_R eAe)$. This means that $Ae \cong \mathrm{Hom}_{eAe}(eA, eAe)$ as (A, eAe)-bimodules and $\mathrm{Ext}^i_{eAe}(eA, eAe) = 0$ for $i \neq 0$.

Applying the functor $\mathrm{Hom}_A(eA, -)$ to the exact sequence of right A-modules $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ with each $I^j \in \mathrm{add}(D(Ae))$, gives an exact sequence $0 \to eAe_{eAe} \to I'^0 \to \cdots \to I'^d \to 0$ with each $I'^j \in \mathrm{add}(D(eAe)_{eAe})$ for $0 \leq j \leq d$. This implies that $\mathrm{injdim}\, eAe_{eAe} \leq d$. As above, we obtain that eAe is d-Gorenstein, and Ae is a Gorenstein projective right eAe-module. $\square$

Now we state the main theorem of this section, which generalises [11, Theorem 3.2] and [23, Section 16]. Note that projective modules always are Gorenstein projective, and thus Gorenstein projective generators exist.

Theorem 4.2. *Let A be an artin R-algebra. Then the following statements are equivalent:*

(1) *A is isomorphic to the endomorphism algebra of a finitely generated Gorenstein projective generator over a d-Gorenstein artin R-algebra for d a non-negative integer.*

(2) *There is an idempotent e of A and a non-negative integer d such that there is an exact sequence of finitely generated right A-modules $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ with each $I^j \in \mathrm{add}(D(Ae))$, and an exact sequence of finitely generated left A-modules $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \mathrm{add}(D(eA))$, and the (A, eAe)-bimodule Ae has the double centraliser property.*

(2′) *There is an idempotent e of A and a non-negative integer d such that there is an exact sequence of finitely generated right A-modules $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ with each $I^j \in \mathrm{add}(D(Ae))$, and an exact sequence of finitely generated left A-modules $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \mathrm{add}(D(eA))$, and the (eAe, A)-bimodule eA has the double centraliser property.*

(3) *There is an idempotent e of A and a non-negative integer d such that eA is a finitely generated left Gorenstein projective eAe-module and $\mathrm{injdim}(eA_A) = \mathrm{injdim}({}_AAe) = d < \infty$, also $D(eA)\text{-}\mathrm{domdim}\, A \geq 2$.*

Proof. (1) $\Longrightarrow$ (2) Let B be a d-Gorenstein artin R-algebra and M a Gorenstein projective generator in $\text{mod-}B$ such that $A = \mathrm{End}_B(M)$. We may write $M = N \oplus B$ for some $N \in \text{mod-}B$. Let $e : M \twoheadrightarrow B$ be the canonical projection, regarded as an element of A. Then $B \cong eAe$ and $M \cong Ae$. This implies $A \cong \mathrm{End}_{eAe}(Ae)$.

Since eAe is d-Gorenstein, there exists an exact sequence $0 \to eAe_{eAe} \to J^0 \to J^1 \to \cdots \to J^d \to 0$ such that each $J^i \in \mathrm{add}\, D({}_{eAe}eAe)$. Since Ae is a finitely generated Gorenstein projective right eAe-module, applying the functor $\mathrm{Hom}_{eAe}(Ae, -)$ yields an exact sequence of right A-modules $0 \to \mathrm{Hom}_{eAe}(Ae, eAe) \to \mathrm{Hom}_{eAe}(Ae, J^0) \to \cdots \to \mathrm{Hom}_{eAe}(Ae, J^d) \to 0$. Since $\mathrm{Hom}_{eAe}(Ae, D({}_{eAe}eAe)) \cong D(Ae)$ as right A-modules, we get that $\mathrm{Hom}_{eAe}(Ae, J^i) \in \mathrm{add}\, D(Ae)_A$ for $0 \leq i \leq d$. Now we claim that $\mathrm{Hom}_{eAe}(Ae, eAe) \cong eA$ as (eAe, A)-bimodules. Indeed, since $A \cong \mathrm{End}_{eAe}(Ae)$, we get from [8, Proposition 2.9] that $\mathrm{Hom}_A(A/AeA, eA) = 0$ and $\mathrm{Ext}^1_A(A/AeA, eA) = 0$. So $\mathrm{Hom}_A(AeA, eA) \cong \mathrm{Hom}_A(A, eA)$ as (eAe, A)-bimodules by the short exact sequence

$0 \to AeA \to A \to A/AeA \to 0$ of A-bimodules. Moreover, the proof of [8, Lemma 2.11] shows that $\mathrm{Hom}_A(AeA, eA) \cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, eA)$ as (eAe, A)-bimodules. Thus, there are the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{eAe}(Ae, eAe) &\cong \mathrm{Hom}_{eAe}\big(Ae, \mathrm{Hom}_A(eA, eA)\big)\\ &\cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, eA)\\ &\cong \mathrm{Hom}_A(AeA, eA)\\ &\cong \mathrm{Hom}_A(A, eA) = eA\end{aligned}$$

of right (eAe, A)-bimodules. Hence eA is a left Gorenstein projective eAe-module, and there is an exact sequence of right A-modules $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ with each $I^j \in \mathrm{add}(D(Ae))$.

Again using that eAe is a d-Gorenstein artin R-algebra, there exists an exact sequence $0 \to {}_{eAe}eAe \to J'^0 \to J'^1 \to \cdots \to J'^d \to 0$ such that each $J'^i \in \mathrm{add}\, D(eAe_{eAe})$. Since eA is a finitely generated Gorenstein projective left eAe-module, applying the functor $\mathrm{Hom}_{eAe}(eA, -)$ yields an exact sequence of left A-modules $0 \to \mathrm{Hom}_{eAe}(eA, eAe) \to \mathrm{Hom}_{eAe}(eA, J'^0) \to \cdots \to \mathrm{Hom}_{eAe}(eA, J'^d) \to 0$. Since $D(eA) \cong \mathrm{Hom}_{eAe}(eA, D(eAe_{eAe}))$ as left A-modules, we get that $\mathrm{Hom}_{eAe}(Ae, J'^i) \in \mathrm{add}_A D(eA)$ for $0 \le i \le d$. Hence by $\mathrm{Hom}_{eAe}(eA, eAe) \cong Ae$, we obtain an exact sequence of left A-modules $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \mathrm{add}(D(eA))$.

(2) $\Longrightarrow$ (1) by Lemma 4.1.

(1) $\Rightarrow$ (3) By the arguments in (1) $\Longrightarrow$ (2) there is an idempotent e such that $A \cong \mathrm{End}_{eAe}(Ae)$, where eAe is a d-Gorenstein algebra and Ae is a finitely generated right eAe-module. Moreover, $eA \cong \mathrm{Hom}_{eAe}(Ae, eAe)$ and $\mathrm{inj.dim}\, eA_A = \mathrm{injdim}_A Ae = d < \infty$. Hence, eA is a finitely generated left Gorenstein projective eAe-module and $A \cong \mathrm{End}_{eAe}(eA)^{\mathrm{op}}$. By Proposition 2.11, $D(eA)\text{-}\,\mathrm{domdim}\, A \ge 2$.

(3) $\Rightarrow$ (1) Since $D(eA)\text{-}\,\mathrm{domdim}\, A \ge 2$, it follows from Proposition 2.11 that $A \cong \mathrm{End}_{eAe}(eA)^{\mathrm{op}}$. Therefore by the dual of [8, Proposition 2.9], $\mathrm{Hom}_A(A/AeA, Ae) = 0$ and $\mathrm{Ext}^1_A(A/AeA, Ae) = 0$. From the proof of the dual of [8, Lemma 2.11] we also get that $\mathrm{Hom}_A(AeA, Ae) \cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, Ae)$ as (A, eAe)-bimodules. Hence we obtain $\mathrm{Hom}_A(AeA, Ae) \cong \mathrm{Hom}_A(A, Ae)$ as (A, eAe)-bimodules, from the short exact sequence $0 \to AeA \to A \to A/AeA \to 0$ of A-bimodules. So, there is an isomorphism $Ae \cong \mathrm{Hom}_{eAe}(eA, eAe)$ of (A, eAe)-bimodules. Since $eA \in eAe\text{-}\mathcal{G}\mathrm{proj}$, we get that $Ae \in \mathcal{G}\mathrm{proj}\text{-}eAe$ and $eA \cong \mathrm{Hom}_{eAe}(Ae, eAe)$, and also $\mathrm{Ext}^i_{eAe}(Ae, eAe) = \mathrm{Ext}^i_{eAe}(eA, eAe) = 0$ for any $i \in \mathbb{Z}$.

For any $M \in \mathrm{mod}\text{-}eAe$, we claim that $\mathrm{Ext}^{d+1}_{eAe}(M, eAe) = 0$. Let $N := M \otimes_{eAe} eA$ and $P^\bullet$ be a projective resolution of N as a right A-module. Then $P^\bullet e$ is a complex in $\mathrm{add}(Ae)_{eAe}$ and $P^\bullet e$ is quasi-isomorphic to $Ne \cong M$. Thus by $\mathrm{injdim}\, eA_A = d < \infty$, there are isomorphisms

$$\begin{aligned}\operatorname{Ext}^{d+1}_{eAe}(M, eAe) &\cong H^{d+1}\big(\operatorname{Hom}_{eAe}\big(P^\bullet e, eAe\big)\big)\\ &\cong H^{d+1}\big(\operatorname{Hom}_A\big(P^\bullet, \operatorname{Hom}_{eAe}(Ae, eAe)\big)\big)\\ &\cong H^{d+1}\operatorname{Hom}_A\big(P^\bullet, eA\big) \cong \operatorname{Ext}^{d+1}_A(N, eA)\\ &= 0\end{aligned}$$

This means that $\operatorname{injdim} eAe_{eAe} \leq d$. For any $Y \in eAe$-mod, we claim that $\operatorname{Ext}^{d+1}_{eAe}(Y, eAe) = 0$. Let $Z := Ae \otimes_{eAe} Y$ and $Q^\bullet$ be a projective resolution of Z as a left A-module. Then $eQ^\bullet$ is a complex in $\operatorname{add}_{eAe}(eA)$ and $eQ^\bullet$ is quasi-isomorphic to $eZ \cong Y$. Thus by $\operatorname{injdim}_A Ae = d < \infty$, there are isomorphisms

$$\begin{aligned}\operatorname{Ext}^{d+1}_{eAe}(Y, eAe) &\cong H^{d+1}\big(\operatorname{Hom}_{eAe}\big(eQ^\bullet, eAe\big)\big)\\ &\cong H^{d+1}\big(\operatorname{Hom}_A\big(Q^\bullet, \operatorname{Hom}_{eAe}(eA, eAe)\big)\big)\\ &\cong H^{d+1}\operatorname{Hom}_A\big(Q^\bullet, Ae\big) \cong \operatorname{Ext}^{d+1}_A(Z, Ae)\\ &= 0\end{aligned}$$

This means that $\operatorname{injdim}_{eAe} eAe \leq d$. Hence eAe is a d-Gorenstein algebra.

(2) $\Longleftrightarrow$ (2′) follows from the proof of Lemma 4.1. □

A special case of Theorem 4.2 can be given a shorter proof using Iyama's results in [16].

Special case: *Let A be a finite dimensional k-algebra over a field k. Then the following statements are equivalent:*

(1) A is isomorphic to the endomorphism algebra of a finite dimensional Gorenstein projective right generator over a finite dimensional d-Gorenstein k-algebra, for d a non-negative integer.

(2) There is an idempotent e of A and a non-negative integer d such that there is an exact sequence of finite dimensional right A-modules $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ with each $I^j \in \operatorname{add}(D(Ae))$, and an exact sequence of finite dimensional left A-modules $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \operatorname{add}(D(eA))$, and the (A, eAe)-bimodule Ae has the double centraliser property.

Proof. (1) $\Longrightarrow$ (2) The isomorphism $A \cong \operatorname{End}_{eAe}(Ae)$ is shown as in the beginning of the proof of Theorem 4.2.

Since eAe is a d-Gorenstein algebra, eAe is a d-cotilting right eAe-module. As Ae is a Gorenstein projective right eAe-module, $\operatorname{Ext}^i_{eAe}(Ae, eAe) = 0$ for any $i \geq 1$. Note that $eAe_{eAe} \in \operatorname{add}(Ae)_{eAe}$. By [16, Proposition 3.4.3(1)], $(\operatorname{Hom}_{eAe}(Ae, eAe), D(Ae))$ is a d-extension pair of right A-modules in the sense of [16]. By [16, Proposition 3.4.1], the two sequences $0 \to \operatorname{Hom}_{eAe}(Ae, eAe) \to I^0 \to \cdots \to I^d \to 0$ with each $I^i \in \operatorname{add} D(Ae)$, and $0 \to P^{-d} \to \cdots \to P^0 \to D(Ae) \to 0$ with each $P^j \in \operatorname{add} \operatorname{Hom}_{eAe}(Ae, eAe)$, are exact. The latter implies that there is an exact sequence $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$

with each $E^j \in \operatorname{add} D\operatorname{Hom}_{eAe}(Ae, eAe)$. Now the proof can be finished as above, using the results of [8] to show that $\operatorname{Hom}_{eAe}(Ae, eAe) \cong eA$ as right A-modules.

(2) $\Longrightarrow$ (1) By assumption and [16, Proposition 3.4.1], $(Ae, D(eA))$ is a d-extension pair of left A-modules. Then by [16, Proposition 3.4.3(2)] and its proof, eAe is a d-cotilting left eAe-module, $\operatorname{Ext}^i_{eAe}(eA, eAe) = 0$ for any $i \geq 1$ and $Ae \cong \operatorname{Hom}_{eAe}(eA, eAe)$ as left A-modules. This means that eAe is a d-Gorenstein algebra, and eA is a Gorenstein projective left eAe-module. Since the (A, eAe)-bimodule Ae has the double centraliser property, we see from the proof of (1) $\Longrightarrow$ (2) in Theorem 4.2 that $Ae \cong \operatorname{Hom}_{eAe}(eA, eAe)$ as right eAe-modules. So, Ae is a Gorenstein projective right eAe-module. □

Definition 4.3. Let A be an artin R-algebra. We say A is a *gendo-d-Gorenstein algebra* for some non-negative integer d, and e is an associated idempotent, if A satisfies one of the equivalent conditions of Theorem 4.2. When d is not specified, A is called a *gendo-Gorenstein algebra*.

Here, as in [12], ‘gendo’ refers to endomorphism ring of a generator.

Remark 4.4. Let A be a gendo-Gorenstein algebra and e an associated idempotent. Then $Ae \cong \operatorname{Hom}_{eAe}(eA, eAe)$ as (A, eAe)-bimodules and $eA \cong \operatorname{Hom}_{eAe}(Ae, eAe)$ as (eAe, A)-bimodules.

5. Properties of gendo-Gorenstein algebras

In this section, we will give further properties of gendo-Gorenstein algebras, including higher Auslander correspondence.

Let A be an artin algebra. Let $\mathcal{X}$ be a full subcategory of mod-A and $M \in$ mod-A. Recall from [3] that a complex $\cdots \to X_2 \to X_1 \to X_0 \to M$ is called a *right $\mathcal{X}$-resolution* of M if $X_i \in \mathcal{X}$ and $\cdots \to \operatorname{Hom}_A(-, X_1) \to \operatorname{Hom}_A(-, X_0) \to \operatorname{Hom}_A(-, M) \to 0$ is exact on $\mathcal{X}$. We write $\operatorname{reldim}_{\mathcal{X}} M \leq n$ if M has a right $\mathcal{X}$-resolution with $X_{n+1} = 0$.

An algebra is *of finite Cohen–Macaulay type*, or simply, CM-finite, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules, see [5]. Clearly, an algebra A is CM-finite if and only if there is an A-module E such that A-$\mathcal{G}$proj $= \operatorname{add} E$.

Now we will consider some special cases of Theorem 4.2.

Corollary 5.1. *Let A be a gendo-d-Gorenstein algebra and e an associated idempotent. If Ae is also a cogenerator of* mod-eAe*, then eAe is a self-injective algebra and A is a Morita algebra. In this case,* $\operatorname{add}(Ae)_{eAe} =$ mod-eAe *if and only if* $\operatorname{gldim} A = 2$.

Proof. Since Ae is a cogenerator of mod-eAe, $D(eAe)$ is a right Gorenstein projective eAe-module. By assumption, eAe is d-Gorenstein. This implies that

$\operatorname{projdim} D(eAe)_{eAe} \leq d$. So $D(eAe)$ is a projective right eAe-module, and eAe is self-injective and $eA \cong D(Ae)$ as right A-modules. Thus, A is a Morita algebra. In this case, $\operatorname{add}(Ae)_{eAe} = \operatorname{mod}\text{-}eAe$ if and only if $\operatorname{reldim}_{Ae} X = 0$ for any $X \in \operatorname{mod}\text{-}eAe$ if and only if $\operatorname{gldim} A = 2$ by [22, Theorem 2.6]. $\square$

Corollary 5.2. *Let A be a gendo-d-Gorenstein algebra for some $d \geq 2$. Let e be an associated idempotent such that $\mathcal{G}\text{proj-}eAe = \operatorname{add}(Ae)_{eAe}$. Then $\operatorname{gldim} A = d$.*

Proof. Since A is gendo-d-Gorenstein and e is an associated idempotent of A, eAe is d-Gorenstein and $A \cong \operatorname{End}_{eAe}(Ae)$. Since eAe is CM-finite by $\mathcal{G}\text{proj-}eAe = \operatorname{add}(Ae)_{eAe}$, [22, Theorem 2.6] implies $\operatorname{gldim} A \leq d$. By Theorem 4.2, $\operatorname{gldim} A \geq d$. Thus, $\operatorname{gldim} A = d$. $\square$

Proposition 5.3. *Let A be a gendo-d-Gorenstein algebra and e an associated idempotent. Then $\operatorname{projdim} \operatorname{Hom}_A(Ae, D(eA)) \leq d$ as both left and right eAe-modules.*

Proof. There is an exact sequence of left A-modules $0 \to Ae \to E^0 \to \cdots \to E^d \to 0$ with each $E^j \in \operatorname{add}(D(eA))$. Applying the functor $\operatorname{Hom}_A(-, D(eA))$ to this sequence, we get an induced exact sequence $0 \to \operatorname{Hom}_A(E^d, D(eA)) \to \cdots \operatorname{Hom}_A(E^0, D(eA)) \to \operatorname{Hom}_A(Ae, D(eA)) \to 0$ with each $\operatorname{Hom}_A(E^i, D(eA)) \in \operatorname{add}(\operatorname{Hom}_A(D(eA), D(eA))) \cong \operatorname{add} eAe_{eAe}$. This implies $\operatorname{projdim} \operatorname{Hom}_A(Ae, D(eA)) \leq d$ as a right eAe-module. By a similar argument, $\operatorname{projdim} \operatorname{Hom}_A(Ae, D(eA)) \leq d$ as a left eAe-module, using the exact sequence $0 \to eA \to I^0 \to \cdots \to I^d \to 0$ of right A-modules with each $I^j \in \operatorname{add}(D(Ae))$. $\square$

Proposition 5.4. *Let A be a gendo-d-Gorenstein algebra and e be its associated idempotent. Let $\operatorname{End}_{\underline{\mathcal{G}\text{proj-}eAe}}(Ae) := \operatorname{End}_{eAe}(Ae)/\langle eAe \rangle$ where $\langle eAe \rangle$ is the ideal of $\mathcal{G}\text{proj-}eAe$ given by all maps which factor through an object in $\operatorname{add}(eAe)_{eAe}$. Then $\operatorname{End}_{\underline{\mathcal{G}\text{proj-}eAe}}(Ae) \cong A/AeA$.*

Proof. Since $\operatorname{End}_{eAe}(Ae) \cong A$ and $\operatorname{Hom}_{eAe}(Ae, eAe) \cong eA$ as right A-modules, by Theorem 4.2, there is an equivalence of categories

$$\operatorname{Hom}_{eAe}(Ae, -) : \operatorname{add}(Ae)_{eAe} \to \text{proj-}A$$

which sends eAe to eA. Thus $\operatorname{End}_{\underline{\mathcal{G}\text{proj-}eAe}}(Ae) = \operatorname{End}_{eAe}(Ae)/\langle eAe \rangle \cong \operatorname{End}_A(A)/\langle eA \rangle \cong A/AeA$. $\square$

Let R be a commutative noetherian ring and A be an associative unital R-algebra over R that is projective as an R-module. We denote the nth Hochschild cohomology group of A with coefficients in A itself by $\operatorname{HH}^n(A)$. It is known that $\operatorname{HH}^n(A) \cong \operatorname{Ext}^n_{A \otimes_R A^{\mathrm{op}}}(A, A)$ as groups.

Lemma 5.5. *Let A be an artin R-algebra that is projective as an R-module, and e an idempotent of A such that (eAe, A)-bimodule eA has the double centraliser property. Let $n \geq 2$ be an integer. Then $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq n-2$ and $\mathrm{Ext}^{n-1}_{eAe}(eA, eA) \neq 0$ if and only if $\mathrm{grade}_A\, A/AeA = n$. In this case, $\mathrm{HH}^i(eAe) \cong \mathrm{HH}^i(A)$ for $0 \leq i \leq n-2$.*

Proof. By Proposition 2.11, $A \cong \mathrm{Hom}_{eAe}(eA, eA)$ and $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq n-2$ if and only if $D(eA)$-domdim $A \geq n$. We claim that $D(eA)$- domdim $A = n$. Otherwise, by Proposition 2.11 again, we get that $\mathrm{Ext}^{n-1}_{eAe}(eA, eA) = 0$, a contradiction. So by Theorem 2.14, $A \cong \mathrm{Hom}_{eAe}(eA, eA)$, $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq n-2$ and $\mathrm{Ext}^{n-1}_{eAe}(eA, eA) \neq 0$ if and only if $\mathrm{grade}_A\, A/AeA = n$. If $\mathrm{grade}_A\, A/AeA = n$, then the dual of [8, Theorem 5.5] implies $\mathrm{HH}^i(eAe) \cong \mathrm{HH}^i(A)$ for $0 \leq i \leq n-2$. $\square$

Recall from [13, 3.2.5] that a k-algebra A over a field k is called *bimodule d-Calabi–Yau* for some integer $d \geq 2$ if $\mathrm{projdim}_A\, A_A < \infty$ and $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}} \otimes_k A}(A, A^{\mathrm{op}} \otimes_k A)[d] \cong A$ in $D(A^{\mathrm{op}} \otimes_k A)$.

Example 5.6. Let A be a finite dimensional k-algebra over a field k, and $e(\neq 1)$ an idempotent of A. Assume that A is a bimodule d-Calabi–Yau algebra for $d \geq 2$. Then:

(1) $\mathrm{grade}_A\, A/AeA = d$;
(2) $\mathrm{HH}^i(eAe) \cong \mathrm{HH}^i(A)$ for $0 \leq i \leq d-2$.

Proof. By [1, Theorem 2.2], A is gendo-d-Gorenstein such that $A \cong \mathrm{Hom}_{eAe}\ (eA, eA)$, and $\mathrm{Ext}^i_{eAe}(eA, eA) = 0$ for $1 \leq i \leq d-2$. Also by the proof of [1, Proposition 2.6] and [1, Lemma 2.5] we see that $\mathrm{Ext}^{d-1}_{eAe}(eA, eA) \neq 0$. Hence by Lemma 5.5 we obtain that $\mathrm{grade}_A\, A/AeA = d$ and $\mathrm{HH}^i(eAe) \cong \mathrm{HH}^i(A)$ for $0 \leq i \leq d-2$. $\square$

In the following we will show higher Auslander correspondence for gendo-d-Gorenstein algebras. We first recall the notion of maximal orthogonal subcategory.

Let $\mathcal{B}$ be a resolving subcategory of an abelian R-category $\mathcal{A}$ with enough projectives. Let $\mathcal{C}$ be a functorially finite subcategory of $\mathcal{B}$ and $l \geq 0$. Recall from [16, Section 2.4] that $\mathcal{C}$ is a maximal l-orthogonal subcategory of $\mathcal{B}$, if $\mathcal{C} \perp_l \mathcal{C}$ and $\mathcal{C} = \mathcal{C}^{\perp l} \cap \mathcal{B} = {}^{\perp l}\mathcal{C} \cap \mathcal{B}$, where $\mathcal{C} \perp_l \mathcal{C}$ means $\mathrm{Ext}^i_{\mathcal{B}}(X, Y) = 0$ for any $X, Y \in \mathcal{C}$ and $0 < i \leq l$, $\mathcal{C}^{\perp l} := \{X \in \mathcal{B} \mid \mathcal{C} \perp_l X\}$ and ${}^{\perp l}\mathcal{C} := \{X \in \mathcal{B} \mid X \perp_l \mathcal{C}\}$.

Lemma 5.7. *Let A be an artin algebra and M be in A-$\mathcal{G}$proj. Put $M^* = \mathrm{Hom}_A(M, A)$. Then $\mathrm{add}_A\, M$ is a maximal l-orthogonal subcategory of A-$\mathcal{G}$proj if and only if $\mathrm{add}\, M^*_A$ is a maximal l-orthogonal subcategory of $\mathcal{G}$proj-A.*

Proof. Since $\mathrm{Hom}_A(-, A)$ is a duality between A-$\mathcal{G}$proj and $\mathcal{G}$proj-A, it is enough to show that $\mathrm{Ext}^i_A(M, E) = 0$ implies $\mathrm{Ext}^i_A(E^*, M^*) = 0$ for any $E \in A$-$\mathcal{G}$proj and $0 < i \leq l$.

Take a minimal projective resolution of M,

$$\cdots \to P_1 \to P_0 \to M \to 0. \tag{$*$}$$

Applying the functor $\mathrm{Hom}_A(-, E)$ to the sequence $(*)$, and using $\mathrm{Ext}_A^i(M, E) = 0$ for $0 < i \leq l$, we get the exact sequence

$$0 \to \mathrm{Hom}_A(M, E) \to \mathrm{Hom}_A(P_0, E) \to \cdots \to \mathrm{Hom}_A(P_{l+1}, E). \tag{$**$}$$

On the other hand, applying the functor $\mathrm{Hom}_A(-, A)$ to the sequence $(*)$, gives an induced exact sequence

$$0 \to M^* \to P_0^* \to P_1^* \to \cdots. \tag{$***$}$$

Again using that $\mathrm{Hom}_A(-, A)$ is a duality between A-$\mathcal{G}$proj and $\mathcal{G}$proj-A, the functor $\mathrm{Hom}_A(E^*, -)$ on the sequence $(***)$ yields the exact sequence

$$0 \to \mathrm{Hom}_A\big(E^*, M^*\big) \to \mathrm{Hom}_A\big(E^*, P_0^*\big) \cdots \to \mathrm{Hom}_A\big(E^*, P_{l+1}^*\big). \tag{$****$}$$

Comparing $(**)$ and $(****)$ implies $\mathrm{Ext}_A^i(E^*, M^*) = 0$ for $0 < i \leq l$. $\square$

Theorem 5.8. *Let A be a gendo-d-Gorenstein algebra for some $d \geq 0$ and e be an associated idempotent. If $\mathrm{add}\,(Ae)_{eAe}$ is a maximal $(n-2)$-orthogonal subcategory of $\mathcal{G}$proj-eAe for some integer $n \geq 2$, then $\mathrm{gldim}\,A \leq n + \max\{0, d-2\}$ and $D(eA)$-$\mathrm{domdim}\,A \geq n$. In particular, if $d \leq 2$, then $\mathrm{gldim}\,A \leq n$.*

Proof. Note that $A \cong \mathrm{End}_{eAe}(eA)$ and $eA \cong \mathrm{Hom}_{eAe}(Ae, eAe)$ as left eAe-modules by Theorem 4.2. Since $\mathrm{add}\,(Ae)_{eAe}$ is a maximal $(n-2)$-orthogonal subcategory of $\mathcal{G}$proj-eAe, it follows from Lemma 5.7 that $\mathrm{add}_{eAe}(eA)$ is a maximal $(n-2)$-orthogonal subcategory of eAe-$\mathcal{G}$proj. Thus $\mathrm{Ext}_{eAe}^i(eA, eA) = 0$ for $1 \leq i \leq n-2$. Hence by Proposition 2.11 we get $D(eA)$-$\mathrm{domdim}\,A \geq n$. By [15, Proposition 2.2.2], $\mathrm{reldim}_{eA}\,X \leq n-2$ for any $X \in eAe$-$\mathcal{G}$proj. Hence by [22, Theorem 2.6], $\mathrm{gldim}\,A \leq n + \max\{0, d-2\}$. Therefore, $d \leq 2$, implies $\mathrm{gldim}\,A \leq n$. $\square$

Theorem 5.9. *Let A be an artin R-algebra. Let e be an idempotent of A such that $Ae \in \mathcal{G}$proj-eAe and $eA \cong \mathrm{Hom}_{eAe}(Ae, eAe)$ as left eAe-modules. If $\mathrm{gldim}\,A \leq n$ and $D(eA)$-$\mathrm{domdim}\,A \geq n$ for some integer $n \geq 2$, then $\mathrm{add}\,(Ae)_{eAe}$ is a maximal $(n-2)$-orthogonal subcategory of $\mathcal{G}$proj-eAe.*

Proof. Since $D(eA)$-$\mathrm{domdim}\,A \geq n$, Proposition 2.11 implies $A \cong \mathrm{End}_{eAe}(eA)$ and $\mathrm{Ext}_{eAe}^i(eA, eA) = 0$ for $1 \leq i \leq n-2$. Since $\mathrm{gldim}\,A \leq n$ and eA is a generator in eAe-mod, it follows from [22, Theorem 2.6] that $\mathrm{reldim}_{eA}\,X \leq n-2$ for any $X \in eAe$-$\mathcal{G}$proj. By assumption, $Ae \in \mathcal{G}$proj-eAe and $eA \cong \mathrm{Hom}_{eAe}(Ae, eAe)$, and thus $eA \in$

eAe-$\mathcal{G}$proj. Now by [16, Proposition 2.4.1], $\mathrm{add}_{eAe}(eA)$ is a maximal $(n-2)$-orthogonal subcategory of eAe-$\mathcal{G}$proj. By Lemma 5.7, the proof is complete. □

Corollary 5.10. *Let A be a gendo-d-Gorenstein algebra for $d \leq 2$ and e an associated idempotent. Then $\mathrm{add}\,(Ae)_{eAe}$ is a maximal $(n-1)$-orthogonal subcategory of $\mathcal{G}$proj-eAe for some integer $n \geq 1$ if and only if $\mathrm{gldim}\, A \leq n+1$ and $D(eA)$-$\mathrm{domdim}\, A \geq n+1$.*

Proof. This is an immediate consequence of Theorems 5.8 and 5.9. □

Before we state the next proposition, we fix some notation. Let A be an artin algebra and e be an idempotent. We denote by $\underline{eAe\text{-}\mathcal{G}\mathrm{proj}}$ the stable category of eAe-$\mathcal{G}$proj modulo projectives. The algebra $\underline{\mathrm{End}}_{eAe}(eA) := \mathrm{End}_{eAe}(eA)/\langle eAe\rangle$ is called the stable endomorphism algebra of eA. Note that if eA is a Gorenstein projective eAe-module, then $\underline{\mathrm{End}}_{eAe}(eA)$ is the same as $\mathrm{End}_{\underline{eAe\text{-}\mathcal{G}\mathrm{proj}}}(eA)$.

Next we show that the stable endomorphism algebra $\underline{\mathrm{End}}_{eAe}(eA)$ has nice properties for $n = 2$.

Proposition 5.11. *Let A be an artin R-algebra. Let e be an idempotent of A such that $eA \in \mathcal{G}$proj-eAe. If $\mathrm{gldim}\, A \leq 3$ and $D(eA)$-$\mathrm{domdim}\, A \geq 3$, then there is an equivalence of categories*

$$\underline{eAe\text{-}\mathcal{G}\mathrm{proj}}/\mathrm{add}(\underline{eA}) \cong \mathrm{mod}\text{-}\underline{\mathrm{End}}_{eAe}(eA)$$

Moreover, eAe is CM-finite if and only if the stable endomorphism algebra $\underline{\mathrm{End}}_{eAe}(eA)$ is of finite representation type. In particular, $\underline{\mathrm{End}}_{eAe}(eA)$ is a 1-Gorenstein algebra.

Proof. The proof of Theorem 5.9 implies that $\mathrm{add}_{eAe}(eA)$ is a maximal 1-orthogonal subcategory of eAe-$\mathcal{G}$proj. It follows from [16, Proposition 2.4.1] that $\mathrm{reldim}_{eA}\, X \leq 1$ for any $X \in eAe$-$\mathcal{G}$proj. Hence by [6, Theorem 7.1] there is an equivalence of categories

$$\underline{eAe\text{-}\mathcal{G}\mathrm{proj}}/\mathrm{add}(\underline{eA}) \cong \mathrm{mod}\text{-}\underline{\mathrm{End}}_{eAe}(eA)$$

and also eAe is CM-finite if and only if the stable endomorphism algebra $\underline{\mathrm{End}}_{eAe}(eA)$ is of finite representation type. By [21, Theorem 4.3], $\underline{\mathrm{End}}_{eAe}(eA)$ is a 1-Gorenstein algebra. □

Now we will establish $n+1$-dimensional Auslander correspondence for gendo-d-Gorenstein algebras.

Proposition 5.12. *Let R be a commutative artin ring and $d \leq 2$ be a non-negative integer. For any $n \geq 1$, there exists a bijection between the set of equivalence classes of finite maximal $(n-1)$-orthogonal subcategories $\mathcal{C}$ of $\mathcal{G}$proj-Λ containing Λ for d-Gorenstein artin R-algebras Λ, and the set of Morita-equivalence classes of gendo-d-Gorenstein artin*

R-algebras Γ with $\operatorname{gldim}\Gamma \leq n+1$ *and* $D(e\Gamma)\text{-}\operatorname{domdim}\Gamma \geq n+1$*, where e is an associated idempotent of Γ. The bijection is given by $\mathcal{C} \mapsto \Gamma := \operatorname{End}_\Lambda(M)$ for an additive generator M of $\mathcal{C}$.*

Proof. Let Λ be a d-Gorenstein artin R-algebra and $\mathcal{C}$ be a finite maximal $(n-1)$-orthogonal subcategory of $\mathcal{G}$proj-Λ containing Λ. Since $\mathcal{C}$ is finite, we may assume that $\mathcal{C} = \operatorname{add} M$ with M an additive generator. Set $\Gamma := \operatorname{End}_\Lambda(M)$. Since Λ is a d-Gorenstein artin R-algebra and M is a finitely generated Gorenstein projective generator in mod-Λ, we may write $M = N \oplus \Lambda$ for some $N \in$ mod-Λ. Let $e : M \twoheadrightarrow \Lambda$ be the canonical projection. Then $\Lambda \cong e\Gamma e$ and $M \cong \Gamma e$. This implies $\Gamma \cong \operatorname{End}_{e\Gamma e}(\Gamma e)$. Thus, by Theorem 4.2, Γ is a gendo-d-Gorenstein artin R-algebra, and e is an associated idempotent. Since $\operatorname{add}_{e\Gamma e}(\Gamma e)$ is maximal $(n-1)$-orthogonal in $\mathcal{G}$proj-$e\Gamma e$, it follows from Corollary 5.10 that $\operatorname{gldim}\Gamma \leq n+1$ and $D(e\Gamma)\text{-}\operatorname{domdim}\Gamma \geq n+1$.

Let Γ be a gendo-d-Gorenstein artin R-algebra such that $\operatorname{gldim}\Gamma \leq n+1$ and $D(e\Gamma)\text{-}\operatorname{domdim}\Gamma \geq n+1$. Then by Theorem 4.2 and Corollary 5.10, $e\Gamma e$ ia a d-Gorenstein artin R-algebra and $\operatorname{add}_{e\Gamma e}(\Gamma e)$ is maximal $(n-1)$-orthogonal in $\mathcal{G}$proj-$e\Gamma e$ such that $e\Gamma e \in \operatorname{add}(\Gamma e)_{e\Gamma e}$. We take $\mathcal{C}$ to be $\operatorname{add}(\Gamma e)_{e\Gamma e}$. $\square$

One implication of a special case of Proposition 5.12 can be derived directly from Iyama's Theorem 4.4.1 in [16]. The other implication is different, since unlike [16] we don't assume the condition: $\operatorname{projdim}_{\Gamma/\Gamma e\Gamma} X \leq n+1$ for any $\Gamma/\Gamma e\Gamma$-module X.

Special case: *Let k be a field and $d \leq 2$ be a non-negative integer. For any $n \geq 1$, there exists a bijection between the set of equivalence classes of finite maximal $(n-1)$-orthogonal subcategories $\mathcal{C}$ of $\mathcal{G}$proj-Λ containing Λ for finite dimensional d-Gorenstein k-algebras Λ, and the set of Morita-equivalence class of finite dimensional gendo-d-Gorenstein k-algebras Γ with* $\operatorname{gldim}\Gamma \leq n+1$ *and* $D(e\Gamma)\text{-}\operatorname{domdim}\Gamma \geq n+1$*, where e is an associated idempotent. It is given by $\mathcal{C} \mapsto \Gamma := \operatorname{End}_\Lambda(M)$ for an additive generator M of $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a finite maximal $(n-1)$-orthogonal subcategory of $\mathcal{G}$proj-Λ containing Λ for a finite dimensional d-Gorenstein k-algebra Λ. Since $\mathcal{C}$ is finite, we may assume that $\mathcal{C} = \operatorname{add} M$ with M an additive generator. Set $\Gamma := \operatorname{End}_\Lambda(M)$. Since Λ is a d-Gorenstein algebra and M is a finite dimensional Gorenstein projective generator in mod-Λ, it follows that Λ is a d-cotilting right Λ-module and $\operatorname{add} M$ is a maximal $(n-1)$-orthogonal subcategory of ${}^\perp\Lambda_\Lambda$ containing Λ. It follows from [16, Definition 4.1] that (Γ, M, Λ) is an Auslander triple of type $(0, d, n)$. By [16, Theorem 4.4.1(2) and Definition 4.4], $\operatorname{gldim}\Gamma \leq \max\{n+1, d\} = n+1$, and $(\operatorname{Hom}_\Lambda(M, \Lambda), DM)$ is a d-extension pair of right Γ-modules and right Γ-module $\operatorname{Hom}_\Gamma(M, \Gamma)$ is n-superprojective. Then [16, Proposition 3.4.1 and 3.5.1] shows that there is an idempotent e of Γ such that $M = \Gamma e$ and $D(\Gamma e)\text{-}\operatorname{domdim}\Gamma \geq n+1$.

Note that $D(\Gamma e)$-domdim $\Gamma \geq n+1$ implies, by Proposition 2.11, that $\mathrm{Ext}^i_{e\Gamma e}(\Gamma e, \Gamma e) = 0$ for $0 < i \leq n-2$. Hence by Lemma 5.7, $\mathrm{Ext}^i_{e\Gamma e}(e\Gamma, e\Gamma) = 0$ for $0 < i \leq n-2$. Proposition 2.11 now implies that $D(e\Gamma)$-domdim $\Gamma \geq n+1$.

Let Γ be a gendo-d-Gorenstein artin R-algebra such that gldim $\Gamma \leq n+1$ and $D(e\Gamma)$-domdim $\Gamma \geq n+1$. Then by Theorem 4.2 and Corollary 5.10, $e\Gamma e$ ia a d-Gorenstein artin R-algebra and $\mathrm{add}_{e\Gamma e}(\Gamma e)$ is maximal $(n-1)$-orthogonal in $\mathcal{G}$proj-$e\Gamma e$ such that $e\Gamma e \in \mathrm{add}\,(\Gamma e)_{e\Gamma e}$. We choose $\mathcal{C}$ to be $\mathrm{add}\,(\Gamma e)_{e\Gamma e}$. $\square$

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