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Canonical complexes associated to a matrix

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ABSTRACT

Let Φ be an $\mathfrak{f} \times \mathfrak{g}$ matrix with entries from a commutative Noetherian ring R, with $\mathfrak{g} \le \mathfrak{f}$. Recall the family of generalized Eagon–Northcott complexes $\{\mathcal{C}^i_\Phi\}$ associated to Φ. (See, for example, Appendix A2 in "Commutative Algebra with a View Toward Algebraic Geometry" by D. Eisenbud.) For each integer i, $\mathcal{C}^i_\Phi$ is a complex of free R-modules. For example, $\mathcal{C}^0_\Phi$ is the original "Eagon–Northcott" complex with zero-th homology equal to the ring $R/I_\mathfrak{g}(\Phi)$ defined by ideal generated by the maximal order minors of Φ; and $\mathcal{C}^1_\Phi$ is the "Buchsbaum–Rim" complex with zero-th homology equal to the cokernel of the transpose of Φ. If Φ is sufficiently general, then each $\mathcal{C}^i_\Phi$, with $-1 \le i$, is acyclic; and, if Φ is generic, then these complexes resolve half of the divisor class group of $R/I_\mathfrak{g}(\Phi)$. The family $\{\mathcal{C}^i_\Phi\}$ exhibits duality; and, if $-1 \le i \le \mathfrak{f} - \mathfrak{g} + 1$, then the complex $\mathcal{C}^i_\Phi$ exhibits depth-sensitivity with respect to the ideal $I_\mathfrak{g}(\Phi)$ in the sense that the tail of $\mathcal{C}^i_\Phi$ of length equal to $\operatorname{grade}(I_\mathfrak{g}(\Phi))$ is acyclic. The entries in the differentials of $\mathcal{C}^i_\Phi$ are linear in the entries of Φ at every position except at one, where the entries of the differential are $\mathfrak{g} \times \mathfrak{g}$ minors of Φ.

This paper expands the family $\{\mathcal{C}^i_\Phi\}$ to a family of complexes $\{\mathcal{C}^{i,a}_\Phi\}$ for integers i and a with $1 \le a \le \mathfrak{g}$. The entries in the differentials of $\{\mathcal{C}^{i,a}_\Phi\}$ are linear in the entries of Φ at every position except at two consecutive positions. At one of the exceptional positions the entries are $a \times a$ minors of Φ, at the other exceptional position the entries are $(\mathfrak{g}-a+1)\times(\mathfrak{g}-a+1)$ minors of Φ.

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The complexes $\{\mathcal{C}_\Phi^i\}$ are equal to $\{\mathcal{C}_\Phi^{i,1}\}$ and $\{\mathcal{C}_\Phi^{i,\mathfrak{g}}\}$. The complexes $\{\mathcal{C}_\Phi^{i,a}\}$ exhibit all of the properties of $\{\mathcal{C}_\Phi^i\}$. In particular, if $-1 \le i \le \mathfrak{f} - \mathfrak{g}$ and $1 \le a \le \mathfrak{g}$, then $\mathcal{C}_\Phi^{i,a}$ exhibits depth-sensitivity with respect to the ideal $I_\mathfrak{g}(\Phi)$.

Contents

1. Introduction

Let R be a commutative Noetherian ring and F and G be free R-modules of rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, with $\mathfrak{g} \le \mathfrak{f}$. Recall that, for each R-module homomorphism

$$\Phi : G^* \to F \tag{1.0.1}$$

there is a family of generalized Eagon–Northcott complexes $\{\mathcal{C}_\Phi^i\}$. (See, for example, Definition 3.1, [13, Appendix A.2], [3, 2.16], or [20]. A more complete history of these complexes may be found in the comments on page 26 in [3].) If

$$-1 \le i \le \mathfrak{f} - \mathfrak{g} + 1, \tag{1.0.2}$$

then $\mathcal{C}_\Phi^i$ has length $\mathfrak{f}-\mathfrak{g}+1$; and, if $\mathfrak{f}-\mathfrak{g}+1 \le \operatorname{grade} I_\mathfrak{g}(\Phi)$, then $\mathcal{C}_\Phi^i$ is acyclic for i satisfying (1.0.2). Furthermore, the complexes $\mathcal{C}_\Phi^i$, for i satisfying (1.0.2), exhibit depth-sensitivity. In particular, if $s \le \operatorname{grade} I_\mathfrak{g}(\Phi)$ for some integer s with $0 \le s \le f-g+1$, then $\mathrm{H}_j(\mathcal{C}_\Phi^i) = 0$ for $\mathfrak{f} - \mathfrak{g} + 2 - s \le j$ and i satisfying (1.0.2). In the generic situation, the complexes $\mathcal{C}_\Phi^i$, with i satisfying (1.0.2), resolve the Cohen–Macaulay elements of the divisor class group of the determinantal ring $R/I_\mathfrak{g}(\Phi)$. The complexes $\mathcal{C}_\Phi^i$, with i in the range (1.0.2), exhibit duality:

$$\mathcal{C}_\Phi^i \cong \text{a shift of } \operatorname{Hom}_R(\mathcal{C}_\Phi^j, R) \text{ in homological degree,}$$

for $i+j=\mathfrak{f}-\mathfrak{g}$. Also, if R is a graded ring, and a matrix representation of Φ is a matrix of linear forms, then the Betti tables for the complexes $\{\mathcal{C}^i_\Phi\}$ are pleasing to the eye. The maps are linear, except at, at most one position where the maps have degree $\mathfrak{g}$. Moreover, the position of non-linearity slides, along a line of slope 1, from the beginning of the complex to the end as i varies from 0 to $\mathfrak{f}-\mathfrak{g}$.

We expand the list of canonical complexes which are associated to the R-module homomorphism (1.0.1). For each pair (i,a) with

$$-1 \le i \le \mathfrak{f}-\mathfrak{g} \quad \text{and} \quad 1 \le a \le \mathfrak{g}, \tag{1.0.3}$$

we consider a complex $\mathcal{C}^{i,a}_\Phi$. The classical generalized Eagon–Northcott complexes

$$\{\mathcal{C}^i_\Phi | (1.0.2) \text{ holds}\} \tag{1.0.4}$$

are included in the set

$$\{\mathcal{C}^{i,a}_\Phi | (1.0.3) \text{ holds}\} \tag{1.0.5}$$

with $\mathcal{C}^i_\Phi = \mathcal{C}^{i,1}_\Phi$ for $-1 \le i \le \mathfrak{f}-\mathfrak{g}$ and $\mathcal{C}^i_\Phi = \mathcal{C}^{i-1,\mathfrak{g}}_\Phi$ for $0 \le i \le \mathfrak{f}-\mathfrak{g}+1$. The complexes of (1.0.5) exhibit many of the properties as the listed properties for the generalized Eagon–Northcott complexes (1.0.4). Each complex of (1.0.5) has length $\mathfrak{f}-\mathfrak{g}+1$; and

$$\text{if } \mathfrak{f}-\mathfrak{g}+1 \le \operatorname{grade} I_\mathfrak{g}(\Phi), \text{ then each } \mathcal{C}^{i,a}_\Phi \text{ is acyclic.}$$

The complexes of (1.0.5) exhibit depth-sensitivity. In the generic situation, the complex $\mathcal{C}^{i,a}_\Phi$ of (1.0.5)

$$\text{resolves a maximal Cohen–Macaulay module over the ring } R/I_\mathfrak{g}(\Phi) \text{ of rank } \tbinom{\mathfrak{g}-1}{a-1}. \tag{1.0.6}$$

The complexes of (1.0.5) exhibit duality:

$$\mathcal{C}^{i,a}_\Phi \cong \text{a shift of } \operatorname{Hom}_R(\mathcal{C}^{j,b}_\Phi, R) \text{ in homological degree,}$$

for $i+j=\mathfrak{f}-\mathfrak{g}-1$ and $a+b=\mathfrak{g}+1$. Also, if R is a graded ring, and Φ is a map of degree 1, then the maps of $C^{i,a}_\Phi$ are linear, except at, at most two adjacent positions where the maps have degree a and $\mathfrak{g}+1-a$. Moreover, the position of non-linearity slides, along a line of slope 1, from the beginning of the complex to the end as i varies from -1 to $\mathfrak{f}-\mathfrak{g}$; see Example 7.6.

The Eagon–Northcott [11] complex $\mathcal{C}^0_\Phi$ and the Buchsbaum–Rim complex [6–8] $\mathcal{C}^1_\Phi$ are very important objects in Commutative Algebra and Algebraic Geometry. (For example, [1,15,16,21,29] are a small sampling of the recent papers about Buchsbaum–Rim multiplicity and its application to equisingularity.) We expect that the rest of the family (1.0.5) will prove to be valuable tools in these fields.

The complexes $\mathcal{C}_\Phi^{i,a}$ arise in the study of the homological properties of the primary components of the content ideal $c(fgh)$ of the product of three generic polynomials f, g, and h. These components have been identified [9, Thm. 4.2] and all but one of the components is known to be Gorenstein [9, Thm. 4.1 and Rem. 4.3]. The complexes $\mathcal{C}_\Phi^{i,a}$ also arise in the study of the resolutions of the symmetric algebra $\operatorname{Sym}(I)$ and the Rees algebra $\mathcal{R}(I)$ of a grade three Gorenstein ideal $I = (g_1, \dots, g_n)$ in a polynomial ring over a field k; see, for example, [23] and [24, Cor. 6.3], where the special fiber ring $\mathcal{F}(I) = k[g_1, \dots, g_n]$ of I is resolved.

The complexes $\mathcal{C}_\Phi^{i,a}$ are straightforward and they are built in a canonical manner. That is, there are no choices; everything is coordinate-free. The modules in $\mathcal{C}_\Phi^{i,a}$ are Schur modules and Weyl modules corresponding to hooks. In other words, the modules in $\mathcal{C}_\Phi^{i,a}$ all are kernels of Koszul complex maps or Eagon–Northcott complex maps; see 4.1 and 4.3. The complex $\mathcal{C}_\Phi^{i,a}$ is obtained by concatenating three finite complexes:

$$\mathbb{K} \to \bigwedge \to \mathbb{L},$$

where $\mathbb{K}$ and $\mathbb{L}$ are standard complexes of Weyl and Schur modules, respectively, and $\bigwedge$ consists of a single exterior power concentrated in one position; see Definition 7.2 for the details. The complexes $\mathbb{K}$ and $\mathbb{L}$ are introduced in Section 5.

The main result of this paper is Theorem 8.4 which states that if Φ is sufficiently general, $-1 \le i$, and $1 \le a \le \mathfrak{g}$, then $\mathcal{C}_\Phi^{i,a}$ is an acyclic complex of free R-modules and $\mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ is a torsion-free $R/I_\mathfrak{g}(\Phi)$-module of rank $\binom{\mathfrak{g}-1}{a-1}$. The most important applications occur when i also satisfies $i \le \mathfrak{f}-\mathfrak{g}$. Indeed, in this situation, $\mathcal{C}_\Phi^{i,a}$ has length $\mathfrak{f}-\mathfrak{g}+1$ and, if $\mathfrak{f}-\mathfrak{g}+1 \le \operatorname{grade} I_\mathfrak{g}(\Phi)$, then $\mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ is a perfect R-module of projective dimension $\mathfrak{f}-\mathfrak{g}+1$ resolved by $\mathcal{C}_\Phi^{i,a}$ and $\operatorname{Ext}_R^{\mathfrak{f}-\mathfrak{g}+1}(\mathrm{H}_0(\mathcal{C}_\Phi^{i,a}), R)$ is a perfect R-module resolved by $\mathcal{C}_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a}$; furthermore, even if $\operatorname{grade} I_\mathfrak{g}(\Phi) < \mathfrak{f}-\mathfrak{g}+1$, the complex $\mathcal{C}_\Phi^{i,a}$ exhibits depth-sensitivity with respect to the ideal $I_\mathfrak{g}(\Phi)$ in the sense that

$$\mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) = 0 \text{ for } \mathfrak{f}-\mathfrak{g}+2-\operatorname{grade} I_\mathfrak{g}(\Phi) \le j, \text{ when } -1 \le i \le \mathfrak{f}-\mathfrak{g} \text{ and } 1 \le a \le \mathfrak{g}. \tag{1.0.7}$$

The depth-sensitivity (1.0.7) allows one to use truncations of various $\mathcal{C}_\Phi^{i,a}$ as acyclic strands in resolutions even when $I_\mathfrak{g}(\Phi)$ is known to be less than $\mathfrak{f}-\mathfrak{g}+1$.

There is a basic similarity between the present paper and the paper [25], which produces a family of complexes $\{\mathcal{D}_\rho^q\}$ for each almost alternating homomorphism ρ. The family $\{\mathcal{D}_\rho^q\}$ shares many properties with the family of generalized Eagon–Northcott complexes $\{\mathcal{C}_\Phi^i\}$. The main difference between [25] and the present paper is that the homomorphism ρ of [25] is special, in the sense that it is an almost alternating homomorphism; whereas, the complexes $\{\mathcal{C}_\Phi^{i,a}\}$ of the present paper and the generalized Eagon–Northcott complexes $\{\mathcal{C}_\Phi^i\}$ are both constructed from an arbitrary homomorphism Φ. Nonetheless, the statement of Theorem 8.4 and some steps in its proof are modeled on [25, Thm. 8.3], although [25] does not contain any analogue to Lemma 8.1, which is the key calculation in the present paper. In place of a result like Lemma 8.1, [25] first treats the generic case and then specializes to the non-generic case. In the present paper, Lemma 8.1 shows that if the image of Φ contains a basis element of F, then

$$\mathcal{C}_{\Phi}^{i,a} \quad \text{and} \quad \mathcal{C}_{\Phi'}^{i,a} \oplus \mathcal{C}_{\Phi'}^{i,a-1}$$

have isomorphic homology for some (“smaller”) R-module homomorphism Φ'. To prove Theorem 8.4, we iterate Lemma 8.1 and apply the acyclicity lemma. The representation theory that is used in the proof of Lemma 8.1 is begun in (6.3.4) and (6.3.5) and carried out in Proposition 6.4.

The complexes $\mathcal{C}_{\Phi}^{i,a}$ are defined in 7.2; examples are given in 7.6 and 7.7; the duality is treated in 7.9; and the zero-th homology

$$\mathrm{H}_0(\mathcal{C}_{\Phi}^{i,a}) = \begin{cases} \frac{\bigwedge^{\mathfrak{f}-\mathfrak{g}+a} F}{\operatorname{im}(\bigwedge^a \Phi)\wedge\bigwedge^{\mathfrak{f}-\mathfrak{g}} F}, & \text{if } i=-1, \\ \operatorname{coker}(\bigwedge^{\mathfrak{g}-a+1} \Phi^*), & \text{if } i=0, \text{ and} \\ \frac{L_{i+1}^{\mathfrak{g}-a} G}{\Phi^*(F)\cdot L_i^{\mathfrak{g}-a} G}, & \text{if } 1 \le i \end{cases}$$

is calculated in 7.11. The Schur module $L_q^p G$ is described in 4.1 and 4.3.

2. Notation, conventions, and preliminary results

There are three subsections: Ground rules, Grade and perfection, and Multilinear algebra.

2.1. Ground rules

2.1. Unless otherwise noted, R is a commutative Noetherian ring and all functors are functors of R-modules; that is, $\otimes$, Hom, $(\_\_)^*$, Sym_i, D_i, and $\bigwedge^i$ mean $\otimes_R$, Hom_R, $\mathrm{Hom}_R(\_\_, R)$, Sym_i^R, D_i^R, and $\bigwedge_R^i$, respectively.

2.2. A complex $\mathcal{C} : \cdots \to C_2 \to C_1 \to C_0 \to 0$ of R-modules is called *acyclic* if $\mathrm{H}_j(\mathcal{C}) = 0$ for $1 \le j$.

2.3. If a complex $\mathcal{C}$ has the form

$$0 \to \mathcal{C}_\ell \to \mathcal{C}_{\ell-1} \to \cdots \to \mathcal{C}_1 \to \mathcal{C}_0 \to 0,$$

with $\mathcal{C}_0 \neq 0$ and $\mathcal{C}_\ell \neq 0$, then we say that the *length of* $\mathcal{C}$ is ℓ and we write $\mathrm{length}(\mathcal{C}) = \ell$.

2.4. If $\mathbb{A}$ is a complex and n is an integer, then $\mathbb{A}[n]$ is a new complex with $\big[\mathbb{A}[n]\big]_j = \mathbb{A}_{n+j}$.

2.5. If $(\mathbb{A}, a) \xrightarrow{\theta} (\mathbb{B}, b)$ is a map of complexes, then the *total complex* (or mapping cone) of θ, denoted $\operatorname{Tot}(\theta)$, is the complex $(\mathbb{T}, t)$ with $\mathbb{T} = \mathbb{A}[-1] \oplus \mathbb{B}$ as a graded module. The differential in $\mathbb{T}$ is given by

$$\mathbb{T}_j = \begin{matrix} \mathbb{A}_{j-1} \\ \oplus \\ \mathbb{B}_j \end{matrix} \xrightarrow{\quad t_j \quad} \begin{matrix} \mathbb{A}_{j-2} \\ \oplus \\ \mathbb{B}_{j-1} \end{matrix} = \mathbb{T}_{j-1},$$

with

$$t_j = \begin{bmatrix} a_{j-1} & 0 \\ \theta_{j-1} & -b_j \end{bmatrix}.$$

2.6. If Φ is a matrix (or a homomorphism of free R-modules), then $I_r(\Phi)$ is the ideal generated by the $r \times r$ minors of Φ (or any matrix representation of Φ).

2.7. If S is a statement then

$$\chi(S) = \begin{cases} 1, & \text{if } S \text{ is true,} \\ 0, & \text{if } S \text{ is false.} \end{cases}$$

2.2. Grade and perfection

2.8. The *grade* of a proper ideal I in a Noetherian ring R is the length of a maximal R-regular sequence in I. The unit ideal R of R is regarded as an ideal of infinite grade.

2.9. Let M be a non-zero finitely generated module over a Noetherian ring R and let $\operatorname{ann}(M)$ be the annihilator of M and $\operatorname{pd}_R M$ be the projective dimension of M. It is well-known that

$$\operatorname{grade} \operatorname{ann}(M) = \min\{j \mid \operatorname{Ext}^j_R(M, R) \neq 0\};$$

therefore, it follows that

$$\operatorname{grade} \operatorname{ann}(M) \le \operatorname{pd}_R M. \tag{2.9.1}$$

If equality holds in (2.9.1), then M is called a *perfect R-module*. Recall, for example, that if R is Cohen–Macaulay and M is perfect, then M is Cohen–Macaulay. (This is not the full story. For more information, see, for example, [3, Prop. 16.19] or [2, Thm. 2.1.5].)

The ideal I in R is called a *perfect ideal* if R/I is a perfect R-module.

2.10. Let M be a finitely generated R-module. The R-module M has *rank* r if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of rank r for all associated primes $\mathfrak{p}$ of $\operatorname{Ass} R$. If every non-zero-divisor in R is regular on M, then M is called *torsion-free*. A proof of the following result may be found in [25, Prop. 1.25].

Proposition 2.10.1. Let M be a non-zero finitely generated R-module with finite projective dimension. Suppose that I is a perfect ideal of R with $IM = 0$. For each integer w, with $1 \le w \le \operatorname{pd}_R M$, let F_w be the ideal of R generated by

$$\{x \in R \mid \operatorname{pd}_{R_x} M_x < w\}.$$

If $w+1 \le \operatorname{grade} F_w$ for all w with $\operatorname{grade} I+1 \le w \le \operatorname{pd}_R M$, then M is a torsion-free R/I module. In particular, if $\operatorname{pd}_R M \le \operatorname{grade} I$, then M is a torsion-free (R/I)-module. $\square$

2.11. The following statement is well-known; see, for example, [19, Cor. 6.10]. It follows from the fact that if M is a perfect module, then the ideals F_w for M (in the sense 2.10.1) and the annihilator of M all have the same radical.

Proposition 2.11.1. Let $A \to R$ be a homomorphism of Noetherian rings and M be a non-zero finitely generated perfect A-module. If $M \otimes_A R \ne 0$, then

$$\operatorname{grade}((\operatorname{ann} M)R) = \operatorname{pd}_A M - \max\{i \mid \operatorname{Tor}_i^A(M, R) \ne 0\}. \qquad \square$$

In our favorite applications of 2.11.1, we focus on the complex $\mathbb{F} \otimes_A R$, where $\mathbb{F}$ is a resolution of M by projective A-modules. We are satisfied with an inequality; and therefore, there is no need for us to assume that $M \otimes_A R \ne 0$. Furthermore, in practice, A is usually a polynomial ring over the ring of integers. We refer to the following statement as depth-sensitivity. Of course, these ideas were first worked out in [12,27,28,17], and especially [18, Cor. 3.1].

Proposition 2.11.2. Let $A \to R$ be a homomorphism of Noetherian rings, M be a non-zero finitely generated perfect A-module, and $\mathbb{F}$ be a resolution of M by projective A-modules with the length of $\mathbb{F}$ equal to the projective dimension of M. Then

$$\mathrm{H}_j(\mathbb{F} \otimes_A R) = 0 \quad \text{for } \operatorname{pd}_A M - \operatorname{grade}((\operatorname{ann} M)R) + 1 \le j. \qquad \square$$

It is worth observing that if $\operatorname{ann} M$ is replaced by any sub-ideal, then the inequality of 2.11.2 continues to hold.

2.3. Multilinear algebra

2.12. Our complexes are described in a coordinate-free manner. Let V be a free module of finite rank d over R. We make much use of the symmetric algebra $\operatorname{Sym}_\bullet V$, the divided power algebra $D_\bullet(V^*)$, and the exterior algebras $\bigwedge^\bullet V$ and $\bigwedge^\bullet(V^*)$. We use the fact that $D_\bullet(V^*)$ is a module over $\operatorname{Sym}_\bullet V$ and the fact that $\bigwedge^\bullet V$ and $\bigwedge^\bullet(V^*)$ are modules over one another. We also use the fact that these module actions give rise to natural perfect pairings

$$\mathrm{ev} : \operatorname{Sym}_i V \otimes D_i(V^*) \to R \quad \text{and} \quad \mathrm{ev} : \bigwedge^i V \otimes \bigwedge^i(V^*) \to R,$$

for each integer i. The duals

$$\mathrm{ev}^* : R \to D_i(V^*) \otimes \mathrm{Sym}_i V \quad \text{and} \quad \mathrm{ev}^* : R \to \bigwedge^i(V^*) \otimes \bigwedge^i V$$

of the above evaluation maps are completely independent of coordinates. It follows that, if $\{m_\ell\}$ is a basis for $\mathrm{Sym}_i V$ (or $\bigwedge^i V$) and $\{m_\ell^*\}$ is the corresponding dual basis for $D_i(V^*)$ (or $\bigwedge^i(V^*)$), then the element

$$\mathrm{ev}^*(1) = \sum_\ell m_\ell^* \otimes m_\ell \in D_i(V^*) \otimes \mathrm{Sym}_i V \quad (\text{or } \bigwedge^i(V^*) \otimes \bigwedge^i V) \tag{2.12.1}$$

is completely independent of coordinates. These elements will also be used extensively in our calculations.

2.12.2. We emphasize a special case of (2.12.1). If ω_{V^*} is a basis for $\bigwedge^d(V^*)$ and ω_V is the corresponding dual basis for $\bigwedge^d V$, then the element $\omega_{V^*} \otimes \omega_V$ is a canonical element of $\bigwedge^d(V^*) \otimes \bigwedge^d V$. This element is also used in our calculations.

The following facts about the interaction of the module structures of $\bigwedge^\bullet V$ on $\bigwedge^\bullet(V^*)$ and $\bigwedge^\bullet(V^*)$ on $\bigwedge^\bullet V$ are well known; see [4, section 1], [5, Appendix], and [22, section 1].

Proposition 2.13. *Let V be a free module of rank d over a commutative Noetherian ring R and let $b_r \in \bigwedge^r V$, $c_p \in \bigwedge^p V$, and $\alpha_q \in \bigwedge^q(V^*)$.*

(a) *If $r = 1$, then $(b_r(\alpha_q))(c_p) = b_r \wedge (\alpha_q(c_p)) + (-1)^{1+q}\alpha_q(b_r \wedge c_p)$.*
(b) *If $q = d$, then $(b_r(\alpha_q))(c_p) = (-1)^{(d-r)(d-p)}(c_p(\alpha_q))(b_r)$.*
(c) *If $p = d$, then $[b_r(\alpha_q)](c_p) = b_r \wedge \alpha_q(c_p)$.*
(d) *If $\Psi : V \to V'$ is a homomorphism of free R-modules and $\delta_{s+r} \in \bigwedge^{s+r}(V'^*)$, then*

$$(\bigwedge^s \Psi^*)[((\bigwedge^r \Psi)(b_r))(\delta_{s+r})] = b_r[(\bigwedge^{s+r} \Psi^*)(\delta_{s+r})]. \qquad \square$$

The next result is an application of Proposition 2.13 and the ideas of 2.12. The result is used in the proof of Observation 7.12.

Proposition 2.14. *Let V be a free module of rank d over a commutative Noetherian ring R and let $b_r \in \bigwedge^r V$, $c_p \in \bigwedge^p V$, and $\alpha_q \in \bigwedge^q(V^*)$. Then $b_r \wedge (\alpha_q(c_p))$ is equal to a sum of elements of the form $b'_{r'} \wedge (\alpha'_{q'}(c_p))$ where*

$$b'_{r'} \in \bigwedge^{r'} V, \quad \alpha'_{q'} \in \bigwedge^{q'}(V^*), \quad r' - q' = r - q, \quad \textit{and} \quad q' \le q + d - r - p.$$

Remark. The element c_p has not been changed; but an upper bound has been imposed on the degree of the $\bigwedge^\bullet(V^*)$ contribution to the expression. Of course, the assertion is only interesting when $d < r + p$.

Proof. Let $\mathrm{ev}^*(1) = \sum_\ell m_\ell^* \otimes m_\ell \in \bigwedge^{d-p}(V^*) \otimes \bigwedge^{d-p} V$, as described in (2.12.1). Notice that

$$\sum_\ell m_\ell^*(m_\ell \wedge c_p) = c_p. \tag{2.14.1}$$

To establish (2.14.1), it suffices to test the proposed equation after multiplying both sides on the left by an arbitrary element x_{d-p} of $\bigwedge^{d-p} F$. The left side becomes

$$\begin{aligned}\sum_\ell x_{d-p} \wedge m_\ell^*(m_\ell \wedge c_p) &= \sum_\ell [x_{d-p}(m_\ell^*)](m_\ell \wedge c_p) \qquad \text{by 2.13.c}\\ &= \Big(\sum_\ell [x_{d-p}(m_\ell^*)] \cdot m_\ell\Big) \wedge c_p\\ &= x_{d-p} \wedge c_p,\end{aligned}$$

as desired. Apply (2.14.1) and 2.13.c to see that

$$b_r \wedge \alpha_q(c_p) = \sum_\ell b_r \wedge (\alpha_q \wedge m_\ell^*)(m_\ell \wedge c_p) = \sum_\ell [b_r(\alpha_q \wedge m_\ell^*)](m_\ell \wedge c_p).$$

Each m_ℓ is a sum of elements of the form $v_1 \wedge \cdots \wedge v_{d-p}$ with $v_i \in V$. Apply 2.13.a numerous times to write

$$\begin{aligned}&[b_r(\alpha_q \wedge m_\ell^*)](v_1 \wedge \cdots \wedge v_{d-p} \wedge c_p)\\ &= \begin{cases} \pm v_1 \wedge [b_r(\alpha_q \wedge m_\ell^*)](v_2 \wedge \cdots \wedge v_{d-p} \wedge c_p)\\ \pm[(v_1 \wedge b_r)(\alpha_q \wedge m_\ell^*)](v_2 \wedge \cdots \wedge v_{d-p} \wedge c_p)\end{cases}\\ &= \begin{cases} \pm v_2 \wedge v_1 \wedge [b_r(\alpha_q \wedge m_\ell^*)](v_3 \wedge \cdots \wedge v_{d-p} \wedge c_p)\\ \pm v_1 \wedge [(v_2 \wedge b_r)(\alpha_q \wedge m_\ell^*)](v_3 \wedge \cdots \wedge v_{d-p} \wedge c_p)\\ \pm v_2 \wedge [(v_1 \wedge b_r)(\alpha_q \wedge m_\ell^*)](v_3 \wedge \cdots \wedge v_{d-p} \wedge c_p)\\ \pm[(v_2 \wedge v_1 \wedge b_r)(\alpha_q \wedge m_\ell^*)](v_3 \wedge \cdots \wedge v_{d-p} \wedge c_p)\end{cases}\\ &= \cdots .\end{aligned}$$

We continue is this manner and express $b_r \wedge \alpha_q(c_p)$ as a sum of elements of the form

$$x \wedge [(x' \wedge b_r)(\alpha_q \wedge m_\ell^*)](c_p),$$

where x and x' are homogeneous elements of $\bigwedge^\bullet F$ and $\deg x + \deg x' = d-p$. The proof is complete because

$$\deg[(x' \wedge b_r)(\alpha_q \wedge m_\ell^*)] = q + d - p - \deg x' - r \le q + d - p - r. \qquad \square$$

3. The classical generalized Eagon–Northcott complexes

We recall the classical generalized Eagon–Northcott complexes $\{\mathcal{C}^i_\Phi \mid i \in \mathbb{Z}\}$ which were introduced at (1.0.1).

Definition 3.1. Let R be a commutative Noetherian ring, F and G be free R-modules of rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, with $\mathfrak{g} \le \mathfrak{f}$, $\Phi : G^* \to F$ be an R-module homomorphism, and i be an integer.

(a) The complex $\mathcal{C}^i_\Phi$ is

$$\begin{aligned}\cdots &\xrightarrow{\eta_\Phi} \textstyle\bigwedge^{\mathfrak{f}-\mathfrak{g}-i-2} F \otimes D_2(G^*) \xrightarrow{\eta_\Phi} \textstyle\bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F \otimes D_1(G^*) \xrightarrow{\eta_\Phi} \textstyle\bigwedge^{\mathfrak{f}-\mathfrak{g}-i} F \otimes D_0(G^*) \\ &\xrightarrow{\bigwedge^{\mathfrak{g}} \Phi} \textstyle\bigwedge^{\mathfrak{f}-i} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \xrightarrow{\operatorname{Kos}_\Phi} \textstyle\bigwedge^{\mathfrak{f}-i+1} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_1 G \xrightarrow{\operatorname{Kos}_\Phi} \cdots,\end{aligned} \tag{3.1.1}$$

with $\bigwedge^{\mathfrak{f}-\mathfrak{g}-i} F \otimes D_0(G^*)$ in position $i+1$. In particular, if j is an integer, then the module $(\mathcal{C}^i_\Phi)_j$ is

$$(\mathcal{C}^i_\Phi)_j = \begin{cases} \bigwedge^{\mathfrak{f}-j} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_{i-j} G, & \text{if } j \le i, \text{ and} \\ \bigwedge^{\mathfrak{f}-\mathfrak{g}-j+1} F \otimes D_{j-i-1}(G^*), & \text{if } i+1 \le j. \end{cases} \tag{3.1.2}$$

(b) The maps η_Φ and Kos_Φ may be found in Definition 5.2.a.

(c) The map $\bigwedge^a F \otimes D_0(G^*) \xrightarrow{\bigwedge^{\mathfrak{g}} \Phi} \bigwedge^{a+\mathfrak{g}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G$ is

$$f_a \mapsto f_a \wedge (\textstyle\bigwedge^{\mathfrak{g}} \Phi)(\omega_{G^*}) \otimes \omega_G,$$

where $\omega_{G^*} \otimes \omega_G$ is the canonical element of $\bigwedge^{\mathfrak{g}}(G^*) \otimes \bigwedge^{\mathfrak{g}} G$ from 2.12.2.

Example 3.2. Adopt the notation of 3.1. We record the classical generalized Eagon Northcott complexes $\mathcal{C}^i_\Phi$ which have the form:

$$0 \to (\mathcal{C}^i_\Phi)_{\mathfrak{f}-\mathfrak{g}+1} \to (\mathcal{C}^i_\Phi)_{\mathfrak{f}-\mathfrak{g}} \to \cdots \to (\mathcal{C}^i_\Phi)_2 \to (\mathcal{C}^i_\Phi)_1 \to (\mathcal{C}^i_\Phi)_0 \to 0.$$

Of course, these complexes have length $\mathfrak{f}-\mathfrak{g}+1$. The corresponding indices i are given in (1.0.4):

$$\begin{aligned}\mathcal{C}^{-1}_\Phi : 0 &\to \textstyle\bigwedge^0 F \otimes D_{\mathfrak{f}-\mathfrak{g}+1}(G^*) \xrightarrow{\eta_\Phi} \textstyle\bigwedge^1 F \otimes D_{\mathfrak{f}-\mathfrak{g}}(G^*) \xrightarrow{\eta_\Phi} \cdots \\ &\xrightarrow{\eta_\Phi} \textstyle\bigwedge^{\mathfrak{f}-\mathfrak{g}+1} F \otimes D_0(G^*) \to 0,\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^0_\Phi :\ & 0 \to \bigwedge^0 F \otimes D_{\mathfrak{f}-\mathfrak{g}}(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes D_{\mathfrak{f}-\mathfrak{g}-1}(G^*) \xrightarrow{\eta_\Phi} \dots \\
& \xrightarrow{\eta_\Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}} F \otimes D_0(G^*) \xrightarrow{\bigwedge^{\mathfrak{g}} \Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \to 0, \\
\mathcal{C}^1_\Phi :\ & 0 \to \bigwedge^0 F \otimes D_{\mathfrak{f}-\mathfrak{g}-1}(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes D_{\mathfrak{f}-\mathfrak{g}-2}(G^*) \xrightarrow{\eta_\Phi} \dots \\
& \xrightarrow{\eta_\Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}-1} F \otimes D_0(G^*) \xrightarrow{\bigwedge^{\mathfrak{g}} \Phi} \bigwedge^{\mathfrak{f}-1} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \\
& \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_1 G \to 0, \\
\mathcal{C}^2_\Phi :\ & 0 \to \bigwedge^0 F \otimes D_{\mathfrak{f}-\mathfrak{g}-2}(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes D_{\mathfrak{f}-\mathfrak{g}-3}(G^*) \xrightarrow{\eta_\Phi} \dots \\
& \xrightarrow{\eta_\Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}-2} F \otimes D_0(G^*) \xrightarrow{\bigwedge^{\mathfrak{g}} \Phi} \bigwedge^{\mathfrak{f}-2} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \\
& \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}-1} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_1 G \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_2 G \to 0, \\
& \qquad\qquad \vdots \\
\mathcal{C}^{\mathfrak{f}-\mathfrak{g}-1}_\Phi :\ & 0 \to \bigwedge^0 F \otimes D_1(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes D_0(G^*) \\
& \xrightarrow{\bigwedge^{g} \Phi} \bigwedge^{\mathfrak{g}+1} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}-2} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_1 G \\
& \xrightarrow{\operatorname{Kos}_\Phi} \dots \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_{\mathfrak{f}-\mathfrak{g}-1} G \to 0, \\
\mathcal{C}^{\mathfrak{f}-\mathfrak{g}}_\Phi :\ & 0 \to \bigwedge^0 (F^*) \otimes D_0(G^*) \\
& \xrightarrow{\bigwedge^{\mathfrak{g}} \Phi} \bigwedge^{\mathfrak{g}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{g}+1} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_1 G \\
& \xrightarrow{\operatorname{Kos}_\Phi} \dots \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_{\mathfrak{f}-\mathfrak{g}} G \to 0, \text{ and} \\
\mathcal{C}^{\mathfrak{f}-\mathfrak{g}+1}_\Phi :\ & 0 \to \bigwedge^{\mathfrak{g}-1} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_0 G \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{g}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_1 G \\
& \xrightarrow{\operatorname{Kos}_\Phi} \dots \xrightarrow{\operatorname{Kos}_\Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{g}} G \otimes \operatorname{Sym}_{\mathfrak{f}-\mathfrak{g}+1} G \to 0.
\end{aligned}$$

The maps η_Φ and Kos_Φ may be found in Definition 5.2.a.

4. Schur and Weyl modules which correspond to hooks

The modules which comprise the complexes $\{\mathcal{C}^{i,a}_\Phi\}$ are Schur and Weyl modules which correspond to hooks. We recall some of the elementary properties of these modules.

4.1. Let V be a non-zero free module of rank d over the commutative Noetherian ring R and let a and b be integers. Define the R-module homomorphisms

$$\begin{aligned}
\kappa^a_b :\ & \bigwedge^a V \otimes \operatorname{Sym}_b V \to \bigwedge^{a-1} V \otimes \operatorname{Sym}_{b+1} V \quad \text{and} \\
\eta^a_b :\ & \bigwedge^a V \otimes D_b V \to \bigwedge^{a+1} V \otimes D_{b-1} V
\end{aligned}$$

to be the compositions

$$\begin{aligned}\textstyle\bigwedge^a V \otimes \operatorname{Sym}_b V &\xrightarrow{\operatorname{ev}^*(1)} (V^* \otimes V) \otimes \textstyle\bigwedge^a V \otimes \operatorname{Sym}_b V \\ &\xrightarrow{\text{rearrange}} (V^* \otimes \textstyle\bigwedge^a V) \otimes (V \otimes \operatorname{Sym}_b V) \\ &\xrightarrow{\text{ModAct}\otimes\text{mult}} \textstyle\bigwedge^{a-1} V \otimes \operatorname{Sym}_{b+1} V \qquad \text{and}\end{aligned} \tag{4.1.1}$$

$$\begin{aligned}\textstyle\bigwedge^a V \otimes D_b V &\xrightarrow{\operatorname{ev}^*(1)} \textstyle\bigwedge^a V \otimes (V \otimes V^*) \otimes D_b V \\ &\xrightarrow{\text{regroup}} (\textstyle\bigwedge^a V \otimes V) \otimes (V^* \otimes D_b V) \\ &\xrightarrow{\text{mult}\otimes\text{ModAct}} \textstyle\bigwedge^{a+1} V \otimes D_{b-1} V,\end{aligned} \tag{4.1.2}$$

respectively. The map $\operatorname{ev}^*(1)$ is discussed in (2.12.1), "mult" is multiplication in the symmetric algebra or the exterior algebra, and "ModAct" is the module action of $\bigwedge^\bullet(V^*)$ on $\bigwedge^\bullet V$ or $\operatorname{Sym}_\bullet(V^*)$ on $D_\bullet V$. Define the R-modules

$$L_b^a V = \ker \kappa_b^a \quad \text{and} \quad K_b^a V = \ker \eta_b^a.$$

In the future, we will often write κ and η in place of κ_b^a and η_b^a.

4.2. The R-modules $L_b^a V$ and $K_b^a V$ have been used by many authors in many contexts. In particular, they are studied extensively in [4]; although our indexing conventions are different than the conventions of [4]; that is,

$$\text{the module we call } L_b^a V \text{ is called } L_b{}^{a+1} V \text{ in [4]}.$$

4.3. The modules $L_b^a V$ and $K_b^a V$ may also be thought of as the Schur modules $L_\lambda V$ and Weyl modules $K_\lambda V$ which correspond to certain hooks λ. We use the notation of Examples 2.1.3.h and 2.1.17.h in Weyman [30] to see that the module we call $L_b^a V$ is also the Schur module $L_{(a+1,1^{b-1})} V$ and the module we call $K_b^a V$ is also the Weyl module $K_{(b+1,1^{a-1})} V$.

4.4. The complex

$$\begin{aligned}0 \to \textstyle\bigwedge^d V \otimes \operatorname{Sym}_{c-d} V &\xrightarrow{\kappa} \textstyle\bigwedge^{d-1} V \otimes \operatorname{Sym}_{c-d+1} V \xrightarrow{\kappa} \\ \cdots \xrightarrow{\kappa} \textstyle\bigwedge^1 V \otimes \operatorname{Sym}_{c-1} V &\xrightarrow{\kappa} \textstyle\bigwedge^0 V \otimes \operatorname{Sym}_c V \to 0,\end{aligned} \tag{4.4.1}$$

which is a homogeneous strand of an acyclic Koszul complex, is split exact for all non-zero integers c; hence, $L_b^a V$ is a projective R-module. In fact, $L_b^a V$ is a free R-module of rank

$$\operatorname{rank} L_b^a V = \binom{d+b-1}{a+b}\binom{a+b-1}{a};$$

see [4, Prop. 2.5]. Similarly, $K_b^a V$ is a free R-module of rank

$$\operatorname{rank} K_b^a V = \binom{d+b}{a+b}\binom{a+b-1}{b}.$$

The perfect pairing

$$(\bigwedge^a V \otimes \operatorname{Sym}_b V) \otimes (\bigwedge^a (V^*) \otimes D_b(V^*)) \to R,$$

induces a perfect pairing

$$L_b^a V \otimes K_{b-1}^{a+1}(V^*) \to R, \quad \text{provided } (a,b) \neq (0,0) \text{ or } (-1,1). \tag{4.4.2}$$

Each assertion in Observation 4.5 is obvious; but it is very convenient to have all of these facts gathered in one place.

Observation 4.5. *Let R be a commutative Noetherian ring, V be a free R-module of rank d, and ℓ be an integer. Then the following modules are canonically isomorphic*:

(a) $L_\ell^0 V = \operatorname{Sym}_\ell V$,
(b) $L_\ell^{d-1} V \cong \bigwedge^d V \otimes \operatorname{Sym}_{\ell-1} V$, *provided* $\ell + d \neq 1$,
(c) $L_\ell^d V = 0$, *provided* $\ell + d \neq 0$,
(d) $L_0^\ell V = 0$, *provided* $\ell \neq 0$,
(e) $L_1^\ell V \cong \bigwedge^{\ell+1} V$, *provided* $\ell \neq -1$,
(f) $K_\ell^0 V = 0$, *provided* $\ell \neq 0$,
(g) $K_\ell^1 V \cong D_{\ell+1} V$, *provided* $\ell \neq -1$,
(h) $K_\ell^d V = \bigwedge^d V \otimes D_\ell V$, *and*
(i) $K_0^\ell V = \bigwedge^\ell V$.

Proof. Assertions (a), (h), and (i) follow from the definitions, and assertions (b), (c), (d), and (e) are immediate consequences of split exact complex (4.4.1). If c is non-zero, then we may apply $\operatorname{Hom}(-, R)$ to the split exact complex (4.4.1) to obtain the split exact complex

$$0 \to \bigwedge^0 (V^*) \otimes D_c(V^*) \xrightarrow{\eta} \bigwedge^1 (V^*) \otimes D_{c-1}(V^*) \xrightarrow{\eta} \dots \xrightarrow{\eta} \bigwedge^d (V^*) \otimes D_{c-d}(V^*) \to 0.$$

Replace V^* with V in order to see that

$$0 \to \bigwedge^0 V \otimes D_c V \xrightarrow{\eta} \bigwedge^1 V \otimes D_{c-1} V \xrightarrow{\eta} \dots \xrightarrow{\eta} \bigwedge^d V \otimes D_{c-d} V \to 0 \tag{4.5.1}$$

is also split exact. Assertions (f) and (g) are consequence of (4.5.1). □

5. The complexes $\mathbb{K}_\Phi$, and $\mathbb{L}_\Phi$ associated to a homomorphism Φ

The complexes $\mathbb{K}$ and $\mathbb{L}$ contain Weyl modules $K_b^a(G^*)$ and Schur modules $L_b^a G$, respectively. The complex $\mathcal{C}_\Phi^{i,a}$, which is the focal point of the present paper, is obtained by concatenating the complexes $\mathbb{K} \to \bigwedge \to \mathbb{L}$; see Definition 7.2.

Data 5.1. Let R be a commutative Noetherian ring, F and G be free R-modules of finite rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, and $\Phi : G^* \to F$ be an R-module homomorphism.

Definition 5.2. Adopt Data 5.1.

(a) If r and q are integers, then define the R-module homomorphisms

$$\begin{aligned} \eta_\Phi &: \textstyle\bigwedge^r F \otimes D_q(G^*) \to \bigwedge^{r+1} F \otimes D_{q-1}(G^*) \qquad \text{and} \\ \mathrm{Kos}_\Phi &: \textstyle\bigwedge^r F \otimes \mathrm{Sym}_q G \to \bigwedge^{r+1} F \otimes \mathrm{Sym}_{q+1} G \end{aligned}$$

to be the compositions

$$\begin{aligned} \textstyle\bigwedge^r F \otimes D_q(G^*) &\xrightarrow{1\otimes \mathrm{ev}^*(1)\otimes 1} \textstyle\bigwedge^r F \otimes G^* \otimes G \otimes D_q(G^*) \\ &\xrightarrow{1\otimes\Phi\otimes \mathrm{ModAct}} \textstyle\bigwedge^r F \otimes \bigwedge^1 F \otimes D_{q-1}(G^*) \\ &\xrightarrow{\mathrm{mult}\,\otimes 1} \textstyle\bigwedge^{r+1} F \otimes D_{q-1}(G^*) \end{aligned}$$

and

$$\begin{aligned} \textstyle\bigwedge^r F \otimes \mathrm{Sym}_q G &\xrightarrow{1\otimes \mathrm{ev}^*(1)\otimes 1} \textstyle\bigwedge^r F \otimes G^* \otimes G \otimes \mathrm{Sym}_q G \\ &\xrightarrow{1\otimes\Phi\otimes \mathrm{mult}} \textstyle\bigwedge^r F \otimes \bigwedge^1 F \otimes \mathrm{Sym}_{q+1} G \\ &\xrightarrow{\mathrm{mult}\,\otimes 1} \textstyle\bigwedge^{r+1} F \otimes \mathrm{Sym}_{q+1} G, \end{aligned}$$

respectively, where $\mathrm{ev}^*(1)$ is described in (2.12.1), ModAct is the module action of $\mathrm{Sym}_\bullet G$ on $D_\bullet(G^*)$ and mult is multiplication in the exterior algebra $\bigwedge^\bullet F$ or the symmetric algebra $\mathrm{Sym}_\bullet G$.

(b) If N and p are integers, then define $\mathbb{K}_\Phi^{N,p}$ to be the maps and modules

$$\begin{aligned} \mathbb{K}_\Phi^{N,p}:\ 0 \to \textstyle\bigwedge^0 F \otimes K_N^p(G^*) &\xrightarrow{\eta_\Phi} \textstyle\bigwedge^1 F \otimes K_{N-1}^p(G^*) \xrightarrow{\eta_\Phi} \dots \\ &\xrightarrow{\eta_\Phi} \textstyle\bigwedge^N F \otimes K_0^p(G^*) \to 0 \end{aligned}$$

with $[\mathbb{K}_\Phi^{N,p}]_j = \bigwedge^{N-j} F \otimes K_j^p(G^*)$; and define $\mathbb{L}_\Phi^{N,p}$ to be the maps and modules

$$0 \to \textstyle\bigwedge^{N+1} F \otimes L_1^p G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^{N+2} F \otimes L_2^p G \xrightarrow{\mathrm{Kos}_\Phi} \cdots \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^{\mathfrak{f}} F \otimes L_{\mathfrak{f}-N}^p G \to 0,$$

with

$$[\mathbb{L}_\Phi^{N,p}]_j = \begin{cases} 0 & \text{if } (p,j) = (0, \mathfrak{f}-N), \text{ and} \\ \bigwedge^{\mathfrak{f}-j} F \otimes L_{\mathfrak{f}-N-j}^p G, & \text{otherwise,} \end{cases} \tag{5.2.1}$$

where the maps are induced by the homomorphisms η_Φ and Kos_Φ of (a) and the modules $K_q^p(G^*)$ and $L_q^p G$ are defined in (4.1.2) and (4.1.1).

Remarks 5.3.

(a) The map

$$\eta_b^a : \bigwedge^a(G^*) \otimes D_b(G^*) \to \bigwedge^{a+1}(G^*) \otimes D_{b-1}(G^*)$$

of (4.1.2) is equal to the map

$$\eta_{\mathrm{id}_{G^*}} : \bigwedge^a(G^*) \otimes D_b(G^*) \to \bigwedge^{a+1}(G^*) \otimes D_{b-1}(G^*)$$

of 5.2.a.

(b) The map

$$\kappa_b^a : \bigwedge^a G \otimes \mathrm{Sym}_a G \to \bigwedge^{a-1} G \otimes \mathrm{Sym}_{b+1} G$$

of (4.1.1) is related to the map $\mathrm{Kos}_{\mathrm{id}_{G^*}}$ of 5.2.a by way of the following commutative diagram:

$$\begin{tikzcd}
\bigwedge^{\mathfrak{g}-a} G^* \otimes \mathrm{Sym}_b G \otimes \bigwedge^{\mathfrak{g}} G \arrow[r, "(-1)^{\mathfrak{g}-a}\mathrm{Kos}_{\mathrm{id}_{G^*}}"] \arrow[d, "\mathrm{ModAct}"', "\cong"] & \bigwedge^{\mathfrak{g}-a+1} G^* \otimes \mathrm{Sym}_{b+1} G \otimes \bigwedge^{\mathfrak{g}} G \arrow[d, "\cong"', "\mathrm{ModAct}"] \\
\bigwedge^a G \otimes \mathrm{Sym}_b G \arrow[r, "\kappa_b^a"] & \bigwedge^{a-1} G \otimes \mathrm{Sym}_{b+1} G.
\end{tikzcd}$$

(c) The maps and modules of $\mathbb{K}_\Phi^{N,p}$ and $\mathbb{L}_\Phi^{N,p}$ form complexes because the diagrams

$$\begin{tikzcd}
0 \to \bigwedge^r F \otimes K_q^p(G^*) \arrow[r, hook] \arrow[d, dashed, "\eta_\Phi"] & \bigwedge^r F \otimes \bigwedge^p(G^*) \otimes D_q(G^*) \arrow[r, "\eta_{\mathrm{id}_{G^*}}"] \arrow[d, "\eta_\Phi"] & \bigwedge^r F \otimes \bigwedge^{p+1}(G^*) \otimes D_{q-1}(G^*) \arrow[d, "\eta_\Phi"] \\
0 \to \bigwedge^{r+1} F \otimes K_{q-1}^p(G^*) \arrow[r, hook] & \bigwedge^{r+1} F \otimes \bigwedge^p(G^*) \otimes D_{q-1}(G^*) \arrow[r, "\eta_{\mathrm{id}_{G^*}}"] & \bigwedge^{r+1} F \otimes \bigwedge^{p+1}(G^*) \otimes D_{q-2}(G^*)
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \to \bigwedge^r F \otimes L_q^p G \arrow[r, hook] \arrow[d, dashed, "\mathrm{Kos}_\Phi"] & \bigwedge^r F \otimes \bigwedge^p G \otimes \mathrm{Sym}_q G \arrow[r, "\kappa"] \arrow[d, "\mathrm{Kos}_\Phi"] & \bigwedge^r F \otimes \bigwedge^{p-1} G \otimes \mathrm{Sym}_{q+1} G \arrow[d, "\mathrm{Kos}_\Phi"] \\
0 \to \bigwedge^{r+1} F \otimes L_{q+1}^p G \arrow[r, hook] & \bigwedge^{r+1} F \otimes \bigwedge^p G \otimes \mathrm{Sym}_{q+1} G \arrow[r, "\kappa"] & \bigwedge^{r+1} F \otimes \bigwedge^{p-1} G \otimes \mathrm{Sym}_{q+2} G
\end{tikzcd}$$

commute.

6. The complexes $\mathbb{K}_\Phi$ and $\mathbb{L}_\Phi$ when Φ is a direct sum of homomorphisms

In this section we assume that the homomorphism $\Phi : G^* \to F$ is a direct sum of homomorphisms. In Proposition 6.4 we relate the complexes $\mathbb{K}_\Phi^{N,p}$ and $\mathbb{L}_\Phi^{N,p}$ of Definition 5.2.b to similar complexes built from smaller data. This result plays a prominent role in the proof of the acyclicity of the complexes $\mathcal{C}_\Phi^{i,a}$; see Theorem 8.4, which is the main result of the paper.

Data 6.1. Let R be a commutative Noetherian ring, F and G be free R-modules of rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, and $\Phi : G^* \to F$ be an R-module homomorphism. Decompose F and G as

$$F = F' \oplus F'' \quad \text{and} \quad G = G' \oplus G'',$$

where F', F'', G' and G'' are free R-modules and $\operatorname{rank} F'' = \operatorname{rank} G'' = 1$ and let

$$F^* = F'^* \oplus F''^* \quad \text{and} \quad G^* = G'^* \oplus G''^*$$

be the corresponding decompositions of F^* and G^*. Assume that

$$\Phi = \begin{bmatrix} \Phi' & 0 \\ 0 & \Phi'' \end{bmatrix},$$

where $\Phi' : G'^* \to F'$ and $\Phi'' : G''^* \to F''$ are R-module homomorphisms.

Notation 6.2. Adopt Data 6.1.

(a) Let $\mathbb{A}(\Phi'')$ and $\mathbb{B}(\Phi'')$ represent the complexes

$$\mathbb{A}(\Phi'') : \quad 0 \to R \xrightarrow{\operatorname{Kos}_{\Phi''}} F'' \otimes G'' \to 0$$

and

$$\mathbb{B}(\Phi'') : \quad 0 \to G''^* \xrightarrow{\Phi''} F'' \to 0.$$

(b) Let $\pi' : F \to F'$ be the projection map which corresponds to the direct sum decomposition $F = F' \oplus F''$.

(c) If a, b, c, d, and e are integers, then let

$$\operatorname{incl}^\dagger : \textstyle\bigwedge^a F' \otimes \bigwedge^b G \otimes \operatorname{Sym}_c G \otimes \bigwedge^d F'' \otimes \operatorname{Sym}_e G'' \to \bigwedge^{a+d} F \otimes \bigwedge^b G \otimes \operatorname{Sym}_{c+e} G$$

be the R-module homomorphism given by

$$(\text{multiplication in } \textstyle\bigwedge^\bullet F) \otimes (\text{the identity map in } \bigwedge^\bullet G) \otimes (\text{multiplication in } \operatorname{Sym}_\bullet G).$$

(d) If a, b, c, d, and e are integers, then let

$$\mathrm{quot}^{\dagger}: \bigwedge^{a+d} F \otimes \bigwedge^{b+e} G \otimes \mathrm{Sym}_c G \to \bigwedge^{a} F' \otimes \bigwedge^{b} G' \otimes \mathrm{Sym}_c G' \otimes \bigwedge^{d} F'' \otimes \bigwedge^{e} G''$$

be the R-module homomorphism given by

$$\begin{pmatrix} \text{the projection map} \\ \bigwedge^{a+d} F \to \bigwedge^{a} F' \otimes \bigwedge^{d} F'' \\ \text{induced by } F = F' \oplus F'' \end{pmatrix} \otimes \begin{pmatrix} \text{the projection map} \\ \bigwedge^{b+e} G \to \bigwedge^{b} G' \otimes \bigwedge^{e} G'' \\ \text{induced by } G = G' \oplus G'' \end{pmatrix} \otimes \begin{pmatrix} \text{the quotient map} \\ \mathrm{Sym}_\bullet G \to \mathrm{Sym}_\bullet G' \\ \text{induced by } G/G'' = G' \end{pmatrix}.$$

(e) If a, b, c, and d are integers, then let

$$\mathrm{incl}^{\ddagger}: \bigwedge^{a} F' \otimes \bigwedge^{b}(G'^*) \otimes D_c(G'^*) \otimes \bigwedge^{d}(G''^*) \to \bigwedge^{a} F \otimes \bigwedge^{b+d}(G^*) \otimes D_c(G^*)$$

be the R-module homomorphism given by

$$(\text{inclusion}) \otimes (\text{multiplication}) \otimes (\text{inclusion}).$$

(f) If a, b, c, d, and e are integers, then let

$$\begin{aligned} \mathrm{quot}^{\ddagger}:\ & \bigwedge^{a+d} F \otimes \bigwedge^{b}(G^*) \otimes D_{c+e}(G^*) \\ & \to \bigwedge^{a} F' \otimes \bigwedge^{b}(G'^*) \otimes D_c(G^*) \otimes \bigwedge^{d} F'' \otimes D_e(G''^*) \end{aligned}$$

be the R-module homomorphism given by

$$\begin{pmatrix} \text{the projection map} \\ \bigwedge^{a+d} F \to \bigwedge^{a} F' \otimes \bigwedge^{d} F'' \\ \text{induced by } F = F' \oplus F'' \end{pmatrix} \otimes \begin{pmatrix} \text{the identity map} \\ \text{on } \bigwedge^{\bullet}(G^*) \end{pmatrix} \otimes \begin{pmatrix} D_{c+e}(G^*) \xrightarrow{\mathrm{ev}^*(1)\otimes 1} \\ D_e(G''^*) \otimes \mathrm{Sym}_e G'' \otimes D_{c+e}(G^*) \\ \xrightarrow{1\otimes \mathrm{ModAct}} D_e(G''^*) \otimes D_c(G^*) \\ \xrightarrow{\text{exchange}} D_c(G^*) \otimes D_e(G''^*) \end{pmatrix},$$

where $\mathrm{ev}^*(1)$ is described in (2.12.1) and ModAct is the module action of $\mathrm{Sym}_\bullet G$ on $D_\bullet(G^*)$.

Proposition 6.4 asserts that (6.4.1) and (6.4.3) are short exact sequences of complexes. Observation 6.3 considers the modules in (6.4.1) and (6.4.3) in the arbitrary position j. Recall the function χ from 2.7.

Observation 6.3. *Adopt Data 6.1 and Notation 6.2.*

(a) *If p, q, r are integers, then*

$$0 \to \begin{matrix} \bigwedge^{r} F' \otimes L_q^p G \\ \oplus \\ \bigwedge^{r-1} F' \otimes L_{q-1}^p G \otimes F'' \otimes G'' \end{matrix} \xrightarrow{\mathrm{incl}^{\dagger}} \bigwedge^{r} F \otimes L_q^p G$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \begin{matrix} \bigwedge^r F' \otimes L_q^p G \\ \oplus \\ \bigwedge^{r-1} F' \otimes L_{q-1}^p G \otimes F'' \otimes G'' \end{matrix} \arrow[r, "\mathrm{incl}^\dagger"] \arrow[d, hook] & \bigwedge^r F \otimes L_q^p G \arrow[r, "\mathrm{quot}^\dagger"] \arrow[d, hook] & \begin{matrix} \bigwedge^{r-1} F' \otimes L_q^p G' \otimes F'' \\ \oplus \\ \bigwedge^{r-1} F' \otimes L_q^{p-1} G' \otimes F'' \otimes G'' \end{matrix} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \begin{matrix} \bigwedge^r F' \otimes \bigwedge^p G \otimes S_q G \\ \oplus \\ \bigwedge^{r-1} F' \otimes \bigwedge^p G \otimes S_{q-1} G \otimes F'' \otimes G'' \end{matrix} \arrow[r, "\mathrm{incl}^\dagger"] \arrow[d, "\kappa"] & \bigwedge^r F \otimes \bigwedge^p G \otimes S_q G \arrow[r, "\mathrm{quot}^\dagger"] \arrow[d, "\kappa"] & \begin{matrix} \bigwedge^{r-1} F' \otimes \bigwedge^p G' \otimes S_q G' \otimes F'' \\ \oplus \\ \bigwedge^{r-1} F' \otimes \bigwedge^{p-1} G' \otimes S_q G' \otimes F'' \otimes G'' \end{matrix} \arrow[r] \arrow[d, "\kappa"] & 0 \\
0 \arrow[r] & \begin{matrix} \bigwedge^r F' \otimes \bigwedge^{p-1} G \otimes S_{q+1} G \\ \oplus \\ \bigwedge^{r-1} F' \otimes \bigwedge^{p-1} G \otimes S_q G \otimes F'' \otimes G'' \end{matrix} \arrow[r, "\mathrm{incl}^\dagger"] \arrow[d, "\kappa"] & \bigwedge^r F \otimes \bigwedge^{p-1} G \otimes S_{q+1} G \arrow[r, "\mathrm{quot}^\dagger"] \arrow[d, "\kappa"] & \begin{matrix} \bigwedge^{r-1} F' \otimes \bigwedge^{p-1} G' \otimes S_{q+1} G' \otimes F'' \\ \oplus \\ \bigwedge^{r-1} F' \otimes \bigwedge^{p-2} G' \otimes S_{q+1} G' \otimes F'' \otimes G'' \end{matrix} \arrow[r] \arrow[d, "\kappa"] & 0 \\
0 \arrow[r] & \begin{matrix} \bigwedge^r F' \otimes L_{q+2}^{p-2} G \\ \oplus \\ \bigwedge^{r-1} F' \otimes L_{q+1}^{p-2} G \otimes F'' \otimes G'' \end{matrix} \arrow[r, "\mathrm{incl}^\dagger"] \arrow[d] & \bigwedge^r F \otimes L_{q+2}^{p-2} G \arrow[r, "\mathrm{quot}^\dagger"] \arrow[d] & \begin{matrix} \bigwedge^{r-1} F' \otimes L_{q+2}^{p-2} G' \otimes F'' \\ \oplus \\ \bigwedge^{r-1} F' \otimes L_{q+2}^{p-3} G' \otimes F'' \otimes G'' \end{matrix} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Fig. 6.3.1. This picture is a commutative diagram which is used in the proof of Observation 6.3.a. The middle two rows are exact. The columns are exact provided $2 \le p + q$. We use "S" as an abbreviation for "Sym".

$$
\xrightarrow{\mathrm{quot}^\dagger} \begin{matrix} \bigwedge^{r-1} F' \otimes L_q^p G' \otimes F'' \\ \oplus \\ \chi\big((p,q) \neq (1,0)\big)\big(\bigwedge^{r-1} F' \otimes L_q^{p-1} G' \otimes F'' \otimes G''\big) \end{matrix} \to 0
$$

is an exact sequence of R-modules.

(b) *If p, q, r are integers, then*

$$
0 \to \begin{matrix} \bigwedge^r F' \otimes K_q^p(G'^*) \\ \oplus \\ \bigwedge^r F' \otimes K_q^{p-1}(G'^*) \otimes G''^* \end{matrix} \xrightarrow{\mathrm{incl}^\ddagger} \bigwedge^r F \otimes K_q^p(G^*)
$$

$$
\xrightarrow{\mathrm{quot}^\ddagger} \begin{matrix} \chi\big((p,q) \neq (0,1)\big)\big(\bigwedge^r F' \otimes K_{q-1}^p(G^*) \otimes G''^*\big) \\ \oplus \\ \bigwedge^{r-1} F' \otimes K_q^p(G^*) \otimes F'' \end{matrix} \to 0
$$

is an exact sequence of R-modules.

Remark 6.3.3. If $r = 1$ and $F' = 0$, then $F = F''$ has rank one and no harm occurs if we set $F = F''$ equal to R (that is, apply $- \otimes F^*$). In this case, 6.3.a is

$$
0 \to L_{q-1}^p G \otimes G'' \xrightarrow{\mathrm{incl}^\dagger} L_q^p G \xrightarrow{\mathrm{quot}^\dagger} L_q^p G' \oplus \chi\big((p,q) \neq (1,0)\big)\big(L_q^{p-1} G' \otimes G''\big) \to 0. \tag{6.3.4}
$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \begin{array}{c} \bigwedge^r F' \otimes K_q^p(G'^*) \\ \oplus \\ \bigwedge^r F' \otimes K_q^{p-1}(G'^*) \otimes G''^* \end{array} \arrow[r, "\mathrm{incl}^\ddagger"] \arrow[d, hook] & \bigwedge^r F \otimes K_q^p(G^*) \arrow[r, "\mathrm{quot}^\ddagger"] \arrow[d, hook] & \begin{array}{c} \bigwedge^r F' \otimes K_{q-1}^p(G^*) \otimes G''^* \\ \oplus \\ \bigwedge^{r-1} F' \otimes K_q^p(G^*) \otimes F'' \end{array} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \begin{array}{c} \bigwedge^r F' \otimes \bigwedge^p(G'^*) \otimes D_q(G'^*) \\ \oplus \\ \bigwedge^r F' \otimes \bigwedge^{p-1}(G'^*) \otimes D_q(G'^*) \otimes G''^* \end{array} \arrow[r, "\mathrm{incl}^\ddagger"] \arrow[d, "\eta_{\mathrm{id}_{G^*}}"] & \bigwedge^r F \otimes \bigwedge^p(G^*) \otimes D_q(G^*) \arrow[r, "\mathrm{quot}^\ddagger"] \arrow[d, "\eta_{\mathrm{id}_{G^*}}"] & \begin{array}{c} \bigwedge^r F' \otimes \bigwedge^p(G^*) \otimes D_{q-1}(G^*) \otimes G''^* \\ \oplus \\ \bigwedge^{r-1} F' \otimes \bigwedge^p(G^*) \otimes D_q(G^*) \otimes F'' \end{array} \arrow[r] \arrow[d, "\eta_{\mathrm{id}_{G^*}}"] & 0 \\
0 \arrow[r] & \begin{array}{c} \bigwedge^r F' \otimes \bigwedge^{p+1}(G'^*) \otimes D_{q-1}(G'^*) \\ \oplus \\ \bigwedge^r F' \otimes \bigwedge^p(G'^*) \otimes D_{q-1}(G'^*) \otimes G''^* \end{array} \arrow[r, "\mathrm{incl}^\ddagger"] \arrow[d, "\eta_{\mathrm{id}_{G^*}}"] & \bigwedge^r F \otimes \bigwedge^{p+1}(G^*) \otimes D_{q-1}(G^*) \arrow[r, "\mathrm{quot}^\ddagger"] \arrow[d, "\eta_{\mathrm{id}_{G^*}}"] & \begin{array}{c} \bigwedge^r F' \otimes \bigwedge^{p+1}(G^*) \otimes D_{q-2}(G^*) \otimes G''^* \\ \oplus \\ \bigwedge^{r-1} F' \otimes \bigwedge^{p+1}(G^*) \otimes D_{q-1}(G^*) \otimes F'' \end{array} \arrow[r] \arrow[d, "\eta_{\mathrm{id}_{G^*}}"] & 0 \\
0 \arrow[r] & \begin{array}{c} \bigwedge^r F' \otimes K_{q-2}^{p+2}(G'^*) \\ \oplus \\ \bigwedge^r F' \otimes K_{q-2}^{p+1}(G'^*) \otimes G''^* \end{array} \arrow[r, "\mathrm{incl}^\ddagger"] \arrow[d] & \bigwedge^r F \otimes K_{q-2}^{p+2}(G^*) \arrow[r, "\mathrm{quot}^\ddagger"] \arrow[d] & \begin{array}{c} \bigwedge^r F' \otimes K_{q-3}^{p+2}(G^*) \otimes G''^* \\ \oplus \\ \bigwedge^{r-1} F' \otimes K_{q-2}^{p+2}(G^*) \otimes F'' \end{array} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Fig. 6.3.2. This picture is a commutative diagram which is used in the proof of Observation 6.3.b. The middle two rows are exact. The columns are exact provided $2 \le p + q$.

If $r = 0$, then 6.3.b is

$$0 \to \begin{array}{c} K_q^p(G'^*) \\ \oplus \\ \left(K_q^{p-1}(G'^*) \otimes G''^*\right) \end{array} \xrightarrow{\mathrm{incl}^\ddagger} K_q^p(G^*) \xrightarrow{\mathrm{quot}^\ddagger} \chi\big((p,q) \ne (0,1)\big)\left(K_{q-1}^p(G^*) \otimes G''^*\right) \to 0. \tag{6.3.5}$$

The exact sequences (6.3.4) and (6.3.5) are results from representation theory; see, for example, [30, 2.3.1] or [14, Exercise 6.11]. The versions we give are stated explicitly, require no assumptions about characteristic, and are precisely the results that we use in the proof of Lemma 8.1. In fact, our proof of Proposition 6.4 was created by starting with proofs of (6.3.4) and (6.3.5).

Proof. (a). If $p < 0$ or $q < 0$, then all of the modules in 6.3.a are zero. If $q = 0$, then 6.3.a is

$$0 \to \chi(p = 0) \bigwedge^r F' \xrightarrow{\mathrm{incl}^\dagger} \chi(p = 0) \bigwedge^r F \xrightarrow{\mathrm{quot}^\dagger} \chi(p = 0) \bigwedge^{r-1} F' \otimes F'' \to 0,$$

which is exact for all p. (Notice that we used the factor $\chi((p, q) \ne (0, 0))$ in our proof that 6.3.a is exact for all p when $q = 0$.) If $p = 0$, then 6.3.a is

$$0 \to \begin{matrix} \bigwedge^r F' \otimes \operatorname{Sym}_q G \\ \oplus \\ \bigwedge^{r-1} F' \otimes \operatorname{Sym}_{q-1} G \otimes F'' \otimes G'' \end{matrix} \xrightarrow{\text{incl}^\dagger} \bigwedge^r F \otimes \operatorname{Sym}_q G$$
$$\xrightarrow{\text{quot}^\dagger} \bigwedge^{r-1} F' \otimes \operatorname{Sym}_q G' \otimes F'' \to 0,$$

which is exact for all q. Henceforth, we assume $2 \le p+q$. Observe that Fig. 6.3.1 is a commutative diagram; each column is split exact; and the middle two rows are split exact. The assertion follows from the snake lemma and the fact that the bottom $\text{incl}^\dagger$ in Fig. 6.3.1 is automatically an injection.

(b). If $p<0$ or $q<0$, then all of the modules in 6.3.b are zero. If $q=0$, then 6.3.b is

$$0 \to \begin{matrix} \bigwedge^r F' \otimes \bigwedge^p (G'^*) \\ \oplus \\ \bigwedge^r F' \otimes \bigwedge^{p-1} (G'^*) \otimes G''^* \end{matrix} \xrightarrow{\text{incl}^\ddagger} \bigwedge^r F \otimes \bigwedge^p (G^*)$$
$$\xrightarrow{\text{quot}^\ddagger} \bigwedge^{r-1} F' \otimes \bigwedge^p (G^*) \otimes F'' \to 0,$$

which is exact for all integers p. If $p=0$, then 6.3.b is

$$0 \to \chi(q=0) \bigwedge^r F' \xrightarrow{\text{incl}^\ddagger} \chi(q=0) \bigwedge^r F \xrightarrow{\text{quot}^\ddagger} \chi(q=0) \bigwedge^{r-1} F' \otimes F'' \to 0,$$

which is exact for all integers q. Henceforth, we assume $2 \le p+q$. Observe that Fig. 6.3.2 is a commutative diagram; each column is split exact; and the middle two rows are split exact. Once again, the assertion follows from the snake lemma and the fact that the bottom $\text{incl}^\ddagger$ in Fig. 6.3.2 is automatically an injection. □

The complexes $\mathbb{L}_\Phi^{N,p}$ and $\mathbb{K}_\Phi^{N,p}$ may be found in Definition 5.2.b; the complexes $\mathbb{A}(\Phi'')$ and $\mathbb{B}(\Phi'')$ are defined in Notation 6.2; and some comments about the total complex (or mapping cone) of a map of complexes is in 2.5.

Proposition 6.4. *Adopt Data 6.1 and Notation 6.2. Let p and N be integers. The following statements hold.*

(a) *Assume that $p \neq 0$. Then there is a canonical short exact sequence of complexes*

$$0 \to \operatorname{Tot}\left(\mathbb{L}_{\pi'\circ\Phi}^{N,p} \otimes \mathbb{A}(\Phi'')\right) \xrightarrow{\text{incl}^\dagger} \mathbb{L}_\Phi^{N,p} \xrightarrow{\text{quot}^\dagger} \begin{matrix} \mathbb{L}_{\Phi'}^{N-1,p} \otimes F'' \\ \oplus \\ \mathbb{L}_{\Phi'}^{N-1,p-1} \otimes F'' \otimes G'' \end{matrix} \to 0. \tag{6.4.1}$$

In particular, if Φ'' is an isomorphism, then $\text{quot}^\dagger$ *induces an isomorphism*

$$\mathrm{H}_j(\mathbb{L}_{\Phi'}^{N-1,p}) \oplus \mathrm{H}_j(\mathbb{L}_{\Phi'}^{N-1,p-1}) \cong \mathrm{H}_j(\mathbb{L}_\Phi^{N,p}), \tag{6.4.2}$$

for all j.

(b) *Assume that $p \neq 0$. Then there is a canonical short exact sequence of complexes*

$$0 \to \begin{matrix} \mathbb{K}_{\Phi'}^{N,p} \\ \oplus \\ (\mathbb{K}_{\Phi'}^{N,p-1} \otimes G''^*) \end{matrix} \xrightarrow{\text{incl}^\ddagger} \mathbb{K}_{\Phi}^{N,p} \xrightarrow{\text{quot}^\ddagger} \operatorname{Tot}\left(\mathbb{K}_{\pi'\circ\Phi}^{N-1,p} \otimes \mathbb{B}(\Phi'')\right) \to 0. \tag{6.4.3}$$

In particular, if Φ'' is an isomorphism, then incl‡ *induces an isomorphism*

$$\mathrm{H}_j(\mathbb{K}_{\Phi'}^{N,p}) \oplus \mathrm{H}_j(\mathbb{K}_{\Phi'}^{N,p-1}) \cong \mathrm{H}_j(\mathbb{K}_{\Phi}^{N,p}), \tag{6.4.4}$$

for all j.

Remark 6.4.5. The conclusion of Proposition 6.4.b does not hold when $p = 0$ and $1 \leq N \leq \mathfrak{f}$. In this case, $\mathbb{K}_{\Phi'}^{N,p}$ and $\mathbb{K}_{\Phi}^{N,p}$ are $0 \to \bigwedge^N F' \to 0$ and $0 \to \bigwedge^N F \to 0$, respectively, with each non-zero module in position zero, and $\mathbb{K}_{\Phi'}^{N,p-1}$ is the zero complex. Thus, (6.4.4) does not hold.

The conclusion of Proposition 6.4.a also does not hold when $p = 0$. In particular, $\mathbb{L}_{\Phi}^{N,0}$ is the complex

$$\begin{aligned} 0 \to {\textstyle\bigwedge^{N+1}} F \otimes \operatorname{Sym}_1 G \xrightarrow{\text{Kos}_\Phi} {\textstyle\bigwedge^{N+2}} F \otimes \operatorname{Sym}_2 G \xrightarrow{\text{Kos}_\Phi} \dots \\ \xrightarrow{\text{Kos}_\Phi} {\textstyle\bigwedge^{\mathfrak{f}}} F \otimes \operatorname{Sym}_{\mathfrak{f}-N} G \to 0. \end{aligned}$$

In the typical situation

$$\mathrm{H}_{\mathfrak{f}-N-1}(\mathbb{L}_{\Phi}^{N,0}) = {\textstyle\bigwedge^N} F, \quad \mathrm{H}_{\mathfrak{f}-N-1}(\mathbb{L}_{\Phi'}^{N-1,0}) = {\textstyle\bigwedge^{N-1}} F',$$

and (6.4.2) fails to hold.

Proof. (a). There is nothing to prove if $p \leq 0$; so we assume that $1 \leq p$. Notice that (6.4.1) in position j is

$$\begin{aligned} 0 \to \begin{matrix} \bigwedge^{\mathfrak{f}-j} F' \otimes L^p_{\mathfrak{f}-j-N} G \\ \oplus \\ \bigwedge^{\mathfrak{f}-j-1} F' \otimes L^p_{\mathfrak{f}-j-N-1} G \otimes F'' \otimes G'' \end{matrix} \xrightarrow{\text{incl}^\dagger} {\textstyle\bigwedge^{\mathfrak{f}-j}} F \otimes L^p_{\mathfrak{f}-j-N} G \\ \xrightarrow{\text{quot}^\dagger} \begin{matrix} \bigwedge^{\mathfrak{f}-j-1} F' \otimes L^p_{\mathfrak{f}-j-N} G' \otimes F'' \\ \oplus \\ \chi((p,j) \neq (1, \mathfrak{f}-N)) \bigwedge^{\mathfrak{f}-j-1} F' \otimes L^{p-1}_{\mathfrak{f}-j-N} G' \otimes F'' \otimes G'' \end{matrix} \to 0, \end{aligned}$$

which is 6.3.a with r replaced by $\mathfrak{f} - j$ and q replaced by $\mathfrak{f} - j - N$; hence, each row of (6.4.1) is a short exact sequence. To see that (6.4.1) is a map of complexes, one verifies that Fig. 6.4.6 is a commutative diagram, and this is straightforward. Thus, (6.4.1) is a short exact sequence of complexes.

$$\begin{tikzcd}
\begin{matrix}\bigwedge^r F'\otimes L_q^pG\\ \oplus\\ \bigwedge^{r-1} F'\otimes L_{q-1}^pG\otimes F''\otimes G''\end{matrix} \arrow[r, "\mathrm{incl}^\dagger"] \arrow[d, "\left[\begin{smallmatrix}\mathrm{Kos}_{\pi'\circ\Phi} & 0\\ \mathrm{Kos}_{\Phi''} & -\mathrm{Kos}_{\pi'\circ\Phi}\end{smallmatrix}\right]"] & \bigwedge^r F\otimes L_q^pG \arrow[r, "\mathrm{quot}^\dagger"] \arrow[d, "\mathrm{Kos}_\Phi"] & \begin{matrix}\bigwedge^{r-1} F'\otimes L_q^pG'\otimes F''\\ \oplus\\ \bigwedge^{r-1} F'\otimes L_q^{p-1}G'\otimes F''\otimes G''\end{matrix} \arrow[d, "\left[\begin{smallmatrix}-\mathrm{Kos}_{\Phi'} & 0\\ 0 & -\mathrm{Kos}_{\Phi'}\end{smallmatrix}\right]"]\\
\begin{matrix}\bigwedge^{r+1} F'\otimes L_{q+1}^pG\\ \oplus\\ \bigwedge^{r} F'\otimes L_{q}^pG\otimes F''\otimes G''\end{matrix} \arrow[r, "\mathrm{incl}^\dagger"] & \bigwedge^{r+1} F\otimes L_{q+1}^pG \arrow[r, "\mathrm{quot}^\dagger"] & \begin{matrix}\bigwedge^{r} F'\otimes L_{q+1}^pG'\otimes F''\\ \oplus\\ \bigwedge^{r} F'\otimes L_{q+1}^{p-1}G'\otimes F''\otimes G''\end{matrix}
\end{tikzcd}$$

Fig. 6.4.6. In the proof of Proposition 6.4.a one verifies that this diagram (with p and q both positive) commutes in order to see that (6.4.1) is a map of complexes.

$$\begin{tikzcd}
\begin{matrix}\bigwedge^r F'\otimes K_q^p(G'^*)\\ \oplus\\ \bigwedge^{r} F'\otimes K_{q}^{p-1}(G'^*)\otimes G''^*\end{matrix} \arrow[r, "\mathrm{incl}^\ddagger"] \arrow[d, "\left[\begin{smallmatrix}\eta_{\pi'\circ\Phi} & 0\\ 0 & \eta_{\pi'\circ\Phi}\end{smallmatrix}\right]"] & \bigwedge^r F\otimes K_q^p(G^*) \arrow[r, "\mathrm{quot}^\ddagger"] \arrow[d, "\eta_\Phi"] & \begin{matrix}\bigwedge^{r} F'\otimes K_{q-1}^p(G^*)\otimes G''^*\\ \oplus\\ \bigwedge^{r-1} F'\otimes K_q^{p}(G^*)\otimes F''\end{matrix} \arrow[d, "\left[\begin{smallmatrix}\eta_{\Phi'} & 0\\ \Phi'' & -\eta_{\Phi'}\end{smallmatrix}\right]"]\\
\begin{matrix}\bigwedge^{r+1} F'\otimes K_{q-1}^p(G^*)\\ \oplus\\ \bigwedge^{r+1} F'\otimes K_{q-1}^{p-1}(G^*)\otimes F''\otimes G''^*\end{matrix} \arrow[r, "\mathrm{incl}^\ddagger"] & \bigwedge^{r+1} F\otimes K_{q-1}^p(G^*) \arrow[r, "\mathrm{quot}^\ddagger"] & \begin{matrix}\bigwedge^{r+1} F'\otimes K_{q-2}^p(G^*)\otimes G''^*\\ \oplus\\ \bigwedge^{r} F'\otimes K_{q-1}^{p}(G^*)\otimes F''\end{matrix}
\end{tikzcd}$$

Fig. 6.4.7. In the proof of Proposition 6.4.b one verifies that this diagram commutes in order to see that (6.4.3) is a map of complexes.

The assertion (6.4.2) is now obvious. Indeed, the hypothesis that Φ'' is an isomorphism ensures that the complex $\operatorname{Tot}\left(\mathbb{L}_{\pi'\circ\Phi}^{N,p}\otimes\mathbb{A}(\Phi'')\right)$ on the left side of (6.4.1) has homology zero and the long exact sequence of homology associated to the short exact sequence of complexes (6.4.1) yields (6.4.2).

(b). Assume that $1\le p$. Notice that (6.4.3) in position j is

$$0\to \begin{matrix}\bigwedge^{N-j} F'\otimes K_j^p(G'^*)\\ \oplus\\ \bigwedge^{N-j} F'\otimes K_j^{p-1}(G'^*)\otimes G''^*\end{matrix} \xrightarrow{\mathrm{incl}^\ddagger} \bigwedge^{N-j} F\otimes K_j^p(G^*)$$
$$\xrightarrow{\mathrm{quot}^\ddagger} \begin{matrix}\bigwedge^{N-j} F'\otimes K_{j-1}^p(G^*)\otimes G''^*\\ \oplus\\ \bigwedge^{N-j-1} F'\otimes K_j^{p}(G^*)\otimes F''\end{matrix}\to 0,$$

which is 6.3.b with r replaced by $N-j$ and q replaced by j. The parameter p is not zero by hypothesis; hence, Observation 6.3.b ensures that each row of (6.4.3) is a short exact sequence. To see that (6.4.3) is a map of complexes, one verifies that Fig. 6.4.7 is a commutative diagram, and this is straightforward. Thus, (6.4.3) is a short exact sequence of complexes. The assertion (6.4.4) is now obvious, as in the proof of (6.4.2). $\square$

7. The definition and elementary properties of the complexes $\mathcal{C}_\Phi^{i,a}$

The maps and modules of $\mathcal{C}_\Phi^{i,a}$ are introduced in 7.2. It is shown in 7.4 that each $\mathcal{C}_\Phi^{i,a}$ is a complex. Examples are given in 7.6 and 7.7. The relationship between the classical generalized Eagon–Northcott complexes $\{\mathcal{C}^i_\Phi\}$ and the complexes $\{\mathcal{C}^{i,a}_\Phi\}$ is examined in 7.8. The duality in the family $\{\mathcal{C}^{i,a}_\Phi\}$ is studied in 7.9. Information about the length of $\mathcal{C}^{i,a}_\Phi$ is recorded in 7.10. The zero-th homology of $\mathcal{C}^{i,a}_\Phi$, for $-1 \le i$, may be found in 7.11. The fact that $I_\mathfrak{g}(\Phi)$ annihilates $\mathrm{H}_0(\mathcal{C}^{i,a}_\Phi)$, for $-1 \le i$, is established in 7.12.

Data 7.1. Let R be a commutative Noetherian ring, F and G be free R-modules of rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, with $\mathfrak{g} \le \mathfrak{f}$, $\Phi : G^* \to F$ be an R-module homomorphism.

Definition 7.2. Adopt Data 7.1. Recall the complexes $\mathbb{K}^{N,p}_\Phi$ and $\mathbb{L}^{N,p}_\Phi$ of Definition 5.2.b and Remark 5.3.c and the homomorphism κ of 4.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Define the maps and modules $(\mathcal{C}^{i,a}_\Phi, d)$ to be

$$0 \to \mathbb{K}^{\mathfrak{f}-\mathfrak{g}-i-1,a}_\Phi[-i-2] \xrightarrow{d} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \xrightarrow{d} \mathbb{L}^{\mathfrak{f}-i-1,\mathfrak{g}-a}_\Phi \to 0, \tag{7.2.1}$$

with $[\mathcal{C}^{i,a}_\Phi]_{i+1} = \bigwedge^{f-g+a-i-1} F$. The differentials

$$[\mathcal{C}^{i,a}_\Phi]_{i+2} \xrightarrow{d_{i+2}} [\mathcal{C}^{i,a}_\Phi]_{i+1} \xrightarrow{d_{i+i}} [\mathcal{C}^{i,a}_\Phi]_i$$

are

$$\begin{aligned}[]
[\mathcal{C}^{i,a}_\Phi]_{i+2} = [\mathbb{K}^{\mathfrak{f}-\mathfrak{g}-i-1,a}_\Phi]_0 = \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F \otimes K^a_0(G^*) = \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F \otimes \bigwedge^a(G^*) \\
\xrightarrow{1\otimes\bigwedge^a \Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F \otimes \bigwedge^a F \xrightarrow{\text{mult}} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F = [\mathcal{C}^{i,a}_\Phi]_{i+1}
\end{aligned}$$

and

$$\begin{aligned}[]
[\mathcal{C}^{i,a}_\Phi]_{i+1} = \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \xrightarrow{1\otimes\mathrm{ev}^*(1)} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \otimes \bigwedge^{\mathfrak{g}+1-a}(G^*) \otimes \bigwedge^{\mathfrak{g}+1-a} G \\
\xrightarrow{1\otimes\bigwedge^{\mathfrak{g}+1-a}\Phi\otimes\kappa} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \otimes \bigwedge^{\mathfrak{g}+1-a} F \otimes L^{\mathfrak{g}-a}_1 G \\
\xrightarrow{\text{mult}\otimes 1} \bigwedge^{\mathfrak{f}-i} F \otimes L^{\mathfrak{g}-a}_1 G = [\mathbb{L}^{\mathfrak{f}-i-1,\mathfrak{g}-a}_\Phi]_i = [\mathcal{C}^{i,a}_\Phi]_i.
\end{aligned}$$

Remark 7.3. We give two other descriptions of the modules in $\mathcal{C}^{i,a}_\Phi$:

$$[\mathcal{C}^{i,a}_\Phi]_j = \begin{cases} \bigwedge^{\mathfrak{f}-j} F \otimes L^{\mathfrak{g}-a}_{i+1-j} G, & \text{if } j \le i, \\ \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F, & \text{if } j = i+1, \text{ and} \\ \bigwedge^{\mathfrak{f}-\mathfrak{g}+1-j} F \otimes K^a_{j-i-2}(G^*), & \text{if } i+2 \le j, \end{cases} \tag{7.3.1}$$

and $\mathcal{C}^{i,a}_\Phi$ looks like

$$\begin{aligned}
\cdots \xrightarrow{\eta_\Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-2} F \otimes K^a_1(G^*) \xrightarrow{\eta_\Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F \otimes K^a_0(G^*) \xrightarrow{\bigwedge^a \Phi} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \\
\xrightarrow{\bigwedge^{\mathfrak{g}+1-a}\Phi} \bigwedge^{\mathfrak{f}-i} F \otimes L^{\mathfrak{g}-a}_1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^{\mathfrak{f}-i+1} F \otimes L^{\mathfrak{g}-a}_2 G \xrightarrow{\mathrm{Kos}_\Phi} \cdots,
\end{aligned} \tag{7.3.2}$$

with $\bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F$ in position $i+1$.

Observation 7.4. *Adopt Data 7.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Then the maps and modules $(\mathcal{C}_\Phi^{i,a}, d)$ of Definition 7.2 form a complex.*

Proof. One obtains $\mathcal{C}_\Phi^{i,a}$ by pasting together two well-known complexes; hence it suffices to show that

$$(\mathcal{C}_\Phi^{i,a})_{i+3} \xrightarrow{d_{i+3}} (\mathcal{C}_\Phi^{i,a})_{i+2} \xrightarrow{d_{i+2}} (\mathcal{C}_\Phi^{i,a})_{i+1} \xrightarrow{d_{i+1}} (\mathcal{C}_\Phi^{i,a})_{i} \xrightarrow{d_{i}} (\mathcal{C}_\Phi^{i,a})_{i-1}$$

is a complex; furthermore, by the duality of Observation 7.9, it suffices to show that

$$(\mathcal{C}_\Phi^{i,a})_{i+2} \xrightarrow{d_{i+2}} (\mathcal{C}_\Phi^{i,a})_{i+1} \xrightarrow{d_{i+1}} (\mathcal{C}_\Phi^{i,a})_{i} \xrightarrow{d_{i}} (\mathcal{C}_\Phi^{i,a})_{i-1}$$

is a complex. Take $f \in \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F = (\mathcal{C}_\Phi^{i,a})_{i+1}$. We compute

$$\begin{aligned}
(d_i \circ d_{i+1})(f) &= d_i\left(\textstyle\sum_\ell f \wedge (\bigwedge^{\mathfrak{g}+1-a}\Phi)(m_\ell^*) \otimes \kappa(m_\ell) \in \bigwedge^{\mathfrak{f}-i} F \otimes L_1^{\mathfrak{g}-a} G = (\mathcal{C}_\Phi^{i,a})_i\right)\\
&= \textstyle\sum_\ell \sum_{\ell'} f \wedge (\bigwedge^{\mathfrak{g}+1-a}\Phi)(m_\ell^*) \wedge \Phi(n_{\ell'}^*) \otimes \kappa(m_\ell)\cdot(1\otimes n_{\ell'})\\
&\quad \in \textstyle\bigwedge^{\mathfrak{f}-i+1} F \otimes L_2^{\mathfrak{g}-a} G = (\mathcal{C}_\Phi^{i,a})_{i-1}\\
&= \textstyle\sum_\ell \sum_{\ell'} \sum_{\ell''} f \wedge (\bigwedge^{\mathfrak{g}+2-a}\Phi)(m_\ell^* \wedge n_{\ell'}^*) \otimes n_{\ell''}^*(m_\ell) \otimes n_{\ell''}\cdot n_{\ell'}\\
&= \textstyle\sum_L \sum_{\ell'} \sum_{\ell''} f \wedge (\bigwedge^{\mathfrak{g}+2-a}\Phi)(M_L^*) \otimes n_{\ell''}^*(n_{\ell'}^*(M)) \otimes n_{\ell''}\cdot n_{\ell'}\\
&= 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{ev}^*(1) &= \textstyle\sum_\ell m_\ell^* \otimes m_\ell && \in \textstyle\bigwedge^{\mathfrak{g}+1-a} G^* \otimes \bigwedge^{\mathfrak{g}+1-a} G,\\
\mathrm{ev}^*(1) &= \textstyle\sum_{\ell'} n_{\ell'}^* \otimes n_{\ell'} = \sum_{\ell''} n_{\ell''}^* \otimes n_{\ell''} && \in G^* \otimes G, \text{ and}\\
\mathrm{ev}^*(1) &= \textstyle\sum_L M_L^* \otimes M_L && \in \textstyle\bigwedge^{\mathfrak{g}+2-a} G^* \otimes \bigwedge^{\mathfrak{g}+2-a} G
\end{aligned}$$

are the canonical elements of (2.12.1). One easily verifies that

$$\begin{aligned}
\textstyle\sum_\ell \sum_{\ell'} m_\ell^* \wedge n_{\ell'}^* \otimes m_\ell \otimes n_{\ell'} &= \sum_L \sum_{\ell'} M_L^* \otimes n_{\ell'}^*(M_L) \otimes n_{\ell'}\\
&\quad \in \textstyle\bigwedge^{\mathfrak{g}+2-a} G^* \otimes \bigwedge^{\mathfrak{g}+1-a} G \otimes G.
\end{aligned}$$

(Merely evaluate both sides at a typical element of $\bigwedge^{\mathfrak{g}+2-a} G \otimes \bigwedge^{\mathfrak{g}+1-a} G^* \otimes G^*$.) It is obvious that

$$\textstyle\sum_{\ell'} \sum_{\ell''} n_{\ell'}^* \wedge n_{\ell''}^* \otimes n_{\ell'} \cdot n_{\ell''} = 0 \in \bigwedge^2 G^* \otimes \mathrm{Sym}_2 G.$$

Take $f \in \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F$ and $\gamma \in \bigwedge^a (G^*) = K_0^a(G^*)$, see 4.5.i. So, $f \otimes \gamma$ is in $(\mathcal{C}_\Phi^{i,a})_{i+2}$. We compute

$$\begin{aligned}(d_{i+1}\circ d_{i+2})(f\otimes\gamma) &= d_{i+1}\left(f\wedge(\textstyle\bigwedge^a\Phi)(\gamma)\in\textstyle\bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1}F=(\mathcal{C}_\Phi^{i,a})_{i+1}\right)\\ &= \textstyle\sum_\ell f\wedge(\bigwedge^a\Phi)(\gamma)\wedge(\bigwedge^{\mathfrak{g}-a+1}\Phi)(m_\ell^*)\otimes\kappa(m_\ell)\\ &\quad\in\textstyle\bigwedge^{\mathfrak{f}-i}F\otimes L_1^{\mathfrak{g}-a}G=(\mathcal{C}_\Phi^{i,a})_i\\ &= \textstyle\sum_\ell f\wedge(\bigwedge^{\mathfrak{g}+1}\Phi)(\gamma\wedge m_\ell^*)\otimes\kappa(m_\ell)\\ &= 0. \qquad\square\end{aligned}$$

Remark 7.5. If a is equal to 0 or $\mathfrak{g}+1$, then it is possible to construct a complex $\mathcal{C}_\Phi^{i,a}$ using the recipe of Remark 7.3. These complexes are

$$\mathcal{C}_\Phi^{i,0}:\quad 0\to\textstyle\bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1}F\xrightarrow{\text{identity map}}\bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1}F\to 0,$$

with the non-zero modules appearing in positions $i+2$ and $i+1$; and

$$\mathcal{C}_\Phi^{i,\mathfrak{g}+1}:\quad 0\to\textstyle\bigwedge^{\mathfrak{f}-i}F\to 0,$$

with the non-zero module appearing in position $i+1$.

Example 7.6. Adopt Data 7.1 with $(\mathfrak{g},\mathfrak{f})=(5,9)$. We record the complexes $\mathcal{C}_\Phi^{i,a}$ of Definition 7.2 and Observation 7.4 which have the form:

$$0\to(\mathcal{C}_\Phi^{i,a})_5\to(\mathcal{C}_\Phi^{i,a})_4\to(\mathcal{C}_\Phi^{i,a})_3\to(\mathcal{C}_\Phi^{i,a})_2\to(\mathcal{C}_\Phi^{i,a})_1\to(\mathcal{C}_\Phi^{i,a})_0\to 0.$$

Of course, these complexes have length $\mathfrak{f}-\mathfrak{g}+1=5$.

$$\mathcal{C}_\Phi^{-1,1}:\ 0\to\textstyle\bigwedge^0F\otimes K_4^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_3^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^2F\otimes K_2^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^3F\otimes K_1^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^4F\otimes K_0^1(G^*)\xrightarrow{\bigwedge^1\Phi}\bigwedge^5F\to 0$$

$$\mathcal{C}_\Phi^{0,1}:\ 0\to\textstyle\bigwedge^0F\otimes K_3^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_2^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^2F\otimes K_1^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^3F\otimes K_0^1(G^*)\xrightarrow{\bigwedge^1\Phi}\bigwedge^4F\xrightarrow{\bigwedge^5\Phi}\bigwedge^9F\otimes L_1^4G\to 0$$

$$\mathcal{C}_\Phi^{1,1}:\ 0\to\textstyle\bigwedge^0F\otimes K_2^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_1^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^2F\otimes K_0^1(G^*)\xrightarrow{\bigwedge^1\Phi}\bigwedge^3F\xrightarrow{\bigwedge^5\Phi}\bigwedge^8F\otimes L_1^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_2^4G\to 0$$

$$\mathcal{C}_\Phi^{2,1}:\ 0\to\textstyle\bigwedge^0F\otimes K_1^1(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_0^1(G^*)\xrightarrow{\bigwedge^1\Phi}\bigwedge^2F\xrightarrow{\bigwedge^5\Phi}\bigwedge^7F\otimes L_1^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_2^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_3^4G\to 0$$

$$\mathcal{C}_\Phi^{3,1}:\ 0\to\textstyle\bigwedge^0F\otimes K_0^1(G^*)\xrightarrow{\bigwedge^1\Phi}\bigwedge^1F\xrightarrow{\bigwedge^5\Phi}\bigwedge^6F\otimes L_1^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^7F\otimes L_2^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_3^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_4^4G\to 0$$

$$\mathcal{C}_\Phi^{4,1}:\ 0\to\textstyle\bigwedge^0F\xrightarrow{\bigwedge^5\Phi}\bigwedge^5F\otimes L_1^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^6F\otimes L_2^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^7F\otimes L_3^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_4^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_5^4G\to 0$$

$$\mathcal{C}_\Phi^{5,1}:\ 0\to\textstyle\bigwedge^4F\otimes L_1^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^5F\otimes L_2^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^6F\otimes L_3^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^7F\otimes L_4^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_5^4G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_6^4G\to 0$$

$$\mathcal{C}_\Phi^{-1,2}:\ 0\to\textstyle\bigwedge^0F\otimes K_4^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_3^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^2F\otimes K_2^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^3F\otimes K_1^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^4F\otimes K_0^2(G^*)\xrightarrow{\bigwedge^2\Phi}\bigwedge^6F\to 0$$

$$\mathcal{C}_\Phi^{0,2}:\ 0\to\textstyle\bigwedge^0F\otimes K_3^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_2^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^2F\otimes K_1^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^3F\otimes K_0^2(G^*)\xrightarrow{\bigwedge^2\Phi}\bigwedge^5F\xrightarrow{\bigwedge^4\Phi}\bigwedge^9F\otimes L_1^3G\to 0$$

$$\mathcal{C}_\Phi^{1,2}:\ 0\to\textstyle\bigwedge^0F\otimes K_2^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_1^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^2F\otimes K_0^2(G^*)\xrightarrow{\bigwedge^2\Phi}\bigwedge^4F\xrightarrow{\bigwedge^4\Phi}\bigwedge^8F\otimes L_1^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_2^3G\to 0$$

$$\mathcal{C}_\Phi^{2,2}:\ 0\to\textstyle\bigwedge^0F\otimes K_1^2(G^*)\xrightarrow{\eta_\Phi}\bigwedge^1F\otimes K_0^2(G^*)\xrightarrow{\bigwedge^2\Phi}\bigwedge^3F\xrightarrow{\bigwedge^4\Phi}\bigwedge^7F\otimes L_1^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_2^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_3^3G\to 0$$

$$\mathcal{C}_\Phi^{3,2}:\ 0\to\textstyle\bigwedge^0F\otimes K_0^2(G^*)\xrightarrow{\bigwedge^2\Phi}\bigwedge^2F\xrightarrow{\bigwedge^4\Phi}\bigwedge^6F\otimes L_1^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^7F\otimes L_2^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_3^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_4^3G\to 0$$

$$\mathcal{C}_\Phi^{4,2}:\ 0\to\textstyle\bigwedge^1F\xrightarrow{\bigwedge^4\Phi}\bigwedge^5F\otimes L_1^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^6F\otimes L_2^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^7F\otimes L_3^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^8F\otimes L_4^3G\xrightarrow{\mathrm{Kos}_\Phi}\bigwedge^9F\otimes L_5^3G\to 0$$

$$\mathcal{C}_\Phi^{-1,3}: \ 0 \to \bigwedge^0 F \otimes K_4^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_3^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_2^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_1^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^4 F \otimes K_0^3(G^*) \xrightarrow{\bigwedge^3 \Phi} \bigwedge^7 F \to 0$$

$$\mathcal{C}_\Phi^{0,3}: \ 0 \to \bigwedge^0 F \otimes K_3^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_2^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_1^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_0^3(G^*) \xrightarrow{\bigwedge^3 \Phi} \bigwedge^6 F \xrightarrow{\bigwedge^3 \Phi} \bigwedge^9 F \otimes L_1^2 G \to 0$$

$$\mathcal{C}_\Phi^{1,3}: \ 0 \to \bigwedge^0 F \otimes K_2^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_1^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_0^3(G^*) \xrightarrow{\bigwedge^3 \Phi} \bigwedge^5 F \xrightarrow{\bigwedge^3 \Phi} \bigwedge^8 F \otimes L_1^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_2^2 G \to 0$$

$$\mathcal{C}_\Phi^{2,3}: \ 0 \to \bigwedge^0 F \otimes K_1^3(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_0^3(G^*) \xrightarrow{\bigwedge^3 \Phi} \bigwedge^4 F \xrightarrow{\bigwedge^3 \Phi} \bigwedge^7 F \otimes L_1^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_2^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_3^2 G \to 0$$

$$\mathcal{C}_\Phi^{3,3}: \ 0 \to \bigwedge^0 F \otimes K_0^3(G^*) \xrightarrow{\bigwedge^3 \Phi} \bigwedge^3 F \xrightarrow{\bigwedge^3 \Phi} \bigwedge^6 F \otimes L_1^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^7 F \otimes L_2^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_3^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_4^2 G \to 0$$

$$\mathcal{C}_\Phi^{4,3}: \ 0 \to \bigwedge^2 F \xrightarrow{\bigwedge^3 \Phi} \bigwedge^5 F \otimes L_1^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^6 F \otimes L_2^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^7 F \otimes L_3^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_4^2 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_5^2 G \to 0$$

$$\mathcal{C}_\Phi^{-1,4}: \ 0 \to \bigwedge^0 F \otimes K_4^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_3^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_2^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_1^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^4 F \otimes K_0^4(G^*) \xrightarrow{\bigwedge^4 \Phi} \bigwedge^8 F \to 0$$

$$\mathcal{C}_\Phi^{0,4}: \ 0 \to \bigwedge^0 F \otimes K_3^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_2^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_1^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_0^4(G^*) \xrightarrow{\bigwedge^4 \Phi} \bigwedge^7 F \xrightarrow{\bigwedge^2 \Phi} \bigwedge^9 F \otimes L_1^1 G \to 0$$

$$\mathcal{C}_\Phi^{1,4}: \ 0 \to \bigwedge^0 F \otimes K_2^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_1^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_0^4(G^*) \xrightarrow{\bigwedge^4 \Phi} \bigwedge^6 F \xrightarrow{\bigwedge^2 \Phi} \bigwedge^8 F \otimes L_1^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_2^1 G \to 0$$

$$\mathcal{C}_\Phi^{2,4}: \ 0 \to \bigwedge^0 F \otimes K_1^4(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_0^4(G^*) \xrightarrow{\bigwedge^4 \Phi} \bigwedge^5 F \xrightarrow{\bigwedge^2 \Phi} \bigwedge^7 F \otimes L_1^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_2^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_3^1 G \to 0$$

$$\mathcal{C}_\Phi^{3,4}: \ 0 \to \bigwedge^0 F \otimes K_0^4(G^*) \xrightarrow{\bigwedge^4 \Phi} \bigwedge^4 F \xrightarrow{\bigwedge^2 \Phi} \bigwedge^6 F \otimes L_1^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^7 F \otimes L_2^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_3^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_4^1 G \to 0$$

$$\mathcal{C}_\Phi^{4,4}: \ 0 \to \bigwedge^3 F \xrightarrow{\bigwedge^2 \Phi} \bigwedge^5 F \otimes L_1^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^6 F \otimes L_2^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^7 F \otimes L_3^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_4^1 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_5^1 G \to 0$$

$$\mathcal{C}_\Phi^{-2,5}: \ 0 \to \bigwedge^1 F \otimes K_5^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_4^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_3^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^4 F \otimes K_2^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^5 F \otimes K_1^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^6 F \otimes K_0^5(G^*) \to 0$$

$$\mathcal{C}_\Phi^{-1,5}: \ 0 \to \bigwedge^0 F \otimes K_4^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_3^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_2^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_1^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^4 F \otimes K_0^5(G^*) \xrightarrow{\bigwedge^5 \Phi} \bigwedge^9 F \to 0$$

$$\mathcal{C}_\Phi^{0,5}: \ 0 \to \bigwedge^0 F \otimes K_3^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_2^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_1^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^3 F \otimes K_0^5(G^*) \xrightarrow{\bigwedge^5 \Phi} \bigwedge^8 F \xrightarrow{\bigwedge^1 \Phi} \bigwedge^9 F \otimes L_1^0 G \to 0$$

$$\mathcal{C}_\Phi^{1,5}: \ 0 \to \bigwedge^0 F \otimes K_2^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_1^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^2 F \otimes K_0^5(G^*) \xrightarrow{\bigwedge^5 \Phi} \bigwedge^7 F \xrightarrow{\bigwedge^1 \Phi} \bigwedge^8 F \otimes L_1^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_2^0 G \to 0$$

$$\mathcal{C}_\Phi^{2,5}: \ 0 \to \bigwedge^0 F \otimes K_1^5(G^*) \xrightarrow{\eta_\Phi} \bigwedge^1 F \otimes K_0^5(G^*) \xrightarrow{\bigwedge^5 \Phi} \bigwedge^6 F \xrightarrow{\bigwedge^1 \Phi} \bigwedge^7 F \otimes L_1^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_2^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_3^0 G \to 0$$

$$\mathcal{C}_\Phi^{3,5}: \ 0 \to \bigwedge^0 F \otimes K_0^5(G^*) \xrightarrow{\bigwedge^5 \Phi} \bigwedge^5 F \xrightarrow{\bigwedge^1 \Phi} \bigwedge^6 F \otimes L_1^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^7 F \otimes L_2^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_3^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_4^0 G \to 0$$

$$\mathcal{C}_\Phi^{4,5}: \ 0 \to \bigwedge^4 F \xrightarrow{\bigwedge^1 \Phi} \bigwedge^5 F \otimes L_1^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^6 F \otimes L_2^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^7 F \otimes L_3^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^8 F \otimes L_4^0 G \xrightarrow{\mathrm{Kos}_\Phi} \bigwedge^9 F \otimes L_5^0 G \to 0$$

Example 7.7. If $\mathfrak{f} = \mathfrak{g} + 1$ and $i = 0$, then $\mathcal{C}_\Phi^{0,a} \otimes \bigwedge^{\mathfrak{f}} F^*$ is

$$\begin{aligned} 0 \to \bigwedge^0 F \otimes \bigwedge^{\mathfrak{f}} F^* \otimes K_0^a(G^*) &\xrightarrow{\bigwedge^a \Phi} \bigwedge^a F \otimes \bigwedge^{\mathfrak{f}} F^* \\ &\xrightarrow{\bigwedge^{\mathfrak{g}+1-a} \Phi} \bigwedge^{\mathfrak{f}} F \otimes \bigwedge^{\mathfrak{f}} F^* \otimes L_1^{\mathfrak{g}-a} G \to 0, \end{aligned}$$

which is naturally isomorphic to

$$0 \to \bigwedge^{\mathfrak{f}} F^* \otimes \bigwedge^a (G^*) \xrightarrow{d_2} \bigwedge^{\mathfrak{f}-a} F^* \xrightarrow{d_1} \bigwedge^{\mathfrak{f}-a} G \to 0, \tag{7.7.1}$$

with $d_2(\omega_{F^*} \otimes \gamma_a) = [(\bigwedge^a \Phi)(\gamma_a)](\omega_{F^*})$ and $d_1(\phi_{\mathfrak{f}-a}) = (\bigwedge^{\mathfrak{f}-a} \Phi^*)(\phi_{\mathfrak{f}-a})$, for $\omega_{F^*} \in \bigwedge^{\mathfrak{f}} F^*$, $\gamma_a \in \bigwedge^a (G^*)$, and $\phi_{\mathfrak{f}-a} \in \bigwedge^{\mathfrak{f}-a} F$. In particular, if $\mathfrak{f} = 4$, $\mathfrak{g} = 3$, $a = 2$, and $\Phi = (\Phi_{i,j})$ is given by a 4×3 matrix, then (7.7.1) is

$$0 \to R^3 \xrightarrow{d_2} R^6 \xrightarrow{d_1} R^3 \to 0$$

with

$$d_1 = \begin{bmatrix} \Delta(1,2;1,2) & \Delta(1,3;1,2) & \Delta(1,4;1,2) & \Delta(3,4;1,2) & \Delta(2,4;1,2) & \Delta(2,3;1,2) \\ \Delta(1,2;1,3) & \Delta(1,3;1,3) & \Delta(1,4;1,3) & \Delta(3,4;1,3) & \Delta(2,4;1,3) & \Delta(2,3;1,3) \\ \Delta(1,2;2,3) & \Delta(1,3;2,3) & \Delta(1,4;2,3) & \Delta(3,4;2,3) & \Delta(2,4;2,3) & \Delta(2,3;2,3) \end{bmatrix}$$

and

$$d_2 = \begin{bmatrix} \Delta(3,4;1,2) & \Delta(3,4;1,3) & \Delta(3,4;2,3) \\ -\Delta(2,4;1,2) & -\Delta(2,4;1,3) & -\Delta(2,4;2,3) \\ \Delta(2,3;1,2) & \Delta(2,3;1,3) & \Delta(2,3;2,3) \\ \Delta(1,2;1,2) & \Delta(1,2;1,3) & \Delta(1,2;2,3) \\ -\Delta(1,3;1,2) & -\Delta(1,3;1,3) & -\Delta(1,3;2,3) \\ \Delta(1,4;1,2) & \Delta(1,4;1,3) & \Delta(1,4;2,3) \end{bmatrix},$$

where

$$\Delta(i,j;k,\ell) = \det \begin{bmatrix} \Phi_{i,k} & \Phi_{i,\ell} \\ \Phi_{j,k} & \Phi_{j,\ell} \end{bmatrix}.$$

Observation 7.8. *Adopt Data 7.1. Recall the complexes $\{\mathcal{C}^i_\Phi\}$ of Definition 3.1 and the complexes $\{\mathcal{C}^{i,a}_\Phi\}$ of Definition 7.2 and Observation 7.4. Then, for each integer i, the complexes*

$$\mathcal{C}^i_\Phi, \quad \mathcal{C}^{i,1}_\Phi, \quad \textit{and} \quad \mathcal{C}^{i-1,\mathfrak{g}}_\Phi \otimes {\textstyle\bigwedge}^{\mathfrak{g}} G$$

are canonically isomorphic.

Proof. Use the formulas (3.1.2) and (7.3.1), Observation 4.5, and the fact that $\bigwedge^{\mathfrak{g}} G \otimes \bigwedge^{\mathfrak{g}} G^*$ is canonically isomorphic to R to see that the modules

$$(\mathcal{C}^i_\Phi)_j, \quad (\mathcal{C}^{i,1}_\Phi)_j, \quad \text{and} \quad (\mathcal{C}^{i-1,\mathfrak{g}}_\Phi)_j \otimes {\textstyle\bigwedge}^{\mathfrak{g}} G$$

are canonically isomorphic for all j. These canonical isomorphisms induce the required canonical isomorphisms of complexes. □

Observation 7.9. *Adopt Data 7.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Then the complexes*

$$\mathcal{C}^{i,a}_\Phi \otimes {\textstyle\bigwedge}^{\mathfrak{f}}(F^*) \quad \textit{and} \quad (\mathcal{C}^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a}_\Phi)^*[-(\mathfrak{f}-\mathfrak{g}+1)]$$

of Definition 7.2 and Observation 7.4 are canonically isomorphic.

Remark. The symbol "$[-(\mathfrak{f}-\mathfrak{g}+1)]$" refers to a shift in homological degree, see 2.4. In particular, for each integer j,

$$((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})^*[-(\mathfrak{f}-\mathfrak{g}+1)])_j = ((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})_{\mathfrak{f}-\mathfrak{g}+1-j})^*.$$

Proof. Use the formula (7.3.1), Observation 4.5, and the fact that $\bigwedge^\ell F^*$ is canonically isomorphic to $\bigwedge^{\mathfrak{f}-\ell} F \otimes \bigwedge^{\mathfrak{f}}(F^*)$, for all integers ℓ to see that the modules

$$(\mathcal{C}_\Phi^{i,a})_j \otimes \bigwedge^{\mathfrak{f}}(F^*) \quad \text{and} \quad ((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})_{\mathfrak{f}-\mathfrak{g}+1-j})^*$$

are canonically isomorphic for all j. Indeed,

$$\begin{aligned}
&((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})_{\mathfrak{f}-\mathfrak{g}+1-j})^* \\
&= \begin{cases} (\bigwedge^{\mathfrak{g}-1+j} F \otimes L_{j-i-1}^{a-1} G)^* & \text{if } i+2 \le j, \\ (\bigwedge^{\mathfrak{g}+i+1-a} F)^*, & \text{if } i+1 = j, \text{ and} \\ (\bigwedge^{j} F \otimes K_{i-j}^{\mathfrak{g}+1-a}(G^*))^*, & \text{if } j \le i \end{cases} \\
&\cong \begin{cases} \bigwedge^{\mathfrak{g}-1+j}(F^*) \otimes K_{j-i-2}^{a}(G^*), & \text{if } i+2 \le j, \\ \bigwedge^{\mathfrak{g}+i+1-a}(F^*), & \text{if } i+1 = j, \text{ and} \\ \bigwedge^{j}(F^*) \otimes L_{i-j+1}^{\mathfrak{g}-a} G, & \text{if } j \le i \end{cases} \qquad \text{by (4.4.2)} \\
&\cong \begin{cases} \bigwedge^{\mathfrak{f}-\mathfrak{g}+1-j} F \otimes K_{j-i-2}^{a}(G^*) \otimes \bigwedge^{\mathfrak{f}}(F^*), & \text{if } i+2 \le j, \\ \bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1+a} F \otimes \bigwedge^{\mathfrak{f}}(F^*), & \text{if } i+1 = j, \text{ and} \\ \bigwedge^{\mathfrak{f}-j} F \otimes L_{i-j+1}^{\mathfrak{g}-a} G \otimes \bigwedge^{\mathfrak{f}}(F^*), & \text{if } j \le i \end{cases} \\
&= (\mathcal{C}_\Phi^{i,a})_j \otimes \bigwedge^{\mathfrak{f}}(F^*).
\end{aligned}$$

The complex $\mathcal{C}_\Phi^{i,a}$ is obtained by patching together two well-known complexes. The duality among the pieces is well understood. We focus on the duality at the patch:

$$\begin{array}{ccccc}
(\mathcal{C}_\Phi^{i,a})_{i+2} \otimes \bigwedge^{\mathfrak{f}}(F^*) & \xrightarrow{d} & (\mathcal{C}_\Phi^{i,a})_{i+1} \otimes \bigwedge^{\mathfrak{f}}(F^*) & \xrightarrow{d} & (\mathcal{C}_\Phi^{i,a})_{i} \otimes \bigwedge^{\mathfrak{f}}(F^*) \\
\downarrow & & \downarrow & & \downarrow \\
((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})_{\mathfrak{f}-\mathfrak{g}-1-i})^* & \xrightarrow{d^*} & ((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})_{\mathfrak{f}-\mathfrak{g}-i})^* & \xrightarrow{d^*} & ((C_\Phi^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a})_{\mathfrak{f}-\mathfrak{g}+1-i})^*,
\end{array}$$

which is the same as

$$\begin{array}{ccccc}
\bigwedge^{\mathfrak{f}-\mathfrak{g}-i-1} F \otimes K_0^{a}(G^*) \otimes \bigwedge^{\mathfrak{f}}(F^*) & \xrightarrow{d} & \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \otimes \bigwedge^{\mathfrak{f}}(F^*) & \xrightarrow{d} & \bigwedge^{\mathfrak{f}-i} F \otimes L_1^{\mathfrak{g}-a} G \otimes \bigwedge^{\mathfrak{f}}(F^*) \\
\downarrow & & \downarrow & & \downarrow \\
(\bigwedge^{\mathfrak{g}+1+i} F \otimes L_1^{a-1} G)^* & \xrightarrow{d^*} & (\bigwedge^{\mathfrak{g}+i+1-a} F)^* & \xrightarrow{d^*} & (\bigwedge^{i} F \otimes K_0^{\mathfrak{g}+1-a}(G^*))^*.
\end{array}$$

The vertical maps are comprised of the canonical isomorphisms $\bigwedge^{\ell} F \otimes \bigwedge^{\mathfrak{f}} F^* \cong \bigwedge^{\mathfrak{f}-\ell} F^*$ and the isomorphisms induced by the perfect pairing of 4.4.2. There is no difficulty in checking that the diagram commutes up to sign. □

Our conventions concerning the length of a complex are given in 2.3.

Observation 7.10. *Adopt Data 7.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Recall the complexes $\{\mathcal{C}_{\Phi}^{i,a}\}$ of Definition 7.2 and Observation 7.4.*

(a) *If $-1 \le i$, then $(\mathcal{C}_{\Phi}^{i,a})_j = 0$ for $j \le -1$ and* $\operatorname{length}(\mathcal{C}_{\Phi}^{i,a}) \le \mathfrak{f}$.
(b) *If $i \le \mathfrak{f}-\mathfrak{g}$ and $\mathfrak{f}-\mathfrak{g}+2 \le j$, then $(\mathcal{C}_{\Phi}^{i,a})_j = 0$.*
(c) *Assume*

$$-1 \le i \le \mathfrak{f}-\mathfrak{g}, \ \textit{or}\ (i,a) = (\mathfrak{f}-\mathfrak{g}+1, 1), \ \textit{or}\ (i,a) = (-2, \mathfrak{g}).$$

Then $(\mathcal{C}_{\Phi}^{i,a})_j = 0$ for $j \le -1$ and for $\mathfrak{f}-\mathfrak{g}+2 \le j$; in particular, $\mathcal{C}_{\Phi}^{i,a}$ is a complex of length at most $\mathfrak{f}-\mathfrak{g}+1$.

Proof. Use (7.3.1). If $j \le -1 \le i$, then the $\bigwedge^{\bullet} F$ contribution to $(\mathcal{C}_{\Phi}^{i,a})_j$ is $\bigwedge^{\mathfrak{f}-j} F = 0$. If $i \le \mathfrak{f}-\mathfrak{g}$ and $\mathfrak{f}-\mathfrak{g}+2 \le j$, then the $\bigwedge^{\bullet} F$ contribution to $(\mathcal{C}_{\Phi}^{i,a})_j$ is $\bigwedge^{\mathfrak{f}-\mathfrak{g}+1-j} F = 0$. The other assertions are checked in a similar manner. □

Observation 7.11. *Adopt Data 7.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Recall the complexes $\{\mathcal{C}_{\Phi}^{i,a}\}$ of Definition 7.2 and Observation 7.4. Then*

$$\mathrm{H}_0(\mathcal{C}_{\Phi}^{i,a}) = \begin{cases} \dfrac{\bigwedge^{\mathfrak{f}-\mathfrak{g}+a} F}{\operatorname{im}(\bigwedge^a \Phi) \wedge \bigwedge^{\mathfrak{f}-\mathfrak{g}} F}, & \textit{if } i=-1, \\ \operatorname{coker}(\bigwedge^{\mathfrak{g}-a+1} \Phi^*), & \textit{if } i=0, \textit{ and} \\ \dfrac{L_{i+1}^{\mathfrak{g}-a} G}{\Phi^*(F) \cdot L_i^{\mathfrak{g}-a} G}, & \textit{if } 1 \le i. \end{cases}$$

Proof. Fix a basis element ω_{F^*} of $\bigwedge^{\mathfrak{f}} F^*$ and, for each j, let $\sigma : \bigwedge^j F \to \bigwedge^{\mathfrak{f}-j} F^*$ be the non-canonical isomorphism which sends f_j to $f_j(\omega_{F^*})$.

For $i = 0$, consider the commutative square

$$\begin{tikzcd} (C_{\Phi}^{0,a})_1 = \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-1} F \arrow[r, "d_1"] \arrow[d, "\sigma", "\cong"'] & (C_{\Phi}^{0,a})_0 = \bigwedge^{\mathfrak{f}} F \otimes L_1^{\mathfrak{g}-a} G \arrow[r] \arrow[d, "\sigma \otimes (4.5.\mathrm{e})", "\cong"'] & \mathrm{H}_0(C_{\Phi}^{0,a}) \arrow[r] & 0 \\ \bigwedge^{\mathfrak{g}+1-a} F^* \arrow[r, "\bigwedge^{\mathfrak{g}+1-a} \Phi^*"] & \bigwedge^{\mathfrak{g}-a+1} G. \end{tikzcd} \tag{7.11.1}$$

For $1 \le i$, consider the commutative square

$$\begin{tikzcd}
(C_\Phi^{i,a})_1 = \bigwedge^{\mathfrak{f}-1} F \otimes L_i^{\mathfrak{g}-a}G \arrow[r, "d_1"] \arrow[d, "\cong"', "\sigma_1\otimes 1"] & (C_\Phi^{i,a})_0 = \bigwedge^{\mathfrak{f}} F \otimes L_{i+1}^{\mathfrak{g}-a}G \arrow[r] \arrow[d, "\cong"', "\sigma_0\otimes 1"] & \mathrm{H}_0(C_\Phi^{i,a}) \arrow[r] & 0 \\
F^* \otimes L_i^{\mathfrak{g}-a}G \arrow[r, "\delta"] & L_{i+1}^{\mathfrak{g}-a}G, & &
\end{tikzcd} \tag{7.11.2}$$

where $\delta(\phi \otimes \sum_\ell A_\ell \otimes B_\ell) = \sum_\ell A_\ell \otimes \Phi^*(\phi) \cdot B_\ell$ for $\phi \in F^*$, $A_\ell \in \bigwedge^{\mathfrak{g}-a} G$, $B_\ell \in \mathrm{Sym}_i G$ and

$$\textstyle\sum_\ell A_\ell \otimes B_\ell \in L_i^{\mathfrak{g}-a}G \subseteq \bigwedge^{\mathfrak{g}-a} G \otimes \mathrm{Sym}_i G.$$

In both diagrams, (7.11.1) and (7.11.2), the top line is exact by 7.10.a. For $i = -1$, the sequence

$$(C_\Phi^{-1,a})_1 = \bigwedge^{\mathfrak{f}-\mathfrak{g}} F \otimes \bigwedge^a G^* \xrightarrow{d_1} (C_\Phi^{-1,a})_0 = \bigwedge^{\mathfrak{f}-\mathfrak{g}+a} F \to \mathrm{H}_0(C_\Phi^{-1,a}) \to 0$$

is exact and d_1 sends $f_{\mathfrak{f}-\mathfrak{g}} \otimes \gamma_a$ to $f_{\mathfrak{f}-\mathfrak{g}} \wedge (\bigwedge^a \Phi)(\gamma_a)$. $\square$

Observation 7.12. *Adopt Data 7.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Recall the complexes $\{\mathcal{C}_\Phi^{i,a}\}$ of Definition 7.2 and Observation 7.4. Then $I_\mathfrak{g}(\Phi)$ annihilates $\mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ for all i with $-1 \le i$.*

Proof. We show that $I_\mathfrak{g}(\Phi) \cdot (\mathcal{C}_\Phi^{i,a})_0 \subseteq \operatorname{im} d_1$ for each relevant i. Consider the arbitrary element $r = [(\bigwedge^\mathfrak{g} \Phi)(\omega_{G^*})](\phi_\mathfrak{g})$ of $I_\mathfrak{g}(\Phi)$, where $\omega_{G^*} \in \bigwedge^\mathfrak{g} G^*$ and $\phi_\mathfrak{g} \in \bigwedge^\mathfrak{g} F^*$.

We prove the assertion for $i = -1$ by showing that $r \cdot \bigwedge^{\mathfrak{f}-\mathfrak{g}+a} F \subseteq \operatorname{im}(\bigwedge^a \Phi) \wedge \bigwedge^{\mathfrak{f}-\mathfrak{g}} F$. Let $f_{\mathfrak{f}-\mathfrak{g}+a} \in \bigwedge^{\mathfrak{f}-\mathfrak{g}+a} F$. Apply Proposition 2.14 in order to write $r \cdot f_{\mathfrak{f}-\mathfrak{g}+a}$, which is equal to

$$f_{\mathfrak{f}-\mathfrak{g}+a} \wedge \phi_\mathfrak{g}\Big((\textstyle\bigwedge^\mathfrak{g}\Phi)(\omega_{G^*})\Big),$$

as a sum of elements of the form

$$f \wedge \phi\Big((\textstyle\bigwedge^\mathfrak{g}\Phi)(\omega_{G^*})\Big),$$

for homogeneous elements $f \in \bigwedge^\bullet F$ and $\phi \in \bigwedge^\bullet F$ with

$$\deg \phi \le \mathfrak{g} + \mathfrak{f} - (\mathfrak{f} - \mathfrak{g} + a) - \mathfrak{g} = \mathfrak{g} - a.$$

Apply Proposition 2.13.d to see that

$$f \wedge \phi\Big((\textstyle\bigwedge^{\mathfrak{g}}\Phi)(\omega_{G^*})\Big) = f \wedge (\textstyle\bigwedge^{\mathfrak{g}-\deg\phi}\Phi)\Big(\Big[(\textstyle\bigwedge^{\deg\phi}\Phi^*)(\phi)\Big](\omega_{G^*})\Big).$$

This completes the proof when $i = -1$ because $a \le \mathfrak{g} - \deg\phi$.

Consider $i = 0$. In light of (7.11.1), it suffices to show that

$$r(\textstyle\bigwedge^{\mathfrak{g}+1-a} G) \subseteq (\textstyle\bigwedge^{\mathfrak{g}+1-a} \Phi^*)(\textstyle\bigwedge^{\mathfrak{g}+1-a} F^*). \tag{7.12.1}$$

Let $g_{\mathfrak{g}+1-a}$ be an element of $\bigwedge^{\mathfrak{g}+1-a} G$. Observe that

$$\big[(\textstyle\bigwedge^{a-1} \Phi)[g_{\mathfrak{g}+1-a}(\omega_{G^*})]\big](\phi_{\mathfrak{g}})$$

is an element of $\bigwedge^{\mathfrak{g}+1-a} F^*$ and $\bigwedge^{\mathfrak{g}+1-a} \Phi^*$ carries this element to

$$\begin{aligned}
&(\textstyle\bigwedge^{\mathfrak{g}+1-a} \Phi^*)\Big(\big[(\textstyle\bigwedge^{a-1} \Phi)[g_{\mathfrak{g}+1-a}(\omega_{G^*})]\big](\phi_{\mathfrak{g}})\Big) \\
&= [g_{\mathfrak{g}+1-a}(\omega_{G^*})]\Big((\textstyle\bigwedge^{\mathfrak{g}} \Phi^*)(\phi_{\mathfrak{g}})\Big) && \text{by 2.13.d} \\
&= g_{\mathfrak{g}+1-a} \wedge \omega_{G^*}\Big((\textstyle\bigwedge^{\mathfrak{g}} \Phi^*)(\phi_{\mathfrak{g}})\Big) && \text{by 2.13.c} \\
&= r \cdot g_{\mathfrak{g}+1-a}.
\end{aligned}$$

The claim (7.12.1) has been established. This completes the proof when $i = 0$.

One further consequence of (7.12.1) is that $rG \subseteq \Phi^*(F^*)$. (Take a to be $\mathfrak{g}$ to obtain this conclusion.)

We prove the assertion for $1 \le i$ by showing that

$$rL_{i+1}^{\mathfrak{g}-a}G \subseteq \delta(F^* \otimes L_i^{\mathfrak{g}-a}G),$$

in the notation of (7.11.2). Observe that the exact sequence (4.4.1) yields

$$L_{i+1}^{\mathfrak{g}-a}G = \kappa(\textstyle\bigwedge^{\mathfrak{g}-a+1} G \otimes \operatorname{Sym}_i G)$$

since $0 < \mathfrak{g} - a + i + 1$. On the other hand,

$$\kappa(\textstyle\bigwedge^{\mathfrak{g}-a+1} G \otimes \operatorname{Sym}_{i-1} G) \subseteq L_i^{\mathfrak{g}-a}G.$$

It follows that

$$\begin{aligned}
rL_{i+1}^{\mathfrak{g}-a}G &\subseteq rG \cdot \Big(\kappa(\textstyle\bigwedge^{\mathfrak{g}-a+1} G \otimes \operatorname{Sym}_{i-1} G)\Big) \subseteq rG \cdot \Big(L_i^{\mathfrak{g}-a}G\Big) \\
&\subseteq \Phi^*(F^*) \cdot L_i^{\mathfrak{g}-a}G = \delta(F^* \otimes L_i^{\mathfrak{g}-a}G),
\end{aligned}$$

where the multiplication "$\cdot$" means multiply into the symmetric algebra factor. The proof is complete for $1 \le i$. $\square$

8. The acyclicity of $\mathcal{C}_\Phi^{i,a}$

Theorem 8.4 is the main result of the paper. It asserts if Φ is sufficiently general, then $\mathcal{C}_\Phi^{i,a}$ is a resolution of $\mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ and $\mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ is a torsion-free $(R/I_\mathfrak{g}(\Phi))$-module, for $-1 \le i$ and $1 \le a \le \mathfrak{g}$. The depth-sensitivity assertion, Corollary 8.5, was promised in (1.0.7), and is in fact our main motivation for writing the paper. Corollary 8.6 records the fact, promised in (1.0.6), that in the generic situation, with $-1 \le i \le \mathfrak{f} - \mathfrak{g}$, then $\mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ is a maximal Cohen–Macaulay module of rank $\binom{\mathfrak{g}-1}{a-1}$ over the determinantal ring $R/I_\mathfrak{g}(\Phi)$. Recall, from 7.11, that

$$\mathrm{H}_0(\mathcal{C}_\Phi^{i,a}) = \begin{cases} \frac{\bigwedge^{\mathfrak{f}-\mathfrak{g}+a} F}{\mathrm{im}(\bigwedge^a \Phi)\wedge\bigwedge^{\mathfrak{f}-\mathfrak{g}} F}, & \text{if } i=-1, \\ \mathrm{coker}(\bigwedge^{\mathfrak{g}-a+1}\Phi^*), & \text{if } i=0, \text{ and} \\ \frac{L_{i+1}^{\mathfrak{g}-a}G}{\Phi^*(F)\cdot L_i^{\mathfrak{g}-a}G}, & \text{if } 1\le i. \end{cases}$$

Theorem 8.4 follows readily from Lemma 8.1 by way of the acyclicity lemma. If $I_1(\Phi) = R$, then it is shown in Lemma 8.1 that

$$\mathcal{C}_\Phi^{i,a} \quad \text{and} \quad \mathcal{C}_{\Phi'}^{i,a} \oplus \mathcal{C}_{\Phi'}^{i,a-1}$$

have isomorphic homology for some smaller R-module homomorphism Φ'.

Lemma 8.1. *Adopt Data 6.1 and Notation 6.2 with $\mathfrak{g} \le \mathfrak{f}$ and Φ'' an isomorphism. Let a and i be integers with $1 \le a \le \mathfrak{g} - 1$. Recall the complex $\mathcal{C}_\Phi^{i,a}$ of Definition 7.2 and Observation 7.4. Then there exists a canonical complex $\mathcal{D}$ of free R-modules and canonical short exact sequences*

$$0 \to \mathrm{Tot}\left(\mathbb{L}_{\pi'\circ\Phi}^{\mathfrak{f}-i-1,\mathfrak{g}-a} \otimes \mathbb{A}(\Phi'')\right) \xrightarrow{\mathrm{incl}^\dagger} \mathcal{C}_\Phi^{i,a} \xrightarrow{\mathrm{quot}^\dagger} \mathcal{D} \to 0 \tag{8.1.1}$$

and

$$0 \to \mathcal{C}_{\Phi'}^{i,a} \oplus \left(\mathcal{C}_{\Phi'}^{i,a-1} \otimes G''^*\right) \xrightarrow{\mathrm{incl}^\ddagger} \mathcal{D} \xrightarrow{\mathrm{quot}^\ddagger} \mathrm{Tot}\left(\mathbb{K}_{\pi'\circ\Phi}^{\mathfrak{f}-\mathfrak{g}-i-2,a}[-i-2] \otimes \mathbb{B}(\Phi'')\right) \to 0. \tag{8.1.2}$$

In particular, there are canonical isomorphisms

$$\mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) \cong \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,a}) \oplus \left(\mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,a-1}) \otimes G''^*\right) \tag{8.1.3}$$

for all integers j.

Remarks 8.1.4.

(a) Lemma 8.1 does apply when $a = 1$ provided one interprets $\mathcal{C}_{\Phi'}^{i,a-1}$ using Remark 7.5. The complex $\mathcal{C}_{\Phi'}^{i,0}$ of Remark 7.5 is split exact; so the ultimate conclusion of Lemma 8.1 when $a = 1$ is

$$\mathrm{H}_j(\mathcal{C}_{\Phi}^{i,1}) \cong \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,1}),$$

for all integers j. In light of Observation 7.8, this conclusion is well-known.

(b) Lemma 8.1 is false when $a = \mathfrak{g}$, even if one interprets $\mathcal{C}_{\Phi'}^{i,\mathfrak{g}}$ using Remark 7.5. The correct statement is that

$$\mathrm{H}_j(\mathcal{C}_{\Phi}^{i,\mathfrak{g}}) \cong \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,\mathfrak{g}-1}) \quad \text{and} \quad \mathrm{H}_j(\mathcal{C}_{\Phi}^{i,\mathfrak{g}}) \not\cong \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,\mathfrak{g}}) \oplus \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,\mathfrak{g}-1}).$$

Indeed,

$$\begin{aligned} \mathrm{H}_j(\mathcal{C}_{\Phi}^{i,\mathfrak{g}}) &\cong \mathrm{H}_j(\mathcal{C}_{\Phi}^{i+1,1}) && \text{by Observation 7.8} \\ &\cong \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i+1,1}) && \text{by (a)} \\ &\cong \mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,\mathfrak{g}-1}) && \text{by Observation 7.8.} \end{aligned}$$

On the other hand, the complex $\mathcal{C}_{\Phi'}^{i,\mathfrak{g}}$ has non-zero homology because the complex is non-zero in exactly one position; see Remark 7.5.

Proof of Lemma 8.1. We know from Definition 7.2 that $\mathcal{C}_{\Phi}^{i,a}$ is the complex

$$0 \to \mathbb{K}_{\Phi}^{\mathfrak{f}-\mathfrak{g}-i-1,a}[-i-2] \xrightarrow{d} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \xrightarrow{d} \mathbb{L}_{\Phi}^{\mathfrak{f}-i-1,\mathfrak{g}-a} \to 0,$$

with $[\mathcal{C}_{\Phi}^{i,a}]_{i+1} = \bigwedge^{f-g+a-i-1} F$. The parameter $\mathfrak{g} - a$ is positive; so, we know from Proposition 6.4.a that

$$0 \to \mathrm{Tot}\left(\mathbb{L}_{\pi'\circ\Phi}^{\mathfrak{f}-i-1,\mathfrak{g}-a} \otimes \mathbb{A}(\Phi'')\right) \xrightarrow{\mathrm{incl}^\dagger} \mathbb{L}_{\Phi}^{\mathfrak{f}-i-1,\mathfrak{g}-a} \xrightarrow{\mathrm{quot}^\dagger} \begin{matrix} \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a} \otimes F'' \\ \oplus \\ \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a-1} \otimes F'' \otimes G'' \end{matrix} \to 0$$

is a short exact sequence of complexes. It follows that

$$\mathrm{Tot}\left(\mathbb{L}_{\pi'\circ\Phi}^{\mathfrak{f}-i-1,\mathfrak{g}-a} \otimes \mathbb{A}(\Phi'')\right) \xrightarrow{\mathrm{incl}^\dagger} \mathcal{C}_{\Phi}^{i,a} \tag{8.1.5}$$

is an injection of complexes. Let $\mathcal{D}$ be the cokernel of (8.1.5). Observe that (8.1.1) is a short exact sequence of complexes and that $\mathcal{D}$ is isomorphic to

$$0 \to \mathbb{K}_{\Phi}^{\mathfrak{f}-\mathfrak{g}-i-1,a}[-i-2] \xrightarrow{d} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \xrightarrow{\mathrm{quot}^\dagger \circ d} \begin{matrix} \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a} \otimes F'' \\ \oplus \\ \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a-1} \otimes F'' \otimes G'' \end{matrix} \to 0.$$

The parameter a is positive; and therefore, we also know from Proposition 6.4.b that

$$0 \to \begin{matrix} \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a} \\ \oplus \\ (\mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a-1} \otimes G''^*) \end{matrix} \xrightarrow{\mathrm{incl}^\ddagger} \mathbb{K}_{\Phi}^{\mathfrak{f}-\mathfrak{g}-i-1,a} \xrightarrow{\mathrm{quot}^\ddagger} \mathrm{Tot}\left(\mathbb{K}_{\pi'\circ\Phi}^{\mathfrak{f}-\mathfrak{g}-i-2,a} \otimes \mathbb{B}(\Phi'')\right) \to 0$$

is a short exact sequence of complexes. It follows that

$$\mathcal{D} \xrightarrow{\mathrm{quot}^\ddagger} \mathrm{Tot}\left(\mathbb{K}_{\pi'\circ\Phi}^{\mathfrak{f}-\mathfrak{g}-i-2,a} \otimes \mathbb{B}(\Phi'')\right)[-i-2]$$

is a surjection of complexes with kernel isomorphic to the complex:

$$\begin{gathered} 0 \to \begin{matrix} \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a}[-i-2] \\ \oplus \\ \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a-1}[-i-2] \otimes G''^* \end{matrix} \xrightarrow{d\circ\mathrm{incl}^\ddagger} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F \\ \xrightarrow{\mathrm{quot}^\dagger\circ d} \begin{matrix} \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a} \otimes F'' \\ \oplus \\ \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a-1} \otimes F'' \otimes G'' \end{matrix} \to 0. \end{gathered} \tag{8.1.6}$$

Thus,

$$0 \to (8.1.6) \xrightarrow{\mathrm{incl}^\ddagger} \mathcal{D} \xrightarrow{\mathrm{quot}^\ddagger} \mathrm{Tot}\left(\mathbb{K}_{\pi'\circ\Phi}^{\mathfrak{f}-\mathfrak{g}-i-2,a} \otimes \mathbb{B}(\Phi'')\right)[-i-2] \to 0 \tag{8.1.7}$$

is a short exact sequence of complexes. Observe that

$$\begin{matrix} \mathcal{C}_{\Phi'}^{i,a} \oplus (\mathcal{C}_{\Phi'}^{i,a-1} \otimes G''^*) \\ \Big\downarrow \\ (8.1.6), \end{matrix} \tag{8.1.8}$$

given by

$$\begin{array}{ccccccc} 0 \to \begin{matrix} \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a}[-i-2] \\ \oplus \\ \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a-1}[-i-2] \otimes G''^* \end{matrix} & \xrightarrow{\left[\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right]} & \begin{matrix} \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F' \\ \oplus \\ \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-2} F' \otimes G''^* \end{matrix} & \xrightarrow{\left[\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right]} & \begin{matrix} \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a-1} \\ \oplus \\ \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a} \otimes G''^* \end{matrix} \to 0 \\ \Big\downarrow \left[\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right] & & \Big\downarrow \left[\begin{smallmatrix} \mathrm{incl} & \mathrm{incl}^\dagger\circ(1\otimes\Phi'') \end{smallmatrix}\right] & & \Big\downarrow \left[\begin{smallmatrix} 0 & 1\otimes\Phi'' \\ 1\otimes\mathrm{Kos}_{\Phi''} & 0 \end{smallmatrix}\right] \\ 0 \to \begin{matrix} \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a}[-i-2] \\ \oplus \\ \mathbb{K}_{\Phi'}^{\mathfrak{f}-\mathfrak{g}-i-1,a-1}[-i-2] \otimes G''^* \end{matrix} & \xrightarrow{d\circ\mathrm{incl}^\ddagger} & \bigwedge^{\mathfrak{f}-\mathfrak{g}+a-i-1} F & \xrightarrow{\mathrm{quot}^\dagger\circ d} & \begin{matrix} \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a} \otimes F'' \\ \oplus \\ \mathbb{L}_{\Phi'}^{\mathfrak{f}-i-2,\mathfrak{g}-a-1} \otimes F'' \otimes G'' \end{matrix} \to 0, \end{array}$$

is an isomorphism of complexes. (The complexes $\mathcal{C}_{\Phi'}^{i,a}$ and $\mathcal{C}_{\Phi'}^{i,a-1}$ have been read from Definition 7.2.) Combine (8.1.7) and (8.1.8) in order to see that (8.1.2) is a short exact sequence of complexes. The isomorphisms of (8.1.3) follow immediately from the short exact sequences (8.1.1) and (8.1.2) because the two total complexes have all homology equal to zero since Φ'' is an isomorphism. □

In Lemma 8.3 we iterate Lemma 8.1. The basic set-up is similar to, but not the same as, the set-up of Data 6.1.

Data 8.2. Let R be a commutative Noetherian ring, F and G be free R-modules of rank $\mathfrak{f}$ and $\mathfrak{g}$, respectively, with $\mathfrak{g} \le \mathfrak{f}$, and $\Phi : G^* \to F$ be an R-module homomorphism. Decompose F and G as

$$F = F' \oplus F'' \quad \text{and} \quad G = G' \oplus G'',$$

where F', F'', G' and G'' are free R-modules and $\operatorname{rank} F'' = \operatorname{rank} G'' = r$ for some integer r with $1 \le r \le \mathfrak{g} - 1$ and let

$$F^* = F'^* \oplus F''^* \quad \text{and} \quad G^* = G'^* \oplus G''^*$$

be the corresponding decompositions of F^* and G^*. Assume that

$$\Phi = \begin{bmatrix} \Phi'_r & 0 \\ 0 & \Phi''_r \end{bmatrix}, \tag{8.2.1}$$

where $\Phi'_r : G'^* \to F'$ is an R-module homomorphism and $\Phi''_r : G''^* \to F''$ is an R-module isomorphism.

Lemma 8.3. *Adopt Data 8.2. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Recall the complex $\mathcal{C}_\Phi^{i,a}$ of Definition 7.2 and Observation 7.4.*

(a) *If $1 \le r \le \mathfrak{g} - 1$, then*

$$\mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) \cong \bigoplus_{\beta=1}^{\mathfrak{g}-r} \mathrm{H}_j(\mathcal{C}_{\Phi'_r}^{i,\beta})^{\binom{r}{a-\beta}}. \tag{8.3.1}$$

(b) *If $r = \mathfrak{g} - 1$, then the following statements hold:*
   - (i) $\mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) \cong \mathrm{H}_j(\mathcal{C}_{\Phi'_{\mathfrak{g}-1}}^{i,1})^{\binom{\mathfrak{g}-1}{a-1}}$,
   - (ii) *if $I_\mathfrak{g}(\Phi) = R$, then the complex $\mathcal{C}_\Phi^{i,a}$ is split exact; and*
   - (iii) *if $I_\mathfrak{g}(\Phi)$ is a proper ideal of grade at least $\mathfrak{f} - \mathfrak{g} + 1$, then $\mathcal{C}_\Phi^{i,a}$ is a resolution of $(R/I_\mathfrak{g}(\Phi))^{\binom{\mathfrak{g}-1}{a-1}}$.*

Proof. (a). We are given

$$\Phi = \begin{bmatrix} \Phi'_r & 0 \\ 0 & \Phi''_r \end{bmatrix},$$

where Φ_r'' is an isomorphism of free modules of rank r. We may rearrange the data so that

$$\Phi = \begin{bmatrix} \Phi_r' & 0 & 0 \\ 0 & \Phi''' & 0 \\ 0 & 0 & \Phi'' \end{bmatrix},$$

where Φ'' is an isomorphism of free modules of rank one and Φ''' is an isomorphism of free modules of rank $r-1$. Let

$$\Phi' = \begin{bmatrix} \Phi_r' & 0 \\ 0 & \Phi''' \end{bmatrix}.$$

Apply Lemma 8.1 and Remarks 8.1.4.a and 8.1.4.b to obtain

$$\mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) \cong \chi(a \le \mathfrak{g}-1)\,\mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,a}) \oplus \chi(2 \le a)\,\mathrm{H}_j(\mathcal{C}_{\Phi'}^{i,a-1}), \tag{8.3.2}$$

where χ is described in 2.7. Notice that (8.3.2) agrees with (8.3.1) when $r = 1$ because

$$\binom{1}{a-\beta} = \begin{cases} 1, & \text{if } \beta = a \text{ and } 1 \le a \le \mathfrak{g}-1, \\ 1, & \text{if } \beta = a-1 \text{ and } 2 \le a \le \mathfrak{g}, \\ 0, & \text{if } \beta \notin \{a-1, a\} \text{ and } 1 \le \beta \le \mathfrak{g}-1. \end{cases}$$

Induction on r applied to (8.3.2) now yields

$$\mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) \cong \chi(a \le \mathfrak{g}-1) \bigoplus_{\beta=1}^{(\mathfrak{g}-1)-(r-1)} \mathrm{H}_j(\mathcal{C}_{\Phi_r'}^{i,\beta})^{\binom{r-1}{a-\beta}} \oplus \chi(2 \le a) \bigoplus_{\beta=1}^{(\mathfrak{g}-1)-(r-1)} \mathrm{H}_j(\mathcal{C}_{\Phi_r'}^{i,\beta})^{\binom{r-1}{a-1-\beta}}.$$

The constraints $\chi(a \le \mathfrak{g}-1)$ and $\chi(2 \le a)$ are redundant. Indeed, if $\mathfrak{g} \le a$, then

$$\beta \le \mathfrak{g} - r \implies r \le \mathfrak{g} - \beta \le a - \beta \implies \tbinom{r-1}{a-\beta} = 0,$$

and if $a \le 1$, then

$$1 \le \beta \implies a - 1 - \beta \le -1 \implies \tbinom{r-1}{a-1-\beta} = 0.$$

Thus,

$$\begin{aligned} \mathrm{H}_j(\mathcal{C}_\Phi^{i,a}) &\cong \bigoplus_{\beta=1}^{\mathfrak{g}-r} \mathrm{H}_j(\mathcal{C}_{\Phi_r'}^{i,\beta})^{\binom{r-1}{a-\beta}} \oplus \bigoplus_{\beta=1}^{\mathfrak{g}-r} \mathrm{H}_j(\mathcal{C}_{\Phi_r'}^{i,\beta})^{\binom{r-1}{a-1-\beta}} \\ &\cong \bigoplus_{\beta=1}^{\mathfrak{g}-r} \mathrm{H}_j(\mathcal{C}_{\Phi_r'}^{i,\beta})^{\binom{r}{a-\beta}}. \end{aligned}$$

(b). Assertion (bi) is a special case of (a). Recall that $\Phi'_{\mathfrak{g}-1} : G'^* \to F'$ is a homomorphism, G' is a free module of rank one, and F' is a free module of rank $\mathfrak{f}-\mathfrak{g}+1$. The complex $\mathcal{C}^{i,1}_{\Phi'_{g-1}}$ is the Koszul complex on a generating set for $I_1(\Phi'_{g-1}) = I_{\mathfrak{g}}(\Phi)$. If $I_{\mathfrak{g}}(\Phi) = R$, then $\mathcal{C}^{i,1}_{\Phi'_{g-1}}$ is split exact. This is (bii). Otherwise, the hypothesis $\mathfrak{f}-\mathfrak{g}+1 \le \operatorname{grade} I_{\mathfrak{g}}(\Phi)$ ensures that $\operatorname{grade} I_{\mathfrak{g}}(\Phi)$ is generated by a regular sequence and therefore $\mathcal{C}^{i,1}_{\Phi'_{g-1}}$ is a resolution of $\mathrm{H}_0(\mathcal{C}^{i,1}_{\Phi'_{g-1}}) = R/I_{\mathfrak{g}}(\Phi)$. This is (biii). $\square$

Theorem 8.4 is the main result of the paper.

Theorem 8.4. *Adopt Data 7.1. Let i and a be integers with $1 \le a \le \mathfrak{g}$. Recall the complex $\mathcal{C}^{i,a}_{\Phi}$ from Definition 7.2 and Observation 7.4. Assume that $I_{\mathfrak{g}}(\Phi)$ is a proper ideal of R with $\mathfrak{f}-\mathfrak{g}+1 \le \operatorname{grade} I_{\mathfrak{g}}(\Phi)$.*

(a) *If $\operatorname{length}(\mathcal{C}^{i,a}_{\Phi}) = \mathfrak{f}-\mathfrak{g}+1$ and $(\mathcal{C}^{i,a}_{\Phi})_j = 0$ for $j \le -1$, then the following statements hold.*
   (i) *The complex $\mathcal{C}^{i,a}_{\Phi}$ is acyclic.*
   (ii) *The R-module $\mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi})$ is perfect of projective dimension $\mathfrak{f}-\mathfrak{g}+1$.*
   (iii) *The $(R/I_{\mathfrak{g}}(\Phi))$-module $\mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi})$ is torsion-free.*
   (iv) *If $\mathfrak{f}-\mathfrak{g}+2 \le \operatorname{grade} I_{\mathfrak{g}-1}(\Phi)$, then the $(R/I_{\mathfrak{g}}(\Phi))$-module $\mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi})$ has rank $\binom{\mathfrak{g}-1}{a-1}$.*

(b) *If $-1 \le i$ and $\mathfrak{f}-t+1 \le \operatorname{grade} I_t(\Phi)$ for all t with $\mathfrak{f}+1-\operatorname{length}(\mathcal{C}^{i,a}_{\Phi}) \le t \le \mathfrak{g}-1$, then*
   (i) *the complex $\mathcal{C}^{i,a}_{\Phi}$ is acyclic,*
   (ii) *$\mathrm{H}_j(\mathcal{C}^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a}_{\Phi}) = \operatorname{Ext}^{\mathfrak{f}-\mathfrak{g}+1-j}_R(\mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi}), R)$ for all j, and*
   (iii) *$\mathrm{H}_j(\mathcal{C}^{\mathfrak{f}-\mathfrak{g}-i-1,\mathfrak{g}+1-a}_{\Phi}) = 0$ for $1 \le j$.*

(c) *If $-1 \le i$, $\mathfrak{f}-\mathfrak{g}+2 \le \operatorname{length}(\mathcal{C}^{i,a}_{\Phi})$, and*

$$\mathfrak{f}-t+2 \le \operatorname{grade} I_t(\Phi), \quad \textit{for all } t \textit{ with } \mathfrak{f}+1-\operatorname{length}(\mathcal{C}^{i,a}_{\Phi}) \le t \le \mathfrak{g}-1, \tag{8.4.1}$$

then $\mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi})$ is a torsion-free $(R/I_{\mathfrak{g}}(\Phi))$-module of rank $\binom{\mathfrak{g}-1}{a-1}$.

Remarks 8.4.2.

(a) Recall from Lemma 8.3.bii that if $I_{\mathfrak{g}}(\Phi) = R$, then $\mathcal{C}^{i,a}_{\Phi}$ is split exact.

(b) Observation 7.10 contains elementary facts about the length of the complexes $\mathcal{C}^{i,a}_{\Phi}$. In particular, the hypotheses of (a) are satisfied when

$$-1 \le i \le \mathfrak{f}-\mathfrak{g}, \quad \text{or} \quad (i,a) = (\mathfrak{f}-\mathfrak{g}+1, 1), \quad \text{or} \quad (i,a) = (-2, \mathfrak{g}).$$

(c) In the generic case (when Φ can be represented by a matrix of variables) all of the grade hypotheses of Theorem 8.4 are automatically satisfied because

$$\mathfrak{f}-t+2 \le (\mathfrak{g}-t+1)(\mathfrak{f}-t+1), \quad \text{whenever } 1 \le t \le \mathfrak{g}-1.$$

(d) The modules $\mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi})$ are recorded in Observation 7.11.

Proof. Throughout this proof, let $\mathcal{C}$ represent $\mathcal{C}_{\Phi}^{i,a}$ and ℓ and represent length$(\mathcal{C})$.

(ai). The complex $\mathcal{C}$ has the form

$$0 \to \mathcal{C}_{\mathfrak{f}-\mathfrak{g}+1} \to \cdots \to \mathcal{C}_1 \to \mathcal{C}_0 \to 0;$$

see Observation 7.10. Let $\mathfrak{p}$ be a prime ideal of R with $\operatorname{grade} \mathfrak{p} \le \mathfrak{f} - \mathfrak{g}$. Observe that

$$\operatorname{grade} \mathfrak{p} \le \mathfrak{f} - \mathfrak{g} < \mathfrak{f} - \mathfrak{g} + 1 \le \operatorname{grade} I_{\mathfrak{g}}(\Phi) \le \operatorname{grade} I_{\mathfrak{g}-1}(\Phi).$$

Thus, there is a $(\mathfrak{g} - 1) \times (\mathfrak{g} - 1)$ minor of Φ which is a unit in $R_{\mathfrak{p}}$ and we may apply Lemma 8.3.biii in order to conclude that $\mathcal{C}_{\mathfrak{p}}$ is acyclic. The acyclicity criterion (see, for example, [2, 1.4.13]) now guarantees that $\mathcal{C}$ is acyclic.

(aii). We know from Observation 7.12 that $I_{\mathfrak{g}}(\Phi) \subseteq \operatorname{ann}(\mathrm{H}_0(\mathcal{C}))$; and therefore,

$$\begin{aligned} \operatorname{pd}_R \mathrm{H}_0(\mathcal{C}) &\le \mathfrak{f} - \mathfrak{g} + 1 && \text{by ai} \\ &\le \operatorname{grade} I_{\mathfrak{g}}(\Phi) && \text{by hypothesis} \\ &\le \operatorname{grade} \operatorname{ann}(\mathrm{H}_0(\mathcal{C})) && \\ &\le \operatorname{pd}_R \mathrm{H}_0(\mathcal{C}) && \text{by (2.9.1).} \end{aligned}$$

The proof of (aii) is complete; see 2.9, if necessary.

Assertion (aiii) is a consequence of Proposition 2.10.1.

(aiv). The R-module $R/I_g(\Phi)$ is perfect of projective dimension $\mathfrak{f} - \mathfrak{g} + 1$. (See, for example, [10, Cor. 5.2], [26, Thm. 1], or [3, 2.7].) Let $\mathfrak{p} \in \operatorname{Spec} R$ be an associated prime of $R/I_{\mathfrak{g}}(\Phi)$. It follows that $\operatorname{grade} \mathfrak{p} = \mathfrak{f} - \mathfrak{g} + 1$ and $I_{\mathfrak{g}-1}(\Phi) \not\subseteq \mathfrak{p}$. Thus, Corollary 8.3.biii may be applied to $\Phi_{\mathfrak{p}}$ in order to conclude that $\mathrm{H}_0(\mathcal{C}_{\Phi}^{i,a})_{\mathfrak{p}} \cong \mathrm{H}_0(R/I_{\mathfrak{g}}(\Phi))_{\mathfrak{p}}^{\binom{\mathfrak{g}-1}{a-1}}$. The proof of (aiv) is complete; see 2.10.

(bi). We induct on ℓ. The base case, $\ell = \mathfrak{f} - \mathfrak{g} + 1$, is established in (ai). We now study the case with $\mathfrak{f} - \mathfrak{g} + 2 \le \ell$. As in the proof of (ai) we apply the acyclicity criterion and prove that $\mathcal{C}_{\mathfrak{p}}$ is acyclic for all prime ideals $\mathfrak{p}$ of R with $\operatorname{grade} \mathfrak{p} < \ell$. Fix such a $\mathfrak{p}$. The hypotheses of (b) with $t = \mathfrak{f} + 1 - \ell$ now ensure that

$$\operatorname{grade} \mathfrak{p} < \ell \le \operatorname{grade} I_{\mathfrak{f}+1-\ell}(\Phi).$$

Thus, there is an $(\mathfrak{f} + 1 - \ell) \times (\mathfrak{f} + 1 - \ell)$ minor of Φ which is a unit in $R_{\mathfrak{p}}$ and, after rearrangement,

$$\Phi_{\mathfrak{p}} = \begin{bmatrix} \Phi'_{\mathfrak{f}+1-\ell} & 0 \\ 0 & \Phi''_{\mathfrak{f}+1-\ell} \end{bmatrix},$$

as described in (8.2.1). Apply Corollary 8.3.a in order to conclude that

$$\begin{aligned}&\mathrm{H}_j(\mathcal{C})_{\mathfrak{p}} \text{ is a direct sum of suitably many copies of modules from the}\\ &\text{set } \{\mathrm{H}_j(\mathcal{C}^{i,\beta}_{\Phi'_{\mathfrak{f}+1-\ell}}) \mid 1 \le \beta \le \mathfrak{g}-(\mathfrak{f}+1-\ell)\}.\end{aligned} \tag{8.4.3}$$

Observe that

$$\mathfrak{f}-t+1 \le \operatorname{grade} I_t(\Phi) \le \operatorname{grade} I_t(\Phi_{\mathfrak{p}}) = \operatorname{grade} I_{t-(\mathfrak{f}+1-\ell)}(\Phi'_{\mathfrak{f}+1-\ell})$$

for all t with $\mathfrak{f}+1-\ell \le t \le \mathfrak{g}-1$. Let

$$\mathfrak{f}' = \mathfrak{f}-(\mathfrak{f}+1-\ell), \quad \mathfrak{g}' = \mathfrak{g}-(\mathfrak{f}+1-\ell), \quad \text{and} \quad t' = t-(\mathfrak{f}+1-\ell).$$

We have shown that

$$\mathfrak{f}'-t'+1 \le \operatorname{grade} I_{t'}(\Phi'_{\mathfrak{f}+1-\ell}) \quad \text{for all } t' \text{ with } 0 \le t' \le \mathfrak{g}'-1.$$

It follows that the hypotheses of (b) apply to $\mathcal{C}^{i,\beta}_{\Phi'_{\mathfrak{f}+1-\ell}}$ for each β with $1 \le \beta \le \mathfrak{g}'$. On the other hand,

$$\operatorname{length}(\mathcal{C}^{i,\beta}_{\Phi'_{\mathfrak{f}+1-\ell}}) \le \operatorname{rank}(\text{the target of } \Phi'_{\mathfrak{f}+1-\ell}) = \mathfrak{f}' = \mathfrak{f}-(\mathfrak{f}+1-\ell) = \ell-1.$$

By induction on ℓ, each $\mathcal{C}^{i,\beta}_{\Phi'_{\mathfrak{f}+1-\ell}}$ is acyclic; and therefore $\mathcal{C}_{\mathfrak{p}}$ is also acyclic by (8.4.3).

(bii) and (biii). Now that we know that $\mathcal{C}$ is a resolution of $\mathrm{H}_0(\mathcal{C})$, assertion (bii) can be read from Observation 7.9; and (biii) follows from (bii) because the grade $\mathfrak{f}-\mathfrak{g}+1$ ideal $I_{\mathfrak{g}}(\Phi)$ is contained in the annihilator of $\mathrm{H}_0(\mathcal{C})$.

(c). We already know from (bi) that

$$\mathcal{C} \text{ is a free resolution of } \mathrm{H}_0(\mathcal{C}) \text{ of length } \ell. \tag{8.4.4}$$

For each integer w, let F_w be the ideal in R generated by:

$$\{x \in R \mid \operatorname{pd}_{R_x} \mathrm{H}_0(\mathcal{C}^{i,a}_{\Phi})_x < w\}.$$

Claim 8.4.5. If $\mathfrak{f}+1-\ell \le t \le \mathfrak{g}-1$, then $I_t(\Phi) \subseteq F_{\mathfrak{f}-t+1}$.

Proof of Claim 8.4.5. If Δ is a $t \times t$ minor of Φ, then one can arrange the data so that

$$\Phi_\Delta = \begin{bmatrix} \Phi'_t & 0 \\ 0 & \Phi''_t \end{bmatrix},$$

where Φ''_t is an isomorphism of free R_Δ-modules of rank t as is described in Data 8.2. Apply Corollary 8.3.a to see that

$$\mathrm{H}_j(\mathcal{C})_\Delta \cong \bigoplus_{\beta=1}^{\mathfrak{g}-t} \mathrm{H}_j(\mathcal{C}^{i,\beta}_{\Phi'_t})^{\binom{t}{a-\beta}}, \quad \text{for all } j. \tag{8.4.6}$$

Apply (8.4.4) to see that each $\mathcal{C}^{i,\beta}_{\Phi'_t}$ which actually appears in (8.4.6) (that is, with $\binom{t}{a-\beta} \neq 0$) is also acyclic. Thus,

$$\operatorname{pd}_{R_\Delta} \mathrm{H}_0(\mathcal{C})_\Delta \le \max\{\operatorname{length}(\mathcal{C}^{i,\beta}_{\Phi'_t}) \mid 1 \le \beta \le \mathfrak{g} - t\} \le \operatorname{rank}(\text{the target of } \Phi'_t) = \mathfrak{f} - t.$$

This completes the proof of Claim 8.4.5.

Combine hypothesis (8.4.1) and Claim 8.4.5 to see that

$$\mathfrak{f} + 1 - \ell \le t \le \mathfrak{g} - 1 \implies (\mathfrak{f} - t + 1) + 1 = \mathfrak{f} - t + 2 \le \operatorname{grade} I_t(\Phi) \le \operatorname{grade} F_{\mathfrak{f}-t+1}.$$

Let $w = \mathfrak{f} - t + 1$. We have shown that

$$\mathfrak{f} - \mathfrak{g} + 2 \le w \le \ell \implies w + 1 \le \operatorname{grade} F_w.$$

Apply Proposition 2.10.1 to conclude that the $R/I_{\mathfrak{g}}(\Phi)$ module $\mathrm{H}_0(\mathcal{C})$ is torsion-free.

We re-use the rank calculation of (aiv). If $\mathfrak{p} \in \operatorname{Ass}_R(R/I_{\mathfrak{g}}(\Phi))$, then

$$\operatorname{grade} \mathfrak{p} = \mathfrak{f} - \mathfrak{g} + 1 < \mathfrak{f} - \mathfrak{g} + 3 \le \operatorname{grade} I_{\mathfrak{g}-1}(\Phi);$$

hence, Corollary 8.3.biii may be applied to $\Phi_{\mathfrak{p}}$, as was done in (aiv), to conclude that $\operatorname{rank} \mathrm{H}_0(\mathcal{C}) = \binom{\mathfrak{g}-1}{a-1}$. □

The next result was promised in (1.0.7), and is our main motivation for writing the paper.

Corollary 8.5. *Adopt Data 7.1. Let and i and a be integers with $1 \le a \le \mathfrak{g}$. Recall the complex $\mathcal{C}^{i,a}_{\Phi}$ from Definition 7.2 and Observation 7.4. If $-1 \le i \le \mathfrak{f} - \mathfrak{g}$, then*

$$\mathrm{H}_j(\mathcal{C}^{i,a}_{\Phi}) = 0 \quad \textit{for} \quad \mathfrak{f} - \mathfrak{g} + 2 - \operatorname{grade} I_{\mathfrak{g}}(\Phi) \le j.$$

Proof. Assertion 8.4.a may be applied in the generic case with the ring equal to the polynomial ring $\mathbb{Z}[\{x_{i,j}\}]$ and the homomorphism given by a matrix of indeterminates. The present assertion is a consequence of Proposition 2.11.2. □

The next result was promised in (1.0.6).

Corollary 8.6. *If k is a field, $\mathfrak{g} \le \mathfrak{f}$ are positive integers, R is the polynomial ring*

$$R = k[\{x_{i,j} \mid 1 \le j \le \mathfrak{g}, 1 \le i \le \mathfrak{f}\}],$$

$\Phi : R^{\mathfrak{g}} \to R^{\mathfrak{f}}$ is the generic map given by

$$\begin{bmatrix} x_{1,1} & \dots & x_{1,\mathfrak{g}} \\ \vdots & & \vdots \\ x_{\mathfrak{f},1} & \dots & x_{\mathfrak{f},\mathfrak{g}} \end{bmatrix},$$

and $M = \mathrm{H}_0(\mathcal{C}_\Phi^{i,a})$ *for some* i *and* a *with* $-1 \le i \le \mathfrak{f} - \mathfrak{g}$ *and* $1 \le a \le \mathfrak{g}$, *then* $\operatorname{ann}_R(M) = I_\mathfrak{g}(\Phi)$ *and* M *is a maximal Cohen–Macaulay* $(R/I_\mathfrak{g}(\Phi))$*-module of rank* $\binom{\mathfrak{g}-1}{a-1}$.

Proof. The ring R is Cohen–Macaulay and the R-module M is perfect (by Theorem 8.4.a); hence M is a Cohen–Macaulay R-module. If $\mathfrak{p}$ is in $\operatorname{Ass}_R M$, then $I_g(\Phi) \subseteq \operatorname{ann}(M) \subseteq \mathfrak{p}$ (see Observation 7.12) and $\mathfrak{f} - \mathfrak{g} + 1 = \operatorname{grade} \mathfrak{p}$ (by [2, 1.4.15]). On the other hand, $I_\mathfrak{g}(\Phi)$ is already a prime ideal of R of grade $\mathfrak{f} - \mathfrak{g} + 1$. It follows that $I_\mathfrak{g}(\Phi) = \mathfrak{p}$, $I_\mathfrak{g}(\Phi) = \operatorname{ann}_R M$, $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(R/I_g(\Phi))$, and $\operatorname{Ass}_R M = \{I_g(\Phi)\}$. $\square$

Remark 8.7. A module over a local Artinian ring has rank only if it is free. Example 7.7 shows that the hypothesis $\mathfrak{f} - \mathfrak{g} + 2 \le \operatorname{grade} I_{\mathfrak{g}-1}(\Phi)$ is needed in Theorem 8.4.aiv. Indeed, if $R = k[x, y]$, for some field k, and

$$\Phi = \begin{bmatrix} x & 0 & 0 \\ y & x & 0 \\ 0 & y & x \\ 0 & 0 & y \end{bmatrix},$$

then $\mathrm{H}_0(\mathcal{C}_\Phi^{0,2})$ is an $R/I_3(\Phi)$-module which does not have any rank. It is easy to see that the length of $\mathrm{H}_0(\mathcal{C}_\Phi^{0,2})$ is twice the length of $R/I_3(\Phi) = k[x, y]/(x, y)^3$; however, $\mathrm{H}_0(\mathcal{C}_\Phi^{0,2})$ has the wrong Betti numbers, as a module over R, to be a free $R/I_3(\Phi)$-module.

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